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Probability Theory and Related Fields

The connected components of the closed support of super Brownian motion

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Summary. The closed support of super Brownian motion in R^d is studied. It is shown that at a fixed time $t>0$ the mass of the process is located in connected components which are single points.

1. Introduction

Super Brownian motion is a measure valued process that arises as a limit of systems of branching Brownian motions. This limiting procedure and an exact definition of super Brownian motion is given in Sect. 2. Let $M_F(R^d)$ be the space of finite measures with the topology of weak convergence. Then a super Brownian motion has continuous paths with values in $M_F(R^d)$. We will write $S(m)$ for the closed support of a measure $m \in M_F(R^d)$. In dimension $d \geqq 2$ it is known (Dawson and Hochberg [2], Perkins [9]) that $S(X_t)$ is a Lebesgue null set for all $t>0$ almost surely. In dimension $d \geqq 3$ it is known that at a fixed $t>0$ the measure X_t can be recovered from its support. Indeed from Dawson and Perkins [3] Theorem 5.2

$$X_t(A) = c_1(d)\,\phi(A \cap S(X_t)) \qquad \text{for all Borel } A \subseteq R^d \tag{1}$$

where $\phi(\cdot)$ is the Hausdorff measure derived from the function $x^2 \log^+ \log^+ (1/x)$. This paper describes the connected components of $S(X_t)$.

The arguments that lead to the proof of the lower bound in (1) use covers of the support that have a structure like a Cantor set and Don Dawson asked the question:

$$\text{For fixed } t>0 \quad \text{is } S(X_t) \text{ a totally disconnected set?}$$

In this paper we give the following partial answer.

Theorem 1 *Let* $\mathrm{Comp}(x)$ *denote the connected component of* $S(X_t)$ *containing* x. *If* $d \geqq 3$ *then for all* $m \in M_F(R^d)$ *and* $t>0$, *with probability one*

$$\mathrm{Comp}(x) = \{x\} \qquad \text{for } X_t-\text{a.a.x.}$$

The proof of Theorem 1 uses a nonstandard model for super Brownian motion involving an infinitesimal branching Brownian motion. This has been used in Perkins [8] and Dawson et al. [4] to study the path properties of the process. In Sect. 2 of this paper we give the notation for the nonstandard model and in Sect. 3 we give the proof of Theorem 1. We finish this introduction with a sketch of the proof.

The proof of Theorem 1 uses an accurate estimate for the probability that super Brownian motion gives mass to a ball at a fixed time $t>0$. Exact asymptotics for such probabilities (as the size of the ball decreases) have been found by Dawson et al. [4]. For $d\geqq 3$ there exists a constant $c_2(d)$ such that for all $x\in R^d$, $m\in M_F(R^d)$, $\varepsilon>0$, $t\geqq\varepsilon^2$

$$P^m(X_t(B(x,\varepsilon))>0)\leqq c_2\varepsilon^{d-2}\int_{R^d} p_{t+\varepsilon^2}(x,y)\,\mathrm{d}m(y) \tag{2}$$

where $p_t(x,y)$ is the transition density for Brownian motion. This estimate comes from the proof of Theorem 1.3a in Dawson et al. [4].

Using the particle picture we consider the measure X_t as a superposition of a number of clusters. Each cluster is the collection of particles descended from a single particle alive at time $t-\delta$. We shall show using (2) that on average at least a fixed proportion of the mass of X_t is contained in clusters that are isolated from the remaining mass. A zero-one law allows to show that this will happen infinitely often along a sequence $\delta_n\to 0$. Hence all the mass at time t can be isolated in arbitrarily small clusters.

2. The nonstandard model

Super Brownian motion can be obtained using a system of binary branching Brownian motions. In these processes a collection of particles move through space. They move according to independent Brownian motions between generation times. At generation times each particle independently splits into two or dies with probability one half. As time proceeds a tree of Brownian motions is traced out by the descendants of each initial particle. We now give a construction for a binary branching Brownian motion that is taken from Dawson et al. [4] including a labelling system first used by Walsh [11] which allow us to point to any of the branches of the tree. This labelling system is used throughout the proof of Theorem 1.

Notation.

$N=\{0,1,2,\dots\}$

$R^d_\Delta=R^d\cup\{\Delta\}$ where Δ is added as a discrete point

$\delta_x=$ point mass at x

$$M_F^\mu(R^d)=\{(1/\mu)\sum_{i=1}^k\delta_{x_i}: x_i\in R^d, k\in N\}\qquad \text{for } \mu=1,2,\dots$$

For any metric space M we write $D(M)$ for the space of right continuous paths with left limits mapping $[0,\infty)\to M$ with the Skorohod topology and $C(M)$

for the space of continuous paths with the topology of uniform convergence on compacts.

Let $((Y_t: t \geqq 0), (P_0^y: y \in R^d))$ be a Brownian motion with state space R^d defined on some probability space $(\Omega_0, \mathscr{F}_0)$. Let e be a coin tossing random variable defined on $(\Omega_1, \mathscr{F}_1, P_1)$ taking the values 0 and 2 each with probability one half.

Let $I = \bigcup_{n \in N} (N \times \{0, 1\}^n)$. The elements of I will label the branches of the branching Brownian motion. If $\beta = (\beta_0, \beta_1, \ldots, \beta_j) \in I$ we write $|\beta| = j$ for the length of the label β. If β is of length j then it will label a branch upto time $(j+1)/\mu$. Write $\beta \sim t$ if $|\beta|/\mu \leqq t < (|\beta|+1)/\mu$ so that β labels a branch upto the first branching time after t. Let $\beta|_i = (\beta_0, \ldots, \beta_i)$ for $i \leqq j$. Call β a descendant of γ and write $\beta \succ \gamma$ if $\gamma = \beta|_i$ for some $i \leqq |\beta|$. Let $\sigma(\beta, \gamma) = |\beta| - \inf(j: \beta|_j \neq \gamma|_j)$ be the number of generations back that β split from γ.

Let $\Omega_2 = (D(R_\Delta^d) \times \{0, 1\})^I$, $\mathscr{F}_2 =$ product σ-field. Writing $\omega \in \Omega_2$ as $\omega = (Y^\alpha, e^\alpha)_{\alpha \in I}$ we define $G_n = \sigma((Y^\alpha, e^\alpha): |\alpha| < n)$. Fix $\mu \in N$ and $x_i \in R_\Delta^d$ $i = 0, 1, \ldots$ We wish to find a probability P on $(\Omega_2, \mathscr{F}_2)$ which satisfies for any measurable $A^\alpha \subseteq D(R_\Delta^d)$, $B^\alpha \subseteq \{0, 1\}$ the initial condition

$$P(\omega: (Y^\alpha, e^\alpha)_{|\alpha|=0} \in \prod_{|\alpha|=0} A^\alpha \times B^\alpha) = \prod_{|\alpha|=0} P_0^{x_{\alpha_0}}(Y_{\cdot \wedge (1/\mu)} \in A^\alpha) \cdot \prod_{|\alpha|=0} P_1(e \in B^\alpha) \tag{3}$$

and for all $n > 0$

$$\begin{aligned} P(\omega: (Y^\alpha, e^\alpha)_{|\alpha|=n} \in \prod_{|\alpha|=n} A^\alpha \times B^\alpha \mid G_n)(\omega) \\ = \prod_{|\alpha|=n} P_0^{x_{\alpha_0}}(Y_{\cdot \wedge ((n+1)/\mu)} \in A^\alpha \mid Y_{\cdot \wedge (n/\mu)} = Y^{\alpha|_{n-1}}_{\cdot \wedge (n/\mu)})(\omega) \prod_{|\alpha|=n} P_1(e \in B^\alpha) \end{aligned} \tag{4}$$

By an adaption of the Kolmogorov extension Theorem there exists a unique probability measure $P = P_2^{(x_i)_i, \mu}$ satisfying (3) and (4). It follows that

$$P(Y^\alpha_\cdot \in A) = P_0^{x_{\alpha_0}}(Y_{\cdot \wedge (|\alpha|+1)/\mu} \in A$$

so that each Y^α has the law of a Brownian motion upto time $(|\alpha|+1)/\mu$ when it is frozen. Also from (3), (4) $(e^\alpha: \alpha \in I)$ are I.I.D. copies of e and are independent of $(Y^\alpha: \alpha \in I)$. The e^α will indicate whether the particles split or die at the branching generations and this will be independent of the spatial motion. Finally from (3), (4) the random variables $(Y_t^\alpha: |\alpha| = n)$ are conditionally independent given G_n indicating that the particles move independently between branching times.

Let

$$\Omega = (R_\Delta^d)^N \times \Omega_2$$
$$\mathscr{F} = \text{product } \sigma\text{-field}.$$
$$P^{(x_i)_i, \mu} = \delta_{(x_i)_i} \times P_2^{(x_i)_i, \mu}.$$

Then for $\omega = ((x_i)_i, (Y^\alpha, e^\alpha)_{\alpha \in I})$ particles will start at those x_i that are not equal to Δ. Define the death times for the branches as

$$\tau^\alpha = \begin{cases} 0 & \text{if } \alpha_0 = \Delta \\ \min((i+1)/\mu: e^{\alpha|_i} = 0) & \text{if this set is nonempty} \\ (|\alpha|+1)/\mu & \text{otherwise} \end{cases}$$

To each branch $\alpha \in I$ we associate a corresponding particle which moves along the branch until the death time. So the position of the particle on the branch α is given by

$$N_t^\alpha = \begin{cases} Y_t^\alpha & \text{for } t < \tau^\alpha \\ \Delta & \text{for } t \geqq \tau^\alpha. \end{cases}$$

Define a filtration where if $j/\mu \leqq t < (j+1)/\mu$

$$\mathscr{A}_t^\mu = \sigma(Y^\alpha, e^\alpha : |\alpha| < j) \vee \bigcap_{u>t} \sigma(Y_s^\beta : |\beta| = j, s \leqq u), \qquad \mathscr{A}^\mu = \bigvee_{t>0} \mathscr{A}_t^\mu.$$

Now we attach mass $1/\mu$ to each particle and define a measure valued process $N^\mu : [0, \infty) \to M_F(E)$ by

$$\begin{aligned} N_t^\mu(A) &= (1/\mu) \times \#(N_t^\alpha \in A : \alpha \sim t) \\ &= (1/\mu) \sum_{\alpha \sim t} I(N_t^\alpha \in A). \end{aligned}$$

For bounded measurable f we write

$$N_t^\mu(f) = \int_E f(x)\, dN_t^\mu(x) = (1/\mu) \sum_{\alpha \sim t} f(N_t^\alpha)$$

where we shall always take $f(\Delta) = 0$. Then $N_t^\mu \in \mathscr{A}_t$ for all t and $N^\mu \in D(M_F(R^d))$ almost surely.

If $m_\mu = (1/\mu) \sum_{i=1}^{K} \delta_{x_i} \in M_F^\mu(E)$ then we extend $(x_i)_{i \leqq K}$ to $(x_i)_{i \in N}$ by setting $x_i = \Delta$ for $i > K$. We write P^{m_μ} for $P^{(x_i)_i, \mu}$. This ignores the order of the $(x_i)_i$ but note that the order does not affect the measure on $\sigma(N_t^\mu : t \geqq 0)$ in which we are mainly interested.

We shall need a strong Markov property. Let $T^\mu = (j/\mu : j = 1, 2, \ldots)$. In Perkins [8] Proposition 2.3 some shift operators are defined and a strong Markov property is proved for stopping times taking values in T^μ. (The construction in Perkins [8] for super stable processes is slightly different but the proposition applies here).

Theorem 2 (Watanabe [12].) *Suppose $m_\mu \in M_F^\mu(R^d) \to m \in M_F(R^d)$ weakly as $\mu \to \infty$. Then*

$$P^{m_\mu}(N^\mu \in \cdot) \to Q^m(\cdot) \quad \text{on } D(M_F(R^d)) \quad \text{as } \mu \to \infty. \tag{5}$$

The law Q^m is supported on the subset of continuous paths. We write Q^m again for the restriction to $C(M_F(R^d))$, X_t for the coordinate process and $\mathscr{F}_t^X$ for the canonical completed right continuous filtration. The family $(Q^m : m \in M_F(E))$ is a strong Markov family.

We now construct the nonstandard model. From Theorem 2 we see that by doing calculations on binary branching Brownian motions and using weak convergence arguments we can obtain results about super Brownian motion. The nonstandard model is simply a binary branching Brownian motion with the inter generation time $1/\mu$ being a positive infintesimal. This gives us a nonstandard object and we have to take its 'standard part' to obtain super Brownian motion. The advantage is that many of the weak convergence arguments are built into the model. Cutland [1] gives an introduction to nonstandard analysis for probabilists which is sufficient for our needs.

The binary branching Brownian motions live in a superstructure $V(S)$ where S contains the reals and various measure spaces $(\Omega_0, \mathscr{F}_0, P_0)$ e.t.c. The nonstandard model will live in an proper extended superstructure $V(^*S)$. We assume the existence of a saturated embedding $*: V(S) \to V(^*S)$ that satisfies the transfer principle. We will write elements of $^*R$ as underscored characters $\underline{x}, \underline{t}, \ldots$ We identify real numbers $r \in R$ with their images $^*r$. If M is a metric space we write $ns(^*M)$ for the set of nearstandard points in $^*M$. For neastandard $x \in {}^*M$ we write $st_M(x)$ for the standard part of x or $^\circ x$ if the metric space M is clear. Indeed for $\underline{r} \in {}^*R$ we shall often write r for the standard part.

We can consider the construction of the binary branching Brownian motion as a map $P:(R^d_\Delta)^N \times N \to M(\Omega)$ where $((x_i)_i, \mu) \to P^{(x_i)_i, \mu}$. Under the embedding we obtain a map $^*P$: $^*((R^d_\Delta)^N \times N) \to {}^*M(\Omega)$. So if $\mu \in {}^*N$, $(x_i)_i \in {}^*(R^d_\Delta)^N$ then $^*P^{(x_i)_i, \mu}$ is an internal probability on $(^*\Omega, {}^*\mathscr{A})$. We also have the embedding of all the particle structure e.g.

$$^*N^{*\beta}_{\underline{t}} \in {}^*R^d_\Delta \qquad \text{for all } {}^*\beta \in {}^*I, \ \underline{t} \in {}^*R_+.$$

To avoid a notational nightmare we drop the $*$ whenever the context makes clear we are talking about a nonstandard object. For instance we write

$$N^\beta_{\underline{t}} \in {}^*R^d_\Delta \qquad \text{for all } \beta \in {}^*I, \ \underline{t} \in {}^*R_+.$$

The transfer principle will allow us to do calculations with the nonstandard branching processes as easily as with their standard equivalents.

For $(X, \mathscr{X}, \nu)$ an internal measure space (for example $(^*R^d, {}^*\mathscr{B}, N_{\underline{t}})$ or $(^*\Omega, {}^*\mathscr{A}, {}^*P^{m_\mu})$), define a real valued set function $^\circ\nu$ on $\mathscr{X}$ by

$$^\circ\nu(A) = {}^\circ(\nu(A)) \qquad \text{for all } A \in \mathscr{X}.$$

Loeb showed that the finitely additive measure $^\circ\nu$ has a σ-additive extension denoted by $L(\nu)$ on the σ-algebra $\sigma(\mathscr{X})$ generated by $\mathscr{X}$. Let $L(\mathscr{X})$ be the completion of $\sigma(\mathscr{X})$ under $L(\nu)$. Then $(X, L(\mathscr{X}), L(\nu))$ is a standard measure space called a Loeb space. We write $(^*\Omega, \mathscr{F}, P^{m_\mu})$ for the Loeb space $(^*\Omega, L(^*\mathscr{A}), L(^*P^{m_\mu}))$. Also when m_μ is fixed we shall often write $P, E, {}^*P, {}^*E$ for $P^{m_\mu}, E^{m_\mu}, {}^*P^{m_\mu}, {}^*E^{m_\mu}$ respectively.

We now state the Theorem that links the nonstandard model to a super Brownian motion. Fix $\mu \in {}^*N \backslash N$.

Theorem 3 *Let $m \in M_F(R^d)$ and choose $m_\mu \in {}^*M^\mu_F(R^d)$ so that $st_{M_F(R^d)}(m_\mu) = m$. Then there is a unique (up to indistinguishability) continuous $M_F(R^d)$ valued process X_t on $(^*\Omega, \mathscr{F}, P^{m_\mu})$ such that P^{m_μ}* — a.s.

(6) $$X_t(A) = L(N^\mu_{\underline{t}})(st^{-1}(A)) \qquad \textit{for all } \underline{t} \in ns(^*[0, \infty)), \textit{ and Borel } A.$$

Moreover

$$P^{m_\mu}(X \in C) = Q^m(C) \qquad \textit{for all } C \in \mathscr{B}(C(M_F(R^d))).$$

For nearstandard $\underline{t} \in {}^*[0, \infty)$ *such that* $t > 0$ *the supports of* X_t *and* $N_{\underline{t}}^{\mu}$ *satisfy*

(7) $$S(X_t) = st_{R^d}(S(N_{\underline{t}}^{\mu})) \quad P^{m_\mu} - \text{a.s.}$$

This theorem is proved in Dawson et al. [4] Theorem 2.3 and Lemma 4.8.

Note that the total number of particles in a binary branching Brownian motion follows a Galton Watson process. Asymptotics for the probability of extinction for a Galton Watson process are well known (see Harris [7] p21–22). The following lemma interprets these results for the nonstandard model.

Lemma 4 *For nearstandard* $\underline{t}, \underline{x} \geqq 0$ *such that* $x, t > 0$

$$\mu^* P^{\mu^{-1}\delta_0}(N_{\underline{t}}^{\mu}(1) > 0) \approx 2t^{-1}$$

where we write $\underline{x} \approx \underline{y}$ *if* $\underline{x}$ *and* $\underline{y}$ *are infinitesimally close.*

3. Proof of Theorem 1

It will be convenient to prove the result first for super Brownian motion started at a point mass. The following result (Evans and Perkins [6] Corollary 2.4) will enable us to extend the result for super Brownian motion started at any $m \in M_F(R^d)$.

Theorem 5 *For any* $m_1, m_2 \in M_F(R^d)$ *and* $s, t > 0$ *the laws* $Q^{m_1}(X_t \in \cdot)$ *and* $Q^{m_2}(X_s \in \cdot)$ *are mutually absolutely continuous.*

Notation. For $\underline{t}, \underline{a}, \underline{\theta} \in T^{\mu}$, $\beta \sim \underline{t}$ let

$$Z^{\beta}(\underline{a}) = \mu^{-1} \sum_{\gamma \sim \underline{t} + \underline{a}^2, \gamma \succ \beta} I(N_{\underline{t}+\underline{a}^2}^{\gamma} \neq \Delta)$$

$$W^{\beta}(\underline{a}, \underline{\theta}) = \mu^{-1} \sum_{\gamma \sim \underline{t} + \underline{a}^2, \gamma \succ \beta} I(|N_{\underline{t}+\underline{a}^2}^{\gamma} - N_{\underline{t}}^{\beta}| > \underline{a}\,\underline{\theta})$$

$Z^{\beta}(\underline{a})$ is the mass of the 'cluster' of particles descended from $N_{\underline{t}}^{\beta}$ that are alive at time $\underline{t} + \underline{a}^2$. The following lemma shows that there is a good chance, independent of $\underline{a}$, that these particles have not spread more than a distance $O(\underline{a})$ from their common root. For $m \in M_F(R^d)$ choose $m_\mu \in {}^*M_F^{\mu}(R^d)$ such that $st_{M_F}(m_\mu) = m$.

Lemma 6 *For nearstandard* $\underline{a}, \underline{\theta} \in {}^*[0, \infty)$ *such that* $a = {}^\circ\underline{a} > 0$, $\theta = {}^\circ\underline{\theta} > 0$

(8) $$P^{m_\mu}(W^{\beta}(\underline{a}, \underline{\theta}) = 0 \mid Z^{\beta}(\underline{a}) > 0) = p(\theta)$$

(9) $$E^{m_\mu}(Z^{\beta}(\underline{a})\, I(W^{\beta}(\underline{a}, \underline{\theta}) = 0) \mid Z^{\beta}(\underline{a}) > 0) = r(\theta)\, a^2$$

where if $\theta \in R$, $\theta > 0$ *then* $p(\theta) > 0$, $r(\theta) > 0$ *and*

(10) $$Q^{(1/2)\delta_0}(X_1(B(0, \theta)^c) = 0) = \exp(p(\theta) - 1)$$

(11) $$E^{(1/2)\delta_0}(X_1(B(0, \theta))\, I(X_1(B(0, \theta)^c) = 0)) = r(\theta) \exp(p(\theta) - 1)$$

Proof. This is essentially due to the space-time-mass scaling property of super Brownian motion. For $\beta>0$ define K_β: $M_F(R^d)\to M_F(R^d)$ as follows

$$\int f(x)\,K_\beta m(\mathrm{d}x)=\int f(\beta x)\,m(\mathrm{d}x) \qquad \text{for all measurable } f.$$

Then for $m\in M_F(R^d)$ the law of the process $(X_t\colon t\geqq 0)$ under Q^m equals the law of the process $(\beta^{-1}K_{\beta_{-1/2}}X_{\beta t}\colon t\geqq 0)$ under $Q^{\beta K_{\beta^{1/2}m}}$. For a proof see Roelly-Coppoletta [10] Proposition 1.8.

Fix nearstandard $\underline{a}$, $\underline{\theta}$ such that $a, \theta>0$. For β such that $\beta|_0 \neq \Delta$ define

$$p(\underline{a}, \underline{\theta})={}^*P_\mu^m(W^\beta(\underline{a}, \underline{\theta})=0\,|\,Z^\beta(\underline{a})>0).$$

The value of $p(\underline{a}, \underline{\theta})$ does not depend on the choice of m_μ or β. Take $x_i=0$ for $i=1, \ldots, [\mu\underline{a}^2/2]$ and $x_i=\Delta$ otherwise, so that $st_{m_F}(m_\mu)=(a^2/2)\,\delta_0$. Then

$$\begin{aligned}
&Q^{(1/2)\delta_0}(X_1(B(0, \theta)^c)=0)\\
&=Q^{(a^2/2)\delta_0}(X_{a^2}(B(0, a\theta)^c=0)\\
&=P^{m_\mu}(N_{\underline{a}^2}(st^{-1}(B(0, a\theta)^c))=0)\\
&=P^{m_\mu}(N_{\underline{a}^2}(B(0, \underline{a}\,\underline{\theta})^c)=0)
\end{aligned}$$

using scaling and the fact that $X_1(\partial B(0, r))=0$, almost surely for any r. So

$$\begin{aligned}
&Q^{(1/2)\delta_0}(X_1(B(0, \theta)^c)=0)\\
&=P^{m_\mu}\left(\bigcap_{i=1}^{[\mu\underline{a}^2/2]}(W^{x_i}(\underline{a}, \underline{\theta})=0)\right)\\
&={}^\circ\prod_{i=1}^{[\mu\underline{a}^2/2]}{}^*P^{m_\mu}(W^{x_i}(\underline{a}, \underline{\theta})=0)\\
&={}^\circ\prod_{i=1}^{[\mu\underline{a}^2/2]}{}^*P^{m_\mu}(W^{x_i}(\underline{a}, \underline{\theta})=0\,|\,Z^{x_i}(\underline{a})>0)\,{}^*P^{m_\mu}(Z^{x_i}(\underline{a})>0)+{}^*P^{m_\mu}(Z^{x_i}(\underline{a})=0)\\
&={}^\circ[(1+(p(\underline{a}, \underline{\theta})-1)\,{}^*P^{m_\mu}(Z^{x^1}(\underline{a})>0))^{[\mu\underline{a}^2/2]}]\\
&=\exp({}^\circ p(\underline{a}, \underline{\theta})-1)
\end{aligned}$$

since $[\mu\underline{a}^2/2]\,{}^*P^{m_\mu}(Z^{x_1}(\underline{a})>0)\sim 1$ from lemma 4. So ${}^\circ p(\underline{a}, \underline{\theta})$ is constant in $\underline{a}$ and (8) and (10) follow taking $p(\theta)={}^\circ p(\underline{a}, \underline{\theta})$.

The calculation for $E^{(1/2)\delta_0}(X_1(B(0, \theta))\,I(X_1(B(0, \theta)^c)=0))$ is similar. Finally $Q^{\delta_0/2}(X_1(R^d)=0)=\exp(-1)$ so that $p(\theta), r(\theta)>0$ will follow if we can show

$$Q^{\delta_0/2}(X_1(B(0, \theta)^c)=0,\ X_1(R^d)\neq 0)>0. \tag{12}$$

But since the support of the process moves with finite speed (see Dawson et al. [4] Theorem 1.1), for small enough s we have

$$Q^{\delta_0/2}(X_s(B(0, \theta)^c)=0,\ X_s(R^d)\neq 0)>0$$

and Theorem 5 implies (12) holds. □

Notation. For $a\in(0, \infty)$ define

$$Q_a=\{(y, r, \delta)\in Q^d\times Q\times Q\colon r, \delta>0,\ |y|<r-\delta,\ |y|+r+\delta<a\}.$$

For $a \in [0, \infty)$, $m \in M_F(R^d)$ define $\mathrm{Ann}(m, a) \subseteq R^d$ by

$$\mathrm{Ann}(m, a) = \bigcup_{(y, r, \delta) \in Q_a} \{x \in R^d : m(z : r - \delta < |z - x - y| < r + \delta)\}.$$

For $\underset{\sim}{a} \in {}^*[0, \infty)$, $m \in {}^*M_F(R^d)$ define $\mathrm{Ann}(m, \underset{\sim}{a}) \subseteq {}^*R^d$ by

$$\mathrm{Ann}(m, \underset{\sim}{a}) = \bigcup_{(y, r, \delta) \in Q_a} \{\underset{\sim}{x} \in {}^*R^d : m(\underset{\sim}{z} : r - \delta < |\underset{\sim}{z} - \underset{\sim}{x} - y| < r + \delta)\}.$$

$\mathrm{Ann}(m, \underset{\sim}{a})$ is defined so that for $\underset{\sim}{x} \in \mathrm{Ann}(m, \underset{\sim}{a})$ there is a mass free annulus of positive standard rational thickness that disconnects $\underset{\sim}{x}$ from $B(\underset{\sim}{x}, \underset{\sim}{a})^c$.

Note that if $x \in \mathrm{Ann}(X_t, a)$ then $\mathrm{Comp}\,(x) \subseteq B(x, a)$. Thus

$$x \in \bigcap_{n=1}^{\infty} \mathrm{Ann}(X_t, n^{-1}) \Rightarrow \mathrm{Comp}(x) \subseteq \{x\}.$$

For $\underset{\sim}{t}, \underset{\sim}{a}, \underset{\sim}{\theta} \in T^\mu$, $\beta \sim \underset{\sim}{t}$, $N_{\underset{\sim}{t}}^\beta \neq \Delta$ define

$$V^\beta(\underset{\sim}{a}, \underset{\sim}{\theta}) = \mu^{-1} \sum_{\gamma \sim \underset{\sim}{t} + \underset{\sim}{a}^2, \gamma \not> \beta} I(|N_{\underset{\sim}{t} + \underset{\sim}{a}^2}^\gamma - N_{\underset{\sim}{t}}^\beta| < 2 \underset{\sim}{a} \underset{\sim}{\theta}).$$

Note that if $\underset{\sim}{a}, \underset{\sim}{\theta}, \underset{\sim}{t} \in T^\mu$, $a > 0$, $0 < \theta < 1/2$, $\beta \sim \underset{\sim}{t}$ are such that $N_{\underset{\sim}{t} - \underset{\sim}{a}^2}^\beta \neq \Delta$, $V^{\beta | \underset{\sim}{t} - \underset{\sim}{a}^2}(\underset{\sim}{a}, \underset{\sim}{\theta}) = W^{\beta | \underset{\sim}{t} - \underset{\sim}{a}^2}(\underset{\sim}{a}, \underset{\sim}{\theta}) = 0$ then there is a particle free annulus surrounding $N_{\underset{\sim}{t}}^\beta$, namely $|N_{\underset{\sim}{t}}^\beta - N_{\underset{\sim}{t} - \underset{\sim}{a}^2}^\beta| \leq a\theta$ and

$$N_{\underset{\sim}{t}}^\mu(\underset{\sim}{z} : 5 \underset{\sim}{a} \underset{\sim}{\theta}/4 < |\underset{\sim}{z} - N_{\underset{\sim}{t} - \underset{\sim}{a}^2}^\beta| < 7 \underset{\sim}{a} \underset{\sim}{\theta}/4) = 0.$$

We may shift the annulus slightly to be centered at a rational and have positive rational thickness so that $N_{\underset{\sim}{t}}^\beta \in \mathrm{Ann}(N_{\underset{\sim}{t}}^\mu, \underset{\sim}{a})$.

The following lemma shows that on average a positive fraction of the initial particles will lie inside $\mathrm{Ann}(N_{\underset{\sim}{t}}^\mu, \underset{\sim}{a})$.

Lemma 7 *For nearstandard $\underset{\sim}{a}^2, \underset{\sim}{t} \in T^\mu$ with $0 < a < 1$, $2a^{1/3} < t < \infty$, $d \geq 3$ if $st_M(m_\mu) = m \in M_F(R^d)$ then there exists a constant $\rho > 0$ depending only on d such that*

$$E^{m_\mu}(\mu^{-1} \sum_{\gamma \sim \underset{\sim}{t}} I(N_{\underset{\sim}{t}}^\gamma \in \mathrm{Ann}(N_{\underset{\sim}{t}}^\mu, \underset{\sim}{a}), N_{\underset{\sim}{t}}^\gamma \neq \Delta)) \geq \rho m(R^d).$$

Proof. The remark following the definition of $V^\gamma(\underset{\sim}{a}, \underset{\sim}{\theta})$ shows that

$$\begin{aligned} (13) \qquad & E(\mu^{-1} \sum_{\gamma \sim \underset{\sim}{t}} I(N_{\underset{\sim}{t}}^\gamma \in \mathrm{Ann}(N_{\underset{\sim}{t}}^\mu, \underset{\sim}{a}), N_{\underset{\sim}{t}}^\gamma \neq \Delta)) \\ & \geq E(\mu^{-1} \sum_{\gamma \sim \underset{\sim}{t}} I(N_{\underset{\sim}{t}}^\gamma \neq \Delta, W^{\gamma | \underset{\sim}{t} - \underset{\sim}{a}^2}(\underset{\sim}{a}, \underset{\sim}{\theta}) = 0, V^{\gamma | \underset{\sim}{t} - \underset{\sim}{a}^2}(\underset{\sim}{a}, \underset{\sim}{\theta}) = 0)) \\ & = E(\sum_{\gamma \sim \underset{\sim}{t} - \underset{\sim}{a}^2} Z^\gamma(\underset{\sim}{a}) I(V^\gamma(\underset{\sim}{a}, \underset{\sim}{\theta}) = W^\gamma(\underset{\sim}{a}, \underset{\sim}{\theta}) = 0)) \\ & = E(\sum_{\gamma \sim \underset{\sim}{t} - \underset{\sim}{a}^2} {}^*E(Z^\gamma(\underset{\sim}{a}) I(W^\gamma(\underset{\sim}{a}, \underset{\sim}{\theta}) = 0) | \mathscr{A}_{\underset{\sim}{t} - \underset{\sim}{a}^2}) {}^*P(V^\gamma(\underset{\sim}{a}, \underset{\sim}{\theta}) = 0 | \mathscr{A}_{\underset{\sim}{t} - \underset{\sim}{a}^2})) \end{aligned}$$

since conditional on $\mathscr{A}_{\underline{t}-\underline{a}^2}$ the random variables $V^\gamma(\underline{a},\underline{\theta})$ and $Z^\gamma(\underline{a})I(W^\gamma(\underline{a},\underline{\theta})=0)$ are *-independent. Now using the *-Markov property (see Perkins [8] Proposition 2.3)

$$\begin{aligned}&{}^\circ{}^*P(V^\gamma(\underline{a},\underline{\theta})=0)\,|\,\mathscr{A}_{\underline{t}-\underline{a}^2})\\&={}^\circ{}^*P^{N^\mu_{\underline{t}-\underline{a}^2}-\mu^{-1}\delta_{N^\gamma_{\underline{t}-a^2}}}(N_{\underline{a}^2}(B(\cdot,2\underline{a}\underline{\theta}))=0)|_{N^\gamma_{\underline{t}-a^2}}\\&=Q^{X_{t-a^2}}(X_{a^2}(B(\cdot,2a\theta))=0)|_{{}^\circ N^\gamma_{\underline{t}-a^2}}\end{aligned}$$

where in the second step we used the continuity in θ of $Q^m(X_s(B(x,2a\theta))=0)$ and the fact that $st_M(N^\mu_{\underline{t}-\underline{a}^2}-\mu^{-1}\delta_{N^\gamma_{\underline{t}-a^2}})=X_{t-a^2}$. Also

$$\begin{aligned}&{}^\circ{}^*E(\mu Z^\gamma(\underline{a})I(W^\gamma(\underline{a},\underline{\theta})=0)\,|\,\mathscr{A}_{\underline{t}-\underline{a}^2})\\&={}^\circ{}^*E(\mu Z^\gamma(\underline{a})I(W^\gamma(\underline{a},\underline{\theta})=0)\,|\,N^\gamma_{\underline{t}-\underline{a}^2}\neq\Delta)\,I(N^\gamma_{\underline{t}-\underline{a}^2}\neq\Delta)\\&={}^\circ[{}^*E(Z^\gamma(\underline{a})I(W^\gamma(\underline{a},\underline{\theta})=0\,|\,Z^\gamma(\underline{a})>0)\,\mu{}^*P(Z^\gamma(\underline{a})>0\,|\,N^\gamma_{\underline{t}-\underline{a}^2}\neq\Delta)]\,I(N^\gamma_{\underline{t}-\underline{a}^2}\neq\Delta)\\&=2r(\theta)\,I(N^\gamma_{\underline{t}-\underline{a}^2}\neq\Delta)\end{aligned}$$

using Lemma 6 and Lemma 4. Substituting into (13) we get

$$\begin{aligned}&E(\mu^{-1}\sum_{\gamma\sim\underline{t}}I(N^\gamma_{\underline{t}}\in\mathrm{Ann}(N^\mu_{\underline{t}},\underline{a}),\,N^\gamma_{\underline{t}}\neq\Delta))\\&\geqq 2r(\theta)\,E[\mu^{-1}\sum_{\gamma\sim\underline{t}}I(N^\gamma_{\underline{t}-\underline{a}^2}\neq\Delta)\,Q^{X_{t-a^2}}(X_{a^2}(B(\cdot,2a\theta))=0)|_{{}^\circ N^\gamma_{\underline{t}-a^2}}]\\&=2r(\theta)\,E[\int_{R^d}Q^{X_{t-a^2}}(X_{a^2}(B(x,2a\theta))=0)\,\mathrm{d}X_{t-a^2}(x)]\\&\geqq 2r(\theta)\,E[\int_{R^d}(1-c_2(2a\theta)^{d-2}(2\pi a^2(1+4\theta^2))^{-d/2}\\&\quad\cdot\int_{R^d}\exp(-(x-y)^2(2a^2(1+4\theta^2))^{-1})\,\mathrm{d}X_{t-a^2}(y))\,\mathrm{d}X_{t-a^2}(x)]\end{aligned}$$

using the estimate on hitting balls in (2). Lemma 8 gives an estimate on the expectation of this double integral and leads directly to

$$\begin{aligned}&E(\mu^{-1}\sum_{\gamma\sim\underline{t}}I(N^\gamma_{\underline{t}}\in\mathrm{Ann}(N^\mu_{\underline{t}},\underline{a}),\,N^\gamma_{\underline{t}}\neq\Delta))\\&\geqq 2r(\theta)\,m(R^d)(1-C\theta^{d-2}(1+4\theta^2)^{1-d/2}(2^{d/2}+m(R^d))).\end{aligned}$$

Now take $\theta>0$ small enough so that the right hand side is strictly positive. □

Lemma 8 *If* $0<a\leqq 1$, $t\geqq a^{1/3}$, $d\geqq 3$, $m\in M_F(R^d)$ *then*

$$E^m[\iint\exp(-(x-y)^2/2a)\,\mathrm{d}X_t\,\mathrm{d}X_t]\leqq a\,m(R^d)(2^{d/2}+m(R^d))$$

Proof. Let $p_t(x)$ be the Brownian transition density with associated semigroup P_t. We have for positive measurable f, g (see Dynkin [5] Theorem 1.1)

$$E^m[X_t(f)\,X_t(g)]=\int P_tf(x)\,\mathrm{d}m(x)\int P_tg(x)\,\mathrm{d}m(x)+\int\mathrm{d}m(x)\int_0^t P_{t-s}(P_sf\,P_sg)(x)\,\mathrm{d}s.$$

By approximating positive measurable $h(x, y)$ by functions of the form $\sum_{i=n}^{n} f_i(x) g_i(x)$ we have

$$(14)\qquad \begin{aligned} & E^m[\iint h(x,y)\,dX_t(x)\,dX_t(y)] \\ &= \int dm(z_1)\int dm(z_2)\int dx\int dy\, h(x,y)\, p_t(x-z_1)\, p_t(y-z_2) \\ &\quad + \int dm(x)\int_0^t ds\int dy\, p_s(x-y)\int dz_1\int dz_2\, p_s(y-z_1)\, p_s(y-z_2)\, h(z_1,z_2) \\ &\geqq (m(R^d))^2 \sup_{z_1,z_2} E[h(B_t^1, B_t^2)] + m(R^d)\int_0^t \sup_{z_1,z_2} E[h(B_s^1, B_s^2)] \end{aligned}$$

where B_t^1, B_t^2 are independent Brownian motions starting at z_1, z_2. For $h(x, y) = \exp(-(x-y)^2/2a)$ we have

$$\sup_{z_1,z_2} E[h(B_s^1, B_s^2)] = \int \exp(-x^2/2a)\, p_{2s}(x)\, dx \leqq (2a/a+2s)^{d/s}.$$

Substituting into (14) and using the bounds on a and t gives the result. □

Proof of Theorem 1. We prove the result first for the nonstandard model with $x_i=0$ for $i=1, \dots \mu$, $x_i=\Delta$ for $i>\mu$ so that if $m_\mu=\mu^{-1}\sum_i \delta_{x_i}$ then $P^{m_\mu}(X\in\cdot) = Q^{\delta_0}(X\in\cdot)$. From Lemma 7 we have for nearstandard $\underline{a}, \underline{t}\in T^\mu$ such that $0<a<1$, $2a^{1/3}<t<\infty$

$$\begin{aligned} 0<\rho &\leqq E(\mu^{-1}\sum_{\gamma\sim\underline{t}} I(N_{\underline{t}}^\gamma\in \mathrm{Ann}(N_{\underline{t}}^\mu, \underline{a}),\, N_{\underline{t}}^\gamma \neq \Delta)) \\ &= E(\mu^{-1}\sum_{\gamma\sim\underline{t}} {}^*P(N_{\underline{t}}^\gamma\in \mathrm{Ann}(N_{\underline{t}}^\mu, \underline{a}) \mid N_{\underline{t}}^\gamma\neq\Delta)\, I(N_{\underline{t}}^\gamma\neq\Delta)) \\ &= P(N_{\underline{t}}^\gamma\in \mathrm{Ann}(N_{\underline{t}}^\mu, \underline{a}) \mid N_{\underline{t}}^\gamma\neq\Delta) \end{aligned}$$

where $P(\cdot\mid N_{\underline{t}}^\gamma\neq\Delta)$ is the Loeb measure induced by ${}^*P(\cdot\mid N_{\underline{t}}^\gamma\neq\Delta)$. Since $\mathrm{Ann}(N_{\underline{t}}^\mu, \underline{a})$ decreases as $\underline{a}$ decreases we have for any $\gamma - \underline{t}$

$$(15)\qquad P(N_{\underline{t}}^\gamma\in \bigcap_{n=1}^{\infty} \mathrm{Ann}(N_{\underline{t}}^\mu, n^{-1}) \mid N_{\underline{t}}^\gamma\neq\Delta) \geqq \rho > 0.$$

We now use a zero-one law to show this probability is in fact 1.

Notation. Fix $\gamma\sim\underline{t}$. For $\underline{u}, \underline{v}\in T^\mu$, $\underline{u}\leqq\underline{v}$ define

$$\begin{aligned} \overline{\mathscr{H}}_{\underline{u},\underline{v}} &= {}^*\sigma(N_{\underline{t}}^\beta - N_{\underline{t}-\mu^{-1}\sigma(\beta,\gamma)}^\beta : \underline{u}<\mu^{-1}\sigma(\beta,\gamma)\leqq\underline{v}) \vee {}^*\sigma(N_{\underline{t}}^\gamma - N_{\underline{t}-\underline{s}}^\gamma : \underline{u}<\underline{s}\leqq\underline{v}) \\ \mathscr{H}_{\underline{u},\underline{v}} &= \sigma(\overline{\mathscr{H}_{\underline{u},\underline{v}}}) \\ \mathscr{H}_{\underline{v}} &= \bigvee_n \mathscr{H}_{n^{-1},\underline{v}} \\ \mathscr{H}_{0+} &= \bigcap_n \mathscr{H}_{n^{-1}}. \end{aligned}$$

The following two results are due to Ed Perkins (personal communication) and are used in the original proof of (1).

Proposition 9 *For* $A \in \mathscr{H}_{0+}$, $P(A \mid N_{\underline{t}}^{\gamma} \neq \Delta) = 0$ *or* 1.

Proposition 10 *If* $0 < \alpha < 2^{-4/d}$ *then* $P(\cdot \mid N_{\underline{t}}^{\gamma} \neq \Delta)$ *almost surely, for r small enough*

$$d(\{N_{\underline{t}}^{\beta} \colon \mu^{-1}\sigma(\beta, \gamma) \geqq (2r)^{\alpha}\}, N_{\underline{t}}^{\gamma}) > r.$$

For $(y, r, \delta) \in Q_a$, $\underline{u}, \underline{v} \in T^{\mu}$, $\underline{u} \leqq \underline{v}$ define

$$\Gamma_{y,r,\delta,\underline{u},\underline{v}}(\gamma) = \{\omega \colon |N_{\underline{t}}^{\beta} - N_{\underline{t}}^{\gamma} - y| \notin (r-\delta, r+\delta) \quad \text{for all } \beta \text{ s.t. } \underline{u} < \mu^{-1}\sigma(\beta, \gamma) \leqq \underline{v}\}$$

$$\Gamma(\gamma) = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{(y,r,\delta) \in Q_{n^{-1}}} \bigcap_{j=k+1}^{\infty} \Gamma_{y,r,\delta,j^{-1},k^{-1}}(\gamma) \in \mathscr{H}_{0+}$$

If $\omega \in \bigcap_{n=1}^{\infty} \{N_{\underline{t}}^{\gamma} \in \operatorname{Ann}(N_{\underline{t}}, n^{-1})\}$ then $\omega \in \Gamma(\gamma)$ and so (15) and the zero-one law imply $P(\Gamma(\gamma) \mid N_{\underline{t}}^{\gamma} \neq \Delta) = 1$. Let

$$\Lambda(\gamma) = \{\omega \colon \text{ for small } r, d(\{N_{\underline{t}}^{\beta} \colon \mu^{-1}\sigma(\beta, \gamma) \geqq (2r)^{\alpha}\}, N_{\underline{t}}^{\gamma}) > r\}$$

so that Proposition 10 says $P(\Lambda(\gamma) \mid N_{\underline{t}}^{\gamma} \neq \Delta) = 1$. Then

$$\begin{aligned}
&E(\mu^{-1} \sum_{\gamma \sim \underline{t}} I(\omega \notin \Gamma(\gamma) \cap \Lambda(\gamma), N_{\underline{t}}^{\gamma} \neq \Delta)) \\
&= E(\mu^{-1} \sum_{\gamma \sim \underline{t}} {}^{*}P(\omega \notin \Gamma(\gamma) \cap \Lambda(\gamma) \mid N_{\underline{t}}^{\gamma} \neq \Delta) I(N_{\underline{t}}^{\gamma} \neq \Delta)) \\
&= P(\omega \notin \Gamma(\gamma) \cap \Lambda(\gamma) \mid N_{\underline{t}}^{\gamma} \neq \Delta) \\
&= 0.
\end{aligned}$$

A global modulus of continuity for the movement of all the particles (see Dawson et al. [4] Theorem 4.5), implies that with probability one all the particles move only an infinitesimal distance in an infinitesimal time. So (7) and the above imply we can pick a single P null set N such that if $\omega \notin N$ we have simultaneously

(16) $${}^{\circ}(\mu^{-1} \sum_{\gamma \sim \underline{t}} I(\omega \notin \Gamma(\gamma) \cap \Lambda(\gamma), N_{\underline{t}}^{\gamma} \neq \Delta)) = 0.$$

(17) For all nearstandard $\underline{s} < \underline{t}$ and $\beta \sim \underline{t}$, $N_{\underline{t}}^{\beta} \neq \Delta$ we have $N_{\underline{s}}^{\beta} \approx N_{\underline{t}}^{\beta}$

(18) $$st(S(N_{\underline{t}})) = S(X_t)$$

Now fix $\omega \notin N$, $\gamma \sim \underline{t}$ such that $N_{\underline{t}}^{\gamma} \neq \Delta$, $\omega \in \Gamma(\gamma) \cap \Lambda(\gamma)$. We claim

$${}^{\circ}N_{\underline{t}}^{\gamma} \in \bigcap_{n=1}^{\infty} \operatorname{Ann}(X_t, n^{-1}).$$

To show this find k so that

$$\omega \in \bigcap_{n=1}^{\infty} \bigcup_{(y,r,\delta)\in Q_{n^{-1}}} \bigcap_{j=k+1}^{\infty} \Gamma_{y,r,\delta,j^{-1},k^{-1}}(\gamma).$$

Find r_0 such that $(2r_0)^{1/2} \leqq k^{-1}$ and

$$d(\{N_{\underline{t}}^{\beta}: \mu^{-1}\sigma(\beta,\gamma) \geqq (2r_0)^{1/2}\}, N_{\underline{t}}^{\gamma}) > r_0. \tag{19}$$

Pick n so that $n^{-1} \leqq r_0$ and find $(y, r, \delta) \in Q_{n^{-1}}$ so that

$$\omega \in \bigcap_{j=k+1}^{\infty} \Gamma_{y,r,\delta,j^{-1},k^{-1}}(\gamma). \tag{20}$$

For $\beta \sim \underline{t}$ such that $\mu^{-1}\sigma(\beta,\gamma) \geqq (2r_0)^{1/2}$, (19) ensures that $|N_{\underline{t}}^{\beta} - N_{\underline{t}}^{\gamma}| > n^{-1}$ and so $|N_{\underline{t}}^{\beta} - N_{\underline{t}}^{\gamma} - y| \notin (r-\delta, r+\delta)$.

For $\beta \sim \underline{t}$ such that $0 < {}^{\circ}(\mu^{-1}\sigma(\beta,\gamma)) \leqq k^{-1}$, (20) and the definition of $\Gamma_{y,r,\delta,j^{-1},k^{-1}}(\gamma)$ ensure $|N_{\underline{t}}^{\beta} - N_{\underline{t}}^{\gamma} - y| \notin (r-\delta, r+\delta)$.
For $\beta \sim \underline{t}$ such that $\mu^{-1}\sigma(\beta,\gamma) \approx 0$ equation (17) ensures $N_{\underline{t}}^{\beta} \approx N_{\underline{t}}^{\gamma}$. So

$$\{z \in {}^*R^d: r-\delta < |z - N_{\underline{t}}^{\gamma} - y| < r+\delta\} \cap \{N_{\underline{t}}^{\beta}: \beta \sim \underline{t}\} = \emptyset.$$

Equation (18) now gives

$$\{z \in R^d: r-\delta < |z - {}^{\circ}N_{\underline{t}}^{\gamma} - y| < r+\delta\} \cap S(X_t) = \emptyset.$$

Since we were free to pick n arbitrarily large this proves the claim. Now Eq. (16) and the claim give

$${}^{\circ}\left[\mu^{-1} \sum_{\gamma \sim \underline{t}} I\left({}^{\circ}N_{\underline{t}}^{\gamma} \notin \bigcap_{n=1}^{\infty} \operatorname{Ann}(X_t, n^{-1}),\ N_{\underline{t}}^{\gamma} \neq \Delta\right)\right] = 0$$

so that for P^{δ_0} – a.a. ω

$$X_t\left(R^d \setminus \bigcap_{n=1}^{\infty} \operatorname{Ann}(X_t, n^{-1})\right) = 0. \tag{21}$$

It is possible to show that the map $m \to m\left(R^d \setminus \bigcap_{n=1}^{\infty} \operatorname{Ann}(m, n^{-1})\right)$ is Borel measurable. So we can apply Theorem 5 and conclude that for any $m \in M_F(R^d)$ equation (21) holds for Q^m – a.a. ω. □

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