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Probability Theory and Related Fields

Random permutations of countable sets

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Summary. The main objective of this paper is a study of random decompositions of random point configurations on R^d into finite clusters. This is achieved by constructing for each configuration Z a random permutation of Z with finite cycles; these cycles then form the cluster decomposition of Z. It is argued that a good candidate for a random permutation of Z is a Gibbs measure for a certain specification, and conditions are given for the existence and uniqueness of such a Gibbs measure. These conditions are then verified for certain random configurations Z.

0. Introduction

Let Z be a countable set and $\xi=(\xi_x)_{x\in Z}$ be a system of independent random variables taking on values in Z with $\mathscr{P}(\xi_x=x)>0$ for all $x\in Z$. The law of ξ is then determined by the matrix $\alpha=(\alpha_{x,y})_{x,y\in Z}$, where

$$\alpha_{x,y}=\frac{\mathscr{P}(\xi_x=y)}{\mathscr{P}(\xi_x=x)}.$$

One of the main problems considered in this paper is to give a meaning to the conditional distribution $\mathscr{P}_\xi(\cdot|E_Z)$ of ξ, where E_Z is the set of bijective mappings of Z onto Z. If Z is finite then there is a simple explicit formula (involving the matrix α) for this conditional distribution. Moreover, it can also be described in terms of certain "local" conditional probabilities of a type similar to those used in the definition of Gibbs states. If Z is infinite then, in general, there is no obvious way to define $\mathscr{P}_\xi(\cdot|E_Z)$ directly. However, the "local" conditional probabilities still make sense, and so the corresponding Gibbs states can be considered as candidates for $\mathscr{P}_\xi(\cdot|E_Z)$. We give a sufficient condition (V3) for the existence of at least one Gibbs state (cf. Theorems 2.2 and 3.3), and a stronger condition (V4) which ensures that there is exactly one such state (cf. Theorems 4.1 and 4.2).

This Gibbs measure P is concentrated on E_Z (cf. Theorem 3.1), i.e. P can be interpreted as probability distribution of a random permutation of the countable set Z. Theorem 3.4 states that this random permutation gives a random partition of Z into finite subsets corresponding to the decomposition of the random permutation into cycles (cf. Theorem 3.2). Moreover the elements of this random partition are independent in a certain sense (cf. Theorem 3.4). In fact, it are these random partitions which are our main object of interest. In particular, we are interested in random partitions of random point configurations on R^d. Hence in Sect. 5 we replace Z by a locally finite random subset of the d-dimensional space R^d with probability distribution Q. We obtain sufficient conditions such that Q-almost surely the conditions (V 3) and (V 4) are fulfilled. This implies existence and uniqueness of a certain random permutation and consequently the existence of the corresponding random partition of the given random set.

This work was motivated by certain mathematical problems connected with equilibrium states in quantum statistical mechanics. For instance, let us consider the (infinite-volume) one-dimensional ideal Bose gas with inverse temperature $\underline{b}$ and activity $\underline{a}$. The position distribution of this quantum particle system was considered in [5]. According to Theorem 3.4 in [5] this position distribution is the probability law of a random point system $Z c R$ which can be constructed as follows:

Let $X=\{x_r: -\infty<r<+\infty\}$ be a stationary Poisson system in R with intensity

$$\lambda=\sum_{n=1}^{\infty} a^n(2n)^{-1}(\pi b n)^{-\frac{1}{2}}.$$

Furthermore, let $W=((w_r(t))_{t\geqq 0})_{r=-\infty}^{\infty}$ be a sequence of independent Wiener processes with parameter $2\underline{b}$. For all $n=1,2,\ldots$ and $-\infty<r<+\infty$ we consider the Brownian bridge given by

$$B_r^{(n)}(t):=w_r(t)-w_r(2bn)\frac{t}{2bn}; \qquad 0\leqq t\leqq 2bn.$$

Finally, let $\eta=(\eta_r)_{r=-\infty}^{\infty}$ be a sequence of independent random numbers such that

$$\mathscr{P}(\eta_r=n)=\lambda^{-1}a^n(2n)^{-1}(\pi b n)^{-\frac{1}{2}}; \qquad n=1,2,\ldots -\infty<r<+\infty$$

We assume that X, W, η are independent. Putting

$$Z_r:=\{x_r+B_r^{(\eta_r)}(2bk): k=1,\ldots,\eta_r\}; \qquad -\infty<r<+\infty$$

we get a partition of $Z=\bigcup_r Z_r$ into independent subsets. This partition $(Z_r)_{r=-\infty}^{\infty}$ corresponds to the cycles of the random permutation G of Z given by

$$G(x_r+B_r^{(\eta_r)}(2bk))=\begin{cases} x_r+B_r^{(\eta_r)}(2b(k+1)); & 1\leqq k\leqq \eta-1 \\ x_r+B_r^{(\eta_r)}(2b); & k=\eta \end{cases}$$

$$;-\infty<r<+\infty$$

In the sense of Theorem 3.4 the probability distribution of G (Z fixed) is related to the matrix $\alpha=\left(\exp\left(-\frac{(x-y)^2}{4b}\right)\right)_{x,y\in Z}$. In a forthcoming paper we will prove that the conditional intensity and, consequently, the local conditional probabilities of the random point system Z in the sense of a Gibbs distribution can be given in terms of this random permutation G. In case of quantum equilibrium states corresponding to an external field we have to consider a more general matrix α.

1. Notations

Let Q be a probability measure on the measurable space $[W^W, \mathfrak{M}_W]$ of mappings from the countable set W into W. For each finite $\Delta \subseteqq W$ we let Q^Δ denote the marginal distribution of Q on W^Δ.

In the following let Z be a countable infinite set. For each pair $Z_1, Z_2 \subseteqq Z$ we denote by E_{Z_1,Z_2} the set of mappings from Z_1 onto Z_2 which are one to one.

If $Z_2 \subseteqq Z_1$ we put

$$D_{Z_1}(Z_2) := \{g \in E_{Z_1,Z_1} : I(x,g) = Z_2 \text{ for all } x \in Z_2\}$$

where $I(x, g)$ denotes the smallest g-invariant set containing x.

Now let $(\alpha_{x,y})_{x,y\in Z}$ be a matrix satisfying the conditions

(V1) $$\alpha_{x,y} \geqq 0; \quad x, y \in Z$$

(V2) $$\alpha_{x,x} = 1; \quad x \in Z$$

For each finite subset $Z' \subseteqq Z$ we define a probability measure $P_{Z'}$ on $(Z')^{Z'}$ by

(1.1) $$P_{Z'}(\{g\}) = \begin{cases} \dfrac{\prod_{x\in Z'} \alpha_{x,g(x)}}{\sum_{h\in E_{Z',Z'}} \prod_{x\in Z'} \alpha_{x,h(x)}}; & g \in E_{Z',Z'} \\ 0; & \text{otherwise} \end{cases}$$

2. A compactness condition

We assume that the matrix $(\alpha_{x,y})_{x,y\in Z}$ fulfills (V1), (V2) and the following condition

(V3) (2.1) $$\sum_{\substack{A\subset Z \\ A \text{ finite} \\ y\in A}} \sum_{g\in D_A(A)} \prod_{x\in A} \alpha_{x,g(x)} < \infty; \quad y \in Z$$

The following theorem illustrates the meaning of (V3).

2.1. Theorem. *Let $(Z_s)_{s=1}^{\infty}$ be a sequence of finite subsets of Z with the following properties*

(E 1) $$Z_s \subseteqq Z_{s+1}: \ s=1,2,\ldots$$

(E 2) $$\bigcup_s Z_s = Z.$$

Then for each $\varepsilon>0$, $y\in Z$ there exists $s(\varepsilon, y)$ such that for all $s>s(\varepsilon, y)$

(2.2) $$P_{Z_s}(I(y,\cdot)\subseteqq Z_{s(\varepsilon,y)})\geqq 1-\varepsilon.$$

The proof can be found in § 6. Using Theorem 2.1 we prove in Sect. 7 the following theorem.

2.2. Theorem. *Let $(Z_s)_{s=1}^{\infty}$ be a sequence of finite subsets of Z with properties* (E 1) *and* (E 2). *There exists a subsequence $(Z_{s_k}^{\infty})_{k=1}$ and a probability measure P on $[Z^Z, \mathfrak{M}_Z)$ such that*

(2.3) $$P=\lim_{k\to\infty} P_{Z_{s_k}}$$

in the sense of convergence of all marginal distributions.

3. Random permutations

Let $(\alpha_{x,y})_{x,y\in Z}$ be any matrix fulfilling the conditions (V 1), (V 2) and (V 3). Furthermore, assume that P is a probability measure on $[Z^Z, \mathfrak{M}_Z]$ obtained as the limit in the sense of Theorem 2.2 i.e. there exists a sequence $(Z_k)_{k=1}^{\infty}$ of finite subsets of Z having the properties (E 1) and (E 2) such that

(3.1) $$P=\lim_{k\to\infty} P_{Z_k}$$

in the sense of convergence of all marginal distributions.

In Sect. 8 we prove the following theorem.

3.1. Theorem. *P is the probability law of a random permutation on Z,* i.e.

(3.2) $$P(E_{Z,Z})=1.$$

In Sect. 9 we prove:

3.2. Theorem. *P-almost surely $I(x,\cdot)$ is finite for all $x\in Z$.*

From Theorems 3.1 and 3.2 we can immediately conclude that a random permutation of the countable set Z with probability distribution P can be decomposed into finite cycles, i.e. we have

(3.3) $$P\Big(\bigcup_{(A_s)}\bigcap_s D_Z(A_s)\Big)=1$$

where (A_s) denotes partitions of Z into finite subsets. Finally, we consider the conditional behaviour of the random permutation on a finite subset Λ of Z under condition that the behaviour outside of Λ is given. In Sect. 10 we prove the following theorem.

3.3. Theorem. *Let $\Lambda \subseteqq Z$, Λ finite and let g_1 be a mapping from Λ into Z. Then the following equality holds for P-a.a. g_2*

(3.4)
$$P(\{g: g(x)=g_1(x), x\in\Lambda/\{g: g(x)=g_2(x), x\in Z\setminus\Lambda\}) = \begin{cases} \dfrac{\prod\limits_{x\in\Lambda} \alpha_{x,g_1(x)}}{\sum\limits_{h\in E_{\Lambda}, Z\setminus g_2(Z\setminus\Lambda)} \prod\limits_{x\in\Lambda} \alpha_{x,h(x)}}; & g_1\in E_{\Lambda}, Z\setminus g_2(Z\setminus\Lambda) \\ 0 & \text{otherwise} \end{cases}$$

From Theorem 3.3 we can immediately conclude that the behaviour of the random permutation under condition $\bigcap_s D_Z(A_s)$ (cf. (3.3)) is independent on different sets A_s. More precisely, we have for all partitions $(A_s)_{s=1}^{\infty}$ of Z, $n=1,2,\ldots$ and $\Delta_s \subseteqq A_s (s=1,\ldots,n)$

(3.5)
$$[P(/\bigcap_s D_Z(A_s))]^{\bigcup_s \Delta_s} = \bigtimes_s P(/\bigcap_s D_Z(A_s))^{\Delta_s}$$

Furthermore, from Theorem 3.3 we can conclude

(3.6)
$$[P(/\bigcap_s D_Z(A_s))]^{\Delta} = [P_{A_k}(/D_{A_k}(A_k))]^{\Delta}; \quad \text{for all } k=1,2,\ldots \text{ and } \Delta\subseteqq A_k$$

(cf. (1.1)). From (3.3), (3.5) and (3.6) we obtain the following theorem.

3.4. Theorem. *There exists a random partition of Z into finite subsets with probability distribution Q characterized by*

(3.7)
$$P = \int Q(\mathrm{d}(A_s)_1^{\infty}) \bigtimes_s P_{A_s}(/D_{A_s}(A_s))$$

Remark. Random partitions of Z into finite subsets may be considered as special random point fields in the space of all finite subsets of Z.

3.5. Example

We consider the d-dimensional lattice

$$Z=\{0, \pm 1, \pm 2, \ldots\}^d \quad (d\geqq 1)$$

and we use the notation

$$e_i = [0, \dots, 0, \underset{\substack{| \\ i\text{-th comp.}}}{1}, 0, \dots, 0]; \qquad i = 1, \dots, d$$

Now let $0 < p < 1$. We consider the following stochastic matrix $(a_{x,y})_{x,y \in Z}$

$$a_{x,y} = \begin{cases} p; & y = x \\ \frac{1}{2d}(1-p); & y = x \pm e_i, \qquad i = 1, \dots, d \\ 0; & \text{otherwise} \end{cases}$$

If we put

$$\alpha_{x,y} := \frac{a_{x,y}}{a_{x,x}}; \qquad x, y \in Z$$

then the matrix $(\alpha_{x,y})_{x,y \in Z}$ fulfills the conditions (V 1) and (V 2). In order to characterize the condition (V 3) we denote by γ_n the number of lattice walks from 0 to 0 of length n.

Then some elementary calculations show:

The matrix $(\alpha_{x,y})_{x,y \in Z}$ *fulfills condition* (V 3) *iff*

(3.8) $$\sum_{n=1}^{\infty} \gamma_n \left[\frac{1}{2d}(p^{-1} - 1)\right]^n < +\infty.$$

If $d = 1$, then $\gamma_n = 0$ for all $n > 4$. Therefore (3.8) holds. Now let $d \geqq 2$. Since $\gamma_{2n} \geqq 1$ for all $n = 1, 2, \dots$ (3.8) implies

$$p > \frac{1}{2d+1}$$

On the other hand, since

$$\gamma_{2n+1} = 0$$

$$\gamma_{2n} \leqq (2d-1)^n; \qquad n = 1, 2, \dots$$

(3.8) follows from

$$p > (2d-1)^{\frac{1}{2}} (2d + (2d-1)^{\frac{1}{2}})^{-1}.$$

4. Uniqueness

In order to prove uniqueness of random permutations considered as Gibbs distributions in the sense of Theorem 3.3 we have to sharpen condition (V 3).

(V 4) *There exists a sequence* $(Z_n)_{n=1}^{\infty}$ *of finite subsets of* Z *having the properties* (E 1) *and* (E 2) *such that the following equation holds*

$$0 = \lim_{n \to \infty} \sum_{y \in Z_n} \sum_{z \in Z \setminus Z_n} \sum_{\substack{A \subseteqq Z \\ A \text{ finite} \\ y, z \in A}} \sum_{g \in D_A(A)} \prod_{x \in A} \alpha_{x, g(x)}$$

Remark. It is easy to check that condition (V 3) follows from (V 4). On the other hand considering the example from the previous section one shows that (V 3) does not imply (V 4) and furthermore, uniqueness fails. For instance, in the case $d=1$ put

$$f_1(x)=x+1; \qquad x\in Z$$
$$f_2(x)=x-1; \qquad x\in Z$$

Then the Dirac measures δ_{f_1} and δ_{f_2} are Gibbs distributions in the sense of Theorem 3.3. In Sect. 11 we will prove the following theorem.

4.1. Theorem. *If the conditions* (V 1), (V 2) *and* (V 4) *are fulfilled then there exists exactly one probability measure P on $[Z^Z, \mathfrak{M}_Z]$ having the following properties*:

a) $P(E_{Z,Z})=1$
b) *P-almost surely $I(x,.)$ is finite for all $x\in Z$.*
c) *For all finite subsets Λ of Z and P-a.a. g_2* (3.4) *holds.*
Immediately from the Theorems 2.2, 3.1, 3.2, 3.3 we get

4.2. Theorem. *If the conditions* (V 1), (V 2) *and* (V 4) *are fulfilled then there exists exactly one probability measure P on $[Z^Z, \mathfrak{M}_Z]$ such that P is the limit of $P_{Z'}$ as Z' runs through the net of all finite subsets of Z.*

5. Locally finite random sets

Let S be a complete separable metric space. $\mathscr{S}$ denotes the γ-algebra of Borel sets in S, $B(S)$ denotes the ring of bounded Borel sets and δ_x the Dirac measure corresponding to $x\in S$. Let $N(S)$ be the set of locally finite simple counting measures on the measurable space $[S, \mathscr{S}]$, i.e. $N(S)$ is the set of all measures φ with properties
– $\varphi(B)\in\{0,1,2,\dots\}$; $B\in B(S)$
– $\varphi(\{x\})\leqq 1$; $x\in S$.
Further, let $\mathfrak{N}(S)$ be the smallest γ-algebra on $N(S)$ which makes the mapping $\varphi\to\varphi(B)$ measurable for each $B\in B(S)$. A probability measure Q on $[N(S), \mathfrak{N}(S)]$ is said to be a (*simple*) *point process* (cf. [3]) on S.
The mapping

$$Z\to\varphi_Z=\sum_{z\in Z}\delta_z \tag{5.1}$$

provides a one-to-one correspondence between $N(S)$ and the set

$$\mathfrak{Z}=\{Z\subseteq S; Z\cap B \text{ is finite for all } B\in B(S)\}.$$

For this reason $\varphi\in N(S)$ may be interpreted as a locally finite point system in S. According to this interpretation of the elements from $N(S)$ a point process may be understood as the distribution law of a locally finite random subset of S. In the following let $\underset{=}{a}$ be a measurable mapping from $S\times S$ into R with property

$$a(x,x)=0; \qquad x\in S \tag{5.2}$$

5.1. *If* $Z \in \mathfrak{Z}$ *and*

(5.3) $$\alpha_{x,y} = \exp[-a(x,y)]; \qquad x, y \in Z$$

then the matrix $(\alpha_{x,y})_{x,y \in Z}$ *fulfills* (V 1) *and* (V 2). *Furthermore, condition* (V 3) *holds iff the counting measure* $\varphi = \varphi_Z$ *given by* (5.1) *fulfills the following condition*

$$(V_\varphi 3) \quad \sum_{n=2}^{\infty} \int_B \varphi(dx_1) \int (\varphi - \delta_{x_1})(dx_2) \ldots \int \left(\varphi - \sum_{i=1}^{n-1} \delta_{x_i}\right)(dx_n)$$
$$\cdot \exp\left[-\sum_{i=1}^{n-1} a(x_i, x_{i+1}) - a(x_n, x_1)\right] < +\infty; \qquad B \in B(S).$$

For this reason we have to look for conditions ensuring that $(V_\varphi 3)$ holds for Q-a.a. φ. Obviously $(V_\varphi 3)$ is trivial if φ is finite. Hence we will consider infinite point processes Q, i.e.

(5.4) $$Q(\{\varphi \in N; \varphi(S) = +\infty\}) = 1.$$

Let $m_Q^{(n)}$ denote the moment measure of n-th order $(n = 1, 2, \ldots)$ characterized by

$$m_Q^{(n)}(B_1 \times \ldots \times B_n)$$
$$= \int Q(d\varphi) \int_{B_1} \varphi(dx_1) \int_{B_2} (\varphi - \delta_{x_1})(dx_2) \ldots \int_{B_n} \left(\varphi - \sum_{i=1}^{n-1} \delta_{x_i}\right)(dx_n);$$
$$B_1, \ldots, B_n \in B(S).$$

A point process Q on S is said to be of *finite intensity* if for all $B \in B(S)$ the value $m_Q^{(1)}(B)$ is finite. Then we get immediately

5.2. Let Q be an infinite point process such that

$$\sum_{n=1}^{\infty} \int_{B \times S^n} m_Q^{(n+1)}(d[x_1, \ldots, x_{n+1}]) \exp$$
$$\cdot \left[-\sum_{i=1}^{\infty} a(x_i, x_{i+1}) - a(x_{n+1}, x_1)\right] < \infty; \qquad B \in B(S)$$

Then $(V_\varphi 3)$ is fulfilled for Q-a.a. φ.

In the following, we consider the special case

(5.5) $$S = R^d \qquad (d = 1, 2, \ldots)$$
$$a(x, y) = U(x - y).$$

We have to assume that U is a measurable mapping from R^d into R with property

$$U(0) = 0.$$

A point process Q on R^d is said to be a *poisson process* if for all pairwise disjoint bounded Borel sets $B_1, \ldots, B_n (n=1,2,\ldots)$ (cf. [3])

$$Q\left(\bigcap_{i=1}^{n} \{\varphi; \varphi(B_i)=k_i\}\right)=\prod_{i=1}^{n}\left(\frac{(m_Q^{(1)}(B_i))^{k_i}}{k_i!}\exp(-m_Q^{(1)}(B_i))\right);$$

$$k_1, \ldots, k_n \in \{0, 1, \ldots\}$$

A poisson process is *stationary* (cf. [3]) if

$$m_Q^{(1)}=\lambda_Q \cdot l^d \tag{5.6}$$

Hereby l^d denotes Lebesgues measure on R^d and $\lambda_Q>0$ is called the *intensity* of Q. In that case the moment measures $m_Q^{(n)}$ are given by (cf. [3])

$$\begin{aligned} m_Q^{(n)} &= (m_Q^{(1)})^{n\times} \\ &= \lambda_Q^n (l^d)^{n\times}; \qquad n=1,2,\ldots \end{aligned}$$

Hence we get the following theorem

5.3. Theorem. *Let Q be a stationary poisson process on R^d with intensity λ. Assume*

$$\inf_x U(x) > -\infty \tag{5.7}$$

$$\lambda \int \exp(-U(x))\, l^d(\mathrm{d}x) < 1 \tag{5.8}$$

Then $(V_\varphi 3)$ holds for Q-a.a. φ.

5.4. Example. We put

$$U([x_1, \ldots, x_d]) = \sum_{i=1}^{d} \frac{x_i^2}{2t_i}; \qquad x_1, \ldots, x_d \in R,$$

$$t_1, \ldots, t_d > 0.$$

Further, let Q be a stationary poisson process with intensity $\lambda>0$. If

$$(2\pi)^d \lambda^2 \prod_{i=1}^{d} t_i < 1 \tag{5.9}$$

then $(V_\varphi 3)$ holds for Q-a.a. φ.

Proof. We get

$$\int \exp(-U(x))\, l^d(\mathrm{d}x) = \left((2\pi)^d \prod_{i=1}^{d} t_i\right)^{\frac{1}{2}}.$$

Hence (5.9) implies (5.8). On the other hand (5.7) holds because $U(x)$ is nonnegative. For this reason we can apply Theorem 5.3. □

Now we are going to verify (V4) for random point fields. We will consider the onedimensional case $d=1$. We put

$$\bar{N} := \{\varphi \in N(R); \varphi((-\infty, 0)) = \varphi((0, +\infty)) = +\infty\}$$
$$\Gamma := \{\dots, -1, 0, 1, \dots\}.$$

We define a sequence $(X_i)_{i\in\Gamma}$ of measurable mappings on $\bar{N}$ by the requirements

$$\varphi = \sum_{i\in\Gamma} \delta_{X_i(\varphi)}; \qquad \varphi \in \bar{N}$$
$$X_i(\varphi) < X_{i+1}(\varphi); \qquad i \in \Gamma, \qquad \varphi \in \bar{N}$$
$$X_0(\varphi) \leqq 0 < X_1(\varphi); \quad \varphi \in \bar{N}.$$

That means the points of φ are numbered in their natural order and the first point in $(0, +\infty)$ gets the number 1. For each $\varphi \in N$ and $n = 1, 2, \dots$ we define a measure $\varphi^{(n)}$ on R^n by setting

$$\varphi^{(n)}(B_1 \times \dots \times B_n) := \int_{B_1} \varphi(dx_1) \int_{B_2} (\varphi - \delta_{x_1})(dx_2) \dots \int_{B_n} \left(\varphi - \sum_{i=1}^{n-1} \delta_{x_i}\right)(\mathrm{d}x_n);$$
$$B_1, \dots, B_n \in B(R).$$

Furthermore, we put

$$H^-(z, \varphi) := \sum_{n=1}^{\infty} \int\limits_{\bigcup_{i=1}^{n} \{[z_0, \dots, z_n] \in R^{n+1};\, z_0 \leqq z,\, z_i > z\}} \varphi^{(n+1)}(d[x_0, \dots, x_n]) \exp\left[-U(x_n - x_0) - \sum_{m=0}^{n-1} U(x_m - x_{m+1})\right]; \qquad z \in R, \qquad \varphi \in \bar{N}$$

$$H^+(z, \varphi) := \sum_{n=1}^{\infty} \int\limits_{\bigcup_{i=1}^{n} \{[z_0, \dots, z_n] \in R^{n+1};\, z_0 \leqq z,\, z_i < z\}} \varphi^{(n+1)}(\mathrm{d}[x_0, \dots, x_n]) \exp\left[-U(x_n - x_0) - \sum_{m=0}^{n-1} U(x_m - x_{m+1})\right]; \qquad z \in R, \qquad \varphi \in \bar{N}$$

$$N_U := \{\varphi \in \bar{N}; \underline{\lim}\, H^+(X_n(\varphi), \varphi) = 0 = \underline{\lim}\, H^-(X_{-n}(\varphi), \varphi)\}.$$

The following lemma one can easily check.

5.5. *Let* $Z \in \mathfrak{Z}$ *and*

$$\alpha_{x,y} = \exp(-U(x-y)); \qquad x, y \in Z$$

Furthermore, let the counting measure φ_Z *be given by* (5.1). *Then the matrix* $(\alpha_{x,y})_{x,y\in Z}$ *fulfills condition* (V4) *if* $\varphi_Z \in N_U$.

Now let $\tilde{N}$ denote the set of all $\Phi \in N(R \times R)$ with property

$$\Phi(\{y\} \times R) = \Phi(R \times \{y\}) \leqq 1; \qquad y \in R.$$

The relation

$$\bar{Q} = \int Q(\mathrm{d}\varphi) \int P^{(\varphi)}(dg)\, \delta_{\int \varphi(\mathrm{d}x)\, \delta_{[x, g(x)]}} \tag{5.10}$$

gives a one-to-one correspondence between point processe $\bar{Q}$ on $R\times R$ with $\bar{Q}(\tilde{N})=1$ and certain pairs $(Q,(P^{(\varphi)})_{\varphi\in N(R)})$ where Q is a point process on R and $P^{(\varphi)}$ denotes the probability distribution of a random permutation of the support of φ. We put

$$\tau(\Phi):=\int\Phi(\mathrm{d}[x_1,x_2])\,\delta_{x_1};\qquad \Phi\in\tilde{N}$$

τ is a measurable mapping from $\tilde{N}$ into $N(R)$ and Q is given by $Q=\bar{Q}\circ\tau^{-1}$. Now for each $\varphi\in N(R)$ let T_φ denote the support of φ, i.e.

$$\varphi=\int_{T_\varphi}\varphi(\mathrm{d}x)\,\delta_x$$

$$\varphi(\{y\})>0;\qquad y\in T_\varphi.$$

For each $\varphi\in N(R)$ there is a one-to-one correspondence between $\Phi\in\tilde{N}$ with $\tau(\Phi)=\varphi$ and mappings $G_\Phi\in E_{T_\varphi,T_\varphi}$ given by

$$\Phi(\{[x,G_\Phi(x)]\})=1;\qquad x\in T_\varphi.$$

We put

$$L_\varphi(\Phi):=G_\Phi;\qquad \Phi\in\tilde{N},\qquad \tau(\Phi)=\varphi,\qquad \varphi\in N(R).$$

If $\bar{Q}_\varphi$ denotes the conditional distribution of $\bar{Q}$ under condition "$\tau(\Phi)=\varphi$", then $P_{\bar{Q},\varphi}$, defined by

$$P_{\bar{Q},\varphi}:=\bar{Q}_\varphi\circ L_\varphi^{-1}$$

is a probability measure on $T_\varphi^{T_\varphi}$ with

$$P_{\bar{Q},\varphi}(E_{T_\varphi,T_\varphi})=1.$$

Furthermore, relation (5.10) holds with $P^{(\varphi)}=P_{\bar{Q},\varphi}$. Now we are going to construct a point process $\bar{Q}$ on $R\times R$ corresponding to a pair $(Q,(P^{\varphi})_{\varphi\in N(R)})$ in the sense of (5.10) where Q is a given point process on R and $P^{(\varphi)}$ corresponds to the matrix $(\alpha_{x,y})_{x,y\in T_\varphi}$ (cf. 5.5) in the sense of Theorem 4.1. We will use the following notations. For $B\in B(R)$, $\varphi\in N(R)$ we put

$$\varphi_B:=\varphi(\ \cap B).$$

Then T_{φ_B} is always finite. $P_{T_{\varphi_B}}$ denotes the corresponding probability measure on

$$T_{\varphi_B}^{T_{\varphi_B}}$$

(cf. (1.1)) with property

$$P_{T_{\varphi_B}}(E_{T_{\varphi_B},T_{\varphi_B}})=1.$$

We define a mapping $\bar{L}_{B,\varphi}$ from $E_{T_{\varphi_B},T_{\varphi_B}}$ into $\tilde{N}$ by setting

(5.11) $$\bar{L}_{B,\varphi}(g):=\int_B\varphi(\mathrm{d}x)\,\delta_{[x,g(x)]};\qquad g\in E_{T_{\varphi_B},T_{\varphi_B}}.$$

If Q is a point process on R then we put

(5.12) $$Q_{(B)} := \int Q(\mathrm{d}\varphi)\,(P_{T_{\varphi B}} \circ \bar{L}_{B,\varphi}^{-1}).$$

Using 5.5 we get from 4.1 and 4.2 the following theorem.

5.6. Theorem. *Let Q be a point process on R with*

$$Q(N_U) = 1.$$

Then there exists exactly one point process $\bar{Q}$ on $R \times R$ having the following properties:

a) $\bar{Q}(\tilde{N}) = 1$
b) $\bar{Q} \circ \tau^{-1} = Q$
c) *For Q-a.a. $\varphi \in N(R)$*

$$P_{\bar{Q},\varphi}(\{g \in E_{T_\varphi, T_\varphi} : \exists A \subseteqq T_\varphi, A \textit{ finite}, x \in A, g(A) = A\}) = 1; \qquad x \in T_\varphi.$$

d) *For Q-a.a. $\varphi \in N(R)$, $A \subseteqq T$, A finite and $P_{\bar{Q},\varphi}$-a.a. g_2 the conditional distribution*

$$P_{\bar{Q},\varphi}(\{g : g(x) = g_1(x), x \in A / \{g : g(x) = g_2(x); x \in T_\varphi \setminus A\})$$

is given by (3.5) *where*

$$\alpha_{x,y} = \exp(-U(x-y)); \qquad x, y \in R.$$

Furthermore, for all sequences $(B_n)_{n=1}^\infty$ of bounded Borel sets with properties

$$B_n \subseteqq B_{n+1}; \qquad n = 1, 2,$$
$$\bigcup_n B_n = R$$

the sequence of point processes $(Q_{B_n})_{n=1}^\infty$ on $R \times R$ converges weakly to $\bar{Q}$ and $\bar{Q}$ is given by (5.10) *with $P^{(\varphi)} = P_{\bar{Q},\varphi}$.*

Because of 5.5 and 5.6 we have to look for conditions ensuring that

$$Q(N_U) = 1.$$

5.7. Definition. Let $\mathfrak{N}$ denote the smallest γ-algebra on $\bar{N}$ which makes the mapping $\varphi \to X_k(\varphi) - X_{k-1}(\varphi)$ measurable for each $k \neq 1$. A point process Q with $Q(\bar{N}) = 1$ is said to be a *(G)-process* if Q-a.s.

$$Q(\{\varphi \in \bar{N}; X_1(\varphi) - X_0(\varphi) > n / \mathfrak{N}) > 0; \qquad n = 1, 2, \ldots$$

5.8. *Remark.* If Q is a (Σ')-process (cf. [2]) with $Q(\bar{N}) = 1$ then Q is a (G)-process. In § 12 we will prove the following theorem.

5.9. Theorem. *Let Q be a stationary (G)-process of finite intensity such that*

(5.13) $$Q(\{\varphi \in \bar{N}; H^-(0, \varphi) < \infty, H^+(0, \varphi) < \infty\}) = 1.$$

Assume that U fulfills the following conditions

$$(5.14) \qquad U(x)=U(-x); \qquad x\in R$$

$$(5.15) \qquad U(x)\leqq U(y); \qquad 0\leqq x\leqq y$$

$$(5.16) \qquad \Delta(x):=\inf_{\substack{y\geqq x\\ z>0}} \frac{U(y+z)-U(y)}{z}>0; \qquad x>0.$$

Then

$$Q(N_U)=1$$

5.10. *Remark.* Immediately from the definitions of H^-, H^+, $m_Q^{(n)}$ we can conclude that (5.13) follows from $Q(\bar{N})=1$ and

$$(5.17) \quad \sum_{n=1}^{\infty} \int m_Q^{(r+1)}(\mathrm{d}[x_0, \ldots, x_n]) \exp\Big[-U(x_n-x_0)-\sum_{i=0}^{n-1} U(x_i-x_{i+1})\Big]<\infty$$
$$\overbrace{\cdot \bigcup_{i=1}^{n} (\{[z_0, \ldots, z_n]; z_0\leqq 0, z_i>0\} \cdot U\{[z_0, \ldots, z_n; z_0\geqq 0, z_i<0\})}$$

5.11. Corollary. *Let Q be a stationary poisson process on R with intensity* $\lambda>0$ *such that the following conditions hold*

$$(5.18) \qquad \lambda \int \exp(-U(x))\, l(\mathrm{d}x)<1$$

$$(5.19) \qquad \int l(\mathrm{d}x)\,|x|\, \exp(-U(x))<\infty$$

Assume that (5.14), (5.15), *and* (5.16) *are fulfilled. Then it holds*

$$Q(N_U)=1$$

The proof can be found in Sect. 13.

5.12. *Example.* We put

$$(5.20) \qquad U(x)=\frac{x^2}{2t}; \qquad x\in R$$
$$t<0$$

Then (5.14), (5.15), (5.16), and (5.19) are fulfilled. (5.18) holds iff

$$2\pi t\lambda^2<1$$

5.13. *Remark.* Let $\bar{U}$ be a measurable mapping from R into R with property $\bar{U}(0)=0$. If $\bar{U}\geqq U$ then

$$Q(N_U)=1$$

implies

$$Q(N_{\bar{U}})=1.$$

This gives some generalizations of 5.9 and 5.11.

5.14. Remark. Let $\Phi_0 = \sum_{i=-\infty}^{\infty} \delta \xi_i$ be a random point system in R with probability distribution Q being a stationary poisson process with intensity $\lambda > 0$. Furthermore, let $W = (W_i)_{i=-\infty}^{\infty}$ be a sequence of independent Wiener processes such that W and $\xi = (\xi_i)_{i=-\infty}^{\infty}$ are independent. We put

$$\Phi_t = \sum_{i=-\infty}^{\infty} \delta_{\xi_i + W_i(t)}; \qquad t \geqq 0.$$

Then $\Phi = (\Phi_t)_{0 \leqq t \leqq T}$ is a measure valued stationary Markov process describing the independent motion of points of the system Φ_0. Now let $2\pi T \lambda^2 < 1$. Then 5.6, 5.9 and 5.12 imply that in case of (5.20) the measure $P_{\bar{Q},\varphi}$ is uniquely defined for Q-a.a. $\varphi \in N$. Now denote $(\mu^T_{x,y})_{x,y \in R}$ the normalized conditional Wiener measures then we put

$$\bar{Q}_\varphi := \int P_{\bar{Q},\varphi}(\mathrm{d}g) \left(\mathop{\times}_{x \in \varphi} \mu^T_{x,g(x)} \right)$$

$$S_\varphi := \bar{Q}_\varphi \circ H^{-1}$$

where the mapping H is characterized by

$$H((\omega_i)_{i=-\infty}^{\infty})(t) = \sum_{i=-\infty}^{\infty} \delta_{\omega_i(t)}; \qquad 0 \leqq t \leqq T.$$

Then S_φ may be interpreted as the conditional distribution of Φ under condition "$\Phi_0 = \varphi = \Phi_T$". In a forthcoming paper we will use this construction of conditional distributions S_φ of certain measure valued processes in order to describe infinite boson systems by generalizing some ideas of the Feynman-Kac formula.

6. Proof of Theorem 2.1

We put

(6.1) $$E(A) := \sum_{g \in D_A(A)} \prod_{x \in A} \alpha_{x,g(x)}$$

(6.2) $$W(A) := \sum_{g \in E_{A,A}} \prod_{x \in A} \alpha_{x,g(x)}$$

for all $A \subseteqq Z$, A finite. Because of (V 2) we get

(6.3) $$W(A_1) \leqq W(A_2); \qquad A_1 \subseteqq A_2.$$

Using (6.3) we obtain

$$P_{Z_s}(D_{Z_s}(A)) = \frac{W(Z_s \setminus A)}{W(Z_s)} E(A) \leqq E(A); \qquad A \subseteqq Z_s.$$

Therefore we can conclude

(6.4) $$P_{Z_s}\Big(\bigcup_{\substack{A\subseteqq Z_s\\ y\in A\\ A\setminus Z_{s'}\neq 0}} D_{Z_s}(A)\Big)\leqq \sum_{\substack{A\subseteqq Z_s\\ y\in A\\ A\setminus Z_{s'}\neq 0}} E(A); \quad s'<s, \quad y\in Z_s.$$

On the other hand it follows from (V3) that for each $y\in Z$ and $\varepsilon>0$ there exists $s(\varepsilon, y)$ such that for all $s>s(\varepsilon, y)$

(6.5) $$\sum_{\substack{A\subseteqq Z_s\\ y\in A}} E(A)-\sum_{\substack{A\subseteqq Z_{s(\varepsilon,y)}\\ y\in A}} E(A)<\varepsilon$$

Putting $s'=s(\varepsilon, y)$ in (6.4) then (6.4) and (6.5) imply (2.2).

7. Proof of Theorem 2.2

For $Z'\subseteqq Z$, $C\in\mathfrak{M}_{Z'}$ and probability measures P on $[(Z')^{Z'}, \mathfrak{M}_{Z'}]$ with $P(C)>0$ we denote by $P_{/C}$ the conditional distribution.

$$P_{/C}(B)=P(B/C); \qquad B\in\mathfrak{M}_{Z'}$$

The following lemma one can easily check.

7.1. Let $A_1, \ldots, A_n$ be a sequence of pairwise disjoint subsets of $Z'\subseteqq Z$ (Z' finite) such that

(7.1) $$P_{Z'}\Big(\bigcap_{i=1}^{n} D_{Z'}(A_i)\Big)>0.$$

Then for all $\Delta\subseteqq A=\bigcup A_i$ the following equation holds:

(7.2) $$P^{\Delta}_{Z'/\cap D_{Z'}(A_i)}=P^{\Delta}_{A/\cap D_A(A_i)}$$

Now for each $Z'\subseteqq Z$ and $\Delta=\{x_1, \ldots, x_r\}\subseteqq Z'$ we put

(7.3) $$J^{\Delta}_{Z'}:=\bigcup_{s=1}^{r}\Big\{\{A_1, \ldots, A_s\}; \Delta\subseteqq\bigcup_{i=1}^{s} A_i\subseteqq Z', A_i \text{ finite},$$
$$A_i\cap\Delta\neq 0, A_1, \ldots, A_s \text{ pairwise disjoint}\Big\}.$$

Under consideration that each permutation of a finite set discomposes in cycles we get

7.2. *Let $Z'\subset Z$ be finite. Then for each $\Delta\subseteqq Z'$ the following equality holds:*

(7.4) $$\sum_{\{A_1,\ldots,A_s\}\in J^{\Delta}_{Z'}} P_{Z'}\Big(\bigcap_{n=1}^{s} D_{Z'}(A_n)\Big)=1.$$

Furthermore, one easily checks

7.3. Let $Z_2 \subset Z$ be finite. For each $Z_1 \subset Z_2$ and $\Delta = \{x_1, \ldots, x_r\} \subseteqq Z_1$ it holds that

(7.5)
$$\bigcup_{\{A_1, \ldots, A_s\} \in J^{\Delta}_{Z_2} \setminus J^{\Delta}_{Z_1}} \bigcap_{i=1}^{s} D_{Z_2}(A_i) = \bigcup_{n=1}^{r} \bigcup_{\substack{A \subseteqq Z_2 \\ x_n \in A \\ A \setminus Z_1 \neq 0}} D_{Z_2}(A).$$

Proof of Theorem 2.2. Using 7.1 and 7.2 we obtain for each $s = 1, 2, \ldots$, $\Delta = \{x_1, \ldots, x_r\} \subset Z_s$ and all $B \subseteqq Z^r$

(7.6)
$$\begin{aligned} P^{\Delta}_{Z_s}(B) &= \sum_{\{A_1, \ldots, A_n\} \in J^{\Delta}_{Z_s}} P^{\Delta}_{Z_s / \bigcap_j D_{Z_s}(A_j)}(B)\, P_{Z_s}(\bigcap_j D_{Z_s}(A_j)) \\ &= \sum_{\{A_1, \ldots, A_n\} \in J^{\Delta}_{Z_s}} P^{\Delta}_{\cup A_i / \bigcap_j D_{\cup A_i}(A_j)}(B)\, P_{Z_s}(\bigcap_j D_{Z_s}(A_j)). \end{aligned}$$

Since the sequence $(P_{Z_s}(\bigcap_j D_{Z_s}(A_j)))_{s=1}^{\infty}$ is always bounded there exists a subsequence $(Z_{s_k})_{k=1}^{\infty}$ of $(Z_s)_{s=1}^{\infty}$ such that for all $\{A_1, \ldots, A_n\} \in J^{\Delta}_Z$ the following limit exists

(7.7)
$$q_{A_1, \ldots, A_n} := \lim_{k \to \infty} P_{Z_{s_k}}(\bigcap_j D_{Z_{s_k}}(A_j)).$$

Using Theorem 2.1 one gets that for each $\varepsilon > 0$ there exists $k(\varepsilon)$ such that for all $k > k(\varepsilon)$ and $m = 1, \ldots, r$

$$P_{Z_{s_k}}\Big(\bigcup_{\substack{A \subseteqq Z_{s_k} \\ x_m \in A \\ A \setminus Z_{s_{k(\varepsilon)}} \neq 0}} D_{Z_{s_k}}(A)\Big) < \frac{\varepsilon}{r}.$$

Using 7.3 we can conclude

$$\sum_{\{A_1, \ldots, A_n\} \in J^{\Delta}_{Z_{s_k}} \setminus J^{\Delta}_{Z_{s_{k(\varepsilon)}}}} P_{\cup A_i / \bigcap_j D_{A_i}(A_j)}(B)\, P_{Z_{s_k}}(\bigcap_i D_{Z_{s_k}}(A_i)) < \varepsilon.$$

Hence we get with (7.6) and (7.7) for all $B \subseteqq Z^r$

(7.8)
$$\lim_{k \to \infty} P_{Z_{s_k}}(B) = \sum_{\{A_1, \ldots, A_n\} \in J^{\Delta}_Z} P_{\cup A_i / \bigcap_j D_{A_i}(A_j)}(B)\, q_{A_1, \ldots, A_n}.$$

Applying (7.12) in case $B=Z^r$ we obtain

$$\sum_{\{A_1,\ldots,A_l\}\in J_Z} {}_{\{x_1,\ldots,x_r\}}q_{A_1,\ldots,A_l}=1.$$

For this reason there exists a probability measure P on $[Z^Z, \mathfrak{M}_Z]$ such that (2.3) holds.

8. Proof of Theorem 3.1

Let $(Z_k)_{k=1}^{\infty}$ be a sequence of finite subsets of Z having properties (E1), (E2) and

$$P=\lim_{k\to\infty} P_{Z_k}. \tag{8.1}$$

From (8.1) and $P_{Z_k}(E_{Z_k,Z_k})=1$ we can conclude

$$P(\{g: \exists x_1, x_2\in Z, g(x_1)=g(x_2), x_1\neq x_2\})=0.$$

Therefore we get

$$P(E_{Z,Z})=P(\{g: g(Z)=Z\}).$$

Now we will prove that

$$P(\{g: g(Z)=Z\})=1. \tag{8.2}$$

From Theorem 2.1 we can conclude that for each $\varepsilon>0$ and $n=1,2,\ldots$ there exists $k(\varepsilon,n)\geq n$ such that

$$\sum_{y\in Z_n} P_{Z_k}\Big(\bigcup_{\substack{A\subseteq Z_k\\ y\in A\\ A\setminus Z_{k(\varepsilon,n)}\neq 0}} D_{Z_k}(A)\Big)<\varepsilon; \qquad k\geq k(\varepsilon,n). \tag{8.3}$$

If $k\geq l\geq k(\varepsilon,n)$ then we get

$$\bigcup_{\substack{A\subseteq Z_k\\ y\in A\\ A\setminus Z_{k(\varepsilon,n)}\neq 0}} D_{Z_k}(A)\supseteq \bigcup_{\substack{A\subseteq Z_k\\ y\in A\\ A\setminus Z_l\neq 0}} D_{Z_k}(A).$$

For this reason (8.3) implies

$$\sum_{y\in Z_n} P_{Z_k}\Big(\bigcup_{\substack{A\subseteq Z_k\\ y\in A\\ A\setminus Z_l\neq 0}} D_{Z_k}(A)\Big)<\varepsilon; \qquad k\geq l, \quad l\geq k(\varepsilon,n). \tag{8.4}$$

On the other hand we get

$$\sum_{y\in Z_n} P_{Z_k}\Big(\bigcup_{\substack{A\subseteq Z_k\\ y\in A\\ A\setminus Z_l\neq 0}} D_{Z_k}(A)\Big)$$

$$\geqq P_{Z_n}(\{g\in E_{Z_k,Z_k};\exists x\in Z_k\setminus Z_l\,\exists y\in Z_n\, g(x)=y\})$$
$$=1-P_{Z_k}(\{g\in E_{Z_k,Z_k};\forall y\in Z_n\,\exists x\in Z_l\, g(x)=y\}).$$

Hence (8.1) and (8.4) imply

(8.5) $$P(\{g;\forall y\in Z_n\,\exists x\in Z_l\, g(x)=y\})\geqq 1-\varepsilon;\qquad n=1,2,\ldots l>k(\varepsilon,n).$$

Furthermore, we get

$$\{g\in Z^Z\colon g(Z)=Z\}$$
$$=\bigcap_{n=1}^{\infty}\bigcup_{l\geqq n}\{g\in Z^Z;\forall y\in Z_n\,\exists x\in Z_l\, g(x)=y\}.$$

For this reason from (8.5) we can conclude

(8.6) $$P(\{g\in Z^Z\colon g(Z)=Z\})$$
$$=\lim_{n\to\infty}\lim_{l\to\infty} P(\{g;\forall y\in Z_n\,\exists x\in Z_l\, g(x)=y\})$$
$$\geqq 1-\varepsilon;\qquad \varepsilon>0$$

i.e. (8.2) holds.

9. Proof of Theorem 3.2

It is assumed that

(9.1) $$P=\lim_{k\to\infty} P_{Z_k}.$$

Furthermore, from Theorem 2.1 we can conclude that for each $\varepsilon>0$ and $x\in Z$ there exists $k(\varepsilon,x)$ such that for all $k>k(\varepsilon,x)$

(9.2) $$P_{Z_k}(I(x,\cdot)\subseteqq Z_{k(\varepsilon,x)})\geqq 1-\varepsilon.$$

Now we put

$$\Delta=\{y_1,\ldots,y_r\}=Z_{k(\varepsilon,x)}.$$

Then there exists $B\subseteqq Z^r$ such that for all $k>k(\varepsilon,x)$

(9.3) $$P_{Z_k}(I(x,\cdot)\subseteqq Z_{k(\varepsilon,x)})$$
$$=P^{\Delta}_{Z_k}(B)$$

(9.4) $$P(I(x,\cdot)\subseteqq Z_{k(\varepsilon,x)})$$
$$=P^{\Delta}(B)$$

(9.1), (9.2), (9.3), and (9.4) imply

$$P(I(x,\cdot)\subseteqq Z_{k(\varepsilon,x)})\geqq 1-\varepsilon.$$

Hence we can conclude that Theorem 3.2 holds.

10. Proof of Theorem 3.3

For all $W_1\subseteqq W_2\subseteqq Z$ we put

$$(\tau_{W_2,W_1}(g))(x):=g(x); \qquad x\in W_1,\, g\in W_2^{W_2}$$

$\mathfrak{M}_{W_2,W_1}$ denotes the smallest σ-algebra on $W_2^{W_2}$ such that the mappings $g\to\tau_{W_2,W}(g)$ ($W\subseteqq W_1$, W finite) are measurable. In the following, let A be a finite subset of Z. We put

$$H_{W_2,W_1}:=\{g\in W_2^{W_2};\exists B\subseteqq W_1\ g(B)=B,\, B\supseteqq A\}; \qquad W_2\supseteqq W_1\supseteqq A.$$

Now let g_1 be a mapping from A into Z such that

$$g_1(x)\neq g_1(y); \qquad x\neq y, \qquad x,y\in A.$$

For each $W\subseteqq Z$, $A\subseteqq W$ we define a mapping $G_W:=W^W\to[0,1]$ by

$$G_W(g):=\begin{cases}\dfrac{\prod\limits_{x\in A}\alpha_{x,g_1(x)}}{\sum\limits_{f\in E_{A,g_1(A)}}\prod\limits_{x\in A}\alpha_{x,f(x)}}; & g(A)=g_1(A)\\[2ex] 0 & \text{otherwise.}\end{cases}$$

Obviously G_Z is $\mathfrak{M}_{Z,Z\setminus A}$-measurable. We have to prove that for each $C\in\mathfrak{M}_{Z,Z\setminus A}$ the following equality holds

$$\int_C G_Z(g)\,P(dg)=P(\tau_{Z,A}^{-1}(\{g_1\})\cap C).$$

If $(Z_s)_{s=1}^{\infty}$ is any sequence of finite subsets of Z having properties (E1) and (E2) then it is sufficient to prove

$$\int_{\tau_{Z,Z_m}^{-1}(Y)} G_Z(g)\,P(dg)=P(\tau_{Z,A}^{-1}(\{g_1\})\cap\tau_{Z,Z_m}^{-1}(Y)) \tag{10.1}$$

for each m with $A\subseteqq Z_m$ and all $Y\in\mathfrak{M}_{Z_m,Z_m\setminus A}$. Now from (3.1) we can conclude

$$\lim_{n\to\infty}P_{Z_n}\circ\tau_{Z_n,Z_k}^{-1}(D)=P\circ\tau_{Z,Z_k}^{-1}(D); \tag{10.2}$$

$$D\in\mathfrak{M}_{Z_k,Z_k}, \qquad k=1,2,\ldots$$

For each $k \geqq m$, $Y \in \mathfrak{M}_{Z_m, Z_m}$, $\Lambda \subseteq Z_m$ (10.2) implies

(10.3)
$$\int_{\tau^{-1}_{Z_k, Z_m}(Y) \cap H_{Z_k, Z_k}} G_{Z_k}(g)\, P \circ \tau^{-1}_{Z, Z_k}(\mathrm{d} g) = \lim_{n \to \infty} \int_{\tau^{-1}_{Z_k, Z_m}(Y) \cap H_{Z_k, Z_k}} G_{Z_k}(g)\, P_{Z_n} \circ \tau^{-1}_{Z_n, Z_k}(\mathrm{d} g)$$

(10.4)
$$P(\tau^{-1}_{Z, Z_m}(Y) \cap H_{Z, Z_k} \cap \tau_Z^{-1}(\{g_1\})) = \lim_{n \to \infty} P_{Z_n}(\tau^{-1}_{Z_n, Z_m}(Y) \cap H_{Z_n, Z_k} \cap \tau^{-1}_{Z_n, \Lambda}(\{g_1\})).$$

If $g \in H_{Z_k, Z_k}$, $\hat{g} \in H_{Z, Z_k}$ with

$$\tau_{Z, Z_k}(\hat{g}) = g$$

then we obtain

$$G_{Z_k}(g) = G_Z(\hat{g}).$$

Therefore we get

(10.5)
$$\begin{aligned} \int_{\tau^{-1}_{Z_k, Z_m}(Y) \cap H_{Z_k, Z_k}} G_{Z_k}(g)\, P \circ \tau^{-1}_{Z, Z_k}(\mathrm{d} g) &= \int_{\tau^{-1}_{Z, Z_k}(\tau^{-1}_{Z_k, Z_m}(Y)) \cap \tau^{-1}_{Z, Z_k}(H_{Z_k, Z_k})} G_Z(g)\, P(\mathrm{d} g) \\ &= \int_{\tau^{-1}_{Z, Z_m}(Y) \cap H_{Z, Z_k}} G_Z(g)\, P(\mathrm{d} g). \end{aligned}$$

Using similar arguments we get

(10.6)
$$\int_{\tau^{-1}_{Z_k, Z_m}(Y) \cap H_{Z_k, Z_k}} G_{Z_k}(g)\, P_{Z_n} \circ \tau^{-1}_{Z_n, Z_k}(\mathrm{d} g) = \int_{\tau^{-1}_{Z_n, Z_m}(Y) \cap H_{Z_n, Z_k}} G_{Z_n}(g)\, P_{Z_n}(\mathrm{d} g); \qquad n \geqq k.$$

Finally, simple calculations show

$$\int_{\tau^{-1}_{Z_n, Z_m}(Y) \cap H_{Z_n, Z_k}} G_{Z_n}(g)\, P_{Z_n}(\mathrm{d} g) = P_{Z_n}(\tau^{-1}_{Z_n, Z_m}(Y) \cap H_{Z_n, Z_k} \cap \tau^{-1}_{Z_n}(\{g_1\})).$$

Therefore (10.3), (10.4), (10.5), (10.6) imply

(10.7)
$$\int_{\tau^{-1}_{Z, Z_m}(Y) \cap H_{Z, Z_k}} G_Z(g)\, P(\mathrm{d} g) = P(\tau^{-1}_{Z, Z_m}(Y) \cap H_{Z, Z_k} \cap \tau^{-1}_{Z, Z_m}(\{g_1\})); \qquad k = 1, 2, \ldots$$

From Theorem 3.2 we can conclude

$$P(\bigcup_k H_{Z, Z_k}) = 1$$

On the other hand we have

$$H_{Z,Z_k} \subseteq H_{Z,Z_{k+1}}; \qquad k=1,2,\ldots$$

For this reason (10.7) implies (10.1).

11. Proof of Theorem 4.1

Immediately from the Theorems 2.1, 3.1, 3.2, 3.3 it follows that there exists a probability measure P with properties a), b) and c). Now we will prove uniqueness.

We will use the notations τ_{W_2,W_1}, $\mathfrak{M}_{W_2,W_1}$ from Sect. 10. Let P_1, P_2 be probability measures on $[Z^Z, \mathfrak{M}_Z]$ with properties a), b) and c). We have to prove that

$$P_1(H) = P_2(H) \tag{11.1}$$

holds for all $k=1,2,\ldots$ and $H \in \mathfrak{M}_{Z,Z_k}$. Thereby the sequence $(Z_n)_{n=1}^{\infty}$ is given by condition (V4). Now let $1 \leqq k \leqq n$, $H \in \mathfrak{M}_{Z,Z_k}$. We put

$$H_n := \{\tau_{Z,Z_n}(g); g \in H\}$$
$$K_n = \{g \in Z^Z; g(Z_n) = Z_n\}.$$

$\bar{K}_n$ denotes the complement of K_n.

Using c) we get

$$\begin{aligned} &P_s(H/K_n) \\ &\quad = P_{Z_n}(H_n); \qquad s=1,2, \qquad n=1,2,\ldots \end{aligned} \tag{11.2}$$

$$\begin{aligned} &|P_1(H) - P_2(H)| \\ &\quad = |P_1(H/K_n)\, P_1(K_n) + P_1(H/\bar{K}_n)\, P_1(\bar{K}_n) \\ &\qquad - P_2(H/K_n)\, P_2(K_n) - P_2(H/\bar{K}_n)\, P_2(\bar{K}_n)| \\ &\quad \leqq P_1(\bar{K}_n) + P_2(\bar{K}_n) \\ &\qquad + |P_1(H/K_n)\, P_1(K_n) - P_2(H/K_n)\, P_2(K_n)|. \end{aligned} \tag{11.3}$$

From (11.2) and (11.3) it follows

$$\begin{aligned} &|P_1(H) - P_2(H)| \\ &\quad \leqq P_1(\bar{K}_n) + P_2(\bar{K}_n) \\ &\qquad + P_{Z_n}(H_n)\, |P_1(K_n) - P_2(K_n)| \\ &\quad \leqq 2(P_1(\bar{K}_n) + P_2(\bar{K}_n)). \end{aligned} \tag{11.4}$$

Using a) and b) we get for $s=1,2$

$$\begin{aligned} P_s(\bar{K}_n) &= P_s\Big(\bigcup_{y \in Z_n} \bigcup_{z \in Z \setminus Z_n} \bigcup_{\substack{A \subseteq Z \\ A \text{ finite} \\ y,z \in A}} D_Z(A)\Big) \\ &\leqq \sum_{y \in Z_n} \sum_{z \in Z \setminus Z_n} \sum_{\substack{A \subseteq Z \\ A \text{ finite} \\ y,z \in A}} P_s(D_Z(A)). \end{aligned} \tag{11.5}$$

Using a) and c) we get for $s=1,2$

(11.6) $$\begin{aligned} P_s(D_Z(A)) &= P_s(D_Z(A)\cap\{g\in E_{Z,Z};\, g(Z\setminus A)=Z\setminus A\}) \\ &\leqq P_s(D_Z(A)/\{g\in E_{Z,Z};\, g(Z\setminus A)=Z\setminus A\}) \\ &= P_A(D_A(A)). \end{aligned}$$

Furthermore, (6.3) implies

(11.7) $$P_A(D_A(A))\leqq \sum_{g\in D_A(A)} \prod_{x\in A} \alpha_{x,g(x)}.$$

From (11.4), (11.5), (11.6), (11.7) we conclude

$$\begin{aligned} &|P_1(H)-P_2(H)| \\ &\leqq 4 \sum_{y\in Z_n} \sum_{z\in Z\setminus Z_n} \sum_{\substack{A\subseteq Z \\ A\text{ finite} \\ y,z\in A}} \sum_{g\in D_A(A)} \prod_{x\in A} \alpha_{x,g(x)}. \end{aligned}$$

Because of (V4) that implies (11.1).

12. Proof of Theorem 5.9

Let Q^0 denote the Palm distribution of Q (cf. [3]). Then we get

12.1. (5.13) *holds iff*

(12.1) $$Q^0(\{\varphi\in\bar{N};\, H^-(0,\varphi)<\infty,\, H^+(0,\varphi)<\infty\})=1$$

12.2. Q *is a* (G)-*process iff* Q^0 *is a* (G)-*process*, i.e. Q^0-*a.s. it holds*

(12.2) $$Q^0(\{\varphi;\, X_1(\varphi)-X_0(\varphi)>n\}/\mathfrak{N})>0; \qquad n=1,2,\dots$$

12.3. $Q(N_U)=1$ *iff*

(12.3) $$Q^0(N_U)=1$$

For that reason we will prove (12.3) using the assumptions (12.1) and (12.2). Putting

(12.4) $$T(\varphi):=\int\varphi(\mathrm{d}x)\,\delta_{x-x_1(\varphi)}; \qquad \varphi\in\bar{N}$$

we get

(12.5) $$Q^0=Q^0\circ T^{-1}$$

Using the results of [4] one can prove that Q^0 can be decomposed into Palm distributions of stationary point processes being ergodic (with respect to T). (12.1) and (12.2) imply that each factor of this decomposition has the properties (12.1) and (12.2). Furthermore, (12.3) is fulfilled if all factors of the decomposition of Q^0 fulfill (12.3). For that reason we can use the additional assumption that Q^0 is ergodic.

Now we are going to prove the following lemma.

12.4. For all $\varepsilon > 0$

$$Q^0(\{\varphi; H^-(0, \varphi) < \varepsilon\}) > 0.$$

Proof. We put

$$N^0 := \{\varphi \in \bar{N}; \varphi(\{0\}) = 1\}$$
$$Y_k(\varphi) := X_k(\varphi) - X_{k-1}(\varphi); \qquad \varphi \in N^0, \qquad k \in \Gamma.$$

Then we get

$$Q^0(N^0) = 1.$$

Furthermore, $\varphi \to [Y_0(\varphi), (Y_k(\varphi))_{k \neq 0}]$ provides a one to one correspondence between N^0 and the set M of all pairs $[z, (z_k)_{k \neq 0}]$ whereby $z > 0$ and $(z_k)_{k \neq 0}$ is a sequence of positive real numbers with property

$$\sum_{k=1}^{\infty} z_{-k} = \sum_{k=1}^{\infty} z_k = +\infty.$$

Now we define a mapping H on M by setting

(12.6) $$H(Y_0(\varphi), (Y_k(\varphi))_{k \neq 0}) := H^-(0, \varphi).$$

Because of (5.14) and (5.15) we get

(12.7) $$\begin{aligned} &H(z, (z_k)_{k \neq 0}) \\ &\quad \leqq H(y, (y_k)_{k \neq 0}); \qquad z, (z_k)_{k \neq 0}, \qquad y, (y_k) \in M, \\ &y \leqq z, \qquad y_k \leqq z_k; \qquad k \neq 0. \end{aligned}$$

From (12.1), (12.2), and (12.5) we can conclude that there exists $d, c > 0$ such that

(12.8) $$Q^0(\{\varphi; H(Y_0(\varphi), (Y_k(\varphi))_{k \neq 0}) \leqq c,\ Y_0(\varphi) \geqq d\}) > 0$$

(12.7) and (12.8) imply

(12.9) $$Q^0(\{\varphi; H(d, (Y_k(\varphi))_{k \neq 0}) \leqq c\}) > 0.$$

Now, let $t > d$ such that

$$c \cdot \exp[-\Delta(d)(t-d)] < \varepsilon.$$

Using (5.16) then we get

(12.10) $$\begin{aligned} &\{\varphi; H(d, (Y_k(\varphi))_{k \neq 0}) \leqq c\} \\ &\quad \subseteqq \{\varphi; H(t, (Y_k(\varphi))_{k \neq 0}) \leqq \varepsilon\}. \end{aligned}$$

From (12.9) and (12.10) it follows

(12.11) $$Q^0(\{\varphi; H(t, (Y_k(\varphi))_{k \neq 0}) < \varepsilon\}) > 0.$$

Finally, using (12.2), (12.5), (12.7) and (12.11) we can show that

$$Q^0(\{\varphi; H(Y_0(\varphi), (Y_k(\varphi))_{k\neq 0}) < \varepsilon) > 0$$

Because of (12.6) this proves 12.4. □

Now we put

$$B^\varepsilon := \bigcap_{k=1}^{\infty} \bigcup_{r \leqq -k} \{\varphi \in N^0; H^-(X_r(\varphi), \varphi) < \varepsilon\}; \qquad \varepsilon > 0$$

Using the well-known Poincaré recurrence theorem one gets

$$Q^0(B^\varepsilon) = 1.$$

This implies

(12.12) $$Q^0(\{\varphi \in N^0; \underline{\lim}\, H^-(X_{-n}(\varphi), \varphi) = 0\}) = 1.$$

Using the same arguments one proves

(12.13) $$Q^0(\{\varphi \in N^0; \underline{\lim}\, H^+(X_n(\varphi), \varphi) = 0\}) = 1.$$

Now (12.3) follows from (12.12) and (12.13).

13. Proof of Corollary 5.11

A stationary poisson process on R with intensity $\lambda > 0$ is a (G)-process. Furthermore, $\lambda > 0$ implies

$$Q(\bar{N}) = 1.$$

Hence (5.13) follows from (5.17). For this reason we are going to prove (5.17).
Using (5.14) we get

(13.1) $$\int l^{n+1}(\mathrm{d}[x_0, \ldots, x_n]) \exp\left[-U(x_n - x_0) - \sum_{m=0}^{n-1} U(x_m - x_{m+1})\right]$$

$$\cdot \bigcup_{i=1}^{n} (\{[z_0, \ldots, z_n]; z_0 \leqq 0, z_i > 0\}$$

$$\cdot U\{[z_0, \ldots, z_n]; z_0 \geqq 0, z_i < 0\})$$

$$\leqq 2 \sum_{i=1}^{n} \int l^{n+1}(\mathrm{d}[x_0, \ldots, x_n]) \exp\left[-U(x_n - x_0) - \sum_{m=0}^{n-1} U(x_m - x_{m+1})\right]$$

$$\cdot \{[z_0, \ldots, z_n], z_0 < 0, z_i > 0\}; \qquad n = 1, 2, \ldots$$

Simple calculations show

$$\int l^{n+1}(\mathrm{d}[x_0, \ldots, x_n]) \exp\left[-U(x_n-x_0)-\sum_{m=0}^{n} U(x_m-x_{m+1})\right] \tag{13.2}$$

$$\cdot\{[z_0, \ldots, z_n]; z_0<0, z_i>0\}$$

$$= \int_{(-\infty,0)} l(\mathrm{d}x_0) \int_{(0,+\infty)} l(\mathrm{d}x_i) f_i(x_0-x_i) f_{n-i+1}(x_i-x_0);$$

$$n=1,2,\ldots, \qquad i=1,\ldots,n.$$

Hereby the sequence of functions $(f_n)_{n=1}^{\infty}$ is given by

$$f_1(x)=\exp[-U(x)]; \qquad x\in R$$

$$f_{n+1}(x)=\int f_n(x-y) f_1(y)\, l(\mathrm{d}y); \qquad x\in R, \qquad n=1,2,\ldots$$

Putting

$$b=\int \exp[-U(x)]\, l(\mathrm{d}x)$$

one can prove

$$\sup_x f_n(x) \leqq (\sup_x f_1(x))\, b^{n-1}; \qquad n=1,2,\ldots \tag{13.3}$$

From (5.14) and (5.15) it follows

$$U(x)\geqq 0; \qquad x\in R.$$

Hence we get

$$\sup_x f_1(x) \leqq 1 \tag{13.4}$$

(13.3) and (13.4) imply

$$\sup_x f_n(x) \leqq b^{n-1}; \qquad b=1,2,\ldots$$

For this reason (13.2) implies

$$\int l^{n+1}(\mathrm{d}x_0, \ldots, x_n) \exp\left[-U(x_n-x_0)-\sum_{m=0}^{n} U(x_m-x_{m+1})\right] \tag{13.5}$$

$$\cdot\{[z_0, \ldots, z_n]; z_0<0, z_i>0\}$$

$$\leqq b^{n-i} \int_{(-\infty,0)} l(\mathrm{d}x) \int_{(0,+\infty)} l(\mathrm{d}y) f_i(x-y);$$

$$n=1,2,\ldots, \qquad i=1,\ldots,n.$$

From (5.14) it follows

$$f_n(x)=f_n(-x); \qquad x\in R.$$

For this reason we get

(13.6)
$$\int_{(-\infty,0)} l(\mathrm{d}x) \int_{(0,+\infty)} l(\mathrm{d}y) f_i(x-y)$$
$$= \int_{(0,+\infty)} l(\mathrm{d}x) \int_{(x,+\infty)} l(\mathrm{d}y) f_i(y)$$
$$= \int_{(0,+\infty)} l(\mathrm{d}x)\, x f_i(x)$$
$$= \tfrac{1}{2} \int l(\mathrm{d}x)\, |x|\, f_i(x); \qquad i=1,2,\ldots$$

The probability measure

$$P_i(dx) := \frac{1}{b^i}\, l(dx)\, f_i(x)$$

is the distribution law of a random variable $\sum_{m=1}^{i} \eta_m$. Hereby $\eta_1, \ldots, \eta_m$ are i.i.d. and the distribution of η_1 is given by $\frac{1}{b} l(dx) \exp[-U(x)]$. For this reason we get

(13.7)
$$\int l(\mathrm{d}x)\, |x| f_i(x) \leqq i b^{i-1} \int l(\mathrm{d}x)\, |x| \exp[-U(x)]; \qquad i=1,2,\ldots$$

Finally, the momentmeasures $m_Q^{(n)}$ of a stationary poisson process on R with intensity λ are given by

(13.8)
$$m_Q^{(n)}(dx) = \lambda^n l^n(dx)$$

From (13.8), (13.1), (13.5), (13.6), and (13.7) we get

$$\sum_{n=1}^{\infty} \int m_Q^{(n)}(\mathrm{d}[x_0, \ldots, x_n]) \exp\left[-U(x_n-x_0) - \sum_{i=0}^{n-1} U(x_i-x_{i+1})\right]$$
$$\cdot \bigcup_{i=1}^{n} (\{[z_0, \ldots, z_n]; z_0 \leqq 0, z_i > 0\}$$
$$\cdot\, U\{[z_0, \ldots, z_n]; z_0 \geqq 0, z_i < 0\})$$
$$\leqq \int |x| \exp[-U(x)]\, l(\mathrm{d}x) \sum_{n=1}^{\infty} \frac{n(n+1)}{2} \lambda^{n+1} b^{n-1}$$

For this reason (5.17) follows from (5.18), (5.19).

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