Probab. Th. Rel. Fields 89, 1–33 (1991)

Probability Theory and Related Fields

The Lifschitz singularity for the density of states on the Sierpinski gasket

Katarzyna Pietruska-Paluba

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA*

Received May 17, 1990; in revised form November 26, 1990

Summary. We prove the existence of the density of states for the Laplacian on the infinite Sierpinski gasket. Then the Lifschitz-type singularity of the density of states is established. We also investigate the long-time asymptotics of the Brownian trajectory on the Sierpinski gasket, getting bounds similar to those in the $\mathbb{R}^d$-case.

1. Introduction

This paper deals with the Brownian motion on the two-dimensional infinite Sierpinski gasket, $\mathscr{G}$. For the construction of the process and its properties, we refer the reader to [1]. However, to make this paper self-contained, we sketch the construction and collect the properties we shall be using in the next section.

Consider a random cloud of points, governed by a Poisson point process with intensity $\nu\, d\mu$ ($\nu>0$ is a fixed number, and μ is the Hausdorff measure on the gasket). We assume that this Poisson point process is independent of the underlying Brownian motion. The points of the Poisson process will constitute the obstacles for our Brownian motion, preventing it from spreading.

One looks at the Brownian motion on a large ball B_M centered at zero (its radius will be equal to 2^M), which is absorbed at the boundary of B_M and at the obstacle points. The generator of this process, $-A$ (with random points at which the Dirichlet boundary conditions are imposed), is a positive self-adjoint operator. By standard theory we obtain that its spectrum consists only of positive eigenvalues, each of finite multiplicity. We build an empirical measure, based on this (random) sequence of eigenvalues, and normalize it by dividing by the volume of the ball B_M. Denote this measure by $l(M, \omega)$.

We are interested in the asymptotic behavior of measures $l(M, \omega)$ as M goes to infinity and then in the behavior of the limiting measure, l (l is called the density of states), near zero. For the survey of results on the density of states in the Euclidean and lattice case, see [2].

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* On leave from Warsaw University, Warsaw, Poland

In present paper, we show the existence of the density of states for the Brownian motion with Poisson obstacles. In the proof, we are not able to profit from the translation invariance of the process (which does not hold) – as one can in the $\mathbb{R}^d$-case, see [2, 10]. The lack of the translation invariance also jeopardizes the attempts to find a close formula for the Laplace transform of the density of states. Such a formula exists in the Euclidean space-case, as well as in some other examples (see, e.g., [11–13]). However, using the methods alike those in [10], we are able to investigate the asymptotics of $l([0, \lambda])$ for λ small. What we get is that there exist two positive constants C and D such that

$$(1) \qquad -C\nu \leqq \liminf_{\lambda \to 0} \frac{\log l([0,\lambda])}{\lambda^{\frac{-d_s}{2}}} \leqq \limsup_{\lambda \to 0} \frac{\log l([0,\lambda])}{\lambda^{\frac{-d_s}{2}}} \leqq -D\nu,$$

where d_s denotes the so-called 'spectral dimension' of the gasket, $d_s = \frac{2\log 3}{\log 5}$. Thus, the behavior near zero, with the obstacles introduced, is exponential (which is the Lifschitz-type singularity). The power that in singled out is, unlike in the $\mathbb{R}^d$-case, not the Hausdorff dimension of the gasket, $d_f = \frac{\log 3}{\log 2}$, but its spectral dimension d_s.

Similar problems for the Brownian motion in the Euclidean space and in the hyperbolic space were treated in [10, 12, 13].

In Sect. 2, we present the results from [1] that will be important for our work.

The construction of the density of states, as well as all the estimates we derive, are performed using the Laplace transforms of underlying measures. This approach is very useful, as it allows us use the trace formula.

Section 3 is devoted to the construction of the density of states. The punchline is alike the one employed in the $\mathbb{R}^d$ case first we show that the expected values of the Laplace transforms of the measures $l(M, \omega)$ converge, and then we derive a property which in virtue of the Fubini's theorem shows the almost sure convergence of $l(M, \omega)$.

Having established this preliminary result we devote the rest of the paper to getting the following inequality:

Denoting by $L(t)$ the Laplace transform of the measure l, we have

$$(2) \qquad -C\nu^{\frac{2}{d_s+2}} \leqq \liminf_{t \to \infty} \frac{\log L(t)}{t^{\frac{d_s}{d_s+2}}} \leqq \limsup_{t \to \infty} \frac{\log L(t)}{t^{\frac{d_s}{d_s+2}}} \leqq -D\nu^{\frac{2}{d_s+2}}$$

with positive C, D. This then can be transformed into (1), using the Minlos-Povzner Tauberian theorem from [5].

The lower bound in (2) (obtained in Sect. 4) is not difficult; one estimates the probability that no obstacles fall into a large ball and that the process does not leave this ball.

For the upper bound part we, as usual in problems of that kind, tend to replace the Brownian motion on the whole Sierpinski gasket with a process with a compact state-space. What we use, is the reflected Brownian motion on the Sierpinski gasket ('reflected' is used for geometrical reasons, or for analogy with one of the constructions of the normally reflected Brownian motion on the finite interval). We have gotten to resort to such a process, since the 'natural guess', i.e. the projection onto a torus of an appropriate size fails in this case by not leading to a Markov process.

The construction of the reflected Brownian motion is carried out in Sect. 5; Sect. 6 is concerned with the upper bound in (2).

A subtle part of the problem is that one must perform all the optimization procedure before passing to the limit with M, and get the bounds independent of M. All way through we also face a technical nuisance of the lack of continuous scaling in the gasket case. Moreover, no scaling can be used after projection onto the compact state-space. This is handled by performing all the optimization before the projection.

Our setting is also suitable for obtaining results for asymptotics of $E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]$, where $Z_{[0,t]}$ denotes the Brownian trajectory from time 0 to t, and μ- the x^{d_f} Hausdorff measure on the Sierpinski gasket. The result we get is a counterpart of the Donsker-Varadhan Wiener sausage estimate in $\mathbb{R}^d$ (see [3]) (with the difference that in our case the trajectory itself has nonzero Hausdorff measure).

The result is:

there exist two positive constants C_2 and D_2 such that:

$$(3)\qquad -C_2\nu^{\frac{2}{d_s+2}} \leq \liminf_{t\to\infty} \frac{\log E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]}{t^{\frac{d_s}{d_s+2}}} \leq \liminf_{t\to\infty} \frac{\log E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]}{t^{\frac{d_s}{d_s+2}}} \leq -D_2\nu^{\frac{2}{d_s+2}}.$$

For the lower bound in (3), one performs some observation on the spectral decomposition of the underlying semigroup. The upper bound can be obtained in the way similar to the upper bound in (2).

Recently, Fukushima and Shima [4], obtained the existence of the density of states in the no-obstacle case. Their paper investigates its asymptotics near zero, which in that case is power law-like. The paper also shows the highly irregular behavior of the density of states, due to the fractal nature of the domain. Independently, Lindstrom in [8], gave the asymptotics of the eigenvalues of the generator of the Brownian motion on an arbitrary nested fractal for large λ. It is consistent with the result from [4].

2. A survey of the properties of the Brownian motion on the Sierpinski gasket

In the sequel, we shall use the notation as in the paper of M.T. Barlow and E.A. Perkins [1]. In what follows, we summarize the notation and list the properties of the Brownian motion on the Sierpinski gasket we shall make use of.

Let $a_0=(0,0)$, $a_1=(0,1)$, $a_2=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, $V_0=\{a_0,a_1,a_2\}$. a_0,a_1,a_2 are the vertices of an equilateral triangle of unit size (see Fig. 1). Let $\mathscr{S}_0$ be this equilateral triangle. We define inductively:

$$V_{M+1}=V_M\cup\{2^M a_1+V_M\}\cup\{2^M a_2+V_M\}$$

and we put:

$$\mathscr{G}_0=\bigcup_{M=0}^{\infty} V_M \cup \bigcup_{M=0}^{\infty} \bar{V}_M,$$

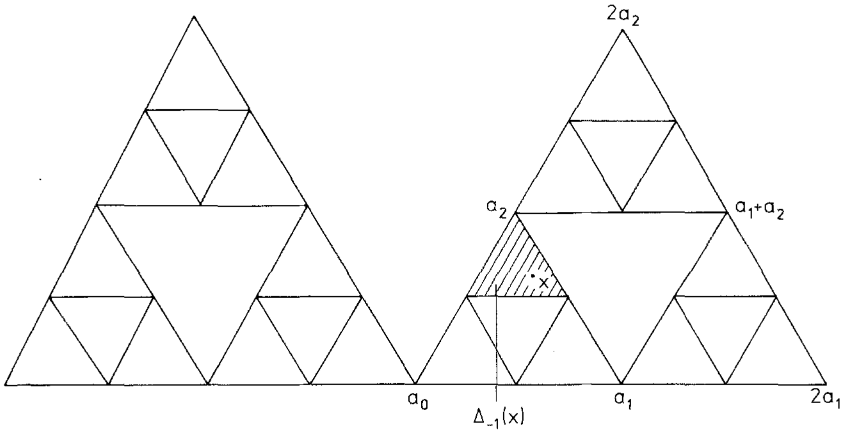

Fig. 1.

where $\bar{V}_M$ denotes the symmetric image of V_M in the symmetry with respect to the y-axis. Now we let

$$\mathscr{G}_M = 2^M \mathscr{G}_0, \qquad M \in \mathbb{Z}$$

and

$$\mathscr{G}_\infty = \bigcup_{M \leq 0} \mathscr{G}_M.$$

$\mathscr{G}_\infty$ is called the (infinite) Sierpinski pre-gasket. Its closure (in the plane topology) is the 2-dimensional Sierpinski gasket and it will be denoted by $\mathscr{G}$.

More notation: a $\mathscr{G}_M$-triangle is the closed set of points in $\mathscr{G}$ that lie inside an equilateral triangle, which is the translation of $2^M \mathscr{I}_0$ and whose vertices are the three neighboring points in $\mathscr{G}_M$. The collection of all closed $\mathscr{G}_M$-triangles will be denoted by $\mathscr{T}_M$. For $x \in \mathscr{G} \setminus \mathscr{G}_M$ we define $\Delta_M(x)$ to be the unique triangle from $\mathscr{T}_M$ containing x (see Fig. 1).

We have the following distance on the gasket: for $x, y \in \mathscr{G}$ we put $d(x, y)$ to be equal to the infimum of the length of all the paths in $\mathscr{G}$, joining x and y.

By B_M we shall denote the closed ball in the gasket metric, of radius 2^M, centered at zero and by $\mathscr{F}_M$ – the intersection $B_M \cap \{(x, y) \in \mathbb{R}_2 : x \geq 0\}$.

We introduce the following numbers:

$$d_f = \frac{\log 3}{\log 2} = 1.58496\ldots \qquad \text{(fractal dimension of } \mathscr{G}\text{)}$$

$$d_s = \frac{2 \log 3}{\log 5} = 1.36521\ldots \qquad \text{(spectral dimension of } \mathscr{G}\text{)}$$

$$d_w = \frac{2 d_f}{d_s} = \frac{\log 5}{\log 2} = 2.32193\ldots \quad \text{(dimension of the walk).}$$

These numbers fulfill:

$$\frac{d_f}{d_f + d_w} = \frac{d_s}{d_s + 2}, \qquad \frac{d_w}{d_f + d_w} = \frac{2}{d_s + 2}. \tag{4}$$

Let μ_M be the measure which puts mass $(\frac{2}{3}) 3^{-M}$ at each point in $\mathscr{G}_M$.

Now we formulate the following lemma (Lemma 1.1 of [1]).

Lemma 1. 1. *There exists unique measure* μ *on* $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$, *supported on* $\mathscr{G}$ *such that* $\mu(\Delta_M)=3^{-M}$ *for all* $\Delta\in\mathscr{T}_M$, $M\in\mathbb{Z}$

2. $\{\mu_M\}$ *converges to* μ *in the vague topology,*
3. μ *is a multiple of the Hausdorff* x^{d_f}*-measure on* $\mathscr{G}$,
4. $\mu(\mathscr{F}_0)=1$.

We introduce a new metric to the gasket, so-called 'gasket metric' – it better suits our purposes:

for $x, y\in\mathscr{G}_\infty$ *define* $d(x, y)$ *to be the infimum over the Euclidean length of all paths, joining x and y on the gasket.*

It extends in an obvious way (limit procedure) to the whole gasket $\mathscr{G}$. This metric is equivalent to the euclidean metric on the plane, in fact

$$|x-y|\leqq d(x, y)\leqq 2|x-y|. \tag{5}$$

[1] gives a construction of the process Z_t, called the Brownian motion on the Sierpinski gasket (the construction of the Brownian motion on the Sierpinski gasket was earlier carried on by Goldstein ([6]) and Kusuoka in [7], but [1] give very precise estimates on the transition density, therefore we choose the approach from that paper). It is a strongly Markov Feller process which has a continuous symmetric density $p(t, x, y)$, satisfying (Theorem 1.5 of [1]):

$$\begin{aligned} C_{1.1}\, t^{\frac{-d_s}{2}} \exp\{-C_{1.2}(d(x, y)\, t^{\frac{-1}{d_w}})^{\frac{d_w}{d_w-1}}\} &\leqq p(t, x, y)\leqq \\ \leqq C_{1.3}\, t^{\frac{-d_s}{2}} \exp\{-C_{1.4}(d(x, y)\, t^{\frac{-1}{d_w}})^{\frac{d_w}{d_w-1}}\}&. \end{aligned} \tag{6}$$

The process admits a discrete scaling, namely, for $\Gamma\in\mathscr{B}(\mathscr{G})$,

$$P_x[Z_t\in\Gamma]=P_{2x}[\tfrac{1}{2}Z_{5t}\in\Gamma].$$

In particular, for the process starting from the origin,

$$\mathscr{L}(Z_t)=\mathscr{L}(\tfrac{1}{2}Z_{5t}).$$

All this translates into terms of density as (Theorem 7.8 of [1])

$$p(t, 2x, 2y)=\tfrac{1}{3}p(t/5, x, y). \tag{7}$$

The last thing we need are the following sample path and hitting time estimates from [1]:

- (Theorem 4.3) For all $x\in\mathscr{G}$, all $t, \delta\in(0, \infty)$

$$P_x[\sup_{s\leqq t} d(Z_s, Z_0)\geqq\delta[\leqq C_{1.5}\exp\{-C_{1.6}(\delta\, t^{\frac{-1}{d_w}})^{\frac{d_w}{d_w-1}}\}. \tag{8}$$

- (Proposition 5.18) Let R_λ be an exponential random variable with mean $1/\lambda$. Then:

$$P_x[T_y\geqq R_\lambda]\leqq C_{1.7}[d(x, y)]^{d_w-d_f}\lambda^{1-\frac{1}{2}d_s}. \tag{9}$$

• (Theorem 5.21)

$$P_x[T_y < R_\lambda] \leqq C_{1.8} \exp\{C_{1.9}\, \lambda^{\frac{1}{d_w}}\, d(x, y)\}, \tag{10}$$

where T_y is the hitting time of the point y.

Estimates (9) and (10) are complementary: inequality (9) is convenient for x, y lying close, inequality (10) – for lying far.

In what follows, when no confusion arises, we shall drop the subscripts – C will denote a generic positive constant.

Finally, let us introduce a notation for the supremum of $p(1, x, y)$:

$$\xi = \sup_{x \in \mathscr{G}} p(1, x, y) < \infty. \tag{11}$$

3. Existence of the density of states

Let $\mathscr{N}$ be a Poisson cloud of points, defined on the probability space $(\Omega, \mathscr{M}, \mathbb{P})$, with intensity $v\,d\mu$ (v is a positive parameter, μ is the Hausdorff measure on the Sierpinski gasket), falling onto the gasket. We assume that $\mathscr{N}$ is independent from the Brownian motion we shall be investigating. By $\mathbb{E}$ we shall be denoting the expected value, corresponding to the probability measure $\mathbb{P}$. The points of this Poisson process will form the obstacles for our Brownian motion, preventing if form spreading.

Denote by $\mathscr{G}^{\mathscr{N}(\omega)}$ the Sierpinski gasket with the obstacles removed; $\mathscr{G}^{\mathscr{N}(\omega)} = \mathscr{G} \setminus \mathscr{N}(\omega)$. In the sequel, when no confusion arises, the dependence on ω will be dropped.

Our goal is to construct the density of states for this process and to investigate its asymptotic properties near 0. To this end, we will denote by B_M the ball of radius 2^M, centered at zero. Let $Z_t^{\mathscr{N},M}$ be the Brownian motion on $B_M \cap \mathscr{G}^{\mathscr{N}}$, killed upon coming to the boundary of the ball or to any of the point-obstacles (it corresponds to the Dirichlet boundary conditions imposed on $\partial B_M \cup \mathscr{N}$). The process $(Z_t^{\mathscr{N},M})_{t \geqq 0}$ is a Markov process with symmetric transition density

$$p^{M,\mathscr{N}}(t, x, y) = \begin{cases} p(t, x, y)\, P_{x,y}^t[T_{\partial B_M} > t,\ T_{\mathscr{N}(\omega)} > t] & \text{for } x, y \in B_M \setminus \mathscr{N} \\ 0 & \text{otherwise.} \end{cases}$$

In the last formula, $P_{x,y}^t$ denotes the bridge measure for the Brownian motion on the Sierpinski gasket; a measure that is concentrated on trajectories which start from x at time 0 and at time t are at site y. Formally:

$P_{x,y}^t$ is a measure on $C([0, t], \mathscr{G})$ such that

$$\textit{for } 0 \leqq s < t \textit{ and } A \in \sigma(Z_u, 0 \leqq u \leqq s)$$

$$P_{x,y}^t[A] = \frac{1}{p(t, x, y)} E_x[1_A\, p(t-s, Z_s, y)].$$

In the sequel, we will need the continuity of the transition density for the non-obstacle problem on a ball, absorbed at its boundary. It follows from the weak continuity of the bridge measures and the continuity of $p(\cdot, \cdot, \cdot)$.

The corresponding semigroup, $(P_t^{M,\mathcal{N}})_{t\geqq 0}$, is a semigroup of trace class operators and has a selfadjoint generator $A^{M,\mathcal{N}}$. A^M is a Laplacian on the Sierpinski gasket (in the Barlow-Perkins sense) with Dirichlet boundary conditions on ∂B_M, $A^{M,\mathcal{N}}$ its counterpart with the Dirichlet boundary conditions on $\partial B_M \cup \mathcal{N}$. Usual arguments give that the spectrum of $-A^{M,\mathcal{N}}$ conisists only of positive eigenvalues, each with finite multiplicity. Let

$$0 \leqq \lambda_1(\omega, M) \leqq \ldots \leqq \lambda_n(\omega, M) \leqq \ldots \tag{12}$$

be the sequence of eigenvalues of $-A^{M,\mathcal{N}}$. We build random empirical measures, based on those spectra:

$$l(M, \omega) = \frac{1}{\mu(B_M)} \sum_{n=1}^{\infty} \delta_{\{\lambda_n(\omega, M)\}}. \tag{13}$$

Let $L(M, \omega)$ be the Laplace transform of $l(M, \omega)$, i.e.:

$$\begin{aligned} L(M, \omega) &= \int_0^\infty \mathrm{e}^{-\lambda t}\, dl(M, \omega)(t) \\ &= \frac{1}{\mu(B_M)} \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n(M,\omega)t} = \frac{1}{\mu(B_M)} \operatorname{Tr} P_t^{M,\mathcal{N}} \end{aligned} \tag{14}$$

By the trace formula one has:

$$\begin{aligned} \operatorname{Tr} P_t^{M,\mathcal{N}} &= \int_{B_M} p^{M,\mathcal{N}}(t, x, x)\, d\mu(x) \\ &= \int_{B_M} p(t, x, x)\, P_{x,x}^t [\{T_{\partial B_M} > t\} \cap \{T_{\mathcal{N}(\omega)} > t\}]\, d\mu(x) \end{aligned} \tag{15}$$

where $T_{\mathcal{N}(\omega)}$ denotes the hitting time of the Poisson-cloud obstacles.

To prove the existence of the density of states, we first prove a lemma stating the convergence of the expected values of the underlying Laplace transforms.

Lemma 2. *For every $t > 0$, $\mathbb{E}L(M, \omega)(t)$ is convergent, as $M \to \infty$, to a finite limit $L(t)$.*

Proof. For simplicity, we will write $L(M, t)$ instead of $L(M, \omega)(t)$. Then, from the trace formula:

$$\begin{aligned} \mathbb{E}L(M, t) &= \mathbb{E} \frac{1}{\mu(B_M)} \int_{B_M} p(t, x, x)\, P_{x,x}^t [\{T_{\partial B_M} > t\} \cap \{T_{\mathcal{N}(\omega)} > t\}]\, d\mu(x) \\ &= \frac{1}{\mu(B_M)} \int_{B_M} p(t, x, x)\, P_{x,x}^t [[T_{\partial B_M} > t]\, \mathbb{P}[T_{\mathcal{N}(\omega)} > t]]\, d\mu(x). \end{aligned} \tag{16}$$

Notice that the event $\{T_{\mathcal{N}(\omega)} > t\}$, meaning that 'the Brownian motion does not hit any obstacle up to time t' is the same as 'no obstacles fall onto the Brownian trajectory up time t', thus

$$\mathbb{P}[T_{\mathcal{N}(\omega)} > t] = \exp\{-\nu \mu(Z_{[0,t]})\}.$$

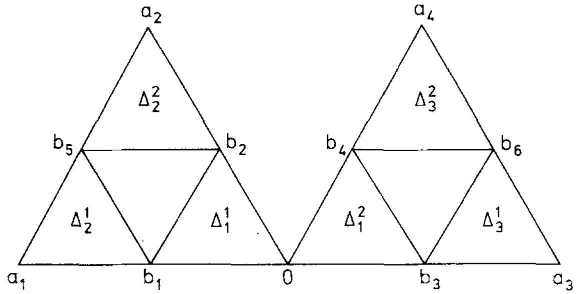

Fig. 2.

It follows:

$$(17)\ \mathbb{E}L(M,t)=\frac{1}{\mu(B_M)}\int_{B_M} p(t,x,x)\,E_{x,x}^t[\exp\{-\nu\mu(Z_{[0,t]})\}\,1\{T_{\partial B_M}>t\}]\,d\mu(x).$$

We shall see that $\mathbb{E}L(M, t)$ is for each t an increasing function of M.

Look at the Fig. 2, representing the balls B_M and B_{M+1}. Notice that in the expression (17), written for $L(M+1,\cdot)$, the quantity under the integral sign corresponds to the absorption imposed at points $\{a_1, a_2, a_3, a_4\}$ (and all the obstacle points). Clearly, this quantity decreases when we impose additional absorbing points $\{b_1, \ldots, b_4\}$.

After introducing the new absorbing points, the process lives in one of the three disjoint sets $\Delta_1=\Delta_1^1\cup\Delta_1^2$, $\Delta_2=\Delta_2^1\cup\Delta_2^2$, $\Delta_3=\Delta_3^1\cup\Delta_3^2$, depending on its starting point, (these sets are in fact balls of radius 2^M, centered at 0, b_5, b_6). Now we can use the fact that the processes, living on Δ_1, Δ_2, Δ_3 (absorbed at their boundary points) are equidistributed. As $\mu(B_{M+1})=3\,\mu(B_M)$ we obtain $\mathbb{E}L(M+1, t)\geqq\mathbb{E}L(M, t)$.

The other point we must check is that for every $t\{\mathbb{E}L(M, t)\}_M$ is a bounded sequence. This is clear from the bound (6) for the density $p(t, x, y)$.

In this way the convergence of expected values is established. □

In what follows, we will denote by $L(t)$ the limit of $\mathbb{E}L(M, t)$.

Theorem 1. $\mathbb{P}$-*almost surely, the random measures $l(M, \omega)$ converge as $M\to\infty$ to some deterministic measure. The limiting measure will be denoted by l.*

Proof. Our proof is a modification of the proof of the theorem (1.1) from [11]. For convenience, we submit the adapted proof here.

It is enough to show that

$$(18)\qquad \sum_M \mathbb{E}([L(M,t)-\mathbb{E}L(M,t)]^2)<\infty.$$

Indeed, then an elementary Borel-Cantelli lemma argument gives (for every t) the almost sure convergence $L(M, t)\to L(t)$. Hence we obtain the almost sure convergence for all rational t. As we know, $\{l(M, \omega)\}_M$ is an almost surely vaguely relatively compact sequence of measures on $[0, \infty)$, thus the theorem will follow.

This way we are led to showing Eq. (18).

To this end, let us introduce the following measures on the probability space $\tilde{\Omega}=\Omega^3\times C([0,t],\mathscr{G})^2$:

(19) $$\nu_M=\mathbb{P}^{\otimes 3}\otimes\left(\frac{1}{\mu(B_M)}\int_{B_M} p(t,x,x)\,P^t_{x,x}\,d\mu(x)\right)^{\otimes 2}$$

Then, using those measures and the trace formula, we obtain that the terms of (18) look like:

(20) $$\begin{aligned}&\mathbb{E}([L(M,t)-\mathbb{E}L(M,t)]^2)\\&=\int_{\tilde{\Omega}}\prod_{i=1}^{2}1\{T_{\partial B_M}(w_i)>t\}\cdot(1\{T_{\mathcal{N}(\omega_0)}(w_i)>t\}-1\{T_{\mathcal{N}(\omega_i)}(w_i)>t\})\\&\quad\cdot d\nu_M(\omega_0,\omega_1,\omega_2,w_1,w_2)\end{aligned}$$

where $(\omega_0,\omega_1,\omega_2,w_1,w_2)$ is an element of $\tilde{\Omega}$, $(\omega_0,\omega_1,\omega_2)$ pertains to the Poisson cloud, (w_1,w_2) – to the Brownian motion on the gasket.

To bound the integral (20), we partition $\tilde{\Omega}$ into two sets: $\mathscr{C}$ and its complement. $\mathscr{C}$ is the subset of $C([0,t],\mathscr{G})^2$ consisting of pairs trajectories with non-intersecting supports (we identify $\mathscr{C}$ with $\Omega^3\times\mathscr{C}$). The random variables on Ω^3:

$$1\{T_{\mathcal{N}(\omega_0)}(w_1)>t\}-1\{T_{\mathcal{N}(\omega_1)}(w_1)>t\},$$
$$1\{T_{\mathcal{N}(\omega_0)}(w_2)>t\}-1\{T_{\mathcal{N}(\omega_2)}(w_2)>t\}$$

are independent if $(w_1,w_2)\in\mathscr{C}$, as they do depend only on events, involving the Poisson measures, restricted to disjoint sets: $Z_{[0,t]}(w_1)$, $Z_{[0,t]}(w_2)$. From the independence we infer that the integral over $\Omega^3\times\mathscr{C}$ equals to zero.

The integral over $\Omega^3\times\mathscr{C}^c$ is estimated by bounding $\nu_M(\Omega^3\times\mathscr{C}^c)$, as the absolute value of the integrand does not exceed 1. Let $\mathscr{B}$ be the set of w's for which

$$d(Z_0(w_1),Z_0(w_2))>2c_M,$$

but $Z_{[0,t]}(w_1)$ and $Z_{[0,t]}(w_2)$ are not disjoint. c_M is some positive constant – it will be chosen later on. It follows that on this set, for $i=1$ or $i=2$ we have

(21) $$\sup_{0<s\leqq t} d(Z_0(w_i),Z_s(w_i))>c_M$$

Using the sample path and density estimates (6, 8) we obtain:

(22) $$\begin{aligned}\nu_M(\mathscr{B})&\leqq 2\sup_{x\in\mathscr{G}}[p(t,x,x)\,P^t_{x,x}[\sup_{0<s\leqq t} d(Z_0,Z_s)>c_m]]\leqq\\&\leqq 4\sup_{x\in\mathscr{G}}E_x\left[1\left\{\sup_{0<s\leqq\frac{t}{2}} d(x,Z_s)>c_M\right\}p\left(\frac{t}{2},Z_{\frac{t}{2}},x\right)\right]\leqq\\&\leqq 4C\sup_{x\in\mathscr{G}}P_x[\sup_{0<s\leqq\frac{t}{2}} d(x,Z_s)>c_M]\cdot\frac{1}{\left(\frac{t}{2}\right)^{\frac{d_s}{2}}}\leqq\\&\leqq 4C\left(\frac{t}{2}\right)^{-\frac{d_s}{2}}\exp\left[-C\left(c_M\left(\frac{t}{2}\right)^{-\frac{1}{d_w}}\right)^{\frac{d_w}{d_w-1}}\right]=C\exp\left[-C(c_M)^{\frac{d_w}{d_w-1}}\right],\end{aligned}$$

where all the constants depend on t only.

We have found that

(23) $$\mathbb{E}[L(M,t)-\mathbb{E}L(M,t)]^2 \leqq C\exp(-C(c_M)^{\frac{d_w}{d_w-1}})+\nu_M(\mathscr{B}^c\cap\mathscr{C}^c).$$

$\mathscr{B}^c\cap\mathscr{C}^c$ is the collection of those pairs of Brownian trajectories which intersect each other, but have started at distance less or equal to $2c_M$.

We have:

(24) $$\nu_M(\mathscr{B}^c\cap\mathscr{C}^c)\leqq C\cdot\bar{\mu}_M\otimes\bar{\mu}_M(\mathscr{D}),$$

where $\bar{\mu}_M$ is the normalized Hausdorff measure on B_M, $\mathscr{D}\subset(B_M)^2$, consists of pairs $\{x_1, x_2\}$ such that $d(x_1,x_2)\leqq 2c_M$ (we came to this set by considering only the starting points of our Brownian trajectories). Thus

(25) $$\bar{\mu}_M\otimes\bar{\mu}_M(\mathscr{D})\leqq\frac{\sup_{x\in\mathscr{G}}\mu(B(x,2c_M))}{\mu(B_M)}.$$

Taking $2c_M=2^{\frac{M}{2}}$ we get from the gasket scaling

(26) $$\bar{\mu}_M\otimes\bar{\mu}_M(\mathscr{B}^c\cap\mathscr{C}^c)\leqq C\cdot 3^{-\frac{M}{2}},$$

thus the right member of (23) is a term of a convergent series. This completes the proof. □

Remark 1. In the proof above the balls that appeared were centered at zero. However, it is not difficult to carry out all the details in the case when we center our balls at some point $x\in\mathscr{G}$. Indeed, if we use the fact that

$$\frac{\mu(B(0,2^M)\div B(x,2^M))}{\mu(B(0,2^M))}$$

goes to zero when $M\to\infty$ and that $p(t,x,x)$ is bounded independently of x (bound depends on t only), we get Lemma 2 immediately. Theorem 1 does not rely on the centering. Hence the density of states can be constructed independently of the centering point.

Remark 2. The other remark is the observation that the limit remains unchanged, if we replace the 'full' ball B_M by its 'half': the triangle $\mathscr{F}_M$. Indeed, one notices that $\{T_{\partial B_M}>t\}=\{T_{\partial\mathscr{F}_M}>1\}\cup\{T_{\partial B_M}>t,\ T_{\{0\}}\leqq t\}$, and this sum is disjoint. Therefore the only thing we must show is that the following quantity:

$$\frac{1}{\mu(\mathscr{F}_M)}\int_{\mathscr{F}_M} p(t,x,x)\,P^t_{x,x}[\{T_{\partial B_M}\leqq t,\ T_{\{0\}}\leqq t\}\cap\{T_{\mathscr{N}(\omega)}>t\}]\,\mathrm{d}\mu(x)$$

goes to 0 when $m\to\infty$.

For fixed, sufficiently large m denote:

$$\mathscr{A}_M=\{x\in\mathscr{F}_M: d(x,0)\leqq 2^{m/2}\},$$

$$\mathscr{B}_M=\{x\in\mathscr{F}_M: d(x,0)>2^{m/2}\}.$$

The integral over $\mathscr{A}_M$ goes to 0, which is obvious from the area comparison (the integrand is bounded and the bound does not depend on M). The other – over $\mathscr{B}_M$ – can also be estimated easily, if we notice that on the paths that start at x

$$\{T_{\{0\}} \leqq t\} \subseteq \{\sup_{s \leqq t} d(Z_0, Z_s) \geqq d(x, 0)\}$$

and take into account the estimate (8). Making m of magnitude $\sqrt{M}$ we get the desired result.

This way we have constructed the density of states for the problem with Poissonian obstacles. It is the deterministic measure l, with Laplace transform $L(t)$. This measure is concentrated on $[0, \infty)$. In what follows, we are interested in the behavior of $l([0, t])$ for t tending to zero. To this end, we shall find some estimates for $L(t)$ as $t \to \infty$ and then use the Tauberian theorem from [5]. In the sequel, we shall rather be using the method of obtaining the density of states which uses the triangles, not balls.

4. Asymptotic lower bound

In this section, we obtain a long time asymptotic lower bound for $\frac{\log L(t)}{t^{\frac{d_s}{d_s+2}}}$ and for

$$\frac{\log E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]}{t^{\frac{d_s}{d_s+2}}}.$$

We get this via a crude estimate: by assuming that the Brownian motion does not leave the ball up to time t, and no obstacles fall onto this ball. Using these methods we obtain

Theorem 2 *There exist positive constants C_1 and C_2 such that:*

$$\liminf_{t \to \infty} \frac{\log L(t)}{t^{\frac{d_s}{d_s+2}}} \geqq -C_1 \cdot \nu^{\frac{2}{d_s+2}}, \tag{27}$$

and, independently of $x \in \mathscr{G}$,

$$\liminf_{t \to \infty} \frac{\log E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]}{t^{\frac{d_s}{d_s+2}}} \geqq -C_2 \cdot \nu^{\frac{2}{d_s+2}} \tag{28}$$

Proof. First, we prove (27).

We know that for every t, $L(t)$ is the increasing limit as $M \to \infty$ of $\mathbb{E}L(M, t)$. Therefore for each M we have

$$\begin{aligned} L(t) &\geqq \frac{1}{\mu(B_M)} \mathbb{E} \int_{B_M} p(t, x, x) P^t_{x,x}[\{T_{\partial B_M} > t\} \cap \{T_{\mathscr{N}(\omega)} > t\}] \, d\mu(x) \\ &\geqq \frac{1}{\mu(B_M)} \int_{B_M} p(t, x, x)(P^t_{x,x} \otimes \mathbb{P})(\mathscr{M}) \, d\mu(x) \end{aligned} \tag{29}$$

where $\mathscr{M}$ is the event 'Z_t does not leave the ball B_M up to time t, no obstacles fall into the ball',

$$\mathscr{M}=\{T_{\partial B_M}>t\}\cap\{\mathcal{N}(\omega, B_M)=0\}.$$

Therefore the quantity (29) is equal to

$$\text{(30)}\qquad \frac{1}{\mu(B_M)}\int_{B_M} p(t,x,x)\,\mathbb{P}[\mathcal{N}(\omega,B_M)=0]\cdot P^t_{x,x}[T_{\partial B_M}>t]\,d\mu(x)$$

$$=\exp[-\nu\mu(B_M)]\frac{1}{\mu(B_M)}\int_{B_M} p(t,x,x)\,P^t_{x,x}[T_{\partial B_M}>t]\,d\mu(x).$$

Now, recognize in the last integral of (30) the trace of the operator P_t^M, (corresponding to the Brownian motion on B_M with no obstacles). It follows that this integral is greater than the principal eigenvalue of P_t^M, $\exp[-t\lambda(B_M)]$, ($\lambda(B)$ denotes the principal eigenvalue of the Laplacian in B, with Dirichlet boundary conditions on ∂B). We get that for every M

$$\text{(31)}\qquad L(t)\geq\frac{1}{\mu(B_M)}\exp[-\nu\mu(B_M)-t\lambda(B_M)]$$

$$=\frac{1}{\mu(B_M)}\exp\left[-\nu\left(\mu(B_M)+\frac{t}{\nu}\lambda(B_M)\right)\right].$$

In what follows we shall be using the scaling:

$$\text{(32)}\qquad \mu(2B_M)=3\mu(B_M),$$

$$\lambda(2B_M)=\tfrac{1}{5}\lambda(B_M),$$

in other words, for all t of the form $t=2^n$

$$\mu(tB_M)=t^{d_f}\mu(B_M),$$

$$\lambda(tB_M)=\frac{1}{t^{d_w}}\lambda(B_M).$$

Although the scaling in our case is only discrete (which makes it different from the Brownian motion in $\mathbb{R}^d$, for example) we shall make use of it: for an arbitrary $t\in\mathbb{R}$, we will replace it with one of the numbers that admit the scaling and then introduce the error. The ideal scaling factor we would like to take is $\left(\frac{t}{\nu}\right)^{\frac{1}{d_f+d_w}}$, instead, if

$$\text{(33)}\qquad 2^n\leqq\left(\frac{t}{\nu}\right)^{\frac{1}{d_f+d_w}}<2^{n+1},$$

then we replace $\left(\frac{t}{v}\right)^{\frac{1}{d_f+d_w}}$ with 2^n and fix $M=n$. Using the scaling (32), (33) and the relation (7) we get that

$$L(t) \geqq \frac{1}{\mu(B_M)} \exp\{t^{\frac{d_s}{d_s+2}}(-v^{\frac{2}{d_s+2}}[\mu(B_0)+5\lambda(B_0)])\}, \tag{34}$$

whence

$$\frac{\log L(t)}{t^{\frac{d_s}{d_s+2}}} \geqq \frac{-\log \mu(B_M)}{t^{\frac{d_s}{d_s+2}}} - v^{\frac{2}{d_s+2}}[\mu(B_0)+5\lambda(B_0)] \tag{35}$$

and,

$$\liminf_{t\to\infty} \frac{\log L(t)}{t^{\frac{d_s}{d_s+2}}} \geqq -v^{\frac{2}{d_s+2}}[\mu(B_0)+5\lambda(B_0)].$$

$\mu(B_0)+5\lambda(B_0)$ is a positive number (easy check, see also Lemma 10), and we set C_1 to be equal to that number. The proof of (27) is complete.

Next we pass to the proof of (28). First notice that for every M

$$\begin{aligned} E_x[e^{-v\mu(Z_{[0,t]})}] &= E_x[\mathbb{P}[\mathcal{N}(\omega, Z_{[0,t]})=0]] \\ &\geqq E_x[\mathbb{P}[\mathcal{N}(\omega, Z_{[0,t]})=0]\, 1\{T_{\partial B_M}>t\}] \\ &= P_x \otimes \mathbb{P}[\{T_{\partial B_M}>t\} \cap \{T_{\mathcal{N}(\omega)}>t\}]. \end{aligned} \tag{36}$$

The last quantity, as in the proof of (27), is greater than or equal to:

$$e^{-v\mu(B_M)} P_x[T_{\partial B_M}>t].$$

In order to single out the behavior in t and ω, we first perform the scaling – as in (33): if t satisfies (33), then we chose $M=n$ in (36).

First, from (4), (33) and the scaling (32) we obtain that:

$$e^{-v\mu(B_M)} = e^{-v\mu(2^M B_0)} \tag{37}$$

$$\geqq e^{-t^{\frac{d_s}{d_s+2}} v^{\frac{2}{d_2+2}} \mu(B_0)}. \tag{38}$$

Now, writing $x=2^M y$:

$$P_x[T_{\partial B_M}>t] = P_{2^M y}[T_{\partial(2^M B_0)}>t]$$

and the scaling of the process, (7), together with (4) give that

$$P_x[T_{\partial B_M}>t] = P_y\left[T_{\partial B_0} > \frac{t}{5^M}\right] \geqq P_y[T_{\partial B_0} > 5\, t^{\frac{d_s}{d_s+2}} v^{\frac{2}{d_s+2}}].$$

For the sake of notation, we shall denote

$$s = 5\, t^{\frac{d_s}{d_s+2}} v^{\frac{2}{d_s+2}}. \tag{39}$$

As the semigroup (P_t^0) is a semigroup of integral operators with continuous kernel $p^0(\cdot,\cdot,\cdot)$, we can use Mercer's theorem (see par. 98 of [9]) to get that the series

$$P_y[T_{\partial B_0}>s]=\sum_{i=1}^{\infty} e^{-\lambda_i(B_0)s}\phi_i(y)\langle\phi_i,\mathbf{1}\rangle_{L^2(B_0,\mu)} \tag{40}$$

is convergent not only in L^2, but also uniformly with respect to y. Here $\{\lambda_i(B_0)\}$ is the sequence of eigenvalues of $-A^0$, $\{\phi_i\}$ is the corresponding sequence of normalized eigenfunctions (which in this case are continuous).

We will show that the principal eigenvalue of A^0 is simple and that it admits a strictly positive eigenfunction.

First we show that every eigenfunction corresponding to the principal eigenvalue $\lambda_1(B_0)=\lambda_1$ has constant sign and, in fact, is never zero inside B_0.

To see this, assume that ϕ is a continuous principal normalized eigenfuncton and that ϕ changes sign. Then $|\phi|>\phi$ on a set of positive measure and, as the kernel $p^0(1,x,y)$ in strictly positive and continuous inside B_0, $(P_1^0\phi,\phi)<(P_1^0|\phi|,|\phi|)$. This yields a contradiction as

$$(P_1^0|\phi|,|\phi|)>(P_1^0\phi,\phi)=e^{-\lambda_1}=\sup_{\|\psi\|=1}(P_1^0\psi,\psi).$$

Strict positivity (or negativity) of a function that never changes sign follows from the relation $\phi(x)=e^{\lambda_1}P_1^0\phi(x)$.

λ_1 must be simple – if it were not, we would have two orthogonal unit eigenfunctions belonging to this eigenvalue, which is impossible, as they have constant sign.

This way we can assume that the eigenfunction ϕ_1 appearing in (40) is strictly positive inside B_0. Therefore there is a positive constant A such that if $|y|<\frac{1}{2}$, then $\phi_1(y)>A$. Moreover, for all $i\geqq 1$

$$\begin{aligned}|e^{-\lambda_i}\phi_i(x)|=|P_1^0\phi_i(x)|&=\Big|\int_{B_0}p^0(1,x,y)\,\phi_i(y)\,d\mu(y)\Big|\\&\leqq\int_{B_0}p^0(1,x,y)\,|\phi_i(y)|\,d\mu(y)\leqq 2\xi\end{aligned}$$

(ξ is the constant from (11))

and

$$|\langle\phi_i,\mathbf{1}\rangle|=\int_{B_0}\phi_i(x)\,d\mu(x)\leqq 2.$$

From the last two estimates it follows

$$\begin{aligned}&\sum_{i=1}^{\infty}e^{-\lambda_i(B_0)s}\phi_i(y)\langle\phi_i,\mathbf{1}\rangle\\&\quad=e^{-\lambda_1 s}\Big[\phi_1(y)\langle\phi_1,\mathbf{1}\rangle+\sum_{i=2}^{\infty}e^{-(\lambda_i s-\lambda_1 s)}\phi_i(y)\langle\phi_i,\mathbf{1}\rangle\Big]\geqq\\&\quad\geqq e^{-\lambda_1 s}\Big[\phi_1(y)\langle\phi_1,\mathbf{1}\rangle-4\xi e^{\lambda_1}\sum_{i=2}^{\infty}e^{-(\lambda_i-\lambda_1)(s-1)}\Big].\end{aligned} \tag{41}$$

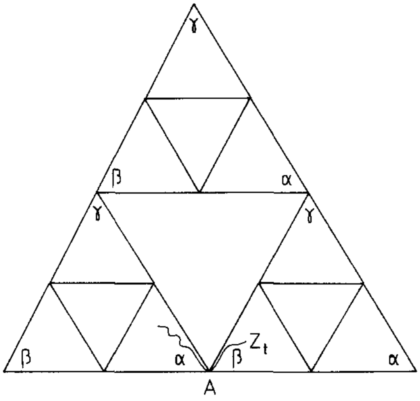

Fig. 3.

Recall now how y was related to x and s to t: if t is large enough, then $|y|<\frac{1}{2}$ and $\phi_1(y)>A$, and as $\sum_{i=2}^{\infty} e^{-\lambda_i(s-1)}$ is finite (trace of a trace-class operator), then $\sum_{i=2}^{\infty} e^{-(\lambda_i-\lambda_1)(s-1)}$ goes to zero as $s\to\infty$. It follows that, for large t, the expression in parenthesis in (41) is bigger than, say,

$$\tfrac{1}{2}A\langle\phi_1,\mathbf{1}\rangle,$$

which is not zero, thus

$$P_x[T_{\partial B_M}>t]\geqq e^{-\nu^{\frac{2}{d_2+2}}t^{\frac{d_s}{d_s+2}}5\lambda(B_0)}\cdot\tfrac{1}{2}A\langle\phi_1,\mathbf{1}\rangle. \tag{42}$$

From (37) and (42), (28) follows as before. The proof of the theorem is completed. □

5. Construction of the reflected Brownian motion

So far, we have obtained an asymptotic lower bound for the Laplace transform of the density of states. To get an upper bound, we should know how to reduce the problem to one with compact state-space. As already written in the introduction, the usual projection onto a torus does not work here; the process that we obtain is non-Markov. Indeed, the Markov property will be destroyed at the vertices, which after the projection get identified to one. If you look at the process before projection (see Fig. 3), then if at some time the process comes to the vertex 'A', then right after it exits through angle 'α' or 'β' – never through 'γ'. But after the projection, this information is lost – to know that the process will stay some time 'far from γ', we need to know not only the present position of the process, but also where it came from. This is not a Markov-type behavior.

However, we overcame this difficulty by using a somewhat different method of 'projection'. The process obtained as a result of this procedure will be called,

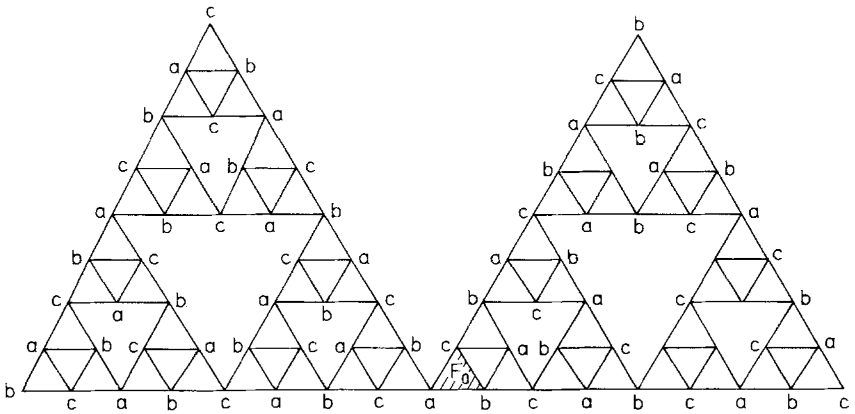

Fig. 4.

for geometrical reasons and for analogy with one of the constructions of the normally reflected process in the real-line case, the reflected Brownian motion on the Sierpinski gasket. We construct its transition density, but do not construct nor investigate its Dirichlet form.

To prepare the ground for our construction, we now introduce two types of labels on the gasket.

5.1. Preparatory labeling of the gasket

5.1.1. Labeling of the vertices. We will introduce a labeling of the gasket (to be precise, we are labeling not the gasket, but the grid of size 1, $\mathscr{G}_0$). Our labeling will distinguish between the vertices of the 0-triangles, although the process is locally symmetric with respect to rotation by an angle 120°. This procedure will allow us to construct the 'reflected Brownian motion' on the Sierpinski gasket.

First observe that $\mathscr{G}_0 \subset \mathbb{Z}e_1 + \mathbb{Z}e_2$ as for every point $x \in \mathscr{G}_0$, $x = ne_1 + me_2$, n, $m \in \mathbb{Z}$ $\left(e_1 = (1, 0),\ e_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right)$ and this representation is clearly well-defined.

We put the labels as follows (see Fig. 4). Consider the commutative 3-group $\mathbb{A}_3$, which consists of even permutations of 3 elements $\{a, b, c\}$. Then $\mathbb{A}_3 = \{id, (a, b, c), (a, c, b)\}$, and we denote $p_1 = (a, b, c)$, $p_2 = (a, c, b)$. Clearly $p_1^3 = id$, $p_2^3 = id$. The mapping

$$\mathscr{G}_0 \ni x = ne_1 + me_2 \mapsto p_1^n \circ p_2^m \in \mathbb{A}_3$$

is well defined. We associate with each point $x = ne_1 + me_2$ the value of $(p_1^n \circ p_2^m)(a)$.

This way, every triangle of size 1 from $\mathscr{T}_0$, with vertices from $\mathscr{G}_0$, has its vertices labeled 'a, b, c', in the way corresponding to the location of this triangle in the gasket.

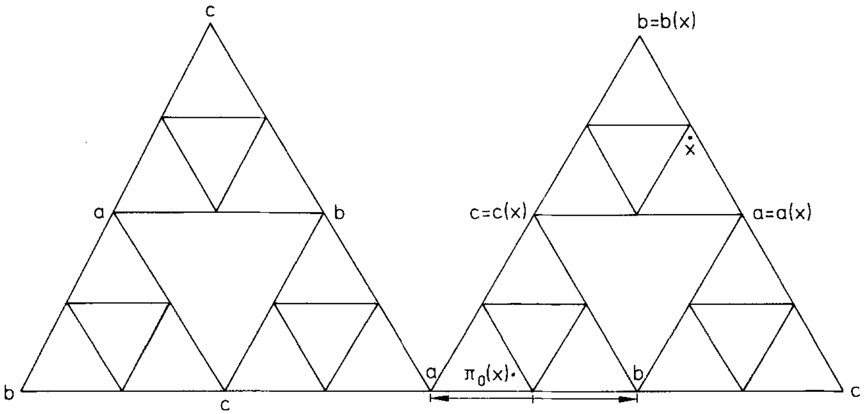

Fig. 5.

For an arbitrary $x \in \mathcal{G} \setminus \mathcal{G}_0$, x belongs to exactly one triangle $\Delta_0(x) \in \mathcal{T}_0$ (see Fig. 5), and x can be written as $x = x_a \cdot a(x) + x_b \cdot b(x) + x_c \cdot c(x)$, where $a(x)$, $b(x)$, $c(x)$ are the corresponding vertices of $\Delta_0(x)$ (with introduced labeling), $x_a, x_b, x_c \in (0, 1)$.

We define a projection map from the Sierpinski gasket onto its intersection with the first triangle, $\mathcal{F}_0$ by setting:

$$\pi_0(x) = x_a \cdot a(0) + x_b \cdot b(0) + x_c \cdot c(0), \tag{43}$$

$$\text{where } a(0) = (0, 0), \quad b(0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad c(0) = (0, 1).$$

If $x \in \mathcal{G}_0$, then x itself has a label and we can map it to a corresponding vertex of the "first" (shaded on Fig. 4) triangle.

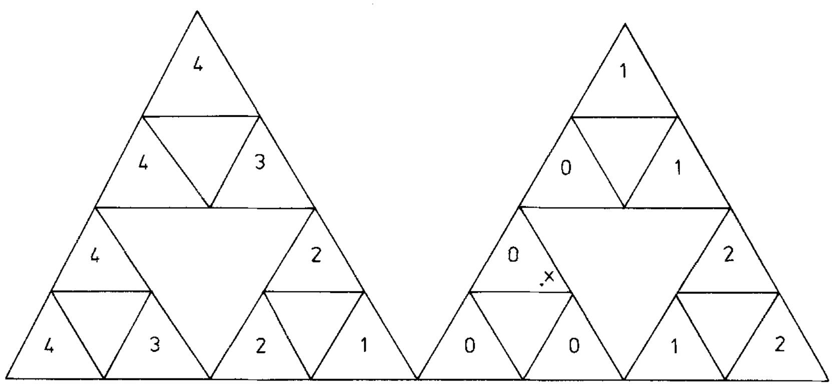

Fig. 6.

5.1.2. Distance labeling of the 0*-triangles.* We will also need some estimates on the number of points from the fiber (with respect to the mapping π_0) of a fixed $x \in \mathcal{F}_0$, lying at particular distance from X. To this end, we shall label the 0-triangles in the manner that takes into account this distance.

For a fixed $x \in \mathscr{G} \setminus \mathscr{G}_0$, we put label '0' on $\Delta_0(x)$ and on the three adjacent triangles (see Fig. 6), then, label '1' on all the triangles that are neighbors of some triangle with label '0', ... If we already have marked the triangles with labels '0, ..., n', then we put the label '$n+1$' on those triangles adjacent to some triangle with label 'n' that have not been labeled so far.

To see this in a more rigorous setting, for $T \in \mathscr{T}_0$, $l_x(T)$ will be the label on this triangle, whereas $\mathscr{L}_{n,x}(\subset \mathscr{T}_0)$ will denote the collection of triangles with label 'n' on them.

For $T \in \mathscr{T}_0$ we put:

1. $l_x(T)=0$ iff $T \cap \Delta_0(x) \neq \emptyset$,

$$\mathscr{L}_{0,x} = \{T \in \mathscr{T}_0 \colon l_x(T)=0\},$$

2. $l_x(T)=n+1$ iff $T \notin \bigcup_{k=0}^{n} \mathscr{L}_{n,x}$ and for some $T_1 \in \mathscr{L}_{n,x}$, $T_1 \cap T \neq \emptyset$,

$$\mathscr{L}_{n+1,x} = \{T \in \mathscr{T}_0 \colon l_x(T)=n+1\}.$$

Notice that if $y \in T \in \mathscr{L}_{n,x}$, then

$$n \leqq d(x,y) \leqq n+2. \tag{44}$$

In the sequel, we will need the following lemma, giving the upper estimate of the cardinality of $\mathscr{L}_{n,x}$:

Lemma 3. *There exists a universal constant C such that*

$$\# \mathscr{L}_{n,x} \leqq C \cdot n^2.$$

Proof. This lemma follows immediately from the crude comparison of the Euclidean distance on the plane and the gasket distance. Notice that if we pick, for example, the south-west vertex of each 0-triangle, we obtain a collection of points, one in each $T \in \mathscr{T}_0$, mutually at distance greater than or equal to 1. Using the inequality (5)

$$|x-y| \leqq d(x,y) \leqq 2|x-y|,$$

we get that all the triangles from $\mathscr{L}_{n,x}$ are included in the annulus

$$\left\{y \in \mathbb{R}^2 \colon \frac{n}{2} \leqq |x-y| \leqq n+2\right\}.$$

Let us now estimate how many points which are mutually at distance greater or equal than 1 can be packed into the annulus

$$\left\{x \in \mathbb{R}^2 \colon \frac{n}{2} \leqq |x| \leqq n+2\right\}.$$

Performing a simple area comparison we get that

$$n_{\max}\cdot\pi\,(\tfrac{1}{2})^2 \leqq 2\pi\left[(n+2)^2-\left(\frac{n}{2}\right)^2\right]$$

(2 comes from the points which may lie too close to the boundary of the annulus) and

$$n_{\max}\leqq 64\,n^2. \tag{45}$$

Such points are in one-to-one correspondence with the 0-triangles (take south-west vertices, as above). The lemma is proved. □

5.2. Construction of the reflected Brownian motion

The goal of constructing this object is to obtain a Markov process on a compact state space, whose local properties remain basically the same as those of the Brownian motion on the whole Sierpinski gasket. The usual projection onto a torus does not work here (imagine how a 0-triangle looks like after a usual projection!), the result of projecting the Brownian motion from the whole gasket turns out to be not Markov. Although the vertices get identified to one, the Markov property would break at this point (as explained above).

What we do, is making use of the mapping π_0, defined in the Sect. 5.1. This mapping behaves as a usual projection with one (crucial!) exception: it distinguishes between the vertices of the projected triangle. By analogy with one of constructions of the normally reflected process on an interval, we shall call this process 'the reflected Brownian motion on $\mathscr{F}_0$'. Its distribution will be defined as a family of measures on $(C(\mathbb{R}_+,\mathscr{F}_0),\mathscr{B}(C(\mathbb{R}_+,\mathscr{F}_0)))$, given by

$$\{Q_x^0\}_{x\in\mathscr{F}_0}=\{\pi_0(P_x)\}_{x\in\mathscr{F}_0},$$

where $(P_x)_{x\in\mathscr{G}}$ is the family of measures, defining the process $(Z_t)_{t\geqq 0}$, and the process itself by

$$X_t^0=\pi_0(Z_t).$$

The transition density for this process will be given by:

$$q_0(t,x,y)=\begin{cases}\sum\limits_{y'\in\pi_0^{-1}(y)} p(t,x,y) & \text{if } x,y\in\mathscr{F}_0,\ y\notin\mathscr{G}_0\\ 2\sum\limits_{y'\in\pi_0^{-1}(y)} p(t,x,y) & \text{if } y\in\mathscr{G}_0.\end{cases} \tag{46}$$

We start by a following technical lemma:

Lemma 4. 1. $\forall\delta>0\ \sum\limits_{y'\in\pi_0^{-1}} p(t,x,y')$ *is uniformly convergent in* $x,y\in\mathscr{G}$, $\quad t\geqq\delta$,

2. q^0 *defined by* (46) *is jointly continuous in* x, y, t,

3. *there exist two positive constants* c_1, θ *such that*

$$\sup_{x, y \in \mathscr{F}_0} q^0(t, x, y) \leqq c_1 \cdot t^{-\theta}.$$

Proof. 1. We use the distance labeling on the gasket from Sect. 5.1.2. We have:

$$\sum_{y' \in \pi_0^{-1}(y)} p(t, x, y') = \sum_{n=0}^{\infty} \sum_{\Delta_0(y') \in \mathscr{L}_{n,x}} p(t, x, y') = \sum_{n=0}^{\infty} a_n. \tag{47}$$

Each term a_n can be estimated as (use (6))

$$a_n = \# \mathscr{L}_{n,x} \cdot \sup_{d(x,y') > n} p(t, x, y') \leqq C \cdot n^2 t^{-\frac{d_s}{2}} \mathrm{e}^{-C[nt^{\frac{-1}{d_w}})]^{\frac{d_w}{d_w - 1}}}.$$

This bound gives a term of a convergent series and 1. and 3. follow.

2. Continuity of q^0. For $y \notin \mathscr{G}_0$ it follows from the uniform convergence of the series and the continuity of $p(\cdot, \cdot, \cdot)$. To prove the continuity at the vertices, suppose $y \in \mathscr{G}_0$, and that y is labeled 'a' (for simplicity, we continue to write a instead of y). We must show

$$\lim_{(t_n, x_n, z_n) \to (t, x, a)} q^0(t_n, x_n, z_n) = q^0(t, x, a).$$

If $d(z_n, a) < \frac{1}{2}$, then for every $a' \in \pi_0^{-1}(a)$ there exist precisely two elements z'_L, $z'_R \in \pi_0^{-1}(z_n)$ such that $d(z'_L, a') < \frac{1}{2}$ and $d(z'_R, a') < \frac{1}{2}$ (L, R stand for 'left' and 'right'. with obvious meaning). It follows:

$$\begin{aligned} q^0(t_n, x_n, z_n) &= \sum_{a' \in \pi_0^{-1}(a)} (p(t_n, x_n, z'_L) + p(t_n, x_n, z'_R)) = \\ &= \sum_{a' \in \pi_0^{-1}(a)} p(t_n, x_n, z'_L) + \sum_{a' \in \pi_0^{-1}(a)} p(t_n, x_n, z'_R). \end{aligned}$$

When $z_n \to a$, then $z'_L \to a'$ and $z'_R \to a'$, and the uniform convergence lets us pass to the limit under the sum,

$$\begin{aligned} &\lim_{(t_n, x_n, z_n) \to (t, x, a)} q^0(t_n, x_n, z_n) = \\ &= \sum_{a' \in \pi_0^{-1}(a)} \lim_{(t_n, x_n, z'_L) \to (t, x, a')} p(t_n, x_n, z'_L) + \sum_{a' \in \pi_0^{-1}(a)} \lim_{(t_n, x_n, z'_R) \to (t, x, a')} p(t_n, x_n, z'_R) = \\ &= 2 \sum_{a' \in \pi_0^{-1}(a)} p(t, x, a'). \end{aligned}$$

The lemma follows. □

Now we are ready to show the basic theorem of this section.

Theorem 3. *Let $x, y \in \mathcal{G}$ be two points in the same 0-fiber, i.e. $\pi_0(x)=\pi_0(y)$. Then the measures $\pi_0(P_x)$ and $\pi_0(P_y)$ on $(C(\mathbb{R}_+, \mathcal{F}_0), \mathcal{B}(C(\mathbb{R}_+, \mathcal{F}_0))$ coincide. Moreover, for every $z \in \mathcal{F}_0$ and x, y as above we have:*

$$\sum_{z' \in \pi_0^{-1}(z)} p(t, x, z') = \sum_{z' \in \pi_0^{-1}(z)} p(t, y, z'). \tag{48}$$

Proof. Let x and y be as in the assumptions of the theorem. It is enough to prove that the finite-dimensional distributions of underlying measures are identical, i.e. that for an arbitrary choice of $0 \leqq t_1 \leqq \ldots \leqq t_n$ and $\Gamma_1, \ldots, \Gamma_n \in \mathcal{B}(\mathcal{F}_0)$ we have:

$$\begin{aligned} & P_x[Z_{t_1} \in \pi_0^{-1}(\Gamma_1), \ldots, Z_{t_n} \in \pi_0^{-1}(\Gamma_n)] = \\ & \quad = P_y[Z_{t_1} \in \pi_0^{-1}(\Gamma_1), \ldots, Z_{t_n} \in \pi_0^{-1}(\Gamma_n)]. \end{aligned} \tag{49}$$

We proceed by induction.

1. $n=1$ (we drop the subscript '1').

For $t=0$ the equality (49) is self-evident:

$$P_x[Z_0 \in \pi_0^{-1}(\Gamma)] = \delta_x(\pi_0^{-1}(\Gamma)) = \delta_y(\pi_0^{-1}(\Gamma)) = P_y[Z_0 \in \pi_0^{-1}(\Gamma)]. \tag{50}$$

For $t>0$, we first observe that if Γ is a subset of $\mathcal{F}_0$, then $\pi_0^{-1}(\Gamma) \setminus \mathcal{G}_0$ is a disjoint sum of sets, each of which lies entirely in one and only one triangle from $\mathcal{T}_0$. Those components will be denoted by Γ^i. Notice that

$$P_x[Z_t \in \pi_0^{-1}(\Gamma)] = \sum_i P_x[Z_t \in \Gamma^i], \tag{51}$$

the summation running over all the components to $\pi_0^{-1}(\Gamma)$. Although we got rid of the possible vertices in introducing the sets Γ^i, this holds as $\mu(\mathcal{G}_0)=0$ and for every t the law of Z_t under P_x is absolutely continuous with respect to μ.

We introduce the following stopping times:

$$T^{(1)} = \inf\{t>0 : Z_t \in \mathcal{G}_0\},$$

$$T^{(n+1)} = \inf\{t > T^{(n)} : Z_t \in \mathcal{G}_0 \setminus \{Z_{T^{(n)}}\}\}, \qquad \text{for } n>1.$$

$\{T^{(n)}\}_n$ is an increasing sequence of stopping times and $\lim_{n \to \infty} T^{(n)} = \infty$ almost surely.

In what follows we shall need the following lemma, coming from the strong Markov property of (Z_t).

Lemma 5. *Let $x, y \in \mathcal{G}$, $\pi_0(x)=\pi_0(y)$. Then the laws of $T^{(n)}$ under P_x and under P_y are identical.*

Proof of the lemma. Induction with respect to n.

For $n=1$ the statement is obvious; if $x \in \mathcal{G}_0$ (therefore automatically $y \in \mathcal{G}_0$), then $T^{(1)}=0$ P_x-a.s. and P_y-a.s. If they are not vertices, then the law of $T_{(1)}$ depends entirely on the laws of (Z_t) up to exit times from $\Delta_0(x)$, $\Delta_0(y)$ respectively, which are identical.

Assume now that for some $n \geqq 1$ the assertion holds, then

$$P_x[T^{(n+1)} < t] = P_x[T^{(n+1)} < t,\, T^{(n)} < t] = E_x[P_{Z_{T^{(n)}}}[\tilde{T}^{(1)} < t - T^{(n)}]]$$

($\tilde{T}^{(1)}$ denotes the first hitting time of $\mathscr{G}_0 \setminus \{Z_0\}$ after $T^{(n)}$). Laws of $T^{(n)}$ are identical under P_x and P_y (inductive assumption); thus the probability under the expectation depends only of these laws. Hence E_x can be replaced by E_y and the proof is concluded. □

Continuation of the proof of the theorem. Now we shall proceed in the standard way: we take into account the vertices that have been visited up to time t. We partition $C([0, \infty), \mathscr{G})$ into the following disjoint sets:

$$V_0 = \{T^{(1)} > t\},$$
$$V_n = \{T^{(n)} \leqq t,\, T^{(n+1)} > t\}, \qquad \text{for } n \geqq 1,$$

and for $n = 1, 2, \ldots$

$$V_n = V_n^a \cup V_n^b \cup V_n^c,$$

where V_n^κ indicates that $Z_{T^{(n)}}$ falls onto a vertex with label κ ($\kappa \in \{a, b, c\}$), i.e. $V_n^\kappa = V_n \cap \{Z_{T^{(n)}} = \kappa\}$, (with meaning that the label on the appropriate point equals to κ). This way we get

$$P_x[Z_t \in \pi_0^{-1}(\Gamma)] = P_x[\{Z_t \in \pi_0^{-1}(\Gamma)\} \cap V_0] + \\ + \sum_{n=1}^{\infty} \sum_{\kappa \in \{a,b,c\}} P_x[\{Z_t \in \pi_0^{-1}(\Gamma)\} \cap V_n^\kappa].$$

Now our goal is to show that the terms of the series remain unchanged if we replace x by y.

$P_x[\{Z_t \in \pi_0^{-1}(\Gamma)\} \cap V_0]$ is equal to $P_y[\{Z_t \in \pi_0^{-1}(\Gamma)\} \cap V_0]$, as the underlying events depend only on the distribution of the processes up to exit times of $\Delta_0(x)$, $\Delta_0(y)$ respectively – those measures are identical.

To get the equality of the latter terms, first assume that x and y are not vertices from $\mathscr{G}_0$. Using the strong Markov property of Z_t we get

$$\begin{aligned} &P_x[\{Z_t \in \pi_0^{-1}(\Gamma)\} \cap V_n^\kappa] = \\ &= E_x[1\{T^{(n)} \leqq t, Z_{T^{(n)}} = \kappa\} \cdot P_x[T^{(n+1)} > 1, Z_t \in \pi_0^{-1}(\Gamma) \mid \mathscr{F}_{T^{(n)}}]] = \\ &= E_x[1\{T^{(n)} \leqq t, Z_{T^{(n)}} = \kappa\} \cdot P_{Z_{T^{(n)}}}[\tilde{T}^{(1)} > t - T^{(n)}, Z_{t - T^{(n)}} \in \pi_0^{-1}(\Gamma)]] = \\ &= \int_0^t P_\kappa[T^{(1)} > t - s, Z_{t-s} \in \pi_0^{-1}(\Gamma)]\, d\mu_n^x(s), \end{aligned}$$

($\tilde{T}^{(1)}$, as before, is the first hitting time of $\mathscr{G}_0 \setminus \{Z_0\}$ after time $T^{(n)}$), where μ_n^x is the distribution of $T^{(n)}$ under P_x. From Lemma 5 we have that $\mu_n^x = \mu_n^y$ and therefore

$$P_x[Z_t \in \pi_0^{-1}(\Gamma) \cap V_n^\kappa] = P_y[Z_t \in \pi_0^{-1}(\Gamma) \cap V_n^\kappa].$$

If x and y are from $\mathscr{G}_0$, we are in an even better shape: we can forget about the particular starting point at once, or we can expand the underlying expression as above.

Having shown that $\pi_0(P_x)=\pi_0(P_y)$, we can establish (48).

For $\Gamma\in\mathscr{B}(\mathscr{F}_0)$

$$\int_\Gamma q^0(t,x,z)\,\mathrm{d}\mu(z)=P_x[Z_t\in\pi_0^{-1}(\Gamma)]$$

$$=P_x[Z_t\in\pi_0^{-1}(\Gamma)]=\int_\Gamma q^0(t,y,z)\,\mathrm{d}\mu(z).$$

But we already know (Lemma 4 above) that q^0 is continuous in z, therefore $q^0(t,x,z)=q^0(t,y,z)$ for all z.

Hence the case $n=1$ is finished.

2. $n>1$.

Assume that the assertion (49) holds for some n, arbitrary choice of $t_1,\dots,t_n$ and $\Gamma_1,\dots,\Gamma_n$. Again, for $\Gamma\in\mathscr{B}(\mathscr{F}_0)$, the superscript '$i$' will stand for a disjoint component of $\pi_0^{-1}(\Gamma\setminus\mathscr{G}_0)$. We have: for all n, $0\leqq t_1\leqq\dots\leqq t_{n+1}$ and $\Gamma_1,\dots,\Gamma_{n+1}\in\mathscr{B}(\mathscr{F}_0)$

(52)
$$P_x[Z_{t_1}\in\pi_0^{-1}(\Gamma_1),\dots,Z_{t_{n+1}}\in\pi_0^{-1}(\Gamma_{n+1})]=\sum_{i_1,\dots,i_{n+1}}P_x[Z_{t_1}\in\Gamma_1^{i_1},\dots,Z_{t_{n+1}}\in\Gamma_{n+1}^{i_{n+1}}],$$

the summation running over all the components of $\pi_0^{-1}(\Gamma_1),\dots,\pi_0^{-1}(\Gamma_{n+1})$. By the Chapman-Kolmogorov equality for Z_t the last sum equals to

$$\sum_{i_1,\dots,i_{n+1}}\int_{\Gamma_1^{i_1}}p(t_1,x,z_1)P_{z_1}[Z_{t_2-t_1}\in\Gamma_2^{i_2},\dots,Z_{t_{n+1}-t_1}\in\Gamma_{n+1}^{i_{n+1}}]\,d\mu(z_1)=$$

$$=\sum_{i_1,\dots,i_{n+1}}\int_{\Gamma_1}p(t_1,x,z_1^{i_1})P_{z_1^{i_1}}[Z_{t_2-t_1}\in\Gamma_2^{i_2},\dots,Z_{t_{n+1}-t_1}\in\Gamma_{n+1}^{i_{n+1}}]\,d\mu(z_1),$$

where $z_1^{i_1}$ is that component of $\pi_0^{-1}(z_1)$, which lie in $\Gamma_1^{i_1}$ (notice that $d\mu(z_1)=d\mu(z_1^{i_1})$). Next, it equals to

$$\sum_{i_1}\Big(\int_{\Gamma_1}p(t_1,x,z_1^{i_1})\cdot\sum_{i_2,\dots,i_{n+1}}P_{z_1^{i_1}}[Z_{t_2-t_1}\in\Gamma_2^{i_2},\dots,Z_{t_{n+1}-t_1}\in\Gamma_{n+1}^{i_{n+1}}]\,d\mu(z_1)\Big).$$

From the inductive assumption we can drop the superscript 'i_1' in the sum under the integral sign. Changing the order of summation we get

$$\int_{\Gamma_1}\Big(\sum_{i_2,\dots,i_{n+1}}P_{z_1}[Z_{t_2-t_1}\in\Gamma_2^{i_2},\dots,Z_{t_{n+1}-t_1}\in\Gamma_{n+1}^{i_{n+1}}]\Big)\cdot\Big(\sum_{i_1}p(t_1,x,z_1^{i_1})\Big)\,\mathrm{d}\mu(z_1).$$

Now, from the already shown (48), we get that

$$\sum_{i_1}p(t_1,x,z_1^{i_1})=\sum_{i_1}p(t_1,y,z_1^{i_1}),$$

and we can turn the formula back to the form as in (52), but with x replaced by y. The theorem follows. □

At this point, we can without difficulty show the Chapman-Kolmogorov identity for the reflected process:

Lemma 6. *For* $t, s>0$, $x, z\in\mathscr{F}_0$

$$q^0(t+s,x,z)=\int_{\mathscr{F}_0} q^0(t,x,y)\,q^0(s,y,z)\,d\mu(y). \tag{53}$$

Proof. For $z\notin\mathscr{G}_0$ we have

$$\begin{aligned}q^0(t+s,x,z)&=\sum_{z'\in\pi_0^{-1}(z)} p(t+s,x,z')=\sum_{z'\in\pi_0^{-1}(z)}\int_{\mathscr{G}} p(t,x,y)\,p(s,y,z)\,d\mu(y)=\\&=\int_{\mathscr{G}} p(t,x,y)\sum_{z'\in\pi_0^{-1}(z)} p(s,y,z')\,d\mu(y)=\int_{\mathscr{G}} p(t,x,y)\sum_{z'\in\pi_0^{-1}(z)} p(s,\pi_0(y),z')\,d\mu(y)=\\&=\int_{\mathscr{F}_0}\sum_{y'\in\pi_0^{-1}(y)} p(t,x,y')\sum_{z'\in\pi_0^{-1}(z)} p(s,y,z')\,d\mu(y)=\int_{\mathscr{F}_0} q^0(t,x,y)\,q^0(s,y,z)\,d\mu(y).\end{aligned}$$

If $z\in\mathscr{G}_0$, then we must introduce obvious changes. □

After having established (48) we can also show that q^0 is symmetric in x, y.

Lemma 7.

$$\forall t\geqq 0\quad \forall x,y\in\mathscr{F}_0\quad q^0(t,x,y)=q^0(t,y,x). \tag{54}$$

Proof. For $x, y\in\mathscr{F}_0\setminus\mathscr{G}_0$ we have:

$$q^0(t,x,y)=\lim_{M\to\infty}\frac{1}{\mu(B_M)}\sum_{A(M,x,y)} p(t,x',y'), \tag{55}$$

where $A(M,x,y)=\{(x',y'): x'\in\pi_0^{-1}(x), y'\in\pi_0^{-1}(y), x', y'\in B_M\}$. As (55) is symmetric in x and y, it proves (54) in the case when x and y are not vertices.

To see (55) we make use of (48). The value of the sum

$$\sum_{y'\in\pi_0^{-1}(y)} p(t,x,y')$$

does not depend on the particular choice of x within the same fiber. It follows

$$\begin{aligned}q^0(t,x,y)&=\frac{1}{\mu(B_M)}\sum_{B(M,x,y)} p(t,x',y')\\&=\frac{1}{\mu(B_M)}\sum_{A(M,x,y)} p(t,x',y')+\frac{1}{\mu(B_M)}\sum_{C(M,x,y)} p(t,x',y')=\alpha_M+\beta_M,\end{aligned}$$

where

$$A(M,x,y)\text{ is defined above,}$$

$$B(M,x,y)=\{x'\in\pi_0^{-1}(x), y'\in\pi_0^{-1}(y), x'\in B_M\},$$

$$C(M,x,y)=\{x'\in\pi_0^{-1}(x), y'\in\pi_0^{-1}(y), x'\in B_M, y'\notin B_M\}.$$

We will be done if we show that β_M goes to zero with $M \to \infty$. We have:

$$\beta_M = \frac{1}{\mu(B_M)} \sum_{\{x' \in B_M \setminus B(0, 2^M - M)\}} \Big(\sum_{y' \notin B_M} p(t, x', y') \Big) \tag{56}$$
$$+ \frac{1}{\mu(B_M)} \sum_{x' \in B(0, 2^M - M)} \Big(\sum_{y' \notin B_M} p(t, x', y') \Big) = \beta_M^1 + \beta_M^2$$

Now: β_M^1 can be estimated using the area comparison –

$$\beta_M^1 = \frac{1}{\mu(B_M)} \# \{x' \in \pi_0^{-1}(x), x' \in B_M \setminus B(0, 2^M - M)\} \cdot \tag{57}$$
$$\cdot \sup_{x, y \in \mathscr{F}_0} q^0(t, x, y) \leqq C \cdot \frac{\mu(B_M \setminus B(0, 2^M - M)}{\mu(B_M)}$$
$$= C \cdot \mu\left(B_0 \setminus B\left(0, 1 - \frac{M}{2^M}\right)\right)$$

which gives that β_M^1 goes to zero as M goes to infinity.

To estimate β_M^2 we notice, that in this case x' and y' are far away:

$$\beta_M^2 \leqq C \cdot \frac{(2^M - M)^{d_f}}{(2^M)^{d_f}} \cdot \sup_{\{x' \in \pi_0^{-1}(x), x' \in B(0, 2^M - M)\}} \sum_{\{y' \notin B_M\}} p(t, x', y') \tag{58}$$
$$\leqq C \cdot \sup_{\{x' \in \pi_0^{-1}(x), x' \in B(0, 2^M - M)\}} \sum_{\{y' : d(x', y') \geqq M\}} p(t, x', y')$$
$$= \sum_{n=M}^{\infty} \sup_{\{x' \in \pi_0^{-1}(x), x' \in B(0, 2^M - M)\}} \sum_{\{y' : d(x', y') \in [n, n+1)\}} p(t, x', y')$$
$$\leqq C \cdot \sum_{n=M}^{\infty} n^2 \exp\{-C \cdot (n t^{-\frac{1}{d_w}})^{\frac{d_w}{d_w - 1}}\},$$

(in the last inequality we used an estimate alike that in Lemma 3) which is the tail of a convergent series – hence goes to zero as M goes to infinity. Combining (57) and (58) we obtain (56).

For arbitrary x and y (54) follows by continuity. The proof of the proposition is finished. □

Now we finally can legitimately define the reflected Brownian motion on $\mathscr{F}_0$ by

$$X_t^0 = \pi_0(Z_t).$$

The corresponding probabilities will be denoted by Q_x^0.

We can collect the properties of X_t^0 in the following theorem:

Theorem 4. $(X_t^0)_{t \geqq 0}$ *is a continuous Markov process with continuous transition density* q^0 *given by* (46). *The transition density* $q^0(t, x, y)$ *is symmetric with respect to* x *and* y.

This way we have constructed the desired Markov process with a compact state-space, $\mathscr{F}_0$. The semigroup associated with this process, denoted by $(T_t^0)_{t\geqq 0}$ is a semigroup of selfadjoint operators on $L^2(\mathscr{F}_0, \mathrm{d}\mu)$.

The properties of the reflected Brownian motion (Markov property and the identity (48)) allow us to construct the bridge measures for this process (definition identical as for the Brownian motion on the whole gasket). They will be denoted by $Q_{x,y}^{t,0}$, and they fulfill the relation (which can be seen by writting down the appropriate measures):

Lemma 8. *For $x, y\in\mathscr{G}\setminus\mathscr{G}_0$, the 'image under π_0' on $C([0,t], \mathscr{F}_0)$ of the measure*

$$\sum_{y'\in\pi_0^{-1}(\pi_0(y))} p(t,x,y')\, P_{x,y'}^t[\cdot]$$

is equal to $q^0(t, \pi_0(x), \pi_0(y))\, Q_{\pi_0(x),\pi_0(y)}^{t,0}[\cdot]$.

Proof. We must check that $\forall A\in\mathscr{B}(C[0,t], \mathscr{F}_0)$

$$q^0(t,\pi_0(x),\pi_0(y))\, Q_{\pi_0(x),\pi_0(y)}^{t,0}[A] = \sum_{y'\{\pi_0^{-1}(\pi_0(y))} p(t,x,y')\, P_{x,y'}^t[\pi_0^{-1}(A)] \tag{59}$$

where $\pi_0(\omega)$, for ω being a trajectory from $C[0,t]$, has the obvious meaning of $[\pi_0(\omega)](t)=\pi_0(\omega(t))$.

Clearly, it is enough to check (59) for cylindrical sets $\{X_{t_1}^0\in\Gamma_1, \dots X_{t_n}^0\in\Gamma_n\}$; for those sets this is a straightforward calculation, which uses only (48). □

Remark 3. The results of this section can be also carried out not for the triangle of unit size, $\mathscr{F}_0$, but also for any triangle $\mathscr{F}_M$. This way we would obtain the reflected Brownian motion on a triangle with sizelength 2^M.

5.3. Recurrence properties for the Brownian motion and for the reflected Brownian motion on $\mathscr{F}_0$

Out further work requires the following recurrence estimates:

Theorem 5. *Let a binary number $b>0$ be fixed. Then there exist $t_0>0$, $\alpha>0$ and two functions:*

$$\Phi: (0,\infty)\to(0,1],$$

$$\psi: (0,1)\to(b,\infty),$$

Φ being a decreasing function; such that for any binary $\varepsilon>0$ and $x, y\in\mathscr{G}$ we have:

1.

$$\text{For } r\in(0,\infty), \text{ if } d(x,y)\leqq\varepsilon r, \text{ then: } P_x[T_{\bar{B}(y,\varepsilon b)}\leqq t_0\,\varepsilon^{d_w}]\geqq\Phi(r). \tag{60}$$

2. *If $d(x,y)\leqq b\varepsilon$ then:*

(a)

$$P_x[T_y\leqq t_0\,\varepsilon^{d_w}]\geqq\alpha, \tag{61}$$

(b)

(62) $$P_x[T_y < T_{B(y,\varepsilon\Psi(\eta))^c}] \geqq 1-\eta, \quad \text{for } 0<\eta<1.$$

Proof. First notice that, as ε was assumed to be binary, we can use the gasket scaling in order to get a reformulation of (60)–(62) – easier to handle. After this, we are led to showing: for $x, y \in \mathscr{G}$,

1.

(63) $$\text{for } r \in (0,\infty), \quad \text{if } d(x,y) \leqq r \quad \text{then} \quad P_x[T_{\bar{B}(y,b)} \leqq t_0] \geqq \Phi(r).$$

2. *If* $d(x,y) \leqq b$, then:

(a)

(64) $$P_x[T_y \leqq t_0] \geqq \alpha,$$

(b)

(65) $$P_x[T_y < T_{B(y,\Psi(\eta))^c}] \geqq 1-\eta.$$

We begin by showing (63), choosing $t_0 = \frac{1}{2}$.

$$\begin{aligned} P_x[T_{\bar{B}(y,b)} \leqq \tfrac{1}{2}] &\geqq P_x[Z_{\frac{1}{2}} \in \bar{B}(y,b)] = \\ &= \int_{\bar{B}(y,b)} p(\tfrac{1}{2}, x, z)\, d\mu(z) \geqq \inf_{\{x,z:\, d(x,z) \leqq b+r\}} p(\tfrac{1}{2}, x, z)\, \mu(\bar{B}(y,b)) \geqq \\ &\geqq C\, e^{-C(b+r)^{\frac{d_w}{d_w-1}}}, \end{aligned}$$

for some positive constant C. Thus we can take $\Phi(r) = C\, e^{-C(b+r)^{\frac{d_w}{d_w-1}}}$.

To prove (65) we must show that that for any $\eta > 0$ we can find $\Psi = \Psi(\eta)$ such that for $x, y \in \mathscr{G}$ with $d(x,y) \leqq b$ one will have

$$P_x[T_y < T_{\bar{B}(y,\Psi(\eta))^c}] > 1-\eta.$$

We begin by showing that, given $\eta > 0$, one can find $\tau_0 = \tau_0(\eta)$ with

(66) $$P_x[T_y < \tau_0] \geqq 1 - \frac{\eta}{2},$$

provided $d(x,y) \leqq b$.

To do this, we introduce R_λ, an independent exponential time with parameter λ ($\lambda > 0$-arbitrary).

Using the estimate (9) we get that for every $\tau > 0$

$$\begin{aligned} P_x[T_y > \tau] &= P_x[T_y > \tau, R_\lambda > \tau] + P_x[T_y > \tau, R_\lambda \leqq \tau] \leqq \\ &\leqq P_x[R_\lambda > \tau] + P_x[T_y \geqq R_\lambda] \leqq e^{-\lambda\tau} + C \cdot d(x,y)^{d_w - d_f} \lambda^{1-\frac{1}{2}d_s} \leqq \\ &\leqq e^{-\lambda\tau} + C b^{d_w-d_f} \lambda^{1-\frac{1}{2}d_s}. \end{aligned}$$

Find now a small λ_0 such that the second term in the last expression is smaller than $\frac{\eta}{4}$, and then an appropriate τ_0 with $e^{-\lambda_0\tau_0} < \frac{\eta}{4}$. (66) is established.

In particular, (66) proves (64).

To prove 2, it is enough to find a $\Psi(\eta)$ such that (τ_0 – as in (66))

$$P_x[T_{B(y,\Psi(\eta))^c} \leqq \tau_0] \leqq \frac{\eta}{2} \tag{67}$$

provided $d(x,y) \leqq b$.

Indeed, this way we obtain, using (67)

$$\begin{aligned} P_x[T_y < T_{B(y,\Psi(\eta))^c}] &\geqq P_x[T_y < \tau_0 < T_{B(y,\Psi(\eta))^c}] \\ &= P_x[T_y < \tau_0] - P_x[\{T_y < \tau_0\} \cap \{T_{B(y,\Psi(\eta))^c} \leqq \tau_0\}] \geqq 1-\eta. \end{aligned}$$

To complete the proof, we must show (67). For given $\rho > b$ we have, using (8)

$$\begin{aligned} P_x[T_{B(y,\rho)^c} \leqq \tau_0] &\leqq P_x[\sup_{s \leqq \tau_0} d(Z_s, Z_0) > \rho - b] \\ &\leqq C \cdot e^{-C[(\rho-b)\tau_0^{-\frac{1}{d_w}}]^{\frac{d_w}{d_w-1}}}. \end{aligned}$$

We choose $\Psi(\eta)$ equal to the ρ such that the last expression is less than $\frac{\eta}{2}$. (67) holds.

The proof of the theorem is finished. □

Remark 4. We established the recurrence properties for the process Z_t. In the sequel we shall need similar relations for X_t^0. They are also true, as the underlying probabilities for X_t^0 are bigger than those for Z_t.

6. Asymptotic upper bound for the Laplace transform

Now we shall present the more delicate part of the question – we shall get an asymptotic upper bound for the Laplace transform of the density of states. The problem basically relies on the long-time asymptotics for the averaged trace of the transformations T_0^t. The other delicate part is that we want to single out the factor $\nu^{\frac{2}{d_s+2}}$. This is done by appropriate scaling. As after projection the good scaling properties of $\lambda(U)$ are destroyed, we shall perform all the necessary scaling first, then project the problem onto the triangle of size 1, and finally use an estimate for the compact problem.

Our goal is the following:

Theorem 6. *There exist positive constants D_1, D_2 such that:*

$$\limsup_{t\to\infty} \frac{\log L(t)}{t^{\frac{d_s}{d_s+2}}} \leqq -D_1 \nu^{\frac{2}{d_s+2}}, \tag{68}$$

and, independently on $x \in \mathcal{G}$,

$$\limsup_{t\to\infty} \frac{\log E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]}{t^{\frac{d}{d+2}}} \leqq -D_2 \nu^{\frac{2}{d_s+2}}. \tag{69}$$

Proof. We start with proof of (68).

Recall that

$$\forall t\ L(t)=\lim_{N\to\infty} L_N(t) \text{ (increasing limit)},$$

where

$$\text{(70)}\quad L_N(t)=\frac{1}{\mu(\mathscr{F}_N)}\mathbb{E}\int_{\mathscr{F}_N} p(t,x,x)\,P^t_{x,x}[T_{\partial\mathscr{F}_N}>t,\ T_{\mathscr{N}(\omega)}>t]\,d\mu(x)$$
$$=\frac{1}{\mu(\mathscr{F}_N)}\int_{\mathscr{F}_N} p(t,x,x)\,E^t_{x,x}[\exp\{-\nu\,\mu(Z_{[0,t]})\}\,1\{T_{\partial\mathscr{F}_N}>t\}]\,d\mu(x).$$

As stated above, we will first perform the necessary scaling.

Let $t>0$ be fixed. Had we have the continuous scaling, we would like to substitute in (70), again,

$$x=\left(\frac{t}{\nu}\right)^{\frac{1}{d_f+d_w}} y$$

However, once more we have the problem that $\left(\frac{t}{\nu}\right)^{\frac{1}{d_f+d_w}}$ is not always a binary number – we proceed as previously: if

$$\text{(71)}\qquad 2^n\leqq\left(\frac{t}{\nu}\right)^{\frac{1}{d_f+d_w}}<2^{n+1}$$

for some n, then we replace $\left(\frac{t}{\nu}\right)^{\frac{1}{d_f+d_w}}$ (the ideal scaling factor) by 2^n.

Substituting $x=2^n y$ in (70) we obtain

$$\text{(72)}\qquad L_N(t)=\frac{1}{\mu(\mathscr{F}_{N-n})}\int_{\mathscr{F}_{N-n}}\frac{p\left(\frac{t}{5^n},y,y\right)}{3^n}\cdot E^{\frac{t}{5^n}}_{y,y}\left[\exp\{-\nu\,3^n\mu(Z_{\left[0,\frac{t}{5^n}\right]})\}\,1\left\{T_{\partial\mathscr{F}_{N-n}}>\frac{t}{5^n}\right\}\right]d\mu(y).$$

This is the point where we are going to perform the projection: we shall project the process onto the triangle of size 1, $\mathscr{F}_0$.

What enable us to do so are Lemma 8, Theorem 3 and the fact that $(X^0_t=\pi_0(Z_t))$ $\mu(Z_{[0,t]})\geqq\mu(X^0_{[0,t]})$ (the Hausdorff measure of the trajectory decreases after the projection π_0, due to possible self-intersections after the projection).

Using those properties we get:

$$\text{(73)}\qquad L_N(t)\leqq\frac{1}{\mu(\mathscr{F}_{N-n})}\int_{\mathscr{F}_{N-n}}\frac{q^0\left(\frac{t}{5^n},\pi_0(y),\pi_0(y)\right)}{3^n}\cdot E^{\frac{t}{5^n},0}_{\pi_0(y),\pi_0(y)}[\exp\{-\nu\,3^n\mu(X^0_{\left[0,\frac{t}{5^n}\right]})\}]\,d\mu(y)=$$
$$=\frac{1}{\mu(\mathscr{F}_0)}\int_{\mathscr{F}_0}\frac{q^0\left(\frac{t}{5^n},y,y\right)}{3^n}E^{\frac{t}{5^n},0}_{y,y}[\exp\{-\nu\,3^n\mu(X^0_{\left[0,\frac{t}{5^n}\right]})\}]\,d\mu(y)$$

($E^{t,0}_{x,y}$ denotes the bridge in time t expectation for the reflected process on $\mathscr{F}_0$). Use now the property (71) and the relations (4), getting that, for all t,

$$\frac{t}{5^n} = t^{\frac{d_s}{d_s+2}} v^{\frac{2}{d_s+2}} f(t)$$

and

$$3^n = t^{\frac{d_s}{d_s+2}} v^{\frac{2}{d_s+2}} \frac{1}{v} g(t),$$

where $f(t)$, $g(t)$ are some numbers fulfilling $1 \leqq f(t) < 5$, $\frac{1}{3} < g(t) \leqq 1$.

Denoting $\tilde{t} = t^{\frac{d_s}{d_s+2}} v^{\frac{2}{d_s+2}} f(t) \left(= \frac{t}{5^n} \right)$ we get, recalling that $\mu(\mathscr{F}_0) = 1$,

$$L_N(t) \leqq \frac{1}{\mu(\mathscr{F}_0)} 15 v \int_{\mathscr{F}_0} \frac{q^0(\tilde{t}, y, y)}{\tilde{t}} \cdot E^{\tilde{t},0}_{y,y}\left[\exp\left\{ -\frac{\tilde{t}}{15} \mu(X_{[0,\tilde{t}]}) \right\} \right] d\mu(x) \leqq \frac{15 v}{\tilde{t}} A(\tilde{t})$$

($A(\tilde{t})$ is defined below).

This upper bound is the averaged trace of the operator associated with the reflected Brownian motion on $\mathscr{F}_0$, absorbed at the obstacle points, but with intensity $\frac{\tilde{t}}{15}$, hence depending on time. In the proof of (69) we will use not the averaged trace of this operator, but the averaged survival time.

The quantities mentioned above are:

(74) $$A(t) = \mathbb{E} \int_{\mathscr{F}_0} q^0(t, x, x)\,{}^{t,0}_{x,x}[T_{\mathscr{N}(\omega)} > t]\, d\mu(x) \quad \text{(averaged trace)},$$

$$B(t) = \mathbb{E} \int_{\mathscr{F}_0} Q_x[T_{\mathscr{N}(\omega)} > t]\, d\mu(x) \quad \text{(averaged survival time)}.$$

As in [10], 1.35, we have that

(75) $$B(t) \leqq A(t).$$

Now we are going to find an upper bound for $\frac{\log A(t)}{t}$ which, in virtue of (75) will give at once the upper bound for $\frac{\log B(t)}{t}$.

Lemma 9.

(76) $$\limsup_{t\to\infty} \frac{\log B(t)}{t} \leqq \limsup_{t\to\infty} \frac{\log A(t)}{t} \leqq -\inf_{U \in \mathbf{U}_0} [\lambda_0(U) + \tfrac{1}{15}\mu(U)],$$

where $\mathbf{U}_0$ *denotes the collection of all open subsets of* $\mathscr{F}_0$.

Proof. Let the binary numbers b and ε be given and such that $b\varepsilon < 1$. We denote, following [10], by $\mathbf{C}_{b\varepsilon}$ the finite cover of the triangle $\mathscr{F}_0$ by smaller closed triangles, obtained by chopping the sides of $\mathscr{F}_0$ into equal parts of length $b\varepsilon$. The cardinality of this cover is equal to

$$\#\mathbf{C}_{b\varepsilon}=\mathbf{N}_{b\varepsilon}=(b\varepsilon)^{-d_f}.$$

Let $\mathbf{U}_{b\varepsilon}$ be the collection of those open subsets of $\mathscr{F}_0$, whose complement in the triangle $\mathscr{F}_0$ is a union of some of the small triangles from the cover $\mathbf{C}_{b\varepsilon}$,

$$\#\mathbf{U}_{b\varepsilon}=2^{\mathbf{N}_{b\varepsilon}}.$$

Suppose now that $R>0$, $\delta>0$, $b>0$ are three fixed binary numbers. In virtue of the Theorem 1.7 of [10], which we can use thanks to the recurrence properties from Theorem 5, there exists $\varepsilon_0=\varepsilon_0(R,\delta,b)$ such that for $\varepsilon\leqq\varepsilon_0$ and $t\geqq 1$ (c_1 is the constant from the estimate in Lemma 4)

$$A(t)\leqq c_1\,2^{\mathbf{N}_{b\varepsilon}}\exp\{R-\inf_{U\in\mathbf{U}_{b\varepsilon}}[t(\lambda_0(U)\wedge R-\delta)+t\tfrac{1}{15}\mu(U)]\}.$$

Now we replace $\mathbf{U}_{b\varepsilon}$ by $\mathbf{U}_0$ (collection of all open subsets of $\mathscr{F}_0$) and use the elementary inequality

$$\forall a,b,c\in\mathbb{R}\quad a\wedge b+c\geqq(a+c)\wedge b,$$

getting that

$$A(t)\leqq c_1\,2^{\mathbf{N}_{b\varepsilon}}\exp\{R-\inf_{U\in\mathbf{U}_0} t[(\lambda_0(U)+\tfrac{1}{15}\mu(U))\wedge R-\delta]\}, \tag{77}$$

and

$$\limsup_{t\to\infty}\frac{\log A(t)}{t}\leqq-\inf_{U\in\mathbf{U}_0}[(\lambda_0(U)+\tfrac{1}{15}\mu(U))\wedge R-\delta]. \tag{78}$$

As (78) is true for arbitrary R, δ, we are allowed to let $\delta\to 0$, $R\to\infty$, getting the right-hand side inequality from (76). The other inequality is obvious. The proof of the lemma is completed. □

What remains to be checked, is that the infimum in the upper bound in (76) is is greater than zero.

Lemma 10.

$$\inf_{U\in\mathbf{U}_0}[\lambda_0(U)+\tfrac{1}{15}\mu(U)]>0.$$

Proof. The lemma follows easily from the spectral decomposition of the semigroup (T_t^0): for any given $t>0$ and $U\in\mathbf{U}_0$

$$\begin{aligned} e^{-t\lambda_0(U)}&\leqq Tr\;T_t^{0,U}=\int_U q^0(t,x,x)\,d\mu(x)\\ &\leqq\mu(U)\cdot\sup_{x\in\mathscr{F}_0} q^0(t,x,x)\leqq\mu(U)c_1\,t^{-\theta} \end{aligned} \tag{79}$$

$$\text{i.e. }\forall t>0\quad \lambda_0(U)\geqq-\frac{\log(\mu(U)c_1\,t^{-\theta}}{t}$$

($T_t^{0,U}$ is the semigroup related to q^0, killed upon coming to the boundary ∂U).

A simple calculation gives that if we choose $t>\max\left\{\left(\frac{15\,c_1}{e}\right)^{\frac{1}{1+\theta}},15\right\}$ then the infimum will be bigger than zero. The estimate is complete. □

Conclusion of the proof of the theorem. After we have estimate (76), the rest is easy. One has:

$$\frac{\log L_N(t)}{t^{\frac{d_s}{d_s+2}}} \leqq \frac{\log\left(\frac{A(\tilde{t})\,15\,\nu}{\tilde{t}}\right)}{\tilde{t}} \cdot 5\,\nu^{\frac{2}{d_s+2}}$$

and, as this upper bound is independent of N,

$$\text{(80)} \qquad \frac{\log L(t)}{\lambda^{\frac{-d_s}{d_s+2}}} \leqq \frac{\log \frac{A(t)}{\tilde{t}\left(\frac{15}{\nu}\right)}}{\tilde{t}}\, 2\,\nu^{\frac{2}{d_s+2}}.$$

Taking into account Lemma 9,

$$\text{(81)} \qquad \limsup_{t\to\infty} \frac{\log\ L(t)}{t^{\frac{d_s}{d_s+2}}} \leqq -2\,\nu^{\frac{2}{d_s+2}} \cdot \inf_{U\in\mathbf{U}_0} [\lambda_0(U)+\tfrac{1}{15}\mu(U)],$$

and we pick $D_1 = 2 \inf_{U\in\mathbf{U}_0} [\lambda_0(U)+\frac{1}{15}\mu(U)]$, which we know to be positive by Lemma 10. (68) is proved.

To see (69), one has to introduce some averaging – we have proved the asymptotic estimate for $B(t)$, not the pointwise estimate of $E_x[\exp\{-\nu\mu(Z_{[0,t]})\}]$.

We have, from the Markov property of Z_t, that

$$\text{(82)} \qquad \begin{aligned} E_x[\exp\{-\nu\mu(Z_{[0,t]})\}] &\leqq E_0\, E_{X_1}[\exp\{-\nu\mu(Z_{[0,t-1]})\}] = \\ &= \int_{\mathscr{G}} p(1,0,x)\, E_x[\exp\{-\nu\mu(Z_{[0,t-1]})\}]\, d\mu(x). \end{aligned}$$

Now we proceed as before – as in the proof of (68). We rescale the whole problem with $x = 2^n y$, if $2^n \leqq \left(\frac{t-1}{\nu}\right)^{\frac{1}{d_f+d_w}} < 2^{n+1}$ getting the last integral in (82) equal to:

$$\text{(83)} \qquad \int_{\mathscr{G}} p\left(\frac{1}{5^n}, 0, y\right) E_y[\exp\{-\nu\, 3^n \mu(Z_{\left[0,\frac{t-1}{5^n}\right]})\}]\, d\mu(y).$$

At this point, as before, we project the problem onto the triangle of size 1, getting

$$\begin{aligned} &\int_{\mathscr{F}_0} \sum_{y'\in\pi_0^{-1}(y)} p\left(\frac{1}{5^n}, 0, y'\right) E_y^0[\exp\{-\nu\, 3^n \mu(X^0_{\left[0,\frac{t-1}{5^n}\right]})\}]\, d\mu(y) \\ &\leqq \sup_{y\in\mathscr{F}_0} q^0\left(\frac{1}{5^n}, 0, y\right) B(\tilde{t}), \end{aligned}$$

with $\tilde{t}=(t-1)^{\frac{d_s}{d_s+2}} \nu^{\frac{2}{d_s+2}} f(t-1)(f(t-1)$ – as before, is a number from $(1, 5])$. $B(\tilde{t})$ is the averaged survival time as in (74), but with intensity $\frac{\tilde{t}}{15}\mu$. Using Lemma 4, we get that

$$\sup_{y \in \mathscr{F}_0} q^0\left(\frac{1}{5^n}, 0, y\right) \leqq c_1 5^{n\theta},$$

i.e. the behavior in t is at most polynomial-like when t goes to infinity – can be neglected. Now (69) follows as before. □

Now our estimates (Theorems 2 and 6) lead us to establishing, via use of the Minlos-Povzner Tauberian theorem of exponential type (Theorem 2.1 in [5]), the asymptotic behavior of $l([0, \lambda))$, as $\lambda \to 0$. This type of behavior is known as the 'Lifschitz-type singularity' for the density of states.

Theorem 7.

$$-C\nu \leqq \liminf_{\lambda \to 0} \frac{\log l([0, \lambda])}{\lambda^{\frac{-d_s}{2}}} \leqq \limsup_{\lambda \to 0} \frac{\log l([0, \lambda])}{\lambda^{\frac{-d_s}{2}}} \leqq -D\nu, \tag{84}$$

where C, D are two positive constants, depending only on the constants in Theorems 2, 6.

Proof. The proof is nothing but use of the Minlos-Povzner type Tauberian theorem mentioned above. □

Acknowledgement. The author wishes to express thanks to Prof. Alain-Sol Sznitman for drawing the attention to the subject and for many helpful discussions.

References

1. Barlow, M.T., Perkins, E.A.: Brownian motion on the Sierpinski gasket. Probab. Th. Rel. Fields **79**, 543–623 (1988)
2. Carmona, R.: Random Schrodinger operators. In: Ecole d'ete de St. Flour. (Lect. Notes Math., vol. 1180, pp. 1–121. Berlin Heidelberg New York: Springer 1986
3. Donsker, M.D., Varadhan, S.R.S.: Asymptotics for the Wiener Sausage. Commun. Pure Appl. Math. **28**, 525–565 (1975)
4. Fukushima, M., Shima, T.: On a spectral analysis for the Sierpinski gasket. Preprint (1989)
5. Fukushima, M.: On the spectral distribution of a disordered system and the range of a random walk. Osaka J. Math. **11**, 73–85 (1974)
6. Goldstein, S.: Random walks and diffusions on fractals. In: Kesten, H. (ed.) Percolation theory and ergodic theory of infinite particle systems. (IMA Math. Appl. vol. 8, pp. 121–128) New York Berlin Heidelberg: Springer 1987
7. Kusuoka, S.: A diffusion process on a fractal. In: Ito, K., Ikeda, N. (eds.) Probabilistic methods in Math. Phys., pp. 251–274 (Kinokuniya). Amsterdam: North Holland 1987
8. Lindstrom, T.: Brownian motion on nested fractals. Memoirs of the AMS, vol. 420, 1989
9. Riesz, F., Sz.-Nagy, B.: Functional analysis. New York: Ungar 1955 (in English)
10. Sznitman, A.S.: Lifschitz tail and Wiener sausage I. Preprint (1988)
11. Sznitman, A.S.: Lifschitz tail on hyperbolic space: Neumann conditions. Commun. Pure Appl. Math. **43**, 1–30 (1990)
12. Sznitman, A.S.: Lifschitz tail and Wiener Sausage on Hyperbolic Space. Commun. Pure Appl. Math. **42**, 1033–1065 (1989)
13. Sznitman, A.S.: Lifschitz tail on hyperbolic space: Neumann conditions. Commun. Pure Appl. Math. **43**, 1–30 (1990)