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Quantum random walk on the dual of SU(n)

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Summary. We study a quantum random walk on $\mathscr{A}(\mathrm{SU}(n))$, the von Neumann algebra of SU(n), obtained by tensoring the basic representation of SU(n). Two classical Markov chains are derived from this quantum random walk, by restriction to commutative subalgebras of $\mathscr{A}(\mathrm{SU}(n))$, and the main result of the paper states that these two Markov chains are related by means of Doob's h-processes.

0. Introduction

In [3], [4], we have studied a quantum generalization of the Bernoulli random walk on $\mathbb{Z}$. This quantum Bernoulli random walk can be interpreted as a quantum Markov chain (in the sense of Accardi et al. [1]) on the dual of the compact group SU(2), which is to be understood as the "non-commutative space" whose algebra of bounded functions is the group von Neumann algebra of SU(2). Natural generalizations of this quantum stochastic process are obtained when one replaces SU(2) by other compact topological groups (see Biane [3], [4]; Parthasarathy [8]). We would like to study probabilistic properties of these quantum stochastic processes, like recurrence, transience, or asymptotic behaviour, but for such problems it is easier to deal with classical (commutative) Markov chains, so it is natural to look for classical Markov chains which can be "imbedded" into these quantum Markov chains.

In [3] it was shown that two interesting classical processes could be obtained from a quantum random walk on the dual of a compact group, by restriction to suitable commutative subalgebras of the group von Neumann algebra. In particular, in the SU(2) case, it was proved in [3], [8], [10] that the restriction to a one-dimensional subgroup algebra is a Bernoulli random walk on $\mathbb{Z}$, while the restriction to the center is a Markov chain on $\mathbb{N}$ obtained from the Bernoulli random walk by conditioning it to "reach ∞ before 0" in the sense of Doob's h-processes. The purpose of this paper is to prove that a similar result holds for SU(n). More precisely, we will construct a quantum Markov chain on the group von Neumann algebra of SU(n) and interpret the restriction of this quantum Markov chain to the algebra of a maximal torus of SU(n) as a random walk on the lattice of integral forms of $\mathrm{SU}(n)$ with respect to this maximal torus. Then, we will see that the restriction of the quantum Markov chain to the center of the von Neumann algebra will be a Markov chain on the same lattice, obtained from the preceding by conditioning it (in Doob's sense) to exit a Weyl chamber by a Martin boundary point at ∞.

This paper is organized as follows:

In the first section, we recall the construction of a quantum random walk on the dual of a compact group, introduced in [3]. In Sect. 2 we recall some useful facts about the Cartan-Weyl theory of root systems and representations of compact Lie groups with a view on the $\mathrm{SU}(n)$ case. In Sect. 3 we introduce the (classical) Markov chains associated to a maximal torus algebra and to the center of the group algebra, and interpret them as Markov chains on the lattice of integral forms of $\mathrm{SU}(n)$ with respect to the maximal torus. In Sect. 4, we state the main result of this paper, that the process on the center can be recovered from the process on the maximal torus by means of a Doob conditionning. The Doob conditioning is then related to the Martin boundary in Sect. 5.

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1. Construction of a quantum Markov chain on the dual of a compact group

We recall here the construction of [3].

Let G be a compact group, and $\mathscr{G}$ be its group von Neumann algebra. This is the von Neuman algebra of operators on $L^2(G)$ generated by the left translation operators λ_g: $\lambda_g(f(h))=f(g^{-1}h)$, $g\in G$ (see Dixmier [6]). Let φ be an irreducible representation of G, of dimension d, and $\mathscr{N}=M_d(\mathbb{C})$.

Let ν and ρ be normal states on $\mathscr{G}$ and $\mathscr{N}$.

Let $\mathscr{W}=\mathscr{G}\otimes\mathscr{N}\otimes\ldots\otimes\mathscr{N}\otimes\ldots$, the tensor product being taken with respect to the product state $\omega=\rho\otimes\ldots\otimes\rho\otimes\ldots$ on $\mathscr{N}\otimes\ldots\otimes\mathscr{N}\otimes\ldots$.

The formula $\tau(\lambda_g)=\lambda_g\otimes\varphi(g)$ extends to a unique morphism of W^* algebras from $\mathscr{G}$ to $\mathscr{G}\otimes\mathscr{N}$.

We define T: $\mathscr{W}\to\mathscr{W}$ by: $T=\tau\otimes s$ where s: $\mathscr{N}^{[1,\infty[}\to\mathscr{N}^{[2,\infty[}$ is the right shift.

We can construct morphisms j_k: $\mathscr{G}\to\mathscr{W}$ by putting $j_k=T^k\circ i$ where i: $\mathscr{G}\to\mathscr{W}$ is the canonical injection.

For $g\in G$, one has: $j_k(\lambda_g)=\lambda_g\otimes\varphi(g)\otimes\ldots\varphi(g)\otimes I\otimes\ldots\otimes I\otimes\ldots$ with k $\varphi(g)$ factors. Let $\mathbf{Q}$ be the completely positive map defined by:

$$\mathbf{Q}=(I\otimes\rho)\circ\tau:\mathscr{G}\to\mathscr{G}.$$

Then one has, for any $\phi_0,\phi_1,\ldots,\phi_n\in\mathscr{G}$:

$$\nu\otimes\omega[j_0(\phi_0)j_1(\phi_1)\ldots j_n(\phi_n)]=\nu(\phi_0\,\mathbf{Q}(\phi_1(\mathbf{Q}\ldots\mathbf{Q}\phi_{n-1}(\mathbf{Q}\phi_n))\ldots)) \tag{1.1}$$

this shows that under the state $\nu\otimes\omega$, the morphisms (j_k) form a quantum Markov chain with generator $\mathbf{Q}$ and initial law ν, in the sense of Accardi et al. [1].

In the rest of this paper, we will use this construction with $G=\mathrm{SU}(n)$, and φ the n-dimensional basic representation of SU(n). Furthermore we will take ρ to be the tracial state $\rho(X)=n^{-1}\operatorname{tr}(X)$.

2. Some facts about compact Lie groups

This section contains some classical facts about compact Lie groups and their representations, specialized to the case of SU(n), which can be found, for example in Bröcker, tom Dieck [5].

Let $\mathbf{T}\subset\mathrm{SU}(n)$ be the subgroup of diagonal matrices. This is a maximal torus in SU(n), of dimension $n-1$. The Lie algebra $L\mathbf{T}$ of $\mathbf{T}$ is composed of purely imaginary diagonal matrices of zero trace.

Let $\mathbf{P}\subset(L\mathbf{T})^*$ be the lattice of integral forms.

$\mathbf{P}$ is an $n-1$ dimensional lattice generated by the element of $(L\mathbf{T})^*$ $e_1, e_2, \ldots, e_n$ where

$$e_i\left(\begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{pmatrix}\right)=\frac{1}{2i\pi}u_j.$$

These elements satisfy the relation: $e_1+\ldots+e_n=0$.

$L\mathbf{T}$ and its dual $(L\mathbf{T})^*$ are endowed with the $(2\pi)^{-2}\times$Killing form, so that $\langle e_i, e_j\rangle=\delta_{ij}-\frac{1}{n}$.

The roots of SU(n) with respect to $\mathbf{T}$ are e_i-e_j, $1\leqq i\leqq n$, $1\leqq j\leqq n$, $i\neq j$.

The Weyl group of SU(n), W is generated by the reflections with respect to the roots. It is isomorphic to the group of permutations of $\{e_1, \ldots, e_n\}$, the reflection with respect to e_i-e_j acting by transposition of e_i and e_j.

For such a reflection of the Weyl group, the hyperplane $H_{i,j}$ generated by the e_k with $k\neq i, j$ is fixed. The complement in $(L\mathbf{T})^*$ of the union of these hyperplanes is composed of $n!$ connected components which are permuted by the Weyl group, and which are called the Weyl chambers; the hyperplanes bounding a chamber are the walls of this chamber.

We shall choose the lexicographic order on $(L\mathbf{T})^*$ with respect to the base $e_1, \ldots, e_{n-1}$, so that the set of positive roots will be e_i-e_j, $1\leqq i<j\leqq n$.

Let $\mathbf{C}_+$ be the Weyl chamber containing the root e_1-e_n, and $\mathbf{C}_+$ its closure. Let $\mathbf{P}_{++}=\mathbf{P}\cap\mathbf{C}_{++}$ and $\mathbf{P}_+=\mathbf{P}\cap\mathbf{C}_+$ then

(2.1) $\mathbf{P}_{++}=\{m_1 e_1+\ldots+m_{n-1}e_{n-1}, m_i\in\mathbb{N}, m_1>m_2>\ldots>m_{n-1}>0\}$ while

$$\mathbf{P}_+=\{m_1 e_1+\ldots+m_{n-1}e_{n-1}, m_i\in\mathbb{N}, m_1\geqq m_2\geqq\ldots\geqq m_{n-1}\geqq 0\}.$$

We know from the Cartan-Weyl theory, that the equivalence class of an irreducible representation of SU(n) is determined by its highest weight which is an element of $\mathbf{P}_+$. In particular the basic n-dimensional representation of SU(n) has weights $e_1, \ldots, e_n$ and highest weight e_1.

Let ϕ be the half sum of positive roots then $\phi=(n-1)e_1+(n-2)e_2\ldots+e_{n-1}$. The following two lemmas will be useful in Sect. 5:

Lemma 2.1 $\mathbf{P}_{++}=\mathbf{P}_{+}+\phi$.

Proof. This follows from (2.1).

Lemma 2.2 i) *If* $x\in\mathbf{P}_{++}$, *then* $x+e_j\in\mathbf{P}_{+}$ *for each* $j=1, \ldots, n$.
ii) *If* $x\in\mathbf{P}_{++}$ *and* $x+e_j\in\mathbf{P}_{+}\backslash\mathbf{P}_{++}$ *then* $j\geqq 2$ *and* $x+e_j\in H_{j,j-1}$.

Proof. i) If $j=1, \ldots, n-1$ this is clear from (2.1).
If $j=n$, recall that $e_n=-(e_1+\ldots+e_{n-1})$ and use again (2.1).
ii) by (2.1) $x=m_1 e_1+\ldots+m_{n-1} e_{n-1}$ with $m_1>m_2>\ldots>m_{n-1}>0$
if $j=2, \ldots, n-1$ then, since $x+e_j\in\mathbf{P}_{+}\backslash\mathbf{P}_{++}$ one has $m_j+1=m_{j-1}$ thus $x+e_j\in H_{j,j-1}$
if $j=n$, m_{n-1} must be one, and thus $x+e_n\in H_{n,n-1}$.

If $x\in\mathbf{P}$ one can define a function $e(x)$ on $\mathbf{T}$ by the formula: $e(x)(\exp X)=\exp(2i\pi x(X))$ for $X\in L\mathbf{T}$. The map $x\to e(x)$ is a group isomorphism between $\mathbf{P}$ and the character group of $\mathbf{T}$.

Let ψ be the irreducible representation of G of highest weight $x\in\mathbf{P}_{+}$, the restriction of its character χ_ψ to $\mathbf{T}$ is given by Weyl's formula:

$$\chi_\psi=\frac{\sum_{\in W}\det w\, e(w(x+\phi))}{\sum_{w\in W}\det w\, e(w(\phi))}. \tag{2.2}$$

One has:

$$\sum_{w\in W}\det w\, e(w(\phi))=\prod_{\alpha>0}(e(\alpha)-e(-\alpha)) \tag{2.3}$$

where the product is taken over the positive roots.

In the sequel we shall use the notation $\sum_{w\in W}\det w\, e(w(\phi))(\theta)=\zeta(\theta)$.

We denote by κ the normalized character of the basic representation of $\mathrm{SU}(n)$. For any $\theta\in\mathbf{T}$:

$$\kappa(\Theta)=\frac{1}{n}(e(e_1)+\ldots+e(e_n))(\theta). \tag{2.4}$$ One has, for $g\in G$, $\mathbf{Q}(\lambda_g)=\kappa(g)\,\lambda_g$.

3. Two classical Markov chains

In this section, we will consider classical Markov chains obtained by restricting the morphisms j_k to commutative subalgebras of $\mathscr{G}$.

Let $\mathscr{T}$ be the commutative subalgebra of $\mathscr{G}$ generated by $\{\lambda_\theta, \theta\in\mathbf{T}\}$.

By decomposing the Haar measure of G, we find an isomorphism $L^2(G)\approx L^2(\mathbf{T})\otimes L^2(\mathbf{T}\backslash G)$ such that λ_θ acts by I on the second factor. It follows that $\mathscr{T}$ is isomorphic to the group von Neumann algebra of $\mathbf{T}$, which is itself isomorphic to the algebra of bounded functions on the dual group of $\mathbf{T}$. Using this and the isomorphism between $\mathbf{P}$ and $\hat{\mathbf{T}}$, we have:

Lemma 3.1 *There is an isomorphism of* W^*-*algebras* $\xi\colon\mathscr{T}\to L^\infty(\mathbf{P})$ *such that* $\xi(\lambda)$ *is the function* $(x\to e(x)(\theta))$.

In this isomorphism, the element of $\mathscr{T}$ defined by $\int \overline{e(y)(\theta)}\,\lambda_\theta\, d\theta$ corresponds to the indicator function of y.

Using this isomorphism we identify $\mathscr{T}$ with $L^\infty(\mathbf{P})$. This allows us to identify the law of the restriction of (j_k) to $\mathscr{T}$:

Theorem 3.1 *The restriction of (j_k) to $\mathscr{T}$ defines a random walk on $\mathbf{P}$ such that the increments have law $n^{-1}(\delta_{e_1}+\ldots+\delta_{e_n})$.*

Proof. Let $\mathscr{M}\subset M_n(\mathbb{C})$ be the algebra of diagonal matrices. Then, the morphism τ of Sect. 1 sends $\mathscr{T}$ into $\mathscr{T}\otimes\mathscr{M}$, and by recurrence, each j_k sends $\mathscr{T}$ into the subalgebra $\mathscr{T}\otimes\mathscr{M}\otimes\ldots\mathscr{M}\otimes\ldots$ of $\mathscr{W}$. Since $\mathscr{T}\otimes\mathscr{M}\otimes\ldots\mathscr{M}\otimes\ldots$ is a commutative algebra, and $\mathbf{Q}$ sends $\mathscr{T}$ into $\mathscr{T}$, we see that the j_k define a commutative process on $\mathscr{T}$, which is, by formula (1.1) a Markov chain with generator $\mathbf{Q}_{|\mathscr{T}}$.

For $\Theta\in\mathbf{T}$, one has $\mathbf{Q}(\lambda_\theta)=\lambda_\theta\frac{1}{n}\operatorname{tr}(\varphi(\theta))$. Using the identification of Lemma 3.1, we see that $\mathbf{Q}(e(\cdot)(\theta))=\frac{1}{n}\sum_{j=1}^{j=n} e(\cdot+e_j)(\theta)$. Since linear combinations of the functions $e(\cdot)(\theta)$ are dense in $L^\infty(\mathbf{P})$, we see that for any $f\in L^\infty(\mathbf{P})$, $\mathbf{Q}(f(\cdot))$ $=\frac{1}{n}\sum_{j=1}^{j=n} f(\cdot+e_j)$, and this proves Theorem 3.1.

We denote by $p_k(x,y)$ for $k\in\mathbb{N}$, $x,y\in\mathbf{P}_{++}$, the transition probabilities of the random walk of Theorem 3.1.

Lemma 3.2 *If $w\in W$, $k\in\mathbb{N}$, $p_k(w(x),w(y))=p_k(x,y)$.*

Proof. By Theorem 3.1, the law of the increments of the random walk is invariant by the Weyl group, and the lemma follows.

Proposition 3.1 $p_k(x,y)=\int_{\mathbf{T}} e(x)(\theta)\,\overline{e(y)(\theta)}\,\kappa^k(\theta)\,d\theta.$

Proof. Denoting the indicator function of $x\in\mathbf{P}$ by 1_x, one has:

$\mathbf{Q}^k(1_y)\,1_x=p_k(x,y)\,1_x$, or

$\mathbf{Q}^k(\int\overline{e(y)(\theta)}\,\lambda_\theta\,d\theta)\circ e(x)=p_k(x,y)\quad e(x)$ ($\circ$ is the product in $\mathscr{G}$), but

$\mathbf{Q}^k(\lambda_\theta)=\kappa^k(\theta)\,\lambda_\theta$ so that $\int_T e(x)(\theta)\,\overline{e(y)(\theta)}\,\kappa^k(\theta)\,d\theta\, e(x)$

$=p_k(x,y)\,e(x)$ which proves the formula.

Let $\mathscr{Z}(\mathscr{G})$ be the center of $\mathscr{G}$.

For each irreducible representation ψ of SU(n) with character χ_ψ and dimension d_ψ, $\prod_\psi=d_\psi\int_G \overline{\chi_\psi(g)}\,\lambda_g\,dg$ is an element of $\mathscr{Z}(\mathscr{G})$, and these elements are the minimal projections of this center (cf. Dixmier [6]). In the sequel, we shall identify an equivalence class of irreducible representations of $SU(n)$ with the element $y\in\mathbf{P}_{++}$ such that $y-\phi$ is the highest weight of the representation, and we will denote by χ_y and d_y, respectively its character and its dimension. With this identification, we see that $\mathscr{Z}(\mathscr{G})$ is isomorphic as a W^*-algebra to $l^\infty(\mathbf{P}_{++})$.

Theorem 3.2 The restriction of (j_k) to $\mathscr{Z}(\mathscr{G})$ is a Markov chain with generator $\mathbf{Q}_{|\mathscr{Z}(\mathscr{G})}$.

Proof. It is enough to prove two things:
i) for any $\phi_0, \phi_1 \in \mathscr{Z}(\mathscr{G})$, and $k, l \in \mathbb{N}$, $j_k(\phi_0)$ and $j_l(\phi_1)$ commute
ii) for any $\phi_0 \in \mathscr{Z}(\mathscr{G})$, $\mathbf{Q}(\phi_0) \in \mathscr{Z}(\mathscr{G})$ because then we can conclude with formula (2.1).

i) Suppose that $k \leq l$, then

$$\begin{aligned} j_k(\phi_0)\, j_l(\phi_1) &= T^k(i(\phi_0))\, T^k(T^{l-k}(i(\phi_1))) \\ &= T^k(i(\phi_0)\, T^{l-k}(i(\phi_1))) \\ &= T^k(T^{l-k}(i(\phi_1))\, i(\phi_0)) \qquad \text{because } i(\phi_0) \text{ is in the center of } \mathscr{W} \\ &= T^l(i(\phi_1))\, T^k(i(\phi_0)) = j_l(\phi_1)\, j_k(\phi_0) \end{aligned}$$

ii) $\mathbf{Q}(\Pi_\psi) = \mathbf{Q}(d_\psi \int_G \overline{\chi_\psi(g)}\, \lambda_g\, dg) = d_\psi \int_G \overline{\chi_\psi(g)}\, \kappa(g)\, \lambda_g\, dg$. Since χ_ψ and κ are central functions on G, their product is also central, so that $\int_G \overline{\chi_\psi(g)}\, \kappa(g)\, \lambda_g\, dg \in \mathscr{Z}(\mathscr{G})$ and $\mathbf{Q}(\Pi_\psi) \in \mathscr{Z}(\mathscr{G})$. Since Π_ψ generate $\mathscr{Z}(\mathscr{G})$, the result follows.

We denote by $q_k(x, y)$ for $k \in \mathbb{N}$, $x, y \in \mathbf{P}_{++}$ the transition probabilities of the Markov chain of Theorem 3.2.

Proposition 3.2. $q_k(x, y) = \frac{d_y}{d_x} \int_G \chi_x(g)\, \overline{\chi_y(g)}\, \kappa^k(g)\, dg.$

Proof. One has, for $x, y \in \mathbf{P}_{++}$:

$$\mathbf{Q}^k(\Pi_y) = d_y \int_G \overline{\chi_y(g)}\, \kappa^k(g)\, \lambda_g\, dg.$$

The central function $\overline{\chi_y}\, \kappa^k$ decomposes into a linear combination $\sum_z \alpha_z \overline{\chi_z}$, where, because of the orthogonality relations of the characters,

$$\alpha_z = \int_G \overline{\chi_y(g)}\, \kappa^k(g)\, \chi_z(g)\, dg.$$

Thus, we have: $\mathbf{Q}^k(\Pi_y) = \sum_x q_k(x, y)\, \Pi_x$ with

$$q_k(x, y) = \frac{d_y}{d_x} \int_G \chi_x(g)\, \overline{\chi_y(g)}\, \kappa^k(g)\, dg.$$

4. A relation between the two Markov chains

In the preceding section, we have derived from the quantum Markov chain (j_k) two classical Markov chains on $\mathbf{P}$ and $\mathbf{P}_{++}$ respectively. We give now the main result of this paper (Theorem 4.1) which gives a relation between these two processes in terms of h-processes. Let us call X and Y Markov chains with transition probabilities as in Propositions 3.1 and 3.2.

We start by killing the process X at the boundary of the Weyl chamber $\mathbf{C}_+$. We obtain a Markov chain on $\mathbf{P}_{++}$ whose generator is given by the submarkovian kernel $p_1(x, y)$ on $\mathbf{P}_{++}$ (of course this kernel can be made markovian by adjoining a cemetary point ∂ in the usual way).

We call $p_k^\circ(x, y)$ the transition probabilities of this Markov chain.

Lemma 4.1 *For each* $k \in \mathbb{N}^*$, $x, y \in \mathbf{P}_{++}$

$$p_k^\circ(x, y) = \sum_{w \in W} \det(w)\, p_k(x, w(y)) = \frac{1}{|W|} \sum_{v \in W} \sum_{w \in W} \det(vw)\, p_k(v(x), w(y)).$$

Proof. This can be proved by counting the paths of k steps from x to y which do not cross the boundary, with the help of the reflection principle for random walks (see Feller [7], Vol. 1), but we give here a direct proof of the result by induction.

First remark that the second equality is a consequence of Lemma 3.2.

By Lemma 2.2 we see that $p_1(x, y) = 0$ if $x \in \mathbf{P}_{++}$ and $y \notin \mathbf{P}_+$, so that in the sum $\sum_{w \in W} \det(w)\, p_1(x, w(y))$ the only term which contributes is $w = \mathrm{Id}$, and the equality $p_k^\circ(x, y) = \sum_{w \in W} \det(w)\, p_k(x, w(y))$ is true for $k = 1$.

Using Chapman-Kolmogorov equation we get

$$\begin{aligned} p_{k+1}^\circ(x, y) &= \sum_{z \in \mathbf{P}_{++}} p_k^\circ(x, z)\, p_1^\circ(z, y) = \sum_{z \in \mathbf{P}_{++}} \sum_{w \in W} \det(w)\, p_k(x, w(z))\, p_1^\circ(z, y) \\ &= \sum_{z \in \mathbf{P}_{++}} \sum_{w \in W} \det(w)\, p_k(x, w(z)) \sum_{v \in W} \det(v)\, p_1(z, v(y)) \\ &= \sum_{z \in \mathbf{P}_{++}} \sum_{w \in W} \det(w)\, p_k(w(x), z) \sum_{v \in W} \det(v)\, p_1(z, v(y)). \end{aligned}$$

By Lemma 3.2 again, one has for any $u \in W$

$$\det(w)\, p_k(w(x), z) \det(v)\, p_1(z, v(y)) = \det(wu)\, p_k(uw(x), u(z)) \det(vu)\, p_1(u(z), uv(y))$$

so that

$$\begin{aligned} p_{k+1}^\circ(x, y) &= \frac{1}{|W|} \sum_{z \in \mathbf{P}_{++}} \sum_{w \in W} \sum_{u \in W} \sum_{v \in W} \det(wu)\, p_k(uw(x), u(z)) \det(vu)\, p_1(u(z), uv(y)) \\ &= \frac{1}{|W|} \sum_{z \in \mathbf{P}_{++}} \sum_{w \in W} \sum_{u \in W} \sum_{v \in W} \det(w)\, p_k(w(x), u(z)) \det(v)\, p_1(u(z), v(y)) \\ &= \frac{1}{|W|} \sum_{u \in W} \sum_{z \in u(\mathbf{P}_{++})} \sum_{w \in W} \sum_{v \in W} \det(w)\, p_k(w(x), z) \det(v)\, p_1(z, v(y)) \end{aligned}$$

if $z \in \mathbf{P} \setminus \bigcup_{u \in W} u(\mathbf{P}_{++})$ then z belongs to one of the H_{ij} so that it is fixed by a reflection v_0 of W, consequently,

$$\sum_{v \in W} \det(v)\, p_1(z, v(y)) = \sum_{v \in W} \det(v)\, p_1(v(z), y) = \sum_{v \in W} \det(v)\, p_1(vv_0(z), y)$$

$$\sum_{v \in W} \det(vv_0)\, p_1(v(z), y) = -\sum_{v \in W} \det(v)\, p_1(v(z), y) = 0.$$

We see that

$$p^\circ_{k+1}(x, y)=\frac{1}{|W|}\sum_{z\in\mathbf{P}}\sum_{w\in W}\sum_{v\in W}\det(w)\,p_k(w(x), z)\det(v)\,p_1(z, v(y))$$

$$=\frac{1}{|W|}\sum_{w\in W}\sum_{v\in W}\det(wv)\,p_{k+1}(w(x), v(y))$$

by Chapman-Kolmogorov equation for p. Lemma 4.1 follows by induction.

Lemma 4.2 $p^\circ_k(x, y)=\int_T \chi_x(\theta)\,\overline{e(y)(\theta)}\,\kappa^k(\theta)\,\zeta(\theta)\,d\theta$

$$=\frac{1}{|W|}\int_T \chi_x(\theta)\,\overline{\chi_y(\theta)}\,\kappa^k(\theta)\,|\zeta(\theta)|^2\,d\theta.$$

Proof. Applying Proposition 3.1 and Lemma 4.1, we obtain

$$p^\circ_k(x, y)=\frac{1}{|W|}\int_T\sum_{v\in W}\sum_{w\in W}\det(vw)\,e(v(x))(\theta)\,\overline{e(w(y))(\theta)}\,\kappa^k(\theta)\,d\theta$$

$$=\frac{1}{|W|}\int_T\Big(\sum_{v\in W}\det(v)\,e(v(x))(\theta)\Big)\Big(\sum_{w\in W}\det(w)\,\overline{e(w(y))(\theta)}\Big)\kappa^k(\theta)\,d\theta$$

$$=\frac{1}{|W|}\int_T \chi_x(\theta)\,\overline{\chi_y(\theta)}\,\kappa^k(\theta)\,|\zeta(\theta)|^2\,d\theta \qquad \text{by Weyl's formula (2.2).}$$

We can now state the main result of this section:

Theorem 4.1 i) *$x\to d_x$ is a p°-harmonic function on $\mathbf{P}_{++}$.*

ii) *$q_k(x, y)=\frac{d_y}{d_x}\,p^\circ_k(x, y)$, so that Y is the h-process of X killed at the boundary of $\mathbf{C}_{++}$, with respect to the harmonic function d.*

Proof. Since $\mathbf{Q}(\lambda_e)=\lambda_e$, q_1 is a Markovian kernel, so that it is enough to prove formula ii). By Lemma 3.2, $q_k(x, y)=\frac{d_y}{d_x}\int_G \chi_x(g)\,\overline{\chi_y(g)}\,\kappa^k(g)\,dg$, where χ_x, χ_y, and κ are central functions, so that, by Weyl's integral formula (see [5]), one has $q_k(x, y)=\frac{1}{|W|}\int_{\mathbf{T}} \chi_x(\theta)\,\overline{\chi_y(\theta)}\,\kappa^k(\theta)\,|\zeta(\theta)|^2\,d\theta$ which is $p^\circ_k(x, y)$, by Lemma 4.2.

We will see in Sect. 5 that the harmonic function d corresponds to a Martin boundary point of $\mathbf{P}_{++}$. The next proposition states that this Martin boundary point is minimal.

Proposition 4.1 *The function d is a minimal p°-harmonic function.*

Proof. Let $n\in\mathbb{N}$ and $\mathfrak{s}$ be a permutation of $\{1, \ldots, n\}$, $\mathfrak{s}$ acts on the algebra $\mathcal{N}\otimes\mathcal{N}\otimes\ldots\otimes\mathcal{N}\ldots$ by the formula:

$$\mathfrak{s}(\phi\otimes x_1\otimes\ldots\otimes x_n\otimes I\otimes\ldots\otimes I\otimes\ldots)=\phi\otimes x_{\mathfrak{s}(1)}\otimes\ldots\otimes x_{\mathfrak{s}(n)}\otimes I\otimes\ldots\otimes I\otimes\ldots.$$

By an easy adaptation of the proof of Hewitt and Savage's 0 or 1 law (see Feller [1]), it is possible to prove that any self-adjoint projection invariant by all the $\mathfrak{s}$ has expectation under $\rho\otimes\ldots\otimes\rho\ldots$ 0 or 1.

Because of Theorem 1 i), Proposition 4.1 is equivalent to the fact that bounded q-harmonic functions are constant. By Revuz [9] Chap. 7, this is equivalent to the fact that the algebra of invariant events of the Markov chain Y is reduced to sets of probability 0 or 1. Using the construction of Sect. 1, we see that this follows from the 0 or 1 law above, since indicator functions of invariant events for Y are permutation invariant self-adjoint projections.

5. Asymptotics of the potential kernel

The purpose of this section is to provide an asymptotic analysis of the potential kernel of the killed process, in order to identify the p°-harmonic function d as the Martin function corresponding to the Martin boundary point of $\mathbf{C}_{++}$ obtained by taking limits inside cones with compact base included in $\mathbf{C}_{++}$.

Let $g^\circ(x, y)$ be the potential kernel $\sum_{k\in\mathbb{N}} p_k^\circ(x, y)$.

Lemma 5.1 $g^\circ(x, y) = \int_{\mathbf{T}} \chi_x(\theta)\,\overline{\mathrm{e}(y)(\theta)}\,\dfrac{\zeta(\theta)}{1-\kappa(\theta)}\,\mathrm{d}\theta.$

Proof. This follows from Lemma 4.2 and the definition of g°.

In the following we use the expression above for $g^\circ(x, y)$ to find its asymptotics. This amounts to the study of a singular integral, and we do it by a method inspired from Babillot [2].

Let η be a $\mathscr{C}^\infty$ function on $\mathbf{T}$ which is 1 in a neighbourhood of I_n.

Lemma 5.2 $\int_{\mathbf{T}} \chi_x(\theta)\,\overline{\mathrm{e}(y)(\theta)}\,\mathrm{d}\theta = o(|y|^{-N})$ *for any* $N\in\mathbb{N}$.

Proof. By (2.4), we have $|\kappa(\theta)|<1$ for $\theta \neq I_n$, so that $\chi_x(\theta)\,(1-\eta(\theta))\dfrac{\zeta(\theta)}{1-\kappa(\theta)}$ is a $\mathscr{C}^\infty$ function on $\mathbf{T}$. Lemma 5.2 follows by well known properties of Fourier series.

We will identify functions on $\mathbf{T}$ with functions on $(L\mathbf{T})^*$ periodic with respect to $\mathbf{P}$, and write the integral $\int_{\mathbf{T}} \chi_x(\theta)\,\overline{\mathrm{e}(y)(\theta)}\,\dfrac{\zeta(\theta)}{1-\kappa(\theta)}\,\mathrm{d}\theta$ as an integral on a fundamental domain of $(L\mathbf{T})^*$. Thus,

$$\int_{\mathbf{T}} \chi_x(\theta)\,\overline{\mathrm{e}(y)(\theta)}\,\frac{\zeta(\theta)}{1-\kappa(\theta)}\,\mathrm{d}\theta = \int_{\Delta} \mathrm{e}^{-i\langle y,\, u\rangle}\,\frac{\zeta(u)}{1-\kappa(u)}\,\mathrm{d}u$$

where Δ is a fundamental domain in $(L\mathbf{T})^*$.

Lemma 5.3 i) $1-\kappa(u) = \dfrac{1}{2}\sum_{i=1}^{i=n} \langle e_i, u\rangle^2 + O(|u|^3)$ in the neighbourhood of 0.

ii) $\sum_{i=1}^{i=n} \langle e_i, u\rangle^2 = |u|^2.$

Proof. i) follows from (2.4).

ii) the quadratic form on $(LT)^* u \to \frac{1}{2} \sum_{i=1}^{i=n} \langle e_i, u\rangle^2$ is invariant by the Weyl group, and so its eigenspaces are also invariant. Since the Weyl group acts irreducibly on $(LT)^*$, this quadratic form is a multiple of $\langle \cdot, \cdot \rangle$. Since $\sum_{i=1}^{i=n} \langle e_i, e_1\rangle^2 = 1 - \frac{1}{n} = \langle e_1, e_1\rangle$, we have the result.

Let C be a cone included in $\mathbf{C}_{++} \cup \{0\}$, with compact base, then there exists a constant $\varepsilon(C) > 0$ such that $\langle y, \alpha\rangle \geq \varepsilon(C)|y|$ for all $y \in C$, and all positive root α. In what follows, we fix such a cone and let y go to ∞ inside this cone.

We will use the notation c_n to denote a constant whose exact value can be computed and depends only on n. In the following each such constant is designated by the same c_n although its value changes from place to place. The rest of this section is devoted to the proof of the following proposition:

Proposition 5.1 *For any $\mathscr{C}^\infty$ function η with compact support on $(LT)^*$, one has:*

$$\lim_{\substack{y \to \infty \\ y \in C}} \frac{|y|^{n^2-3}}{\prod_{\alpha > 0} \langle y, \alpha\rangle} \int_{(LT)^*} e^{-i\langle y, u\rangle} \eta(u) \frac{\zeta(u)}{1-\kappa(u)} du = c_n \eta(0).$$

Combining this proposition with Lemmas 5.1 and 5.2, and using $\chi_x(0) = d_x$ we see that

$$g^\circ(x, y) \sim c_n \frac{\prod_{\alpha > 0} \langle y, \alpha\rangle}{|y|^{n^2-3}} d_x \quad \text{so that} \quad \frac{g^\circ(x, y)}{g^\circ(\phi, y)} \to d_x \quad \text{when } y \to \infty \text{ inside the cone } C.$$

This proves that d is the harmonic function corresponding to the Martin boundary point of $\mathbf{P}_{++}$ obtained by taking limit at ∞ inside any cone with compact base in $\mathbf{C}_{++}$. It is plausible that Proposition 5.1 is still true if y is not restricted to stay in a cone, and so that there exists only one Martin boundary point at infinity for the kernel p°, but we have not been able to obtain sufficiently precise estimates on the potential kernel g° to prove this.

We now proceed with the proof of Proposition 5.1. We will begin by comparing

$$\int_{(LT)^*} e^{-i\langle y, u\rangle} \eta(u) \frac{\zeta(u)}{1-\kappa(u)} du \quad \text{with} \quad \int_{(LT)^*} e^{-i\langle y, u\rangle} \eta(u) \frac{\zeta(u)}{r(u)} du$$

where $r(u) = \frac{1}{2}|u|^2$.

The difference of these two expressions is

$$\int_{(LT)^*} e^{-i\langle y, u\rangle} \eta(u) \frac{\zeta(u)(r(u) - 1 + \kappa(u))}{(1-\kappa(u))\, r(u)} du. \tag{5.1}$$

In order to evaluate 5.1, we remark that the function

$$\eta(u) \frac{\zeta(u)(r(u) - 1 + \kappa(u))}{(1-\kappa(u))\, r(u)} \quad \text{is } \mathscr{C}^\infty \text{ in } (LT)^* \backslash \{0\}. \text{ Furthermore:}$$

Lemma 5.4 *Let* $D^{(k)}=\frac{\partial}{\partial u_{i_1}}\frac{\partial}{\partial u_{i_2}}\cdots\frac{\partial}{\partial u_{i_k}}$ *be a differential operator of order* k *on* $(LT)^*$, *one has the following bounds in the neighbourhood of* 0:

i) $D^{(k)}(\zeta(u))=O(|u|^{\frac{n(n-1)}{2}-k})$

ii) $D^{(k)}\left(\frac{1}{1-\kappa(u)}\right)=O(|u|^{-2-k})$

iii) $D^{(k)}\left(\frac{1}{r(u)}\right)=O(|u|^{-2-k})$

iv) $D^{(k)}(r(u)-1+\kappa(u))=O(|u|^{3-k})$

v) $D^{(k)}\left(\eta(u)\frac{\zeta(u)(r(u)-1+\kappa(u))}{(1-\kappa(u))\,r(u)}\right)=O\left(|u|^{\frac{n(n-1)}{2}-k-1}\right).$

Proof. i) follows from formula (2.3) and iv) from Lemma 5.3.

ii), iii) and v) can be proved using Leibnitz rule as in Babillot [2] Proposition (2.31).

We can conclude from Babillot [2], Corollaire 2.18, that (5.1) is

$$O(|u|^{\frac{n(n-1)}{2}-n+3})\text{ at }\infty.$$

We now study the expression: $\int\limits_{(LT)^*} e^{-i\langle y,u\rangle}\eta(u)\frac{\zeta(u)}{r(u)}\,du.$

Lemma 5.5 *For* $n\geqq 4$

$$\int\limits_{(LT)^*} e^{-i\langle y,u\rangle}\eta(u)\frac{\zeta(u)}{r(u)}\,du=c_n\int\limits_{\mathbb{R}^{n-1}}\hat{\eta}(z)\sum_{w\in W}\det(w)\frac{1}{|z-y-w(\phi)|^{n-3}}\,dz$$

for $n=3$

$$\int\limits_{(LT)^*} e^{-i\langle y,u\rangle}\eta(u)\frac{\zeta(u)}{r(u)}\,du=c_3\int\limits_{\mathbb{R}^2}\hat{\eta}(z)\sum_{w\in W}\det(w)\log|z-y-w(\phi)|\,dz.$$

Proof. This follows from Plancherel formula, and the fact that the Fourier transform (in the distribution sense) of $\frac{1}{|u|^2}$, is $\frac{c_n}{|z|^{n-3}}$ if $n\geqq 4$, and $c_2\log|z|$, if $n=3$.

In the following we consider the case $n\geqq 4$, but the case $n=3$ can be treated in the same way.

Lemma 5.6 i) *For any* u *in* $(LT)^*$:

$$\sum_{w\in W}\det(w)\langle w(\phi),u\rangle^l=0\qquad\text{if } l<\tfrac{1}{2}n(n-1)$$

ii) *for any polynomial function* P *on* $(LT)^*$ *of degree* $<n(n-1)$, *one has* $\sum\limits_{w\in W}\det(w)\,P(w(\phi))=0.$

Proof. i) Expand relation (2.3) near 0.

ii) follows from i) by polarization.

We now use Lemma 5.6 to obtain an asymptotic of $\sum_{w\in W} \det(w)\dfrac{1}{|z-y-w(\phi)|^{n-3}}$.

Lemma 5.7. *Let $N\in\mathbb{N}$ be $>n(n-1)$, then*

$$\sum_{w\in W} \det(w)\frac{1}{|z-y-w(\phi)|^{n-3}}=c_n\frac{\prod_{\alpha>0}\langle\alpha, y\rangle}{|y|^{n^2-3}}+o(|y|^{-\frac{n(n-1)}{2}-n+3})$$

and this estimate holds uniformly on $|z|<|y|^{1/N}$.

Proof. We expand

$$\frac{1}{|z-y-w(\phi)|^{n-3}}=|y|^{3-n}\left(1+\frac{2\langle y, z-w(\phi)\rangle+|z-w(\phi)|^2}{|y|^2}\right)^{-\frac{3-n}{2}}$$
$$=|y|^{3-n}\left(\sum_{k=0}^{2} a_k\left(\frac{2\langle y, z-w(\phi)\rangle+|z-w(\phi)|^2}{|y|^2}\right)^k+o(|y|^{-\frac{n(n-1)}{2}})\right)$$

where the $o(|y|^{-\frac{n(n-1)}{2}})$ is uniform on $|z|<|y|^{1/N}$ and the a_k are nonzero coefficients.

If $k<\frac{1}{2}n(n-1)$ $\sum_{w\in W}\det(w)\left(\dfrac{2\langle y, z-w(\phi)\rangle+|z-w(\phi)|^2}{|y|^2}\right)^k=0$

by Lemma 5.6 ii). In the term

$$\sum_{w\in W}\det(w)\left(\frac{2\langle y,z-w(\phi)\rangle+|z-w(\phi)|^2}{|y|^2}\right)^{\frac{n(n-1)}{2}}$$

we can develop and obtain

$$\sum_{w\in W}\det(w)\left(\frac{2\langle y-z, w(\phi)\rangle}{|y|^2}\right)^{\frac{n(n-1)}{2}}+o(|y|^{-\frac{n(n-1)}{2}}),$$

but

$$\sum_{w\in W}\det(w)\left(\frac{2\langle y-z, w(\phi)\rangle}{|y|^2}\right)^{\frac{n(n-1)}{2}}=c_n\frac{\prod_{\alpha>0}\langle 2(y-z), \alpha\rangle}{|y|^{n(n-1)}} \quad \text{by Lemma 5.6}$$
$$=c_n\frac{\prod_{\alpha>0}\langle 2y, \alpha\rangle}{|y|^{n(n-1)}}+o(|y|^{-\frac{n(n-1)}{2}})$$

uniformly on $|z|<|y|^{1/N}$.

Lemma 5.8

$$\lim_{\substack{y\to\infty\\ y\in C}} c_n\frac{|y|^{n^2-3}}{\prod_{\alpha>0}\langle\alpha, y\rangle}\int_{\mathbb{R}^{n-1}}\hat{\eta}(z)\sum_{w\in W}\det(w)\frac{1}{|z-y-w(\phi)|^{n-3}}\,dz=\int_{\mathbb{R}^{n-1}}\hat{\eta}(z)\,dz.$$

Proof. Since $y \in C \langle y, \alpha \rangle \geqq \varepsilon(C)|y|$, so that $\prod_{\alpha>0} \langle y, \alpha \rangle \geqq (\varepsilon(C)|y|)^{\frac{n(n-1)}{2}}$

(5.2) $$\frac{|y|^{n^2-3}}{\prod_{\alpha>0} \langle \alpha, y \rangle} \int_{|z|<|y|^{1/N}} \hat{\eta}(z) \sum_{w \in W} \det(w) \frac{1}{|z-y-w(\phi)|^{n-3}} \, dz = \int_{|z|<|y|^{1/N}} \hat{\eta}(z) \, dz + o(1) \quad \text{by Lemma 5.7.}$$

Let B_y be the union of the balls $B(y-w(\phi), 1)$, $w \in W$, then:

$$\left| \int_{\{|z|>|y|^{1/N}\} \setminus B} \hat{\eta}(z) \sum_{w \in W} \det(w) \frac{1}{|z-y-w(\phi)|^{n-3}} \right| \leqq |W| \int_{|z|>|y|^{1/N}} |\hat{\eta}(z)| \, dz$$

since η is $\mathscr{C}^\infty$, $|\hat{\eta}(z)|$ is $o(|z|^{-K})$ for any $K \geqq 0$, so that the expression above is $o(|y|^{-K})$ for all $K \geqq 0$.

(5.3) $$\left| \int_{\{|z|>|y|^{1/N}\} \setminus B_y} \hat{\eta}(z) \sum_{w \in W} \det(w) \frac{1}{|z-y-w(\phi)|^{n-3}} \right| = o(|y|^{-K}) \quad \text{for all } K \geqq 0.$$

Finally, for all $K \geqq 0$, on B_y $|\hat{\eta}(z)|$ is uniformly $o(|y|^{-K})$ so that:

(5.4) $$\int_{B_y} \hat{\eta}(z) \frac{1}{|z-y-w(\phi)|^{n-3}} \, dz = o(|y|^{-K}) \quad \text{for all } K \geqq 0.$$

The lemma follows from (5.2), (5.3), (5.4).

Proposition 1 follows from Lemma 5.8 and Fourier inversion formula for $\hat{\eta}$.

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