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Probability Theory and Related Fields

A probabilistic approach to one class of nonlinear differential equations*

E.B. Dynkin

Department of Mathematics, White Hall, Cornell University, Ithaca, NY 14853, USA

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Summary. We establish connections between positive solutions of one class of nonlinear partial differential equations and hitting probabilities and additive functionals of superdiffusion processes. As an application, we improve results on superprocesses by using the recent progress in the theory of removable singularities for differential equations.

1. Introduction

1.1 We consider positive solutions of a differential equation

$$-Lv(x)+\psi(x, v(x))=\rho(x) \quad \text{for } x\in D \tag{1.1}$$

where L is a strongly elliptic differential operator in $\mathbb{R}^d$, D is a domain in $\mathbb{R}^d$, $\rho \geqq 0$ and ψ belongs to a convex cone Ψ which contains, in particular, all functions

$$\psi(x, z)=\gamma(x)\, z^\alpha, \quad 1<\alpha \leqq 2 \tag{1.2}$$

with positive bounded Borel γ.

The differential operator L is the generator of a diffusion process $\xi=(\xi_t, \Pi_x)$ in $\mathbb{R}^d$ (see Theorem 0.1 in the Appendix)). If $\psi=0$, then (1.1) is a linear equation which can be studied probabilistically using paths of ξ. Analogously, the Eq. (1.1) for any $\psi\in\Psi$ can be investigated by using the superprocess corresponding to (ξ, ψ). In particular, we establish (under some restrictions on L and ψ) that a compact set K is a removable singularity for (1.1) if and only if the superprocess started outside K does not hit K. Removable singularities for (1.1) have been studied in [LN], [BV], [L], [V1]–[V4], [BP], [VV], [RV] ... (see the survey of literature in Sect. 5). Independently, hitting probabilities for superprocesses have been investigated in [I], [DIP], [DP], [D4], [P] .... It seems that both theories can gain from an interplay between probabilistic and analytic methods.

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The cone Ψ mentioned in the first paragraph is the set of functions

$$\psi(x, z)=a(x)\,z+b(x)\,z^2+\int_0^\infty (e^{-uz}-1+uz)\,n_x(du) \tag{1.3}$$

where n is a kernel from $\mathbb{R}^d$ to $(0, +\infty)$ and $a(x)$, $b(x)$ and $A(x)=\int_0^\infty u \wedge u^2\, n_x(du)$ are positive bounded Borel functions. The function (1.2) corresponds to $a=n=0$, $b=\gamma$ if $\alpha=2$, and to $a=b=0$, $n_x(du)=\frac{1}{c}\gamma(x)\,u^{-1-\alpha}\,du$ where $c=\int_0^\infty (e^{-u}-1+u)\,u^{-1-\alpha}\,du$ if $1<\alpha<2$.

1.2 The superprocesses can be constructed in a very general setting (see, [D2], [D4], [F]). Here we describe a particular case of this construction assuming that $\xi=(\xi_t, \Pi_x)$ is a right Markov process in a locally compact Hausdorff space E with a countable base (all diffusion processes are in this class). Denote by $\mathscr{E}$ the Borel σ-algebra in E, by M the set of all finite measures on $\mathscr{E}$ and by $\mathscr{M}$ the σ-algebra in M generated by the functions $f_B(\mu)=\mu(B)$, $B\in\mathscr{E}$. There exists a Markov process (X_t, P_μ) in $(M, \mathscr{M})$ such that:

1.2.A If f is a bounded continuous function, then $\langle f, X_t\rangle$ is right continuous in t on $\mathbb{R}^+=[0, \infty)$ (writing $\langle v, \mu\rangle$ means the integral of v with respect to μ).

1.2.B For every $\mu\in M$,

$$P_\mu \exp\langle -f, X_t\rangle=\exp\langle -v_t, \mu\rangle, \tag{1.4}$$

where v is the unique solution of the integral equation

$$v_t(x)+\int_0^t \Pi_x\, \psi(\xi_s, v_{t-s}(\xi_s))\,ds=\Pi_x f(\xi_t). \tag{1.5}$$

Moreover, to every set $D\in\mathscr{E}$ there correspond random measures X_τ and Y_τ on $(E, \mathscr{E})$ associated with the first exit time $\tau=\inf\{t\colon \xi_t\notin D\}$ from D by the formula

$$P_\mu \exp\{-\langle \rho, Y_\tau\rangle-\langle f, X_\tau\rangle\}=\exp\langle -v, \mu\rangle, \tag{1.6}$$

where

$$v(x)+\Pi_x \int_0^\tau \psi(\xi_s, v(\xi_s))\,ds=\Pi_x\left[\int_0^\tau \rho(\xi_s)\,ds+f(\xi_\tau)\right] \tag{1.7}$$

(see Theorem 1.4 in [D4]). The superprocess X with parameters (ξ, ψ) is the collection $(X_t, X_\tau, Y_\tau; P_\mu)$. [In fact, X_τ and Y_τ subject to conditions (1.6), (1.7) can be defined for all coanalytic sets D.] If ξ is a diffusion with the generator L, then we call X *the superdiffusion with parameters* (L, ψ).

The heuristic meaning of random measures X_t and X_τ and Y_τ can be explained in terms of branching particle systems. Particles are distributed at time 0 according to the Poisson point process with intensity measure $\mu\in M$. The motion of each particle is governed by the process ξ. The life time is distributed exponentially with parameter k. A dying particle gives birth to n offsprings with probability $p_n(x)$ depending on the death place x. We assume that $\sum n p_n(x) \leqq 1$ and we put $\varphi(x, z) = \sum p_n(x) z^n$. The historical path w_t^a of a particle a consists of its own trajectory and the trajectories of all its ancestors (the law of w^a is identical to the law of ξ). If particles have mass β, then

$$X_t^\beta(B) = \beta \sum_a 1_B(w_t^a)$$

is the mass distribution at time t. (The sum is taken over all particles which live at time t.) We set

$$X_\tau^\beta(B) = \beta \sum_a 1_B(w_{\tau_a}^a),$$

$$Y_\tau^\beta(B) = \beta \sum_a \int_0^{\tau_a} 1_B(w_s^a)\, ds$$

where $\tau_a = \inf\{t\colon w_t^a \notin D\}$. (The terms in these two sums are in a 1–1 correspondence with particles which exit from D assuming that each particle does not move, die or procreate after time τ_a.) Random measures X_t^β, X_τ^β and Y_τ^β converge weakly to X_t, X_τ and Y_τ as $\beta \to 0$ assuming that $k_\beta = 1/\beta$, $\mu_\beta = \mu/\beta$ and that

$$[\varphi_\beta(x, 1-\beta z) - (1-\beta z)]\, \beta^{-2} \to \psi(x, z)$$

uniformly on $E \times [0, c]$ for every c.

1.3 Consider the first exit time τ of ξ from D and put

$$u(x) = \Pi_x \left[\int_0^\tau \rho(\xi_s)\, ds + f(\xi_\tau) \right] \tag{1.8}$$

($f(\xi_\tau) = 0$ if $\tau = \infty$). Under broad assumptions (see Theorems 0.2 and 0.3) u is the unique solution of the Dirichlet problem

$$-Lu = \rho \quad \text{in } D, \tag{1.9}$$

$$u(x) \to f(a) \quad \text{as} \quad x \to a \in \partial D, \qquad x \in D. \tag{1.10}$$

Under the same assumptions, the solution of the Dirichlet problem for the equation (1.1) is given by the formula

$$v(x) = -\log P_{\delta_x} \exp\{-\langle \rho, Y_\tau \rangle - \langle f, X_\tau \rangle\} \tag{1.11}$$

where δ_x is the unit measure concentrated at x. More precisely, we have:

Theorem 1.1 *Let X be a superdiffusion with parameters (L, ψ) where L is a differential operator with properties* 0.2.A, B *and $\psi \in \Psi$ satisfies the condition:*

1.3.A *For every compact set $K \subset D$ and for every N, there exists a constant C such that*

$$(1.12) \quad |\psi(x_1, z) - \psi(x_2, z)| \leqq C\,|x_1 - x_2|^\lambda \quad \textit{for all } x_1, x_2 \in K,\ z \in [0, N]$$

(*the exponent $\lambda \in (0, 1]$ is independent of K and N*).

If D is a bounded regular domain, ρ is bounded and belongs to $C^{0,\lambda}(D)$ and f is a positive continuous function on ∂D, then formula (1.11) *defines a unique solution of* (1.1) *which satisfies the boundary condition*

$$(1.13) \qquad v(x) \to f(a) \quad \textit{as} \quad x \to a \in \partial D, \qquad x \in D.$$

1.4 *Starting from this point we assume that ψ is given by the formula* (1.2) *with a Hölder continuous function $\gamma(x)$ subject to the condition*

$$(1.14) \qquad \inf_x \gamma(x) > 0.$$

Theorem 1.2 *Under the conditions of Theorem* 1.1,

$$(1.15) \qquad v(x) = -\log P_{\delta_x}\{X_\tau = 0\}$$

is the minimal positive solution of the problem

$$(1.16) \qquad Lv(x) = \gamma(x)\, v(x)^\alpha \quad \textit{in } D,$$

$$(1.17) \qquad v(x) \to +\infty \quad \textit{as} \quad x \to a \in \partial D, \qquad x \in D$$

(*that is $v \leqq u$ for every $u \geqq 0$ subject to* (1.16), (1.17)).

The uniqueness of a solution for the problem (1.16)–(1.17) has been established for some particular cases in [LN] and [I] (see more details in Sect. 5). Under rather mild conditions on L and D, it has been proved recently by Veron [V4] and by Kondrat'yev and Nikishkin [KN].

1.5 For every $\varepsilon \geqq 0$, we denote by $\mathcal{R}_\varepsilon$ the minimal closed set which contains the supports S_t of X_t for all $t \geqq \varepsilon$. The set $\mathcal{R} = \mathcal{R}_0$ is called *the range of X*.

Theorem 1.3 *Under conditions of Theorem* 1.2, *the range $\mathcal{R}$ of X is compact P_μ-a.s. for every $\mu \in M$ with compact support. For an arbitrary open set D formula*

$$(1.18) \qquad v(x) = -\log P_{\delta_x}\{\mathcal{R} \subset D\}$$

determines the maximal positive solution of the Eq. (1.16) *in D* (*i.e., $v \geqq u$ for every positive solution u of* (1.16)).

1.6 In Sect. 2.5 we show that, if B is an analytic set, then the set $\{\omega: \mathcal{R} \cap B \neq \phi\}$ belongs to the universal completion $\mathcal{G}^*$ of the σ-algebra $\mathcal{G}$ generated by $\{X_t, t \in \mathbb{R}^+\}$. We say that B is *S-polar* if, for every $\mu \in M$ and every $\varepsilon > 0$, there exists an analytic set $A \supset B$ such that $P_\mu\{\mathcal{R}_\varepsilon \cap A \neq \phi\} = 0$. Clearly,

$$(1.19) \qquad P_\mu\{\mathcal{R}_\varepsilon \cap B \neq \phi\} = 0 \qquad \text{for all } \varepsilon > 0,\ \mu \in M.$$

For the superdiffusions in this paper we prove in Lemmas 2.4 and 2.5 that an analytic set B is S-polar if and only if

$$P_{\delta_x}(R \cap B \neq \phi) = 0 \qquad \text{for all } x \notin B.$$

Theorem 1.4 *Each of the following conditions is necessary and sufficient for a closed set Γ to be S-polar:*

1.6.A *If $v \geqq 0$ satisfies* (1.16) *in $D = \Gamma^c$, then $v = 0$.*

1.6.B *The maximal solution of* (1.16) *in D is bounded.*

Example. Put

$$\kappa_\alpha = \frac{2\alpha}{\alpha - 1}. \tag{1.20}$$

If $d < \kappa_\alpha$, then

$$v(x) = [(\alpha-1)^{-1}(\kappa_\alpha - d)]^{1/(\alpha-1)} |x|^{-2/(\alpha-1)} \tag{1.21}$$

is a positive solution of the Eq.

$$\tfrac{1}{2}\Delta v = v^\alpha \quad \text{in } \mathbb{R}^d \setminus \{0\} \tag{1.22}$$

and it tends to $+\infty$ as $|x| \to 0$. Hence singletons are not S-polar sets if $d < \kappa_\alpha$. If $d \geqq \kappa_\alpha$, then, according to [BV], every positive solution of (1.22) is bounded near 0 and therefore the singletons are S-polar. (See Corollary 1 to Theorem 1.6 for an analogous result for more general equations.)

Theorem 1.5 *Suppose a compact set K is contained in an open set D. If K is S-polar, then the maximal solution $\tilde{v}$ in $\tilde{D} = D \setminus K$ coincides in $\tilde{D}$ with the maximal solution v in D. Hence $\tilde{v}$ is bounded in a neighborhood of K. Conversely, if $\tilde{v}$ is bounded near K, then K is S-polar.*

1.7 The Newton potential of a measure η is defined by the formula

$$n_\eta(y) = \int_{\mathbb{R}^d} \eta(dx)\, k_d(x, y) \tag{1.23}$$

where

$$k_d(x, y) = |x - y|^{2-d} \qquad \text{for } d > 2. \tag{1.24}$$

For $d = 2$, an analogous role is played by the logarithmic potential which corresponds to

$$k_2(x, y) = \log^+ |x - y|^{-1}. \tag{1.25}$$

Put $A \in Q_\infty$ if there exist no finite measure η, concentrated on A (except $\eta = 0$) such that n_η is bounded. Let ξ be the Brownian motion in $\mathbb{R}^d$. If $A \in Q_\infty$, then

$$\Pi_x\{\xi_t \notin A \text{ for all } t > 0\} = 1 \qquad \text{for all } x \in \mathbb{R}^d; \tag{1.26}$$

conversely, if A satisfies (1.26), then $A \in Q_\infty$ (see, for instance, [C]).

The class Q_∞ does not change if we replace k_d by the Bessel kernel

$$\tilde{k}_d(x, y) = \int_0^\infty e^{-t/2}\, p_t(x, y)\, dt \tag{1.27}$$

where

$$p_t(x, y) = (2\pi t)^{-d/2} \exp\{-|x-y|^2/2t\} \tag{1.28}$$

is the Brownian transition density in $\mathbb{R}^d$. The Bessel potential $\tilde{n}_\eta$ of a measure η is defined by formula (1.23) with k_d replaced with $\tilde{k}_d$.

We introduce, for every $\alpha > 1$, a class Q_α by replacing the condition "n_η is bounded" by the condition $\tilde{n}_\eta \in L^\alpha(\mathbb{R}^d)$. Class Q_α consists of all sets A such that $\mathrm{Cap}_\alpha A = 0$ where

$$\mathrm{Cap}_\alpha A = \sup \nu(A) \tag{1.29}$$

with the supremum taken over all measures ν concentrated on A and such that

$$\int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} \nu(dx)\, \tilde{k}_d(x, y)\Big]^\alpha dy \leqq 1. \tag{1.30}$$

Let $g(x, y)$ be Green's function for L in D. If a compact $K \subset D$, then there exist constants $0 < c_1 < c_2 < \infty$ such that

$$c_1\, \tilde{k}_d(x, y) \leqq g(x, y) \leqq c_2\, \tilde{k}_d(x, y) \qquad \text{for all } x, y \in K$$

(see [Mi], Sects. 8–10). Therefore $K \notin Q_\alpha$ if and only if $\int_D \eta(dx)\, g(x, y)$ belongs to $L^\alpha(D)$ for some $\eta \neq 0$ concentrated on K.

Theorem 1.6 *Let X be a superdiffusion with parameters (L, ψ) where L has the divergence form described in Sect.* 0.4 *and $\psi = z^\alpha$. Then the class of S-polar sets coincides with Q_α.*

Corollary 1 *A singleton $\{c\}$ is an S-polar set if and only if $d \geqq \kappa_\alpha$ where κ_α is given by formula* (1.20).

Indeed, $\tilde{n}_\eta \in L^\alpha(\mathbb{R}^d)$ for a measure $\eta \neq 0$ concentrated at $\{c\}$ if and only if $\int_0^1 r^{d-1+(2-d)\alpha}\, dr < \infty$ which is equivalent to the condition $d < \kappa_\alpha$.

Corollary 2 *Let K be a compact S-polar set and let D be an open set which contains K. Then every positive solution u of the equation*

$$Lu = u^\alpha \quad \text{in } D \setminus K$$

can be continued to a function which belongs to class $C^{1,\lambda}(D)$ for all $\lambda < 1$.

In combination with Theorem 1.5 this result can be interpreted as follows:

A compact set K is a removable singularity for the equation $Lu = u^\alpha$ if and only if K is S-polar.

1.8 Put $\eta \in \mathscr{K}_\alpha(D)$ if η is a finite measure on D which charges no set $A \in Q_\alpha$. Denote by M_α the set of all measures $\mu \in M$ of the form $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$ with $f \in L^{\alpha'}(\mathbb{R}^d)$ $\left(\text{here } \frac{1}{\alpha} + \frac{1}{\alpha'} = 1\right)$. The proof of Theorem 1.6 is based on the following result which is also of independent interest.

Theorem 1.7 *Let X satisfy the conditions of Theorem 1.6. To every domain D and to every $\eta \in \mathscr{K}_\alpha(D)$ there corresponds a positive $\mathscr{G}^*$-measurable function Y_D^η such that:*

1.8.A *The function*

$$v(x) = -\log P_{\delta_x} \exp(-Y_D^\eta) \tag{1.31}$$

satisfies the equations

$$v(x) + \int_D g(x, y)\, v(y)^\alpha\, \mathrm{d}y = \int_D g(x, y)\, \eta(\mathrm{d}y) \; a.e. \quad on\ D \tag{1.32}$$

where $g(x, y)$ is Green's function for ξ in D. If D is bounded and has a smooth boundary and if the function in the right side of (1.32) *belongs to $L^\alpha(D)$, then* (1.32) *determines $v \geqq 0$ uniquely up to equivalence.*

1.8.B *If $\mu \in M_\alpha$, then*

$$P_\mu \exp(-Y_D^\eta) = \mathrm{e}^{-\langle v, \mu \rangle}. \tag{1.33}$$

1.8.C *If $\eta(\mathrm{d}x) = f(x)\,\mathrm{d}x$, then $Y_D^\eta = \langle f, Y_\tau \rangle$ where τ is the first exit time of ξ from D.*

1.8.D $Y_D^{\eta' + \eta''} = Y_D^{\eta'} + Y_D^{\eta''}$ *and* $Y_D^{c\eta} = c\, Y_D^\eta$ *for any constant* $c \geqq 0$.

1.8.E *If $D \subset D'$, $\eta \leqq \eta'$ then $Y_D^\eta \leqq Y_{D'}^{\eta'}$.*

1.8.F *If $\eta_n \uparrow \eta$, $D_n \uparrow D$, then $Y_{D_n}^{\eta_n} \uparrow Y_D^\eta$ P_μ-a.s. for all $\mu \in M_\alpha$.*

1.9 Suppose h is an increasing positive function on an interval $[0, \varepsilon_0]$ such that $h(0) = 0$. For any $A \in \mathbb{R}^d$ and any $0 < \varepsilon \leqq \varepsilon_0$ we set

$$H^\varepsilon(A) = \inf \sum_i h(r_i)$$

where infimum is taken over all countable coverings of A by open balls $U(x_i; r_i)$ of center x_i and radius $r_i \leqq \varepsilon$. The Hausdorff measure H corresponding to h is defined by the formula

$$H(A) = \lim_{\varepsilon \to 0} H^\varepsilon(A).$$

Let H_γ and $H_{\gamma,\beta}$ be the Hausdorff measures corresponding to the functions

$$h_\gamma(r) = r^\gamma, \quad \gamma > 0 \quad \text{and} \quad h_{\gamma,\beta}(r) = r^\gamma \left(\log^+ \frac{1}{r}\right)^{-\beta} \quad \gamma \geqq 0,\ \beta > 0. \tag{1.33}$$

The Hausdorff dimension $H-\dim A$ is defined as the supremum of γ such that $H_\gamma(A)>0$. The Carleson logarithmic dimension $L-\dim A$ is the supremum of β such that $H_{0,\beta}(A)>0$.

Theorem 1.8 *Put $\gamma=d-\kappa_\alpha$. If $\gamma<0$, then the only element of Q_α is the empty set. If $\gamma>0$, then Q_α contains all sets $A\subset\mathbb{R}^d$ for which $H_\gamma(A)<\infty$; on the other hand, $H_{\gamma,\beta}(A)=0$ for all $A\in Q_\alpha$ and all $\beta>(\alpha-1)^{-1}$. Finally, if $\gamma=0$, then Q_α contains all sets A for which $H_{0,\beta}(A)<\infty$ with $\beta=(\alpha-1)^{-1}$; and $H_{0,\beta}(A)=0$ for all $A\in Q_\alpha$ and all $\beta>(\alpha-1)^{-1}$.*

Corollary. *If $d>\kappa_\alpha$, then the sets $A\subset\mathbb{R}^d$ with $H-\dim A<d-\kappa_\alpha$ are S-polar and the sets with $H-\dim A>d-\kappa_\alpha$ are not S-polar. In the case $d=\kappa_\alpha$, A is S-polar if $L-\dim A<(\alpha-1)^{-1}$ and A is not S-polar if $L-\dim A>(\alpha-1)^{-1}$.*

1.10 A general part of theory which is applicable to arbitrary right processes is presented in Sect. 2. In Sect. 3 we deal with classical solutions of (1.1) and in Sect. 4 with generalized solutions in Sobolev spaces. In Sect. 5, the present results are compared with those in literature. We also state a few open problems. The Appendix contains some basic facts on elliptic equations and their relation to diffusion processes and on Sobolev spaces.

2. Range of X and S-polar sets

2.1 In this section ξ is a right process in a locally compact Hausdorff space E with a countable base and X is the corresponding right superprocess which implies condition 1.2.A. Let $\mathscr{G}$ be the σ-algebra generated by X_t, $t\in\mathbb{R}^+$. It follows from 1.2.A that, for every Borel set B, $X_t(B)$ is $\mathscr{B}(\mathbb{R}^+)\times\mathscr{G}$-measurable in (t,ω) and therefore

(2.1) $$Y(B)=\int_0^\infty X_t(B)\,dt$$

is $\mathscr{G}$-measurable in ω.

2.2 The range $\mathscr{R}$ of X was introduced in Sect. 1.5. We claim that

(2.2) $$\{\mathscr{R}\subset\Gamma\}=\{Y(\Gamma^c)=0\}\in\mathscr{G}$$

for every closed set Γ. Indeed, $\{S_t\subset\Gamma\}=\{X_t(\Gamma^c)=0\}$ and therefore

(2.3) $$\{\mathscr{R}\subset\Gamma\}=\{X_t(\Gamma^c)=0 \text{ for all } t\in\mathbb{R}^+\}.$$

There exists a positive bounded continuous function f such that $\Gamma=\{f=0\}$. By 1.2.A, $\{\langle f,Y\rangle=0\}=\{\langle f,X_t\rangle=0$ for all $t\in\mathbb{R}^+\}$. Thus $\{Y(\Gamma^c)=0\}=\{X_t(\Gamma^c)=0$ for all $t\in\mathbb{R}^+\}$ and (2.2) follows from (2.3).

Lemma 2.1 *Let τ be the first exit time from an open set D. If $B\subset D\subset\Gamma$ and if B, Γ are closed, then*

(2.4) $$\{\mathscr{R}\subset B\}\subset\{X_\tau=0\}\subset\{\mathscr{R}\subset\Gamma\}\ \text{a.s.}$$

(Writing "a.s." means "P_μ-a.s. for all finite measures μ".)

Proof. By the special Markov property (see Theorem 1.5 in [D4]),

$$P_\mu\{X_\tau=0,\ Y(\Gamma^c)>0\}=P_\mu\{X_\tau=0;\ P_{X_\tau}[Y(\Gamma^c)>0]\}. \tag{2.5}$$

Since $P_0\{Y=0\}=1$ by (1.4), the right side vanishes, and the second part of (2.4) follows from (2.2).

Let $B=\{f=0\}$ where f is a positive bounded continuous function. By Theorem 1 in the Addendum to [D4],

$$\{\langle f, X_t\rangle=0 \text{ for all } t\geqq 0\}\subset\{\langle f, X_\tau\rangle=0\} \text{ a.s.}$$

for every stopping time τ. Therefore $\{\mathscr{R}\subset B\}\subset\{X_\tau(B^c)=0\}$ a.s. Since $X_\tau(B)=0$ a.s. by (1.6), (1.7), we get the first part of (2.4).

2.3 For every open set D, we denote by Ω_D the union of the sets $\{\mathscr{R}\subset\Gamma\}$ over all compact sets $\Gamma\subset D$. Note that, if $\mathscr{R}$ is compact, then

$$\Omega_D=\{\mathscr{R}\subset D\}. \tag{2.6}$$

Consider an arbitrary sequence D_n with the properties:

(a) D_n are open sets;
(b) $\Gamma_n=\bar{D}_n$ is compact and $\Gamma_n\subset D_{n+1}$;
(c) $D_n\uparrow D$.

Clearly,

$$\Omega_D=\bigcup_n\{\mathscr{R}\subset\Gamma_n\}. \tag{2.7}$$

Let τ_n be the first exit time from D_n. Since $\Gamma_{n-1}\subset D_n\subset\Gamma_n$, we have, by Lemma 2.1, that $\{\mathscr{R}\subset\Gamma_{n-1}\}\subset\{X_{\tau_n}=0\}\subset\{\mathscr{R}\subset\Gamma_n\}$ a.s. and therefore

$$\Omega_D=\bigcup_n\{X_{\tau_n}=0\} \text{ a.s.} \tag{2.8}$$

By Theorem 2 in the Addendum to [D4], (X_{τ_n}, P_μ) is a Markov process with an absorbing state 0. Therefore

$$P_\mu\{X_{\tau_n}=0\}\uparrow P_\mu(\Omega_D). \tag{2.9}$$

2.4 If τ is the first exit time from an open set D, then

$$P_\mu\exp\langle -k, X_\tau\rangle\downarrow P_\mu\{X_\tau=0\} \quad\text{as}\quad k\uparrow\infty. \tag{2.10}$$

By (1.6),

$$P_\mu\exp\langle -k, X_\tau\rangle=\exp\langle -v_k, \mu\rangle \tag{2.11}$$

where

$$v_k(x)=-\log P_{\delta_x}\exp\langle -k, X_\tau\rangle. \tag{2.12}$$

By (2.10),

$$v_k(x)\uparrow v(x)=-\log P_{\delta_x}\{X_\tau=0\} \tag{2.13}$$

and

$$P_\mu\{X_\tau=0\}=e^{-\langle v,\mu\rangle}. \tag{2.14}$$

By applying this to stopping times τ_n introduced in the previous section, we get

$$P_\mu\{X_{\tau_n}=0\}=e^{-\langle v^n,\mu\rangle}, \qquad v^n(x)=-\log P_{\delta_x}\{X_{\tau_n}=0\}. \tag{2.15}$$

By (2.9) and (2.15),

$$v^n(x)\downarrow v_D(x)=-\log P_{\delta_x}(\Omega_D). \tag{2.16}$$

If for some n, $P_\mu\{X_{\tau_n}=0\}>0$, then $\langle v^n, \mu\rangle<\infty$, $\langle v^n, \mu\rangle\to\langle v_D, \mu\rangle$ by the dominated convergence theorem and

$$P_\mu(\Omega_D)=\exp\langle -v_D, \mu\rangle. \tag{2.17}$$

Formula (2.17) holds if v^n are locally bounded in D_n and if the support of μ is compact and is contained in D (we write $\mu\in M_c(D)$).

Suppose that Z_n is a sequence of mappings from Ω to a measurable space $(M, \mathscr{M})$ and let Y be a function from Ω to $[0, +\infty]$. We say that Y is a shift-invariant functional of $\{Z_n\}$ if there exists a measurable function F on $(M^\infty, \mathscr{M}^\infty\}$ such that

$$Y(\omega)=F(Z_m(\omega), Z_{m+1}(\omega), \ldots) \qquad \text{for all} \quad \omega \quad \text{and all} \quad m.$$

Put $\omega\in\tilde{\Omega}_D$ if $X_{\tau_n}=0$ for all sufficiently large n. Clearly, $1_{\tilde{\Omega}_D}$ is an invariant functional of $\{X_{\tau_n}\}$. Since $(X_{\tau_n}, P_{\delta_x})$ is a Markov process, and since $1_{\Omega_D}=1_{\tilde{\Omega}_D}$ P_{δ_x}-a.s. for $x\in D$, we have

$$\lim_{n\to\infty} P_{X_{\tau_n}}(\Omega_D)=0 \quad \text{or} \quad 1\ P_{\delta_x}\text{-a.s.} \tag{2.18}$$

which implies

$$\lim_{n\to\infty} \langle v_D, X_{\tau_n}\rangle=0 \quad \text{or} \quad \infty\ P_{\delta_x}\text{-a.s.} \tag{2.19}$$

Suppose that v_D is bounded. Then

$$\langle v_D, X_{\tau_n}\rangle\leqq\langle N, X_{\tau_n}\rangle \tag{2.20}$$

for some constant N. It follows from (1.6), (1.7) that $P_\mu\langle 1, X_\tau\rangle\leqq\langle 1, \mu\rangle$ for every $\mu\in M$ and every τ and, by Fatou's lemma,

$$P_\mu \liminf\langle v_D, X_{\tau_n}\rangle\leqq N. \tag{2.21}$$

By (2.19) and (2.21),

$$\langle v_D, X_{\tau_n}\rangle\to 0\ P_{\delta_x}\text{-a.s.} \tag{2.22}$$

2.5 Let $\omega\to F(\omega)$ be a mapping from a measurable space $(\Omega, \mathscr{G})$ to the space of all closed sets in E. For every $A\subset E$ we put $\Omega^A=\{\omega: F(\omega)\cap A\neq\phi\}$. We say that F is *a random closed set* if $\Omega^K \in \mathscr{G}$ for all compact sets K. This is equivalent to the condition $\Omega^U \in \mathscr{G}$ for all open sets U. It is known (see, e.g., [MT], Chap. 2) that, for every analytic set B in E, $\Omega^B \in \mathscr{G}^*$ and, for every probability measure P on $\mathscr{G}$,

$$P(\Omega^B) = \sup P(\Omega^K) = \inf P(\Omega^U) \tag{2.23}$$

where K runs over all compact subsets of B and U runs over all open sets which contain B.

We say that an analytic set A is *B-polar* if

$$\Pi_\mu\{\xi_t \notin B \text{ for all } t>0\} = 1 \qquad \text{for all } \mu \in M. \tag{2.24}$$

The diffusion processes considered in this paper have the following property:

2.5.A There exists a measure m and a strictly positive measurable function $p_t(x, y)$ such that, for every $t>0$ and every $B \in \mathscr{E}^*$,

$$\Pi_x\{\xi_t \in B\} = \int_B p_t(x, y)\, m(dy).$$

Lemma 2.2 *Under condition* 2.5.A, *an analytic set* $B \neq E$ *is B-polar if and only if*

$$\Pi_x\{\xi_t \notin B \textit{ for all } t \geqq 0\} = 1 \qquad \textit{for all } x \notin B. \tag{2.25}$$

The condition (2.25) *is equivalent to*

$$P_{\delta_x}\{X_\tau = 0\} = 1 \qquad \textit{for all } x \notin B \tag{2.26}$$

where $\tau = \inf\{t\colon \xi_t \in B\}$.

Proof. Clearly (2.24) implies (2.25). On the other hand, by the Markov property,

$$\Pi_\mu\{\xi_t \in B \text{ for some } t \geqq \varepsilon\} = \Pi_\mu \Pi_{\xi_\varepsilon}\{\xi_t \in B \text{ for some } t \geqq 0\}. \tag{2.27}$$

By 2.5.A, if $\Pi_x\{\xi_\varepsilon \in B\} = 0$ for some x, then $\Pi_\mu\{\xi_\varepsilon \in B\} = 0$ for all $\mu \in M$ and (2.24) follows from (2.25) by (2.27).

By (1.6) and (1.7),

$$P_{\delta_x} \exp\langle -1, X_\tau\rangle = e^{-v(x)}$$

where

$$v(x) + \Pi_x \int_0^\tau \psi(\xi_s, v(\xi_s))\, ds = \Pi_x\{\tau < \infty\}.$$

Clearly, (2.25) and (2.26) are both equivalent to the condition: $v(x) = 0$ for all $x \notin B$.

2.6 For every $\varepsilon \geqq 0$, $\mathscr{R}_\varepsilon$ introduced in Sect. 1.5 is a random closed set. We denote by Ω_D^ε the union of sets $\{\mathscr{R}^\varepsilon \subset \Gamma\}$ over all compact $\Gamma \subset D$ and we put $v_D^\varepsilon(x) = -\log P_{\delta_x}(\Omega_D^\varepsilon)$.

Lemma 2.3 *For every* $\mu \in M$,

$$P_\mu(\Omega_D^\varepsilon) = e^{-\langle v_D^\varepsilon, \mu\rangle}. \tag{2.28}$$

Proof. Consider a sequence D_n described in Sect. 2.3 and put $\tau_n(\varepsilon)=\inf\{t: t \geqq \varepsilon, \xi_t \notin D_n\}$. By the Markov property of X,

$$P_\mu\{X_{\tau_n(\varepsilon)}=0\}=P_\mu P_{X_\varepsilon}\{X_{\tau_n}=0\} \geqq P_\mu\{X_\varepsilon=0\}. \tag{2.29}$$

The right side is equal to the limit as $k \to \infty$ of

$$P_\mu \exp\langle -k, X_\varepsilon\rangle=\exp\langle -v_\varepsilon^k, \mu\rangle \tag{2.30}$$

where v_t^k is the solution of (1.5) with $f=k$. By computation we check that

$$v_t^k=[(\alpha-1)\,t+k^{1-\alpha}]^{-1/(\alpha-1)}.$$

Clearly,

$$\lim_{k\to\infty} \langle v_\varepsilon^k, \mu\rangle=\langle 1, \mu\rangle[(\alpha-1)\,\varepsilon]^{-1/(\alpha-1)}<\infty.$$

By (2.29) and (2.30), $P_\mu\{X_{\tau_n(\varepsilon)}=0\}>0$, and (2.28) follows by the same arguments as in Sects. 2.3 and 2.4.

It will be proved in Sect. 3.4 that all superdiffusion have the property:

2.6.A For every x, $P_{\delta_x}\{\mathcal{R}$ is compact$\}=1$.

Lemma 2.4 *Under conditions* 2.5.A *and* 2.6.A, *every analytic set* $B \neq E$ *such that*

$$P_{\delta_x}\{\mathcal{R} \cap B \neq \phi\}=0 \quad \textit{for all } x \notin B \tag{2.31}$$

is S-polar.

Proof. Let $x^0 \notin B$. For every $s>0$,

$$0=P_{\delta_{x^0}}\{\mathcal{R}_s \cap B \neq \phi\} \geqq P_{\delta_{x^0}}\{X_s(B) \neq 0\}.$$

By (1.4), (1.5), this implies: $\Pi_{x^0}\{\xi_s \in B\}=1$ for all $s>0$. By 2.5.A, for every $\mu \in M$, $\Pi_\mu\{\xi_s \in B\}=1$ and therefore, by (1.4), (1.5), $P_\mu\{X_s(B)=0\}=1$.

Let $s<\varepsilon$ and let $t=\varepsilon-s$. By the Markov property

$$\begin{aligned} P_\mu\{\mathcal{R}_\varepsilon \cap B=\phi\} &= P_\mu\{X_s(B)=0, \mathcal{R}_\varepsilon \cap B=\phi\} \\ &= P_\mu\{X_s(B)=0, P_{X_s}(\mathcal{R}_t \cap B=\phi)\}. \end{aligned} \tag{2.32}$$

Let K be a compact subset of B and let $D=K^c$. By Lemma 2.3, for every $\nu \in M$,

$$P_\nu\{\mathcal{R}_t \cap K=\phi\} \geqq P_\nu(\Omega_D^t)=\exp\langle -v_D^t, \nu\rangle.$$

By 2.6.A and (2.6), for every $x \notin B$,

$$\exp\{-v_D^t(x)\} \geqq P_{\delta_x}\{\mathcal{R} \subset D\} \geqq P_{\delta_x}\{\mathcal{R} \cap B=\phi\}=1 \qquad \text{and therefore } v_D^t(x)=0.$$

Hence, if $\nu(B)=0$, then $P_\nu\{\mathcal{R}_t \cap K=\phi\}=1$ for all compact $K \subset B$ and therefore $P_\nu\{\mathcal{R}_t \cap B=\phi\}=1$. The statement of lemma follows from (2.32).

2.7 Now we introduce the following condition:

2.7.A For every $x \in E$ and every neighborhood U of x,

$$\inf\{t\colon X_t(U^c)>0\}>0 \; P_{\delta_x}\text{-a.s.}$$

If X is a superdiffusion and if $\psi=\frac{1}{2}z^2$, then 2.7.A follows from the results in Sect. 8 of [DP]. The arguments in [DP] can be extended to all ψ described at the beginning of Sect. 1.4.

Lemma 2.5 *Under condition* 2.7.A, *every S-polar analytic set B has the property* (2.31).

Proof. By 2.7.A, $\mathscr{R}\subset S_0\cup\mathscr{R}_+$ a.s. where $\mathscr{R}_+$ is the union of $\mathscr{R}_\varepsilon$ over all $\varepsilon>0$. Therefore

$$P_{\delta_x}\{\mathscr{R}\cap B\neq\phi\}\leqq P_{\delta_x}\{\mathscr{R}_+\cap B\neq\phi\}+P_{\delta_x}\{S_0\cap B\neq\phi\}.$$

If B is S-polar, then the first term is equal to 0. The second term vanishes for all $x\notin B$ because $P_{\delta_x}X_0(E\backslash x)\leqq\Pi_x\{\xi_0\in E\backslash x\}=0$.

3. Proofs of Theorems 1.1 through 1.5

3.1 *Proof of Theorem 1.1* We get (1.11) by setting $\mu=\delta_x$ in (1.6). The Eq. (1.7) can be rewritten as

$$v+F_1=h+F \tag{3.1}$$

where h and F are given by (0.3) and (0.6) and

$$F_1(x)=\Pi_x\int_0^\tau \psi(\xi_s, v(\xi_s))\,ds.$$

By 0.3, F is bounded and therefore v is also bounded. It is clear from (1.3) that $\rho_1(x)=\psi(x, v(x))$ is bounded. By 0.3.B, F and F_1 belong to $C^{0,\lambda}(D)$. By Theorem 0.2, $h\in C^{2,\lambda}(D)$. We conclude from (3.1) that $v\in C^{0,\lambda}(D)$. By 1.3.A, $\rho_1\in C^{0,\lambda}(D)$. Hence $F_1\in C^{2,\lambda}(D)$. Now 0.3.C, (0.4) and (3.1) imply (1.1). Formula (1.13) follows from 0.3.A and (0.5). The uniqueness of the solution follows from the Maximum Principle (Theorem 0.5).

3.2 Lemma 3.1 *Let* $U=\{x:|x-x^0|<R\}$ *and*

$$u(x)=\lambda(R^2-r^2)^{-2/(\alpha-1)} \tag{3.2}$$

where λ *is a positive constant and* $r=|x-x^0|$. *We have*

$$\lim_{x\to a,x\in U} u(x)=\infty \quad \text{for all } a\in\partial U \tag{3.3}$$

and, under conditions 0.2.A, B *and* (1.14),

$$Lu-\gamma u^\alpha\leqq 0 \quad \text{in } U \tag{3.4}$$

for

$$\lambda=c(1\vee R)^{3/(\alpha-1)} \tag{3.5}$$

where c is a constant which depends only on α, *the dimension d and the upper bounds for* $\tilde{a}_{ij}=a_{ij}/\gamma$ *and* $\tilde{b}_i=b_i/\gamma$ *in U.*

Proof. By a direct computation we get

$$(3.6)\quad Lu-\gamma u^{\alpha} = \lambda(R^2-r^2)^{-2\alpha/(\alpha-1)}\{c_1\sum a_{ij}z_iz_j+c_2(R^2-r^2)(\sum a_{ii}+\sum b_i z_i)-\gamma\lambda^{\alpha-1}\}$$

where $z_i=x_i-x_i^0$, $c_1=8(\alpha+1)(\alpha-1)^{-2}$, $c_2=4(\alpha-1)^{-1}$. Let $\Lambda(x)$ be the biggest eigenvalue of the matrix $a_{ij}(x)$ and let $B(x)^2=\sum b_i(x)^2$. Let Λ and B be upper bounds for $\Lambda(x)$ and $B(x)$ in U. Note that $\sum \tilde{a}_{ii}\leqq \Lambda d$ in U. Clearly (3.6) implies (3.4) if

$$(3.7)\qquad c_1\Lambda r^2+c_2 d\Lambda(R^2-r^2)+c_2BR^3-\lambda^{\alpha-1}\leqq 0$$

for all $0\leqq r\leqq R$. The condition (3.7) holds if

$$\lambda^{\alpha-1}\geqq(c_1+c_2d)\Lambda R^2+c_2BR^3$$

which is true for λ given by (3.5).

3.3 *Proof of Theorem 1.2* By Theorem 1.1, the function v_k determined by (2.12) satisfies (1.16) and the boundary condition

$$(3.8)\qquad \lim_{x\to a,\, x\in D} v_k(x)=k.$$

Let $U=\{|x-x^0|<R\}\subset D$ and let u be defined as in Lemma 3.1. By the Maximum Principle (Theorem 0.5), $v_k\leqq u$ in U. By (2.13), $v_k\uparrow v$ where v is defined by (1.15). The functions v_k are uniformly bounded on every compact set $\Gamma\subset U$ and the same is true for their partial derivatives of the first and second order. Hence $v_k(x)$ converge uniformly on Γ to a solution of (1.16) (see, e.g., Chap. 6 in [GT]). Since $v\geqq v_k$ for all k, we get

$$\liminf_{x\to a,\, x\in D} v(x)\geqq k$$

for all k which implies (1.17).

Let $u\geqq 0$ satisfy (1.16) and (1.17). Then, by the Maximum Principle, $v_k\leqq u$ in D and therefore $v\leqq u$.

Remark. The argument in the last paragraph is applicable to any function $u\geqq 0$ which satisfies the inequality $Lu-\gamma u^{\alpha}\leqq 0$ in D and the boundary condition (1.17). In particular, if $D=\{|x-x^0|<R\}$ and if u is the function given by (3.2) and (3.5), then, for every $\mu\in M_c(D)$,

$$(3.9)\qquad P_\mu\{X_\tau=0\}\geqq e^{-\langle u,\mu\rangle}.$$

3.4 *Proof of Theorem 1.3* Let τ_R be the first exit time from $U_R=\{|x|<R\}$. By Lemma 2.1, $\{\mathcal{R}\subset U_{2R}\}\supset\{X_{\tau_R}=0\}$. Consider the function $u=u_R$ defined by (3.2) and (3.5) (with $x^0=0$). For every x, $u_R(x)\to 0$ as $R\to\infty$ uniformly on every compact set. If follows from (3.9) that $\mathcal{R}$ is compact P_μ-a.s. if μ has compact support.

Consider a sequence of regular domains D_n which satisfy conditions (a), (b), (c) of Sect. 2.3. By (2.15), (2.16) and (2.6), $v^n=-\log P_{\delta_x}\{X_{\tau_n}=0\}\downarrow v$ where v is given by (1.18). By Theorem 1.2, v^n satisfies (1.16) and (1.17) in D_n. Same arguments as in proof of Theorem 1.2 show that v satisfies (1.16) in D. If u is an arbitrary positive solution in D, then $u \leqq v_n$ in D_n by Theorem 0.5. Hence $u \leqq v_D$.

Remark. For the sake of brevity, we call v *the maximal solution in D*. Note that $v \geqq u$ for every positive u such that $Lu - \gamma u^\alpha \geqq 0$ in D.

3.5 *Proof of Theorem 1.4* By Lemmas 2.4 and 2.5, the condition (1.19) is equivalent to 1.6.A. Clearly, 1.6.A implies 1.6.B. On the other hand, if 1.6.B holds, then, by (2.22), $\langle v_D, X_{\tau_n}\rangle \to 0$ P_{δ_x}-a.s. for all $x \in D$ where v_D is the maximal solution in D. We can assume that the open sets D_n are regular and then, by Theorem 1.1,

$$\exp[-v_D(x)] = P_{\delta_x} \exp\langle -v_D, X_{\tau_n}\rangle \quad \text{in } D$$

for every n. Passing to the limit as $n \to \infty$, we get that $v_D(x) = 0$. Hence 1.6.B implies 1.6.A.

3.6 Lemma 3.2 *If K is a compact set then the maximal solution v in $D = K^c$ tends to* 0 *as* $|x| \to \infty$.

Proof. Since K is contained in a ball $B = \{x: |x| \leqq \rho\}$ for some ρ and, since the maximal solution in a larger domain is smaller, it is sufficient to prove the lemma for $K = B$. If $|x^0| > 2R$ and $R > \rho$, then $U = \{|x - x^0| < R\} \subset K^c$ and, by Theorem 0.5,

$$v(x) \leqq u(x) \quad \text{in } U$$

where u is given by (3.2) and (3.5). In particular, $v(x^0) \leqq u(x^0) \leqq (1 \vee R)^{-1/(\alpha-1)}$. Hence $v(x^0) \to 0$ as $|x^0| \to \infty$.

Proof of Theorem 1.5 Clearly, $\{\mathcal{R} \subset \tilde{D}\} = \{\mathcal{R} \subset D, \mathcal{R} \subset K^c\}$. If K is S-polar, then $P_{\delta_x}\{\mathcal{R} \subset K^c\} = 1$ for all $x \in K^c$ and, by (1.18), $v = \tilde{v}$ in $\tilde{D}$.

On the other hand, the maximal solution u in K^c does not exceed $\tilde{v}$ in $\tilde{D}$. If $\tilde{v}$ is bounded near K, then u is bounded by Lemma 3.2, and K is S-polar by Theorem 1.4.

4. Proof of Theorems 1.6, 1.7 and 1.8

4.1 It is known (see [BP]) that $W^{-2,\alpha}(D) \subset \mathcal{K}_\alpha(D)$. Note that $W^{-2,\alpha}(D)$ contains all measures η for which the function

$$u(x) = \int_D \eta(dy)\, g(x, y)$$

belongs to $L^\alpha(D)$. Indeed, if $\varphi \in C_0^\infty(D)$ and if $f = -L\varphi$, then, by Theorem 0.3,

$$\varphi(x) = \int_D g(x, y) f(y)\, dy$$

and

$$\left|\int_D \varphi(x)\, \eta(dx)\right| = \left|\int_D u(y) f(y)\, dy\right| \leqq \|u\|_\alpha \, \|f\|_{\alpha'}$$

by Hölder's inequality. Since $\|f\|_{\alpha'} \leqq \text{const.}\, \|\varphi\|_{2,\alpha'}$, we conclude that $\eta \in W^{-2,\alpha}(D)$.

4.2 *Proof of Theorem 1.7* 1°. Consider a function $\gamma \geqq 0$ of class $C_0^\infty(\mathbb{R}^d)$ such that $\int \gamma \, dx = 1$. For every finite measure η, the function

(4.2) $$\rho_\varepsilon(x) = \varepsilon^{-d} \int_{\mathbb{R}^d} \gamma\left(\frac{x-y}{\varepsilon}\right) \eta(dx)$$

belongs to $C^\infty(\mathbb{R}^d)$. Put $\rho_{\beta\varepsilon} = (\rho_\beta + \rho_\varepsilon)/2$.

Suppose that D is bounded and ∂D is smooth. By Theorem 1.1,

(4.3) $$v_{\beta\varepsilon}(x) = -\log P_{\delta_x} \exp\langle -\rho_{\beta\varepsilon}, Y_\tau \rangle$$

satisfies the conditions

(4.4) $$-L v_{\beta\varepsilon} + v_{\beta\varepsilon}^\alpha = \rho_{\beta\varepsilon} \quad \text{in } D,$$
$$v_{\beta\varepsilon}(x) \to 0 \quad \text{as} \quad x \to a \in \partial D, \quad x \in D.$$

It is known (see [ADN] and [Ko]) that $\|u\|_{2,\alpha} \leqq c_\alpha \|Lu\|_\alpha$. Since $\rho_{\beta\varepsilon}$ and $v_{\beta\varepsilon}$ are bounded in $\bar{D}$, we conclude from (4.4) that $v_{\beta\varepsilon} \in W^{2,\alpha}$. Clearly, $v_{\beta\varepsilon}$ is a solution of the problem

(4.5) $$v_{\beta\varepsilon} \in W_0^{1,1}(D) \cap L^\alpha(D),$$
$$-L v_{\beta\varepsilon} + v_{\beta\varepsilon}^\alpha = \rho_{\beta\varepsilon} \quad \text{in } C_0^\infty(D).$$

Note that $\|\rho_{\beta\varepsilon}\|_1 = \eta(\mathbb{R}^d) < \infty$. By Theorem 0.4, $v_{\beta\varepsilon}$ is the unique solution of (4.5) and

(4.6) $$\|v_{\beta\varepsilon}\|_{1,q} + \|v_{\beta\varepsilon}\|_\alpha^\alpha \leqq C(q)\, \eta(\mathbb{R}^d)$$

for an arbitrary $q \in \left[1, \frac{d}{d-1}\right)$. Let $\beta_k \downarrow 0$ and $\varepsilon_k \downarrow 0$. One can choose a sequence $k_n \to \infty$ such that $v^n = v_{\beta_{k_n}\varepsilon_{k_n}}$ converges weakly both in $W_0^{1,1}(D)$ and in $L^\alpha(D)$ to a $\tilde{v} \in W_0^{1,1}(D) \cap L^\alpha(D)$. By Kondrashov's theorem (see [GT], Theorem 7.22), $W_0^{1,1}(D)$ is compactly imbedded in $L^q(D)$ and therefore there is a subsequence of v^n which converges in $L^q(D)$. Taking a subsequence once more, we get a sequence which converges a.e. in D. Call it $\tilde{v}_n$.

Suppose that $\eta \in W^{-2,\alpha}$. Arguments in [BP], pp. 195, 196 show that: (a) $\tilde{v}_n^\alpha$ are uniformly integrable and therefore $\|\tilde{v} - \tilde{v}_n\|_\alpha \to 0$; (b) $\tilde{v}$ is the unique solution of the problem

(4.7) $$\tilde{v} \in W_0^{1,1}(D) \cap L^\alpha(D),$$
$$-L\tilde{v} + \tilde{v}^\alpha = \eta \quad \text{in } C_0^\infty(D).$$

This is sufficient to conclude that $\|v_{\beta\varepsilon} - \tilde{v}\|_\alpha \to 0$ as $\beta, \varepsilon \downarrow 0$.

By (1.6),

(4.8) $$P_\mu[\exp\langle -\tfrac{1}{2}\rho_\beta, Y_\tau\rangle - \exp\langle -\tfrac{1}{2}\rho_\varepsilon, Y_\tau\rangle]^2$$
$$= \exp\langle -v_{\beta\beta}, \mu\rangle + \exp\langle -v_{\varepsilon\varepsilon}, \mu\rangle - 2\exp\langle -v_{\beta\varepsilon}, \mu\rangle.$$

If $\mu \in M_\alpha$, then $\langle v_{\beta\beta}, \mu\rangle$, $\langle v_{\varepsilon\varepsilon}, \mu\rangle$ and $\langle v_{\beta\varepsilon}, \mu\rangle$ converge to $\langle \tilde{v}, \mu\rangle$ and therefore $\langle \rho_\varepsilon, Y_\tau\rangle$ converges in P_μ-probability. The limit can be chosen independently of μ by the formula

$$Y_D^\eta = \lim \operatorname{med} \langle \rho_{1/n}, Y_\tau\rangle \tag{4.9}$$

(see, e.g., the Appendix in [D4]). We can choose a sequence $\varepsilon_k = 1/n_k$ such that, for almost all x, $v_{\varepsilon_k \varepsilon_l}(x) \to \tilde{v}(x)$ as k, $l \to \infty$. If, for some x, $v_{\varepsilon_k \varepsilon_l}(x) \to \tilde{v}(x)$, then $\langle \rho_{\varepsilon_k}, Y_\tau\rangle \to Y_D^\eta$ in P_{δ_x}-probability and therefore

$$P_{\delta_x} \exp\langle -\rho_{\varepsilon_k}, Y_\tau\rangle \to P_{\delta_x} \exp(-Y_D^\eta)$$

which implies that $\tilde{v}(x) = v(x)$ where v is defined by (1.31). Let us prove that v satisfies (1.32). Let $\varphi \in C^{0,\lambda}(\bar{D})$. Then by Theorem 0.3 (applied to L^*),

$$F(y) = \int_D \mathrm{d}x\, \varphi(x)\, g(x, y)$$

belongs to $C^{2,\lambda}(D)$. By (4.4),

$$\int_D \varphi(x)\, v_\varepsilon(x)\, \mathrm{d}x + \int_D F(y)\, v_\varepsilon^\alpha(y)\, \mathrm{d}y = \int_D F(y)\, \rho_\varepsilon(y)\, \mathrm{d}y. \tag{4.10}$$

By passing to the limit as $\varepsilon \to 0$, we get

$$\int_D \varphi(x)\, \tilde{v}(x)\, \mathrm{d}x + \int_D F(y)\, \tilde{v}^\alpha(y)\, \mathrm{d}y = \int_D F(y)\, \eta(\mathrm{d}y). \tag{4.11}$$

Therefore (1.32) holds for $\tilde{v}$. Since $v = \tilde{v}$ a.e., (1.32) holds also for v.

Now suppose that v is an arbitrary positive solution of (1.32) such that the right side belongs to $L^\alpha(D)$. Then $v \in L^\alpha(D)$ too and $v(x) = \int_D g(x, y)\, \tilde{\eta}(\mathrm{d}y)$ where $\tilde{\eta}(\mathrm{d}y) = \eta(\mathrm{d}y) - v(y)^\alpha \mathrm{d}y$ is a finite signed measure. The arguments used for $v_{\beta\varepsilon}$ can be applied to $u_\varepsilon(x) = \int_D g(x, y)\, \rho_\varepsilon(y)\, \mathrm{d}y$ and they demonstrate that $v \in W_0^{1,1}(D)$ and that $-Lv = \tilde{\eta}$ in $C_0^\infty(D)$ which implies that v is a solution of the problem (4.7). Since the solution of (4.7) is unique, the same is true for (1.32) as well.

Clearly, $P_\mu \exp(-Y_D^\eta) = \lim_{\varepsilon \downarrow 0} P_\mu \exp\langle -\rho_\varepsilon, Y_\tau\rangle = \lim_{\varepsilon \downarrow 0} \exp\langle -v_{\varepsilon\varepsilon}, \mu\rangle$ which implies 1.8.B.

2°. For every bounded domain D with smooth boundary, formula (4.9) defines a family Y_D^η, $\eta \in W^{-2,\alpha}$ which satisfies 1.8.A, B and D. Let η be an arbitrary measure of class $\mathscr{K}_\alpha(D)$. By Lemma 4.2 in [BP], there exists a sequence $\eta_n \in W^{-2,\alpha}$ such that $\eta_n \uparrow \eta$. By 1.8.D, the corresponding sequence $Y_D^{\eta_n}$ is monotone increasing and therefore there exists a limit

$$Y_D^\eta = \lim Y_D^{\eta_n}. \tag{4.12}$$

We claim that Y_D^η is defined by this formula uniquely up to P_μ-equivalence for all $\mu \in M_\alpha$. Indeed, let $\eta'_n \in W^{-2,\alpha}$, $\eta'_n \uparrow \eta$. First we assume that $\eta'_n \leq \eta_n$. Let $Y_n = Y_D^{\eta_n} \uparrow Y$, $Y'_n = Y_D^{\eta'_n} \uparrow Y'$. For every $\mu \in M_\alpha$, by 1.8.A, B,

$$P_\mu \exp(-Y_n) = \exp\langle -v_n, \mu\rangle \tag{4.13}$$

where

(4.14) $$v_n(x) = -\log P_{\delta_x} \exp(-Y_n)$$

satisfies the equation

(4.15) $$v_n(x) + \int_D g(x, y)\, v_n(y)^\alpha \,\mathrm{d}y = \int_D g(x, y)\, \eta_n(\mathrm{d}y) \text{ a.e.} \quad \text{in } D.$$

Clearly,

(4.16) $$P_\mu \mathrm{e}^{-Y} = \mathrm{e}^{-\langle v, \mu\rangle}$$

where $v = \lim v_n$ satisfies (1.32). Analogously,

(4.17) $$P_\mu \mathrm{e}^{-Y'} = \mathrm{e}^{-\langle v', \mu\rangle}$$

where $v' = \lim v'_n$ also satisfies (1.32). By Theorem 0.3 (applied to L^*), $\int_D \mathrm{d}x\, g(x, y)$ is bounded and therefore $\int_D g(x, y)\, \eta(\mathrm{d}y) < \infty$ a.e. in D for every finite measure η. Since $v' \leqq v$, we conclude that $v' = v$ a.e. in D. By 1.8.D, $Y' \leqq Y$ and, by (4.16), (4.17), $Y = Y'$ P_μ-a.s.

Let now $\eta'_n, \eta''_n \in W^{-2,\alpha}$ and $\eta'_n \uparrow \eta$, $\eta''_n \uparrow \eta$. Denote by η_n a measure defined by the formula $\mathrm{d}\eta_n = a' \vee a''\, \mathrm{d}\eta$ where $a' = \mathrm{d}\eta'/\mathrm{d}\eta$, $a'' = \mathrm{d}\eta''/\mathrm{d}\eta$. Clearly, $\eta_n \in W^{-2,\alpha}$ and $\eta_n \uparrow \eta$. Therefore

$$\lim Y_D^{\eta'_n} = \lim Y_D^{\eta''_n} = \lim Y_D^{\eta_n} \; P_\mu\text{-a.s.}$$

It is easy to see that Y_D^η defined by (4.12) satisfy conditions 1.8.A, B, D, E, F (to get 1.8.E, note that Y_τ is monotone increasing in τ by the Addendum to [D4] (see (2)); and to obtain 1.8.F, use that $g_n(x, y) \uparrow g(x, y)$ if $D_n \uparrow D$). 1.8.C follows from (4.9) if f is bounded and continuous because in this case $\rho_\varepsilon \to \varphi$ as $\varepsilon \downarrow 0$. By 1.8.D and F, 1.8.C holds for all positive Borel f.

3°. If D is an arbitrary domain and if $\eta \in \mathscr{K}_\alpha(D)$, then we consider a sequence of bounded domains with smooth boundaries which satisfy conditions (a)–(c) of Sect. 2.3 and we denote by η_n the restriction of η to D_n. By 1.8.E, $Y_{D_n}^{\eta_n}$ increase and we put

(4.18) $$Y_D^\eta = \lim Y_{D_n}^{\eta_n}.$$

Clearly, the limit does not depend on the choice of D_n and Y_D^η satisfy conditions 1.8.A through F.

4.3 Proof of Theorem 1.6 Class Q_α has the properties: (i) if $B \in Q_\alpha$, then Q_α contains all subsets of B; (ii) if B is a Borel set and if Q_α contains all compact subsets of B, then $B \in Q_\alpha$. The class of S-polar sets also possesses these properties. Therefore it is sufficient to prove the theorem for compact sets K.

Suppose that $K \in Q_\alpha$ and let $K \subset U \subset \bar{U} \subset D$ where U and D are bounded domains with smooth boundaries. By Theorem 3.1 in [BP] every positive solution v of the equation

(4.19) $$Lv = v^\alpha \quad \text{in } D \backslash K$$

belongs to $W^{2,p}(U)$ for all $p \geqq 1$ and, by the Sobolev imbedding theorem (see, e.g., [A], p. 97) $v \in C^{1,\lambda}(U)$ for every $\lambda < 1$. In particular, the maximal positive solution in $D \setminus K$ is bounded near K, and K is S-polar by Theorem 1.5.

Now suppose that K is S-polar and consider the domains $D_n = \left\{x: d(x, K) < \frac{1}{n}\right\}$ and the first exit time τ from $D = D_1$. We have

(4.20) $$\{\mathscr{R} \subset K^c\} \subset \bigcup_n \{\mathscr{R} \subset D_n^c\} = \bigcup_n \{X_t(D_n) = 0 \text{ for all } t \geqq 0\}.$$

Let f_n be a bounded positive continuous function such that $\{f_n > 0\} = D_n$. By Theorem 1 in the Addendum to [D4],

(4.21) $$\begin{aligned}\{X_t(D_n) = 0 \text{ for all } t \geqq 0\} &= \{\langle f_n, X_t\rangle = 0 \text{ for all } t \geqq 0\} \\ &\subset \{\langle f_n, Y_\tau\rangle = 0\} = \{Y_\tau(D_n) = 0\} \text{ a.s.}\end{aligned}$$

Suppose that $\eta \in W^{-2,\alpha}$ is concentrated on K and let ρ_ε be given by (4.2). By (4.20), (4.21) and (4.9)

$$\{\mathscr{R} \subset K^c\} \subset \bigcup_n \{Y_\tau(D_n) = 0\} = \bigcup_n \{\langle \rho_{1/n}, Y_\tau\rangle = 0\} \subset \{Y_D^\eta = 0\} \text{ a.s.}$$

Since K is S-polar, $P_{\delta_x}\{Y_D^\eta = 0\} \geqq P_{\delta_x}\{\mathscr{R} \subset K^c\} = 1$ for all $x \notin K$. By 1.8.A, $\int_D g(x, y)\, \eta(dy) = 0$ a.e. on D. Since $g(x, y) > 0$ for all $x, y \in D$, we conclude that $\eta = 0$. By Sect. 4.1, this implies $K \in Q_\alpha$.

Proof of Corollary 2. By Theorem 3.1 in [BP], u can be continued to a function v which belongs to the class $W^{2,p}(U)$ for every $p \geqq 1$ and every bounded open set U such that $\bar{U} \subset D$. If U has a smooth boundary, then $v \in C^{1,\lambda}(U)$ by the Sobolev imbedding theorem cited in the proof of Theorem 1.6.

4.4 *Proof of Theorem 1.8* is based on results in [Mey] and [AM] about the relationship between Hausdorff dimension and the Bessel capacity $B_{\alpha,p}$. For every analytic set A, $\mathrm{Cap}_\alpha(A)$ defined by (1.29)–(1.30) is equal to $[B_{2,\alpha'}(A)]^{1/\alpha'}$ where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Therefore $A \in Q_\alpha$ if and only if $B_{2,\alpha'}(A) = 0$.

By Theorem 20 in [Mey], the empty set is the sole element of Q_α if $\gamma = d - \kappa_\alpha = d - 2\alpha' < 0$.

If $\gamma > 0$, then, by Theorem 4.2 in [AM], $H_\gamma(A) < \infty$ implies $A \in Q_\alpha$ and, by Theorem 4.3 there, $H_{\gamma,\beta}(A) = 0$ for all $A \in Q_\alpha$, $\beta > a' - 1$.

Finally, by Theorem 4.1 in [AM], $H_{0,\alpha'-1}(A) < \infty$ implies that $A \in Q_\alpha$; and $H_{0,\beta}(A) = 0$ if $A \in Q_\alpha$ and $\beta > \alpha' - 1$.

Corollary follows immediately from Theorems 1.5 and 1.7 and the following remark: if $H-\dim A = b$, then $H_\gamma(A) = 0$ for $\gamma > b$ and $H_\gamma(A) = \infty$ for $0 < \gamma < b$; and if $L-\dim A = c$, then $H_{0,\beta}(A) = 0$ for $\beta > c$ and $H_{0,\beta}(A) = \infty$ for $0 < \beta < c$.

5. Survey of literature. Concluding remarks

5.1 In 1974 Loewner and Nirenberg [LN] have established a number of interesting properties of positive solutions of the equation

(5.1) $$\Delta v = v^{(d+2)/(d-2)};$$

(a) If D is a bounded domain with a smooth boundary, then (5.1) has a unique positive solution in D which tends to $+\infty$ at the boundary
(b) For an arbitrary domain D, there exists the maximal positive solution v_D.
(c) Suppose that ∂D is compact and $\tilde{D}=D\setminus K$ where $K\subset D$ is compact. If $H-\dim K<\frac{d}{2}-1$, then $v_{\tilde{D}}$ is bounded near K; if K is a smooth hypersurface with of dimension $>\frac{d}{2}-1$, then $v_{\tilde{D}}\to+\infty$ as x tends to K. (To compare this with the results in Sect. 1.9, note that $\frac{d}{2}-1=d-\kappa_\alpha$ for $\alpha=(d+2)/(d-2)$.)

5.2 Isolated singularities of the Eq.

(5.2) $$\Delta u=|u|^{\alpha-1}u, \qquad \alpha>1$$

have been studied in [BV], [L], [V2]. If we consider only positive solutions (which is natural from the probabilistic point of view), then (5.2) is equivalent to

(5.2a) $$\Delta v=v^\alpha.$$

[Recall that a probabilistic interpretation is known only for $1<\alpha\leqq 2$.] It was established in [BV] and [L] that the singularity is removable for $d\geqq\kappa_\alpha$. The most complete results for $d<\kappa_\alpha$ have been obtained in [V2]. In particular, it is proved that, if $3\leqq d<\kappa_\alpha$, then every positive solution v in $D\setminus\{0\}$ has either the form

(5.3) $$q_{\alpha,d}|x|^{-2/(\alpha-1)}[1+\varepsilon(x)]$$

or the form

(5.4) $$c|x|^{2-d}[1+\varepsilon(x)]$$

where $\varepsilon(x)\to 0$ as $x\to 0$, $q_{\alpha,d}=[2(\alpha-1)^{-1}(\kappa_\alpha-d)]^{1/(\alpha-1)}$ and $c\geqq 0$ is a constant.

A solution of the form (5.3) can be obtained by the probabilistic formula

(5.5) $$v(x)=-\log P_{\delta_x}\{\mathscr{R}\subset D\setminus\{0\}\}.$$

In particular,

(5.6) $$-\log P_{\delta_x}\{\mathscr{R}\subset\mathbb{R}^d\setminus\{0\}\}=q_{\alpha,d}|x|^{-2/(\alpha-1)}$$

(this follows from Lemma 3.2 and a slight modification of Theorem 0.5). A solution of the form (5.4) is given by the formula

(5.7) $$v(x)=-\log P_{\delta_x}\exp\{-c\,Y_D^{\delta_0}\}$$

where $Y_D^{\delta_0}$ corresponds to Dirac's measure by Theorem 1.7.

Isolated singularities for the equation $\Delta v=\psi(v)$ with a continuous increasing function ψ have been studied in [VV], [V3] and [RV]. (Note that all functions of the form (1.3) satisfy these conditions.)

5.3 In the present paper we use extensively a general theory of singularities developed by Baras and Pierre in [BP]. Most results are obtained for the case

$\psi(z)=z^{\alpha}$ but one sufficient condition of removability is given in terms of behavior of $\psi(z)$ as $z\to+\infty$. (Earlier an analogous criterion was established in [V1] in a more classical setting.)

5.4 The class of measure-valued Markov processes which we call superprocesses was introduced by S. Watanabe and D. Dawson. An enriched model including measures X_{τ} and Y_{τ} at random times τ was considered, first, in [D4]. A number of authors (Dawson, Perkins, Iscoe, Fitzsimmons, Roelly-Copoletta, Le Gall, Le Jan and others) contributed to the recent progress of the theory of superprocesses. We refer to bibliographical notes in [D3], [D4], [DIP] and [DP] for more details.

Dawson et al. have studied hitting properties of X for the super Brownian motion with the branching mechanism determined by $\psi(z)=\frac{1}{2}z^2$. The Eq. (1.16) has in this case the form

$$\Delta v=v^2. \tag{5.8}$$

Using rotation invariance and self-similarity arguments, Iscoe has established in [I] that (5.8) has a unique solution u in the unit ball $B=\{|x|<1\}$ which tends to $+\infty$ as $|x|\to 1$ and that, for every finite measure μ on B,

$$P_{\mu}\{X_t\{|x|>R\}=0 \text{ for all } t\geqq 0\}=\exp\langle -u_R, \mu\rangle$$

where $u_R(x)=R^{-2}\,u(x/R)$.

In [DIP] a sufficient conditions for S-polarity was given in terms of a Hausdorff measure. In [P] Perkins proved that A is S-polar if $\mathrm{Cap}_2 A=0$ and he has conjectured that the converse statement is also true. Dawson et al. have studied also the set of "k-multiple points" of X (which coincides with the range $\mathcal{R}$ for $k=1$). An extension of these results to the case $\alpha<2$ is a challenging open problem.

5.5 Suppose that L and ψ satisfy the conditions of Theorem 1.1 and, in addition, $a=0$ in formula (1.3). It follows from (1.6)–(1.7) that

$$P_{\mu}\langle f, X_{\tau}\rangle=\Pi_{\mu}f(\xi_{\tau}). \tag{5.9}$$

Let D be an arbitrary open set and let τ_n be a sequence of exit times constructed in Sect. 2.3. If v is a positive solution of the equation

$$Lv(x)=\psi(x, v(x)) \quad \text{in } D, \tag{5.10}$$

then $\Pi_x\, v(\xi_{\tau})\geqq v(x)$ and $\langle v, X_{\tau_n}\rangle$ is a submartingale relative to P_{μ}. The same is true for $\exp\langle -v, X_{\tau_n}\rangle$. Therefore the limit

$$Z=\lim_{n\to\infty}\langle v, X_{\tau_n}\rangle \tag{5.11}$$

in topology of $[0, +\infty]$ exists a.s. Clearly:

5.5.A Z is a shift-invariant functional of $\{X_{\tau_n}\}$.

5.5.B For every $\mu\in M_c(D)$, $P_{\mu}\,e^{-Z}=e^{-\langle v,\mu\rangle}$ where

$$v(x)=-\log P_{\delta_x}\,e^{-Z}. \tag{5.12}$$

5.5.C $P_{\delta_x}\{Z<\infty\}>0$ for all $x\in D$.

Conversely, if a function Z from Ω to $[0,+\infty]$ satisfies conditions 5.5.A, B, C, then v given by (5.12), is a positive solution of the Eq. (5.10). Indeed, by the special Markov property and 5.5.B, $P_{\delta_x}e^{-Z}=P_{\delta_x}\exp\langle -v, X_{\tau_n}\rangle$ for $x\in D_n$, and (5.10) follows from Theorem 1.1.

We see that description of all positive solutions of (5.10) can be reduced to the description of all random variables Z subject to conditions 5.5.A, B, C. This class contains $Z=\langle f, X_\tau\rangle$ for every Borel function f from ∂D to $[0, +\infty]$. Suppose that D is bounded and regular and that f is continuous in the topology of $[0,+\infty]$. It can be proved that

$$v(x)\to f(a) \quad \text{as} \quad x\to a\in\partial D, \qquad x\in D.$$

Theorem 1.2 is a particular case of this result.

Appendix

0.1 Let D be a domain in $\mathbb{R}^d$. We denote by ∂D the boundary of D and we put $\bar{D}=D\cup\partial D$. The class of all k times continuously differentiable functions in D is denoted by $C^k(D)$.

Let f be a real-valued function on $\bar{D}$ and let $0<\lambda\leqq 1$. We put $f\in C^{0,\lambda}(\bar{D})$ if f is Hölder continuous on $\bar{D}$ with exponent λ that is if there exists a constant c such that $|f(x)-f(y)|\leqq c|x-y|^\lambda$ for all $x, y\in\bar{D}$. Put $f\in C^{k,\lambda}(\bar{D})$ if $f\in C^k(D)$ and if the derivatives of order $\leqq k$ have extensions to $\bar{D}$ which belong to $C^{0,\lambda}(\bar{D})$. The class $C^{k,\lambda}(D)$ consists of functions f which belong to $C^{k,\lambda}(\bar{D}_1)$ for every domain D_1 such that $\bar{D}_1\subset D$.

We say that ∂D is of class $C^{k,\lambda}$ if it can be described locally by functions of class $C^{k,\lambda}$. As usual, a condition "∂D is smooth" means that $\partial D\in C^{k,\lambda}$ for a sufficiently big k.

0.2 Theorem 0.1 *Put $D_i=\partial/\partial x_i$. Let*

$$L=\sum_{i,j} a_{ij}D_iD_j+\sum_i b_iD_i \tag{0.1}$$

be a differential operator in $\mathbb{R}^d$ such that:

0.2.A *Functions $a_{ij}=a_{ji}$ and b_i are bounded and belong to $C^{0,\lambda}(\mathbb{R}^d)$.*

0.2.B *There exists a constant $\gamma>0$ such that*

$$\sum a_{ij}(x)u_iu_j\geqq\gamma\sum u_i^2$$

for all $x\in\mathbb{R}^d$ and all $u_1, \ldots, u_d$.

Then there exists a Markov process $\xi=(\xi_t, \Pi_x)$ in $\mathbb{R}^d$ with continuous paths such that, for every bounded continuous function f on $\mathbb{R}^d$,

$$F_t(X)=\Pi_x f(\xi_t)$$

is the unique solution of the equation

$$\partial F/\partial t = LF$$

with the property $F_t(x) \to f(x)$ *as* $t \downarrow 0$.

(Cf. Theorem 5.11 in [D].)

We call ξ *the diffusion with the generator L.*

0.3 Let

(0.2) $$\tau = \inf\{t: t>0, \xi_t \notin D\}.$$

A point $a \in \partial D$ is called *regular* if $\Pi_a\{\tau=0\}=1$. A domain D is regular if all points $a \in \partial D$ are regular. Every smooth domain is regular.

Theorem 0.2 *Let f be a bounded Borel function on* ∂D *and let*

(0.3) $$h(x) = \Pi_x f(\xi_\tau).$$

Then $h \in C^{2,\lambda}(D)$ *and*

(0.4) $$Lh = 0 \quad \text{in } D.$$

If a is a regular point of ∂D *and if f is continuous at a, then*

(0.5) $$h(x) \to f(a) \quad \text{as} \quad x \to a, \quad x \in D.$$

(Cf. Theorems 12.12, 13.3 and 13.9 in [D].)

Theorem 0.3 *Let* ρ *be a bounded Borel function in a bounded domain D and let*

(0.6) $$F(x) = \Pi_x \int_0^\tau \rho(\xi_s)\, ds.$$

Then:

0.3.A *F is bounded and*

(0.7) $$F(x) \to 0 \quad \text{as} \quad x \to a, \quad x \in D$$

for every regular point a of ∂D.

0.3.B $F \in C^{0,\lambda}(D)$.

0.3.C *If* $\rho \in C^{0,\lambda}(D)$, *then* $F \in C^{2,\lambda}(D)$ *and*

(0.8) $$LF = -\rho \quad \text{in } D.$$

0.3.D *If D belongs to class* $C^{1,\lambda}$ *and if* $\rho \in C^{0,\lambda}(\bar{D})$, *then* $F \in C^{2,\lambda}(\bar{D})$.

Proof. By Theorem 13.7 [D], the function $m(x) = \Pi_x \tau$ is bounded and $m(x) \to 0$ as $x \to a$, $x \in D$ if a is regular. This implies 0.3.A. According to Sect. 13.23 in [D], (0.6) is equivalent to

(0.9) $$F(x) = \int_D g(x, y)\, \rho(y)\, dy$$

where g is a continuous function from $D \times D$ to $(0, +\infty]$ called Green's function of L in D. For a smooth domain D, Giraud constructed $g(x, y)$ as a solution of integral equations. It follows from his construction that F satisfies 0.3.B, C and D (see [Mi], Sects. 21 and 13; for the case $L = \Delta$, a very clear presentation can be found in [GT], Chap. 4).

To extend 0.3.B and C to an arbitrary bounded domain D, consider a smooth domain U such that $\bar{U} \subset D$. Let σ be the first exit time from U. By the strong Markov property, $F = \tilde{F} + h$ in U where

$$\tilde{F}(x) = \Pi_x \int_0^\sigma \rho(\xi_s)\, ds, \qquad h(x) = \Pi_x F(\xi_\sigma).$$

By Theorem 0.2, $h \in C^{2,\lambda}(\bar{U})$. If $\rho \in C^{0,\lambda}(D)$, then, by 0.3.D, $C^{2,\lambda}(\bar{U})$ contains $\tilde{F}$ and therefore it contains F. Hence (0.8) holds in U which implies 0.3.C. For an arbitrary bounded Borel function ρ, $\tilde{F} \in C^{0,\lambda}(\bar{U})$. Hence $F \in C^{0,\lambda}(\bar{U})$ which implies 0.3.B.

0.4 If the coefficients a_{ij} and b_i are sufficiently smooth, then the operator L can be represented in divergence form

$$Lv = \sum_{i,j} D_j [a_{ij} D_i v] - \sum_i D_i [\tilde{b}_i v] - c v \tag{0.10}$$

where

$$\tilde{b}_i = -b_i + \sum_j D_j a_{ij}$$

and

$$c = -\sum_i D_i \tilde{b}_i.$$

We assume that $c \geqq 0$ and that a_{ij} and $\tilde{b}_i$ belong to $C^1(\mathbb{R})$. It is known (see [Mi], Sects. 10 and 21) that Green's functions of the adjoint operator

$$L^* u = \sum_{i,j} D_i [a_{ij} D_j u] + \sum_i \tilde{b}_i D_i u - c u, \tag{0.11}$$

is connected with Green's function for L by the formula $g^*(x, y) = g(x, y)$.

0.5 The Sobolev space $W^{k,\alpha}(D)$ is defined as the Banach space of all functions in $L^\alpha(D)$ which have weak derivatives up to order k belonging to $L^\alpha(D)$. (Two functions represent the same element of $W^{k,\alpha}$ if they coincide a.e. in D.) We denote by $C_0^\infty(D)$ the class of all infinitely differentiable functions in D, with compact supports and by $W_0^{k,\alpha}(D)$ the closure of $C_0^\infty(D)$ in $W^{k,\alpha}(D)$. We use notations $\|\cdot\|_\alpha$ and $\|\cdot\|_{k,\alpha}$ for the norms in L^α and in $W^{k,\alpha}$.

Let η be a finite measure on D. We set $\eta \in W^{-k,\alpha}$ if

$$\left| \int_D \varphi(x)\, \eta(dx) \right| \leqq C(\eta) \, \|\varphi\|_{k,\alpha'}$$

for all $\varphi \in C_0^\infty(D)$ (here $1/\alpha + 1/\alpha' = 1$).

Writing

(0.12) $$Lv=f \quad \text{in } C_0^\infty(D)$$

means

(0.13) $$\int_D L^* \varphi(x)\, v(x)\,\mathrm{d}x = \int_D \varphi(x) f(x)\,\mathrm{d}x \qquad \text{for all } \varphi \in C_0^\infty(D).$$

(Writing $Lv=\eta$ in $C_0^\infty(D)$ has an analogous meaning with $f(x)\,\mathrm{d}x$ replaced by $\eta(\mathrm{d}x)$ in (0.13).)

We need the following result proved in [BS], p. 577.

Theorem 0.4 *Suppose that L satisfies the conditions in Sect.* 0.4, *that D is bounded and has a sufficiently smooth boundary and that $g(u)$ is a continuous increasing function with $g(0)=0$. If $\rho \in L^1(D)$, then there exists a unique solution of the problem*

(0.14) $$u \in W_0^{1,1}(D), \qquad g(u) \in L^1(D),$$
$$-Lu+g(u)=\rho \quad \textit{in } C_0^\infty(D).$$

Moreover, for every $1 \leqq q < d/(d-1)$, u belongs to $W_0^{1,q}(D)$ and

(0.15) $$\|u\|_{1,q} + \|g(u)\|_1 \leqq C(q)\, \|\rho\|_1 .$$

0.6 Theorem 0.5 (Maximum principle) *Let L be an elliptic operator in a bounded domain D and let $\psi: D \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the condition*

(0.16) $$\psi(x,u) \geqq \psi(x,v) \quad \textit{for every } u \geqq v \in \mathbb{R}^+ \quad \textit{and every} \quad x \in D.$$

If $u, v \geqq 0$ belong to $C^2(D)$ and satisfy the conditions:

(0.17) $$Lu(x) - \psi(x, u(x)) \geqq Lv(x) - \psi(x, v(x)) \quad \textit{in } D$$

and

(0.18) $$\limsup_{x \to a,\, x \in D} [u(x)-v(x)] \leqq 0 \quad \textit{for all } a \in \partial D,$$

then $u(x) \leqq v(x)$ in D.

Proof. Let $w=u-v$. If our statement is false, then $\tilde{D}=\{x: x \in D,\ w(x)>0\}$ is not empty. By (0.16) and (0.17), $Lw(x) \geqq \psi(x,u(x)) - \psi(x,v(x)) \geqq 0$ in $\tilde{D}$. By (0.18), $\limsup w(x) \leqq 0$ as $x \to a \in \partial \tilde{D}$ which contradicts the Maximum Principle for linear elliptic equations (see, e.g., [GT], p. 32).

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