Probab. Th. Rel. Fields 89, 61–74 (1991)

Probability Theory and Related Fields

Limit theorems for transient diffusions on the line

D.G. Hobson and L.C.G. Rogers *

Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, UK

Received October 17, 1990

Summary. Let X be a diffusion in natural scale on $(0, 1]$, with 1 reflecting, and let $c(x) \equiv \mathbb{E}(H_x)$ and $v(x) \equiv \operatorname{var}(H_x)$, where $H_x = \inf\{t: X_t = x\}$. Let $\sigma_x = \sup\{t: X_t = x\}$. The main results of this paper are firstly that (i) c is slowly varying; (ii) $c(X_t)/t \xrightarrow{\mathbb{P}} 1$; (iii) $H_x/c(x) \xrightarrow{\mathbb{P}} 1$; (iv) $\sigma_x/c(x) \xrightarrow{\mathbb{P}} 1$ are all equivalent: and secondly that (v) $c(X_t)/t \xrightarrow{\text{a.s.}} 1$; (vi) $H_x/c(x) \xrightarrow{\text{a.s.}} 1$; (vii) $\sigma_x/c(x) \xrightarrow{\text{a.s.}} 1$ are all equivalent, and are implied by the condition $\int_{0+} c(x)^{-2}\, dv(x) < \infty$. Other partial results for more general limit theorems are proved, and new results on regular variation are established.

1. Introduction and notation

Limit theorems for transient one-dimensional diffusions have been studied before by Gikhman and Skorokhod [3], Keller et al. [5] amongst others. The starting point in these works is to take the diffusion in the form of the solution of a stochastic differential equation (SDE)

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt$$

and impose conditions on the coefficients σ and b. Broadly speaking, these conditions amount to saying that the drift b will drive a transient ordinary differential equation when σ is replaced by 0, and the diffusion co-efficient σ is small enough not to perturb the solution of the SDE significantly from the deterministic trajectory. The case where both σ and b are positive constants is the paradigm example. However, the conditions one obtains change in appearance whenever one changes scale, and this seems an unnatural property for a limit theorem; two diffusions related by a scale change are essentially the same, and any theorem one proves for one should hold for the other. Motivated by these considerations, our aim here has been to establish limit results for

* *Current address:* School of Mathematical Sciences, Queen Mary + Westfield College, Mile End Rd., London E14NS, UK

one-dimensional diffusions in natural scale, which can then be used on the general one-dimensional diffusion. Specifically, we shall investigate the limit behaviour of $f(X_t)/g(t)$, where f and g are suitable functions. It turns out that there is a very close link between limit behaviour of the diffusion X and of the first-passage and last-exit times.

We shall consider a one-dimensional diffusion X in natural scale with speed measure m supported in (0, 1). Suppose that X starts at 1 and tends to zero, but does not hit zero in finite time. This is no real loss of generality, because if the diffusion is in natural scale in $(0, \infty)$, say, it will almost surely spend only finite time in $(1, \infty)$, and by time-changing out this finite amount of time, we get a diffusion on (0, 1] which will obey the same limit laws as the original. We shall assume that 1 is instantaneously reflecting. For background on one-dimensional diffusions, see Chap. V of Rogers and Williams [8]. The condition on m for 0 to be approached but not reached in finite time is simply $\int_{0+} x\,m(dx) = \infty$. For $0 < x \leq 1$ define

$$H_x = \inf\{t: X_t = x\}, \qquad \sigma_x = \sup\{t: X_t = x\}.$$

We shall frequently use the abbreviations

$$\mathbb{E}(H_x) \equiv c(x), \qquad \operatorname{var}(H_x) \equiv v(x).$$

The following expressions in terms of the speed measure m will also be used:

$$\mathbb{E}(H_x) \equiv c(x) = 2 \int_x^1 (y-x)\, m(dy) \tag{1}$$

$$= 2 \int_x^1 m(y)\, dy \tag{2}$$

and

$$\operatorname{var}(H_x) \equiv v(x) = 8 \int_x^1 (y-x)\, m(y)^2\, dy \tag{3}$$

$$= 8 \int_x^1 dy \int_y^1 dz\, m(z)^2. \tag{4}$$

Here, we have used the shorthand $m(x) = m(x, 1)$. For a proof of these relations, see Sect. 4. It follows easily from (2) and (4) and the fact that $m(x)$ is a decreasing function of x that $v(x) \leq c(x)^2$.

Throughout, we shall take f: $(0, 1] \to \mathbb{R}^+$ and g: $\mathbb{R}^+ \to \mathbb{R}^+$ to be unbounded continuous functions, respectively decreasing and increasing; we shall frequently take g to be the identity function. Here are our main results.

Theorem 1.1 *The following are equivalent:*

i) $c(x) \equiv \mathbb{E}(H_x)$ *is slowly varying,*

ii) $$\frac{H_x}{\mathbb{E}(H_x)} \xrightarrow{\mathbb{P}} 1,$$

iii) $$\frac{c(X_t)}{t} \xrightarrow{\mathbb{P}} 1,$$

iv) $$\frac{\sigma_x}{\mathbb{E}(H_x)} \xrightarrow{\mathbb{P}} 1.$$

Moreover, if for F non-degenerate

$$\frac{f(x)}{H_x} \xrightarrow{\mathscr{D}} F, \quad \frac{f(x)}{\sigma_x} \xrightarrow{\mathscr{D}} F$$

then

$$\frac{f(x)}{c(x)} \to \lambda \in (0, \infty)$$

and F is a point mass.

Keller et al. [5] showed that if $\mathrm{var}(H_x) \to \infty$ and

(5) $$\mathrm{var}(H_x)/c(x)^2 \to 0$$

then H_x obeys a Central-Limit Theorem type result:

$$\frac{H_x - c(x)}{\sqrt{\mathrm{var}\, H_x}} \xrightarrow{\mathscr{D}} N(0, 1).$$

We show using Theorem 1.1 that the condition of Eq. (5) is equivalent to $c(x)$ slowly varying.

Theorem 1.2 *The following are equivalent:*

i) $$\frac{c(X_t)}{t} \xrightarrow{\text{a.s.}} 1,$$

ii) $$\frac{H_x}{\mathbb{E}(H_x)} \xrightarrow{\text{a.s.}} 1,$$

iii) $$\frac{\sigma_x}{\mathbb{E}(H_x)} \xrightarrow{\text{a.s.}} 1.$$

A necessary condition for any (all) of the above is $c(x)$ slowly varying; a sufficient condition is $c(x)$ slowly varying and

$$\int_{0+} \frac{\int_x^1 m(y, 1)^2 \, dy}{c(x)^2} \, dx < \infty.$$

These are our results for the case $g=\mathrm{id}$, but we also establish a number of 'structural' results for the convergence of $f(X_t)/g(t)$, $f(x)/g(H_x)$, and $f(x)/g(\sigma_x)$. Typical of these is the following.

Proposition 1.1 i) *Suppose any two of the following conditions hold for a non-degenerate distribution function* F:
(a) f *is slowly varying,*

(b) $f(x)/g(H_x) \xrightarrow{\mathscr{D}} F$,

(c) $f(x)/g(\sigma_x) \xrightarrow{\mathscr{D}} F$.

Then so does the third, and $f(X_t)/g(t) \xrightarrow{\mathscr{D}} F$.

(ii) *Consider the conditions:*

(d) $f(X_t)/g(t) \xrightarrow{\text{a.s.}} Z$,

(e) $f(x)/g(H_x) \xrightarrow{\text{a.s.}} Z$,

(f) $f(x)/g(\sigma_x) \xrightarrow{\text{a.s.}} Z$.

Then (d) *implies* (e) & (f), *and* (e) & (f) *implies* (d).

Remark 1.1 In i) the other implications fail in general. For an example in which $f(X_t)/g(t)$, $f(x)/g(H_x)$ and $f(x)/g(\sigma_x)$ have different weak limits let X be BES(3), $f(x)=x^2$ and $g=\mathrm{id}$. Then it is well known that (with $\theta=(2\lambda)^{1/2}$)

$$\mathbb{E}\exp(-\lambda H_x)=\theta x/\sinh\theta x;$$

$$\mathbb{E}\exp(-\lambda\sigma_x)=\exp(-\theta x);$$

$$\mathbb{E}\exp(-\lambda X_t^2/t)=(1+2\lambda)^{-3/2};$$

and thus the three limit laws exist, but are distinct.

Remark 1.2 Theorem 1.2 improves on Proposition 1.1 in the special case where g is the identity function; then (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f). It is natural to ask whether this improvement is possible in the general case. Remark 3.1 answers this question in the negative by providing a counterexample to (d) $\Leftrightarrow$ (e). Remark 3.1 also includes an example, this time with $g=\mathrm{id}$, of a weak limit which is not an almost sure limit.

Remark 1.3 Rösler [9]; Fristedt and Orey [2] showed that if $\mathscr{T}$ is the tail σ-field of X then $\mathscr{T}$ is non trivial if and only if $\lim_{x\to 0}\mathrm{var}(H_x)<\infty$. (See also Rogers [7] for a survey of this area). In this case $f(X_t)/g(t)$ has a non-degenerate almost sure limit with $g(t)=e^t$ and $f(x)=e^{c(x)}$.
The proof of Proposition 1.1 is based on the observation that

$$\forall\varepsilon\in(0,1),\ \forall x\in(0,1),\ \mathbb{P}(H_{\varepsilon x}<\sigma_x)=\varepsilon$$

and the simple implications $H_x > t \Rightarrow X_t > x \Rightarrow \sigma_x > t$. The method is then to 'sandwich' the family of distributions of interest between two families with a common limit.

The contents of the remainder of the paper are as follows. In Sect. 2 we deal with weak convergence results, and in Sect. 3 with almost-sure-convergence results. The final section, Sect. 4, contains a number of useful results on slow variation and a derivation of the expressions for $c(x)$ and $v(x)$ quoted above.

2. Weak convergence

In this section we detail our results concerning the convergence in distribution of $f(X_t)/g(t)$. Our approach is first to consider the laws of $f(x)/g(H_x)$ and $f(x)/g(\sigma_x)$; we deduce our general results by comparing their respective limits. It is natural to consider these families of distributions because, provided a suitable transformation is applied to the function f, they are independent of the scale of the diffusion X.

Recall that $f\colon (0,1] \to \mathbb{R}^+$ and $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ are assumed to be unbounded continuous functions, respectively decreasing and increasing. A hypothesised limiting distribution F is to be assumed non-degenerate, by which we mean that F has no mass at infinity and F is not the unit mass at 0.

We wish to find conditions on our functions f and g, which will be necessary and/or sufficient for convergence of $f(X_t)/g(t)$.

Proposition 2.1 *Suppose any two of the following conditions hold:*

i) *f is slowly varying,*

ii) $f(x)/g(H_x) \xrightarrow{\mathcal{D}} F$,

iii) $f(x)/g(\sigma_x) \xrightarrow{\mathcal{D}} F$.

Then so does the third, and $f(X_t)/g(t) \xrightarrow{\mathcal{D}} F$.

Proof. Let d be a continuity point of F, and let $t_x = g^{-1}(f(x)/d)$. Then

$$\mathbb{P}\left[\frac{f(x)}{g(H_x)} \leqq d\right] = \mathbb{P}[H_x \geqq t_x] \leqq \mathbb{P}[\sigma_x \geqq t_x] = \mathbb{P}\left[\frac{f(x)}{g(\sigma_x)} \leqq d\right].$$

For any $0 < \varepsilon < 1$

$$\begin{aligned}\mathbb{P}[\sigma_x \geqq t_x] &\leqq \varepsilon + \mathbb{P}[\sigma_x \geqq t_x;\ \sigma_x < H_{\varepsilon x}] \\ &\leqq \varepsilon + \mathbb{P}\left[\frac{f(\varepsilon x)}{g(H_{\varepsilon x})} \leqq d\,\frac{f(\varepsilon x)}{f(x)}\right];\end{aligned}$$

whence if f is slowly varying then ii) and iii) are equivalent. Conversely, to prove that ii) & iii) implies i) note that

$$\begin{aligned}\mathbb{P}[\sigma_x \geqq t_x] &= \mathbb{P}[\sigma_x \geqq t_x;\ \sigma_x < H_{\varepsilon x}] + \mathbb{P}[\sigma_x \geqq t_x;\ \sigma_x > H_{\varepsilon x}] \\ &\geqq \mathbb{P}[H_x \geqq t_x;\ \sigma_x < H_{\varepsilon x}] + \mathbb{P}[H_{\varepsilon x} \geqq t_x;\ \sigma_x > H_{\varepsilon x}] \\ &= (1-\varepsilon)\,\mathbb{P}[H_x \geqq t_x] + \varepsilon\,\mathbb{P}[H_{\varepsilon x} \geqq t_x] \\ &= (1-\varepsilon)\,\mathbb{P}\left[\frac{f(x)}{g(H_x)} \leqq d\right] + \varepsilon\,\mathbb{P}\left[\frac{f(\varepsilon x)}{g(H_{\varepsilon x})} \leqq d\,\frac{f(\varepsilon x)}{f(x)}\right].\end{aligned}$$

For the final part:

$$\mathbb{P}\left[\frac{f(x)}{g(H_x)} \leqq d\right] = \mathbb{P}[H_x \geqq t_x] \leqq \mathbb{P}[X_{t_x} \geqq x] = \mathbb{P}\left[\frac{f(X_{t_x})}{g(t_x)} \leqq d\right] \tag{6}$$

and

$$\mathbb{P}\left[\frac{f(x)}{g(\sigma_x)} \leqq d\right] = \mathbb{P}[\sigma_x \geqq t_x] \geqq \mathbb{P}[X_{t_x} \geqq x] = \mathbb{P}\left[\frac{f(X_{t_x})}{g(t_x)} \leqq d\right]. \quad \square \tag{7}$$

We now specialize to the case $g \equiv \mathrm{id}$. In this case the natural choice for f is $f(x) \equiv c(x)$. Indeed, up to asymptotic equivalence and a multiplication by a scalar, this is the only suitable choice.

Lemma 2.1 *Suppose* $H_x/f(x)$ and $\sigma_x/f(x) \xrightarrow{\mathscr{D}} F$. *Then*

$$\frac{f(x)}{\mathbb{E}(H_x)} \to \alpha \in (0, \infty).$$

In particular $\mathbb{E}(H_x)$ *is slowly varying.*

Proof. Since $v(x) \leqq c(x)^2$ the random variables $\{H_x/c(x)\colon 0<x<1\}$ are bounded in L^2 and hence uniformly integrable. If $f(x_n)/c(x_n) \to \alpha \in (0, \infty)$, where $x_n \downarrow 0$, then $\{H_{x_n}/f(x_n)\colon n \in \mathbb{N}\}$ is again a uniformly integrable family, and

$$\mathbb{E}(H_{x_n}/f(x_n)) = \frac{c(x_n)}{f(x_n)} \to \int x F(\mathrm{d}x).$$

Thus $\alpha = (\int x F(\mathrm{d}x))^{-1}$ is the only possible limit value of $f(x)/c(x)$. $\square$

Lemma 2.2 *If* $\mathbb{E}(H_x)$ *is slowly varying then*

$$\frac{H_x}{\mathbb{E}(H_x)} \xrightarrow{L^2} 1.$$

Proof.

$$(\mathbb{E}H_x)^2 \equiv c(x)^2 = 4 \int_x^1 \int_x^1 \mathrm{d}y\, \mathrm{d}z\, m(y)\, m(z) = 8 \int_x^1 \left(\int_z^1 m(y)\, \mathrm{d}y\right) m(z)\, \mathrm{d}z,$$

so from (3),

$$\frac{\mathrm{Var}(H_x)}{c(x)^2} = \frac{\int_x^1 (z-x)\, m(z)^2\, \mathrm{d}z}{\int_x^1 \left(\int_z^1 m(y)\, \mathrm{d}y\right) m(z)\, \mathrm{d}z}.$$

But

$$\frac{(z-x)\, m(z)}{\int_z^1 m(y)\, \mathrm{d}y} \leqq 2\, \frac{z\, m(z)}{c(z)} \to 0$$

by Lemma 4.1. $\square$

Theorem 2.1 *Suppose*

$$\frac{f(x)}{H_x} \xrightarrow{\mathscr{D}} F \quad \text{and} \quad \frac{f(x)}{\sigma_x} \xrightarrow{\mathscr{D}} F,$$

then

$$\frac{f(X_t)}{t} \xrightarrow{\mathscr{D}} F$$

and F is a point mass.

Proof. Proposition 2.1 and Lemmas 2.1 and 2.2. □

Henceforth we may (and indeed we shall) assume that $f(x) \equiv c(x)$. If $c(x)$ is not slowly varying then $H_x/c(x)$ and $\sigma_x/c(x)$ do not share a common non-degenerate limiting distribution. However, if $c(x)$ is slowly varying, then $H_x/c(x)$, $\sigma_x/c(x)$ and $c(X_t)/t$ all converge weakly to the unit mass at one. Our main Theorem of this section states that the reverse implications also hold.

Theorem 2.2 *The following are equivalent:*
i) $c(x) \equiv \mathbb{E}(H_x)$ *is slowly varying,*

ii)
$$\frac{H_x}{\mathbb{E}(H_x)} \xrightarrow{\mathbb{P}} 1,$$

iii)
$$\frac{c(X_t)}{t} \xrightarrow{\mathbb{P}} 1,$$

iv)
$$\frac{\sigma_x}{\mathbb{E}(H_x)} \xrightarrow{\mathbb{P}} 1.$$

Proof.
i) ⇒ ii)
Lemma 2.2.
ii) ⇒ i)
Fix $\lambda > 1$, and let $x_n \equiv x_n^\lambda = c^{-1}(\lambda^n)$. Let $H_n = H_{x_n}$. Then

$$\alpha_n = \frac{x_n - x_{n+1}}{x_{n-1} - x_{n+1}} = \mathbb{P}_{x_n}(\text{hit } x_{n-1} \text{ before hitting } x_{n+1}).$$

We prove that $\alpha_n \to 0$; then Lemma 4.1 implies that $c(x)$ is slowly varying. Fix n, and take i.i.d. copies $Y_1, Y_2, \ldots$ of $H_n - H_{n-1}$. By considering only time spent on downcrossings from x_{n-1} to x_n we see that

$$H_{n+1} - H_n \text{ is stochastically larger than } \sum_{j=1}^{N} Y_j$$

where N is independent of the Y_j, and $\mathbb{P}(N=k) = \alpha_n^k(1-\alpha_n)$.
Suppose that $\limsup \alpha_n = 2\delta > 0$. By hypothesis

$$\lambda^{-n}(H_{n+1} - H_n) \xrightarrow{\mathbb{P}} (\lambda - 1) \quad \text{and} \quad \lambda^{-n}(H_n - H_{n-1}) \xrightarrow{\mathbb{P}} (\lambda - 1)/\lambda.$$

Choose $m > 3\lambda$, and n_0 so large that, for $n \geqq n_0$

$$\mathbb{P}\left(\left|\frac{H_{n+1}-H_n}{\lambda^n}-(\lambda-1)\right|>\frac{\lambda-1}{2}\right)\leqq\frac{1}{2m}.$$

Choose $n > n_0$ such that $\alpha_n > \delta$. Let $\varepsilon = \delta^m$; then $\mathbb{P}(N \geqq m) \geqq \varepsilon$ and

$$\mathbb{P}\left(\frac{H_{n+1}-H_n}{\lambda^n}>\frac{3(\lambda-1)}{2}\right)\geqq\mathbb{P}\left(\sum_{j=1}^{N} Y_j>\frac{3\lambda^n(\lambda-1)}{2}\right)$$

$$\geqq\mathbb{P}\left(N\geqq m;\, Y_1, \ldots, Y_m>\frac{\lambda^n(\lambda-1)}{2\lambda}\right)$$

$$\geqq\frac{\varepsilon}{2}.$$

But this contradicts the assumption that $\lambda^{-n}(H_{n+1}-H_n)\xrightarrow{\mathbb{P}}\lambda-1$.

i) and ii) $\Rightarrow$ iii) and iv)
Proposition 2.1.

iii) $\Rightarrow$ ii)
By inequality (6), any limit distribution of $H_x/c(x)$ is concentrated in $[0,1]$. But $\{H_x/c(x)\colon x\in(0,1]\}$ is an L^2 bounded and therefore uniformly integrable family, each member of which has mean 1.

iv) $\Rightarrow$ ii)
As above but combining inequality (7) with (6) for the first stage. □

3. Almost sure convergence

Recall that f and g are assumed unbounded continuous functions, decreasing and increasing respectively. Recall also that $f(X_t)/g(t)$ has a non-trivial almost sure limit only if $\lim_{x\to 0} v(x) < \infty$. Throughout this section, the hypothesis that a limiting random variable Z exists contains the implicit assumption that Z is non-degenerate, in particular $\mathbb{P}(Z>0)>0$.

Suppose $f(X_t)/g(t)\xrightarrow{\text{a.s.}}Z$; clearly $f(x)/g(H_x)\xrightarrow{\text{a.s.}}Z$ and $f(x)/g(\sigma_x)\xrightarrow{\text{a.s.}}Z$.

Conversely if $f(x)/g(H_x)\xrightarrow{\text{a.s.}}Z$ and $f(x)/g(\sigma_x)\xrightarrow{\text{a.s.}}Z$, then

$$\frac{f(X_t)}{g(H_{X_t})}\geqq\frac{f(X_t)}{g(t)}\geqq\frac{f(X_t)}{g(\sigma_{X_t})}$$

and $f(X_t)/g(t)\xrightarrow{\text{a.s.}}Z$.

Consider first the case $g=\mathrm{id}$ where we have our most complete results.

Theorem 3.1 *The following are equivalent:*

i) $$\frac{c(X_t)}{t} \xrightarrow{\text{a.s.}} 1,$$

ii) $$\frac{H_x}{\mathbb{E}(H_x)} \xrightarrow{\text{a.s.}} 1,$$

iii) $$\frac{\sigma_x}{\mathbb{E}(H_x)} \xrightarrow{\text{a.s.}} 1.$$

A necessary condition for any (all) of the above is $c(x)$ slowly varying; a sufficient condition is $c(x)$ slowly varying and

$$\int_{0+} \frac{\int_x^1 m(y,1)^2\,dy}{c(x)^2}\,dx < \infty.$$

Proof. ii) ⇒ i) Fix $\rho \in (0,1)$, and let $y = y(x) \equiv c^{-1}(\rho\, c(x)) > x$. Since c is necessarily slowly varying $x/y(x) \to 0$, and, in particular, $x < y(x)/2$ eventually. Since $H_x/c(x) \to 1$ a.s. we have both

(8) $$\frac{H_{x-} - H_x}{c(x)} \xrightarrow{\text{a.s.}} 0,$$

and

$$\frac{H_x - H_{y(x)}}{c(x)} \xrightarrow{\mathbb{P}} (1-\rho).$$

Then, given $\varepsilon > 0$, pick x_0 so small that, $\forall\, x \leqq x_0$, $x \leqq y(x)/2$ and

$$\mathbb{P}\left\{\frac{H_x - H_{y(x)}}{c(x)} \geqq (1-\varepsilon)(1-\rho)\right\} \geqq 1-\varepsilon.$$

Let $U_t = \inf\{X_s : s \leqq t\}$, $Z_t = X_t - U_t$. Consider the sequence of excusions of Z_t from 0, starting when X_t first reaches x_0. If there were infinitely many excursions during which X rose to $y(U)$ then on infinitely many excursions $H_{x-} - H_x \geqq (1-\varepsilon)(1-\rho)\, c(x)$, contradicting (8). Thus ultimately $X_t \leqq y(U_t)$, and

$$\rho \xleftarrow{\text{a.s.}} \rho\, \frac{c(U_t)}{H_{U_t-}} \leqq \frac{c(y(U_t))}{t} \leqq \frac{c(X_t)}{t} \leqq \frac{c(U_t)}{t} \leqq \frac{c(U_t)}{H_{U_t}} \xrightarrow{\text{a.s.}} 1.$$

Since ρ is arbitrary the result follows.

iii) ⇒ i) This case is similar. In reversed time X is a time change of a BES(3) process. Defining, for $\lambda > 1$, $y = y(x) = c^{-1}(\lambda\, c(x)) < x$ and $V_t = \sup\{X_s : s \geqq t\}$ it is possible to obtain a contradiction similar to that above and to deduce $X_t \geqq y(V_t)$ eventually, and the result follows.

The necessity of $c(x)$ slowly varying is a corollary of 1.1; to prove that the stated conditions are sufficient we show that $H_x/c(x) \xrightarrow{\text{a.s.}} 1$. Since $c(.)$ is slowly varying this will follow if there is almost sure convergence down the sequence $x_n = 2^{-n}$. Let $c_n \equiv c(2^{-n})$ and $H_n \equiv H_{x_n}$. Kronecker's Lemma implies that it is sufficient to show that $\sum_{k \geqq 1} Y_k$ converges almost surely where

$$Y_k = \frac{H_k - H_{k-1}}{c_k} - \frac{(c_k - c_{k-1})}{c_k}.$$

Then $M_n = \sum_1^n Y_k$ is a martingale, and, to show that M converges it is sufficient to show that M is L^2 bounded; in particular that

$$\sum_{k \geqq 1} \operatorname{var}(Y_k) = \sum_{k \geqq 1} \frac{\operatorname{var}(H_k - H_{k-1})}{c_k^2} < \infty.$$

But

$$\operatorname{var}(H_x) = \int_x^1 V(u)\, du$$

where

$$V(u) = 8 \int_u^1 m(y)^2\, du;$$

thus

$$\sum_{k \geqq 1} \frac{\operatorname{var}(H_k - H_{k-1})}{c_k^2} = \sum_{k \geqq 1} \frac{\int_{x_k}^{x_{k-1}} V(u)\, du}{c_k^2} \leqq \int_0^1 \frac{V(u)}{c(u)^2}\, du.$$

Since $c_k / c_{k-1} \to 1$, the sum and integral in the above equation converge (and diverge) together. □

We now return to the general case. Define

$$\Phi = \{\phi \colon (0, 1] \to \mathbb{R}, \text{ increasing, continuous } \phi(x) > x\}.$$

Let

$$\Phi = \Phi_C \cup \Phi_D$$

where Φ_C and Φ_D are disjoint, and

$$\Phi_C = \left\{ \phi \in \Phi;\ \int_{0+} \frac{dx}{\phi(x) - x} < \infty \right\}.$$

Lemma 3.1 *Suppose $\phi \in \Phi$. Then*

$$H_x > \sigma_{\phi(x)} \textit{ eventually} \Leftrightarrow \phi \in \Phi_C.$$

Proof. Since X is a time-change of a Brownian motion B started at 1 and stopped at 0, it suffices to prove the result for that process.
For each $0 < x < 1$ define

$$p_x \equiv \sup\{B_u - x;\ H_x \leqq u \leqq H_{x-}\}.$$

Then $\{(x, p_x): x \in (0,1), p_x > 0\}$ is a Poisson point process with characteristic measure $dx \times p^{-2} dp$. Now '$H_x > \sigma_{\phi(x)}$ eventually' is equivalent to the statement that, for all sufficiently small x, $p_x < \phi(x) - x$. The number of points of the Poisson point process for which $p_x > \phi(x) - x$ is a Poisson variable with mean

$$\int_0^1 dx \int_0^\infty p^{-2}\, dp\, I_{\{p > \phi(x) - x\}} = \int_0^1 \frac{dx}{\phi(x) - x}.$$

The result now follows. □

Theorem 3.2 i) *Suppose* $f(X_t)/g(t) \xrightarrow{\text{a.s.}} Z$. *Then,* $\forall \phi \in \Phi_D$, $\limsup f(\phi(x))/f(x) = 1$.

ii) *Suppose* $f(x)/g(H_x) \xrightarrow{\text{a.s.}} Z$ *and* $\exists \phi \in \Phi_C$ *such that* $f(\phi(x))/f(x) \to 1$. *Then*

$f(X_t)/g(t) \xrightarrow{\text{a.s.}} Z$.

Proof. i) We have both $f(x)/g(H_x) \xrightarrow{\text{a.s.}} Z$ and $f(x)/g(\sigma_x) \xrightarrow{\text{a.s.}} Z$. Let $\phi \in \Phi_D$. $H_x < \sigma_{\phi(x)}$ infinitely often, and for such an x,

$$\frac{f(x)}{g(H_x)} > \frac{f(x)}{g(\sigma_{\phi(x)})} = \frac{f(\phi(x))}{g(\sigma_{\phi(x)})} \frac{f(x)}{f(\phi(x))} > \frac{f(\phi(x))}{g(\sigma_{\phi(x)})}.$$

The limits of the two outside terms are (a.s.) identical so $f(\phi(x))/f(x) \to 1$.
ii) For the ϕ in the hypothesis $H_x > \sigma_{\phi(x)}$ eventually, and then

$$\frac{f(x)}{g(H_x)} < \frac{f(x)}{g(\sigma_{\phi(x)})} = \frac{f(\phi(x))}{g(\sigma_{\phi(x)})} \frac{f(x)}{f(\phi(x))}.$$

But $H_{\phi(x)} < \sigma_{\phi(x)}$ so

$$\frac{f(\phi(x))}{f(x)} \frac{f(x)}{g(H_x)} < \frac{f(\phi(x))}{g(\sigma_{\phi(x)})} < \frac{f(\phi(x))}{g(H_{\phi(x)})}$$

and the outer two terms both have limit Z. Then $f(x)/g(\sigma_x) \xrightarrow{\text{a.s.}} Z$. □

Remark 3.1 As promised in Remark 1.2 we provide examples to show that:
i) almost sure convergence does not follow from convergence in distribution, and

ii) $f(x)/g(H_x) \xrightarrow{\text{a.s.}} Z$ does not imply that $f(X_t)/g(t) \xrightarrow{\text{a.s.}} Z$.

i) Take $c(x) = \exp\{(\log 1/x)(\log\log 1/x)^{-1}\}$. Then $c(x)$ is slowly varying and $c(X_t)/t \xrightarrow{\mathbb{P}} 1$ by Theorem 2.2. However if $\phi(x) = x + x(\log 1/x)$ then $\phi \in \Phi_D$ and $c(\phi(x))/c(x) \to e^{-1}$; almost sure convergence of $c(X_t)/t$ is then ruled out by Theorem 3.2.
ii) Motivated by the above example set $f(x) = \exp\{(\log 1/x)(\log\log 1/x)^{-1}\}$. It is then sufficient to find a pair (X_t, g) such that $f(x)/g(H_x) \xrightarrow{\text{a.s.}} Z$.

Suppose

$$c(x)=\exp\left(\exp\left\{\frac{\log(1/x)}{\log\log(1/x)}\right\}\right);$$

set $g(t)=\log(t)$ and $f(x)=\log c(x)$. Define $Y_x=H_x/c(x)$. The construction of the Laplace transform of Y_x (see Mandl [6]) yields the inequalities

$$e^{-\lambda}\leqq\mathbb{E}(e^{-\lambda Y_x})\leqq(1+\lambda)^{-1}.$$

Then for $\gamma>1$

$$\mathbb{P}(Y_x\leqq\gamma^{-1})\leqq e\gamma^{-1},$$

and trivially

$$\mathbb{P}(Y_x\geqq\gamma)\leqq\gamma^{-1};$$

so for $\delta>0$

$$\mathbb{P}(|\log Y_x|\geqq\delta)\leqq(e+1)e^{-\delta}.$$

Let $H_k=H_{2^{-k}}$ and $c_k=c(2^{-k})$, then

$$\mathbb{P}\left(\left|\log\left(\frac{H_k}{c_k}\right)\right|\geqq\delta\log c_k\right)\leqq(e+1)(c_k)^{-\delta}.$$

Now $(c_k)^{-\delta}<k^{-2}$ eventually so

$$\mathbb{P}\left(\left|\frac{g(H_k)}{f(2^{-k})}-1\right|\geqq\delta,\text{ infinitely often}\right)=0.$$

Since $f(.)$ is slowly varying the limit exists as required.

4. Slow variation

This section contains derivations of results quoted in preceeding sections. First we define and derive properties of slowly varying functions; then we provide a proof of the relations for $\mathbb{E}(H_x)$ and $\mathrm{var}(H_x)$ cited in §1 and used throughout.

Our slow variation results are largely abstracted from Bingham, Goldie and Teugels [1]. However parts iv) and v) of Lemma 4.1 are not contained therein.

Recall that a slowly varying function is defined as follows:

Definition 1. Let l be a positive measurable function. Then l is slowly varying if for all $\lambda>0$

$$\lim_{x\to 0}\frac{l(\lambda x)}{l(x)}=1. \tag{9}$$

Lemma 4.1 *For $\lambda>1$ define $x_n\equiv x_n^\lambda=c^{-1}(\lambda^n)$. The following are then equivalent:*

i) *$c(x)\equiv\mathbb{E}(H_x)$ is slowly varying,*

ii) *For some $\lambda>0$,* $$\lim_{x\to 0}\frac{c(\lambda x)}{c(x)}=1,$$

iii) $$\lim_{x\to 0} \frac{x\,m(x,1)}{c(x)}=0,$$

iv) $\forall \lambda>1$, $$\frac{x_{n+1}^{\lambda}}{x_n^{\lambda}}\to 0,$$

v) $\forall \lambda>1$, $$\frac{x_{n-1}^{\lambda}-x_n^{\lambda}}{x_{n-2}^{\lambda}-x_{n-1}^{\lambda}}\to 0.$$

Proof. i) $\Leftrightarrow$ ii)

Since $c(.)$ is monotonic it is sufficient to check the slow variation condition (9) for a single $\lambda>0$.

i) $\Leftrightarrow$ iii)

The Representation Theorem for slowly varying functions: $c(x)$ is slowly varying if and only if it can be represented in the form

$$c(x)=a(x)\exp\left(\int_x^1 \eta(y)\,\frac{dy}{y}\right)$$

where $a(x)\to a$ and $\eta(y)\to 0$.

i) $\Rightarrow$ iv)

Fix $0<\delta<1$, take $\lambda>1$. Choose x' such that $\forall x<x'$, $c(\delta x)/c(x)<\lambda$.

Then $\forall n$ such that $x_n<x'$, $\delta x_n>x_{n+1}$.

iv) $\Rightarrow$ ii)

Fix $0<\delta<1$, take $\lambda>1$. Choose n_0 such that $\forall n>n_0$, $x_{n+1}\leqq \delta x_n$. Then for any $x<x_{n_0}$, $\exists k$ such that $x_k>x\geqq \delta x\geqq x_{k+2}$. Then

$$1\leqq \frac{c(\delta x)}{c(x)}\leqq \frac{c(x_{k+2})}{c(x_k)}=\lambda^2.$$

ii) follows since λ is arbitrary.

iv) $\Rightarrow$ v)

Immediate.

v) $\Rightarrow$ iv)

Set $z_n=x_n-x_{n+1}$. Then for each $\delta>0$, for sufficiently large n, $z_{n+1}\leqq \delta z_n$. Thus $z_{n+k}\leqq \delta^k z_n$ and

$$x_n=\sum_{k\geqq 0} z_{n+k}\leqq (1-\delta)^{-1} z_n=(1-\delta)^{-1}(x_n-x_{n+1}). \quad \square$$

As promised in Sect. 1 we now derive the expressions for $c(x)$ and $v(x)$ quoted therein. By considering X as a time change of Brownian motion and using the Ray-Knight characterisation of Brownian local time one obtains an explicit expression for the distribution of the first hitting time H_x; namely

$$H_x \overset{\mathcal{D}}{=} \int_x^1 Z_{\xi-x}\, m(d\xi)$$

where $Z_.$ is the solution of the SDE

$$Z_u = 2\int_0^u (Z_v^+)^{1/2}\,\mathrm{d}W_v + 2u$$

and $\overset{\mathscr{D}}{=}$ signifies identity in law.

(For explanation of the time substitution techniques see Itô and McKean [4] Sect. V; Rogers and Williams [8] has details of the representation of the local time as an occupation density (IV.45) and of the Ray-Knight Theorem (VI.52).)
Since $\mathbb{E}(Z_u) = 2u$ we have immediately

$$\mathbb{E}(H_x) = c(x) = 2\int_x^1 (y-x)\, m(\mathrm{d}y).$$

Similarly

$$H_x - \mathbb{E}(H_x) = \int_x^1 m(dy)\left(2\int_0^{y-x} (Z_v^+)^{1/2}\,\mathrm{d}W_v\right)$$

$$= 2\int_0^{1-x} (Z_v^+)^{1/2}\, dW_v \left(\int_{x+v}^1 m(\mathrm{d}y)\right)$$

and then

$$\mathrm{Var}(H_x) = v(x) = \int_x^1 (y-x)\, m(y)^2\,\mathrm{d}y.$$

References

1. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular variation. Cambridge: Cambridge University Press 1987
2. Fristedt, B., Orey, S.: The tail σ-field of one dimensional diffusions. Friedman, A., Pinsky, M. (ed.) In: Stochastic analysis. New York: Academic Press 1978
3. Gikhman, I.I. and Skorokhod, A.V.: Stochastic differential equations. Berlin Heidelberg New York: Springer 1972
4. Itô, K., McKean, H.P.: Diffusion processes and their sample paths. Berlin Heidelberg New York: Springer 1965
5. Keller, G., Kersting, G., Rösler, U.: On the asymptotic behaviour of first passage times for diffusions. Probab. Th. Rel. Fields **77**, 379–395 (1988)
6. Mandl, P.: Analytical treatment of one-dimensional Markov processes. Berlin Heidelberg New York: Springer 1968
7. Rogers, L.C.G.: Coupling and the tail σ-field of a one-dimensional diffusion. In: Norris, J.R. (ed.) Stochastic calculus in application. London: Longman 1988
8. Rogers, L.C.G., Williams, D.: Diffusions, Markov processes and martingales. Vol. 2. Chichester: Wiley 1987
9. Rösler, U.: The tail σ-field of a time homogeneous one-dimensional diffusion process. Ann. Probab., 7, 847–857 (1979)