Probab. Theory Relat. Fields 94, 119–133 (1992)

Probability Theory and Related Fields

An Erdős–Révész type law of the iterated logarithm for stationary Gaussian processes*

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Received October 11, 1991; in revised form March 23, 1992

Summary. Let $\{X(t), t \geqq 0\}$ be a stationary Gaussian process with $EX(t) = 0$, $EX^2(t) = 1$ and covariance function satisfying (i) $r(t) = 1 - C|t|^\alpha + o(|t|^\alpha)$ as $t \to 0$ for some $C > 0$, $0 < \alpha \leqq 2$; (ii) $r(t) = O(t^{-2\gamma})$ as $t \to \infty$ for some $\gamma > 0$ and (iii) $\sup_{t \geqq s}|r(t)| < 1$ for each $s > 0$. Put $\xi(t) = \sup\{s: 0 \leqq s \leqq t, X(s) \geqq (2\log s)^{1/2}\}$. The law of the iterated logarithm implies $\limsup_{t\to\infty}(\xi(t) - t) = 0$ a.s. This paper gives the lower bound of $\xi(t)$ and obtains an Erdős–Rèvèsz type LIL, i.e., $\liminf_{t\to\infty}(\xi(t) - t)/(t(\log t)^{(\alpha-2)/(2\alpha)}\log\log t) = -(2+\alpha)\sqrt{\pi}/(\alpha H_\alpha (2C)^{1/\alpha})$ a.s. if $0 < \alpha < 2$ and $\liminf_{t\to\infty}\log(\xi(t)/t)/\log\log t = -2\pi/\sqrt{2C}$ a.s. if $\alpha = 2$. Applications to infinite series of independent Ornstein-Uhlenbeck processes and to fractional Wiener processes are also given.

Mathematics Subject Classification (1991): 60 G 15, 60 F 15

1 Introduction and main result

Let $W(t)$, $t \geqq 0$ be a standard Wiener process and define $\eta(t) = \sup\{s:0 \leqq s \leqq t, W(s) \geqq (2s\log\log s)^{1/2}\}$, $t \geqq 0$. From the law of the iterated logarithm it follows immediately that

$$\limsup_{t\to\infty} \eta(t)/t = 1 \quad \text{a.s.}$$

Erdős and Révész [3] considered the lower bound of $\eta(t)$ and obtained a new law of the iterated logarithm:

$$\liminf_{t\to\infty} \frac{(\log_2 t)^{1/2}}{\log_3 t \cdot \log t} \cdot \log \frac{\eta(t)}{t} = -C_0 \quad \text{a.s.} \tag{1.1}$$

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* Research supported by the Fok Yingtung Education Foundation of China and by Charles Phelps Taft Postdoctoral Fellowship of the University of Cincinnati

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for some constant C_0 with $\frac{1}{4} \leqq C_0 \leqq 2^{14}$, while Shao [8] gave the exact value $3\sqrt{\pi}$ of C_0, where $\log_2 t = \log\log t$, $\log_3 t = \log\log_2 t$. Put

$$\bar{\eta}(t) = \sup\{s : 1 \leqq s \leqq t, \quad W(s) \geqq (2s\log\log s)^{1/2}\}, \quad \text{for} \quad t \geqq 1 ,$$

$$\hat{\eta}(t) = \sup\{s : 0 \leqq s \leqq t : \frac{W(e^s)}{e^{s/2}} \geqq (2\log s)^{1/2}\}, \quad \text{for } t \geqq 0 .$$

It is easy to see that

$$\hat{\eta}(t) = \log \bar{\eta}(e^t) \quad \text{a.s. for every } t \geqq 0$$

and hence, by (1.1)

$$\liminf_{t\to\infty} \frac{(\log t)^{1/2}}{t\cdot\log_2 t}\cdot(\hat{\eta}(t) - t) = -3\sqrt{\pi} \quad \text{a.s.} \tag{1.2}$$

Clearly, $W(e^s)/e^{s/2}$, $s \geqq 0$ is an Ornstein-Uhlenbeck process, a stationary Gaussian process. This promotes us to study the corresponding problem to $\hat{\eta}(t)$ for general stationary Gaussian process.

Let $\{X(t), t \geqq 0\}$ be a real separable stationary Gaussian process with $EX(t) = 0$ and $EX^2(t) = 1$ for each $t \geqq 0$. Denote the correlation function

$$r(t) = EX(t+s)X(s) \quad \text{for } s \geqq 0 \text{ and } t \geqq 0 . \tag{1.3}$$

Consider the process

$$\xi(t) = \sup\{s : 0 \leqq s \leqq t, X(s) \geqq (2\log s)^{1/2}\}, \quad t \geqq 0 . \tag{1.4}$$

The upper class of law of the iterated logarithm implies

$$P(X(s) \geqq (2\log s)^{1/2}, \text{i.o.}) = 1$$

under certain conditions on $r(t)$ (cf. Qualls and Watanable [7]). Hence, we have

$$\lim_{t\to\infty} \xi(t) = \infty , \quad \text{a.s.}$$

and

$$\limsup_{t\to\infty} (\xi(t) - t) = 0 \quad \text{a.s.}$$

The aim of this paper is to give the lower bound of $\xi(t)$. We will state our main result in this section, while its proof is given in Sect. 2. Sections 3 and 4 will be devoted to two special Gaussian processes, infinite series of independent Ornstein-Uhlenbeck processes and fractional Wiener process, respectively.

Throughout this paper we will use the following notations: $[x]$ denotes the integer part of x; $x^+ = \max(0, x)$; $\log x = \ln x$ if $x > 0$ and $\log x = 1$ if $x \leqq 0$, where ln is the natural logarithm; $\log_2 x = \log\log x$, and $\log_3 x = \log\log_2 x$; $\sum_{i=x}^{y}$ stands for $\sum_{i=[x]}^{[y]}$ and $\max_{x\leqq i\leqq y}$ means $\max_{[x]\leqq i\leqq [y]}$ for $y \geqq x$.

In what follows we always assume that $X(t)$ is a stationary Gaussian process with $EX(t) = 0$, $EX^2(t) = 1$ and correlation function $r(t)$.

Theorem 1.1 *Assume that the following conditions are satisfied:*

$$r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \text{ as } t \to 0 \text{ for some } C > 0, \quad 0 < \alpha \leqq 2\,, \tag{1.5}$$

$$r(t) = O(t^{-2\gamma}) \text{ as } t \to \infty \text{ for some } \gamma > 0\,, \tag{1.6}$$

$$\sup_{t \geqq s} |r(t)| < 1 \text{ for each } s > 0. \tag{1.7}$$

Then

$$\liminf_{t\to\infty} \frac{\xi(t) - t}{t(\log t)^{(\alpha-2)/(2\alpha)} \cdot \log_2 t} = -\frac{(2+\alpha)\sqrt{\pi}}{\alpha H_\alpha (2C)^{1/\alpha}} \quad \text{a.s. if } 0 < \alpha < 2 \tag{1.8}$$

$$\liminf_{t\to\infty} \frac{\log(\xi(t)/t)}{\log_2 t} = -\frac{2\sqrt{\pi}}{H_2\sqrt{2C}} \quad \text{a.s. if } \alpha = 2 \tag{1.9}$$

where $0 < H_\alpha \equiv \lim_{T\to\infty} T^{-1} \int_0^\infty e^s P(\sup_{0 \leqq t \leqq T} Y(t) > s)\,ds < \infty$, *and* $Y(t)$ *is a non-stationary Gaussian process with mean* $EY(t) = -|t|^\alpha$ *and covariance function* $\mathrm{Cov}(Y(s), Y(t)) = -|t-s|^\alpha + |s|^\alpha + |t|^\alpha$.

The value of H_α is unknown except $H_1 = 1$ and $H_2 = 1/\sqrt{\pi}$ (cf. [5, p. 232]). (1.8) shows that for any t big enough there exists an s in

$$[t - t(\log t)^{(\alpha-2)/(2\alpha)} \cdot \log_2 t, t]$$

such that $X(s) \geqq (2\log s)^{1/2}$ and that the length of the interval $t(\log t)^{(\alpha-2)/(2\alpha)} \log_2 t$ is smallest possible. Moreover, the bigger the parameter α is, the wider the interval will be.

2 Proof

We start with some preliminary lemmas.

Lemma 2.1 *Suppose* $\xi_1, \ldots, \xi_n$ *are standard normal variables with covariuance matrix* $\Lambda^1 = (\Lambda^1_{ij})$ *and* $\eta_1, \ldots, \eta_n$ *similarly with covariance matrix* $\Lambda^0 = (\Lambda^0_{ij})$, *and let* $\rho_{ij} = \max(|\Lambda^1_{ij}|, |\Lambda^0_{ij}|)$. *Further, let* $u_1, \ldots, u_n$ *be real numbers. Then*

$$P\left(\bigcap_{j=1}^n \{\xi_j \leqq u_j\}\right) - P\left(\bigcap_{j=1}^n \{\eta_j \leqq u_j\}\right)$$
$$\leqq \frac{1}{2\pi} \sum_{1 \leqq i < j \leqq n} (\Lambda^1_{ij} - \Lambda^0_{ij})^+ (1 - \rho^2_{ij})^{-1/2} \exp\left(-\frac{u_i^2 + u_j^2}{2(1+\rho_{ij})}\right).$$

This is Theorem 4.2.1 of [5].

Lemma 2.2 *If* (1.5) *and* (1.7) *hold, then*

$$\lim_{x\to\infty} \frac{P(\sup_{0 \leqq s \leqq 1} X(s) > x)}{x^{2/\alpha}\psi(x)} = C^{1/\alpha} H_\alpha \tag{2.1}$$

$$\lim_{x\to\infty} \frac{P(\max_{0 \leqq j \leqq x^{2/\alpha}/\theta} X(j\theta x^{-2/\alpha}) > x)}{x^{2/\alpha}\psi(x)} = C^{1/\alpha} \frac{H_\alpha(\theta)}{\theta} \tag{2.2}$$

for each $\theta > 0$ *and*

$$\lim_{\theta\to 0}\frac{H_\alpha(\theta)}{\theta} = H_\alpha \tag{2.3}$$

where H_α *is defined as in Theorem* 1.1 *and* $\psi(x) = (2\pi)^{-1/2}x^{-1}e^{-x^2/2}$.

This is Lemma 2.5 of [6].

From the proof of Lemma 12.2.5 of [5], one can see that the next lemma holds.

Lemma 2.3 *If the conditions of Theorem* 1.1 *are satisfied, then there exist constant* K_0 *and* x_0 *such that*

$$P\left(\max_{0\le j\le x^{2/\alpha}/\theta} X(j\theta x^{-2/\alpha}) \le x - \theta^{\alpha/4}/x,\ \sup_{0\le s\le 1} X(s) > x\right) \le K_0 x^{2/\alpha}\psi(x)\theta^{\frac{\alpha}{2}-1}\exp(-\theta^{-\alpha/4}/K_0) \tag{2.4}$$

for each $\theta > 0$ *and* $x \geqq x_0$.

In what follows we define

$$\tilde{\alpha} = (2+\alpha)/(2\alpha),\quad \hat{\alpha} = (2-\alpha)/(2\alpha)\quad \text{for}\quad 0<\alpha\leqq 2. \tag{2.5}$$

Lemma 2.4 *Under the condition of Theorem* 1.1, *for each* $0 < \varepsilon < 1$, *there exist positive constants* N *and* ρ *depending only on* ε, α *and* γ *such that*

$$P\left(\sup_{a\le s\le b}\frac{X(s)}{(2\log s)^{1/2}} \leqq 1\right) \leqq \exp\left(-\frac{(1-\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(1+\varepsilon)(2+\alpha)\sqrt{\pi}}(\log^{\tilde{\alpha}} b - \log^{\tilde{\alpha}}(a+1))\right) + Na^{-\rho} \tag{2.6}$$

for each $b \geqq a+1 \geqq N$, *where* $\tilde{\alpha}$ *is defined as in* (2.5).

Proof. By Lemma 2.2, there exist $\theta > 0$ and x_0 such that

$$P\left(\max_{0\le j\le x^{2/\alpha}/\theta} X(j\theta x^{-2/\alpha}) \geqq x\right) \geqq (1-\varepsilon)H_\alpha C^{1/\alpha}x^{2/\alpha}\psi(x) \tag{2.7}$$

for each $x \geqq x_0$. Put

$$b(a,\varepsilon) = [(b-a-1)/(1+\varepsilon)],$$

$$x_i = (2\log(a+1+i(1+\varepsilon)))^{1/2},\quad \hat{x}_i = [x_i^{2/\alpha}/\theta],\quad i = 0, 1, \ldots .$$

Then

$$\begin{aligned}
&P\left(\sup_{a\le s\le b}\frac{X(s)}{(2\log s)^{1/2}} \leqq 1\right)\\
&\leqq P\left(\max_{0\le i\le (b-a-1)/(1+\varepsilon)}\ \sup_{a+i(1+\varepsilon)\le s\le a+1+i(1+\varepsilon)}\frac{X(s)}{(2\log s)^{1/2}} \leqq 1\right)\\
&\leqq P\left(\max_{0\le i\le b(a,\varepsilon)}\ \sup_{a+i(1+\varepsilon)\le s\le a+1+i(1+\varepsilon)}\frac{X(s)}{(2\log(a+1+i(1+\varepsilon)))^{1/2}} \leqq 1\right)\\
&\leqq P\left(\max_{0\le i\le b(a,\varepsilon)}\ \max_{0\le j\le \hat{x}_i}\frac{X(a+i(1+\varepsilon)+j\theta x_i^{-2/\alpha})}{x_i} \leqq 1\right).
\end{aligned} \tag{2.8}$$

Let

$$\mathbf{X}_i = (X(a + i(1 + \varepsilon) + j\theta x_i^{-2/\alpha}), \quad 0 \leqq j \leqq \hat{x}_i), \; i = 0, 1, \ldots, b(a, \varepsilon),$$

$\{\mathbf{Y}_i, 0 \leqq i \leqq b(a, \varepsilon)\}$ be independent normal random vectors and $\mathbf{Y}_i$ and $\mathbf{X}_i$ have the same distribution for each $0 \leqq i \leqq b(a, \varepsilon)$. Applying Lemma 2.1 yields

$$P\left(\max_{0 \leqq i \leqq b(a,\varepsilon)} \max_{0 \leqq j \leqq \hat{x}_i} \frac{X(a + i(1 + \varepsilon) + j\theta x_i^{-2/\alpha})}{x_i} \leqq 1\right)$$

$$\leqq \prod_{i=0}^{b(a,\varepsilon)} P\left(\max_{0 \leqq j \leqq \hat{x}_i} X(a + i(1 + \varepsilon) + j\theta x_i^{-2/\alpha}) \leqq x_i\right)$$

$$+ \sum_{0 \leqq i < j \leqq b(a,\varepsilon)} \sum_{u=0}^{\hat{x}_i} \sum_{v=0}^{\hat{x}_j} \frac{|r(i, j, u, v)|}{\sqrt{1 - r^2(i, j, u, v)}} \exp\left(-\frac{\frac{1}{2}(x_i^2 + x_j^2)}{1 + |r(i, j, u, v)|}\right)$$

$$:= I_1 + I_2 \tag{2.9}$$

where

$$r(i, j, u, v) = r((j - i)(1 + \varepsilon) + v\theta x_j^{-2/\alpha} - u\theta x_i^{-2/\alpha}),$$

$$I_1 = \prod_{i=0}^{b(a,\varepsilon)} P\left(\max_{0 \leqq j \leqq \hat{x}_i} X(a + i(1 + \varepsilon) + j\theta x_i^{-2/\alpha}) \leqq x_i\right),$$

$$I_2 = \sum_{0 \leqq i < j \leqq b(a,\varepsilon)} \sum_{u=0}^{\hat{x}_i} \sum_{v=0}^{\hat{x}_j} \frac{|r(i, j, u, v)|}{\sqrt{1 - r^2(i, j, u, v)}} \exp\left(-\frac{\frac{1}{2}(x_i^2 + x_j^2)}{1 + |r(i, j, u, v)|}\right).$$

Put

$$r^*(s) = \sup_{t \geqq s} |r(t)|, \; s > 0. \tag{2.10}$$

Noting that

$$(j - i)(1 + \varepsilon) + v\theta x_j^{-2/\alpha} - u\theta x_i^{-2/\alpha} \geqq (j - i)\varepsilon \geqq \varepsilon$$

for every $j > i$, $0 \leqq u \leqq \hat{x}_i$, $\quad 0 \leqq v \leqq \hat{x}_j$, we have

$$|r(i, j, u, v)| \leqq r^*((j - i)\varepsilon) \leqq r^*(\varepsilon) < 1 \tag{2.11}$$

by (1.7). From (1.6) it follows that there is a t_0 such that

$$r^*(t) \leqq t^{-\gamma} \leqq \min(1, \gamma)/4 \text{ for every } t \geqq t_0. \tag{2.12}$$

In what follows, for the sake of simplicity of statement, we will use K to denote the constant which is independent of a and b, and may be different from line to line. We have

$$I_2 \leqq \frac{4}{\theta^2\sqrt{1 - r^*(\varepsilon)}} \sum_{0 \leqq i < j \leqq b(a,\varepsilon)} x_i^{2/\alpha} x_j^{2/\alpha} r^*((j - i)\varepsilon) \exp\left(-\frac{x_i^2 + x_j^2}{2(1 + r^*((j - i)\varepsilon))}\right)$$

$$= \frac{4}{\theta^2\sqrt{1 - r^*(\varepsilon)}} \left(\sum_{0 < j - i \leqq t_0/\varepsilon} + \sum_{j - i > t_0/\varepsilon}\right)(\cdot)$$

$$\leqq K\Bigg(\sum_{0\leqq i\leqq b(a,\varepsilon)} x_i^{4/\alpha}\exp\left(-\frac{x_i^2}{1+r^*(\varepsilon)}\right)$$

$$+\sum_{j-i>t_0/\varepsilon,\ j\leqq b(a,\varepsilon)} x_i^{2/\alpha}x_j^{2/\alpha}(j-i)^{-\gamma}\exp\left(-\frac{x_i^2+x_j^2}{2(1+\gamma/4)}\right)\Bigg)$$

$$\leqq K\Bigg(\sum_{i=0}^{\infty}(a+i(1+\varepsilon))^{-2/(1+r^*(\varepsilon))}\log^{2/\alpha}(a+i)$$

$$+\sum_{j-i>t_0/\varepsilon,\ j\leqq b(a,\varepsilon)}(a+i)^{-\frac{1}{1+\gamma/4}}(a+j)^{-\frac{1}{1+\gamma/4}}\log^{1/\alpha}(a+i)\log^{1/\alpha}(a+j)\Bigg)$$

$$\leqq K\left(a^{-\frac{1-r^*(\varepsilon)}{4}}+\sum_{i=0}^{\infty}\frac{\log^{1/\alpha}(a+i)}{(a+i)^{1+\gamma/4}}\right)$$

$$\leqq K(a^{-(1-r^*(\varepsilon))/4}+a^{-\gamma/6})$$

$$\leqq Ka^{-\rho}$$

where $\rho=\min((1-r^*(\varepsilon))/4,\gamma/6)$.

Noting that $X(\cdot)$ is a stationary process, we derive from (2.7) that

$$I_1=\prod_{i=0}^{b(a,\varepsilon)}P\left(\max_{0\leqq j\leqq\hat{x}_i}X(j\theta x_i^{-2/\alpha})\leqq x_i\right)$$

$$\leqq\exp\left(-\sum_{i=0}^{b(a,\varepsilon)}P\left(\max_{0\leqq j\leqq\hat{x}_i}X(j\theta x_i^{-2/\alpha})>x_i\right)\right)$$

$$\leqq\exp\left(-\sum_{i=0}^{b(a,\varepsilon)}(1-\varepsilon)C^{1/\alpha}H_\alpha x_i^{2/\alpha}\psi(x_i)\right)$$

$$=\exp\left(-(1-\varepsilon)C^{1/\alpha}H_\alpha\sum_{i=0}^{b(a,\varepsilon)}\frac{(2\log(a+1+i(1+\varepsilon)))^{\hat{\alpha}}}{\sqrt{2\pi}(a+1+i(1+\varepsilon))}\right)$$

$$=\exp\left(-\frac{(1-\varepsilon)(2C)^{1/\alpha}H_\alpha}{2\sqrt{\pi}}\sum_{i=0}^{b(a,\varepsilon)}\frac{\log^{\hat{\alpha}}(a+1+i(1+\varepsilon))}{a+1+i(1+\varepsilon)}\right)$$

provided that a is sufficiently large, where $\hat{\alpha}$ is defined as in (2.5). An elementary calculation implies

$$\sum_{i=0}^{b(a,\varepsilon)}\frac{\log^{\hat{\alpha}}(a+1+i(1+\varepsilon))}{a+1+i(1+\varepsilon)}\geqq\int_0^{\frac{b-a-1}{1+\varepsilon}}\frac{\log^{\hat{\alpha}}(a+1+y(1+\varepsilon))}{a+1+y(1+\varepsilon)}\,\mathrm{d}y$$

$$=\frac{2\alpha}{(1+\varepsilon)(2+\alpha)}(\log^{\tilde{\alpha}}b-\log^{\tilde{\alpha}}(a+1))\,.$$

Hence

$$I_1\leqq\exp\left(-\frac{(1-\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(1+\varepsilon)(2+\alpha)\sqrt{\pi}}(\log^{\tilde{\alpha}}b-\log^{\tilde{\alpha}}(a+1))\right).\tag{2.15}$$

Putting the above inequalities together yields that (2.6) holds true. This proves the lemma.

Lemma 2.5 *Under the condition of Theorem* 1.1, *for each* $0 < \varepsilon < 1$, *there exist positive constants* N *and* τ *depending only on* ε, α *and* γ *such that*

$$P\left(\bigcap_{0 \leqq i \leqq [b-a]} \left\{ \max_{0 \leqq j \leqq y_i^{2/\alpha}/\theta_i} X(a + i + j\theta_i y_i^{-2/\alpha}) < y_i - \theta_i^{\alpha/4}/y_i \right\}\right)$$

$$\geqq \frac{1}{4}\exp\left(- \frac{(1+\varepsilon)\alpha(2C)^{1/\alpha}H_\alpha}{(2+\alpha)\sqrt{\pi}}(\log^{\tilde{\alpha}} b - \log^{\tilde{\alpha}} a)\right) - Na^{-\tau} \tag{2.16}$$

for each $b \geqq a + 1 \geqq N$, *where* $y_i = (2\log(a+i))^{1/2}$, $\theta_i = \log^{-8/\alpha}(a+i)$, $\tilde{\alpha}$ *is as in* (2.5).

Proof. Put

$$\bar{y}_i = y_i - \theta_i^{\alpha/4}/y_i, \quad \hat{y}_i = [y_i^{2/\alpha}/\theta_i], \quad i = 0, 1, \ldots .$$

Applying Lemma 2.1, one can find that

$$P\left(\bigcap_{0 \leqq i \leqq [b-a]} \left\{ \max_{0 \leqq j \leqq y_i^{2/\alpha}/\theta_i} X(a + i + j\theta_i y_i^{-2/\alpha}) < y_i - \theta_i^{\alpha/4}/y_i \right\}\right)$$

$$\geqq \prod_{i=0}^{[b-a]} P\left(\max_{0 \leqq j \leqq \hat{y}_i} X(a + i + j\theta_i y_i^{-2/\alpha}) < \bar{y}_i \right)$$

$$- \frac{1}{2\pi} \sum_{0 \leqq i < j \leqq b-a} \sum_{u=0}^{\hat{y}_i} \sum_{v=0}^{\hat{y}_j} \frac{(\tau(i,j,u,v))^+}{\sqrt{1 - \tau^2(i,j,u,v)}} \exp\left(- \frac{\frac{1}{2}(\bar{y}_i^2 + \bar{y}_j^2)}{1 + |\tau(i,j,u,v)|}\right)$$

$$:= J_1 - J_2, \tag{2.17}$$

where $\tau(i,j,u,v) = -r(j - i + v\theta_j y_j^{-2/\alpha} - u\theta_i y_i^{-2/\alpha})$.

Clearly, for $j \geqq i + 2$, $0 \leqq u \leqq \hat{y}_i$ and $0 \leqq v \leqq \hat{y}_j$, by (1.7)

$$|\tau(i,j,u,v)| \leqq r^*(j - i - 1) \leqq r^*(1) < 1. \tag{2.18}$$

On the other hand, by (1.5), there exists a constant $0 < t_1 < 1$ such that

$$r(t) \geqq 1 - C|t|^\alpha/2 > 0 \text{ for every } 0 \leqq t \leqq t_1 .$$

Hence

$$(\tau(i,j,u,v))^+ = 0, \quad \text{if } j = i + 1,\ 1 + v\theta_j y_j^{-2/\alpha} - u\theta_i y_i^{-2/\alpha} \leqq t_1 \tag{2.19}$$

and

$$|\tau(i,j,u,v)| \leqq r^*(t_1) < 1 \quad \text{if } j = i + 1,\ 1 + v\theta_j y_j^{-2/\alpha} - u\theta_i y_i^{-2/\alpha} > t_1 . \tag{2.20}$$

Therefore, by (2.18), (2.19) and (2.20) we obtain

$$J_2 \leqq \sum_{0 \leqq i \leqq b-a,\ j=i+1} \sum_{u=0}^{\hat{y}_i} \sum_{v=0}^{\hat{y}_j} \frac{1}{\sqrt{1 - r^*(t_1)}} \exp\left(- \frac{\frac{1}{2}(\bar{y}_i^2 + \bar{y}_j^2)}{1 + r^*(t_1)}\right)$$

$$+ \sum_{0 \leqq i + 2 \leqq j \leqq b-a} \sum_{u=0}^{\hat{y}_i} \sum_{v=0}^{\hat{y}_j} \frac{r^*(j-i-1)}{\sqrt{1 - r^*(1)}} \exp\left(- \frac{\frac{1}{2}(\bar{y}_i^2 + \bar{y}_j^2)}{1 + r^*(j-i-1)}\right).$$

Completely similar to the estimation of I_2, we can arrive that there exist positive constants K and τ such that

$$J_2 \leqq K a^{-\tau} \tag{2.21}$$

for every a sufficiently large. Using Lemma 2.2, one can also obtain that

$$\begin{aligned} J_1 &= \prod_{i=0}^{[b-a]} P\left(\max_{0 \leqq j \leqq \hat{y}_i} X(j\theta_i y_i^{-2/\alpha}) < \bar{y}_i \right) \\ &\geqq \prod_{i=0}^{[b-a]} \left(1 - P\left(\sup_{0 \leqq s \leqq 1} X(s) \geqq \bar{y}_i \right) \right) \\ &\geqq \frac{1}{4} \exp\left(- \frac{(1+\varepsilon)\alpha(2C)^{1/\alpha} H_\alpha}{(2+\alpha)\sqrt{\pi}} \left(\log^{\tilde{\alpha}} b - \log^{\tilde{\alpha}} a \right) \right) \end{aligned} \tag{2.22}$$

provided that a is sufficiently large, along the same line of the proof of I_1 in Lemma 2.4. This proves (2.16), by (2.17), (2.21) and (2.22).

Proof of Theorem 1.1 We formulate the proof in three steps.

Step 1 Assume $0 < \alpha < 2$. Then

$$\liminf_{t \to \infty} \frac{\xi(t) - t}{t \log^{-\hat{\alpha}} t \cdot \log_2 t} \geqq -(1+2\varepsilon)^2 C_1 \quad \text{a.s.} \tag{2.23}$$

for every $0 < \varepsilon < \frac{1}{4}$, where $C_1 = (2+\alpha)\sqrt{\pi}/(\alpha H_\alpha (2C)^{1/\alpha})$, $\hat{\alpha}$ is as in (2.5).

Proof. Put

$$t_k = \exp(k^{1/\tilde{\alpha}}), \quad s_k = t_k - (1+2\varepsilon) C_1 t_k \log^{-\hat{\alpha}} t_k \cdot \log_2 t_k, \quad k = 1, 2, \ldots .$$

Then

$$\begin{aligned} \log^{\tilde{\alpha}} t_k - \log^{\tilde{\alpha}} s_k &\sim \log^{\tilde{\alpha}} t_k - (\log t_k - (1+2\varepsilon)^2 C_1 (\log t_k)^{-\hat{\alpha}} \cdot \log_2 t_k)^{\tilde{\alpha}} \\ &\sim \frac{(1+2\varepsilon)^2 (2+\alpha) C_1}{2\alpha} \log_2 t_k . \end{aligned} \tag{2.24}$$

Noting that

$$\{\xi(t) \leqq a\} = \left\{ \sup_{a < s \leqq t} \frac{X(s)}{(2 \log s)^{1/2}} < 1 \right\}$$

for every $0 < a < t$, and using Lemma 2.4 and (2.24), we obtain that

$$\begin{aligned} &P\left(\frac{\xi(t_k) - t_k}{t_k \log^{-\hat{\alpha}} t_k \cdot \log_2 t_k} \leqq -(1+2\varepsilon) C_1 \right) \\ &= P\left(\sup_{s_k < s \leqq t_k} \frac{X(s)}{(2 \log s)^{1/2}} < 1 \right) \\ &\leqq \exp\left(- \frac{(1-\varepsilon)\alpha(2C)^{1/\alpha} H_\alpha}{(1+\varepsilon)(2+\alpha)\sqrt{\pi}} (\log^{\tilde{\alpha}} t_k - \log^{\tilde{\alpha}} (s_k + 1)) \right) + N s_k^{-\rho} \end{aligned}$$

$$\leq \exp\left(-\frac{(1-\varepsilon)(2C)^{1/\alpha}(1+2\varepsilon)^2 C_1}{(1+2\varepsilon)2\sqrt{\pi}}\log_2 t_k\right) + 2Nt_k^{-\rho}$$

$$\leq \exp\left(-\frac{(1+\varepsilon/2)(2+\alpha)\log_2 t_k}{2\alpha}\right) + 2Nt_k^{-\rho}$$

$$\leq 2k^{-(1+\varepsilon/2)}$$

for every k sufficiently large. Hence, by the Borel-Cantelli lemma, we have

$$\liminf_{k\to\infty} \frac{\xi(t_k) - t_k}{t_k \log^{-\hat{\alpha}} t_k \cdot \log_2 t_k} \geq -(1+2\varepsilon)^2 C_1 \quad \text{a.s.} \tag{2.25}$$

Since $\xi(t)$ is a non-decreasing random function of t, for every $t_k \leq t \leq t_{k+1}$, we have

$$\frac{\xi(t) - t}{t\log^{-\hat{\alpha}} t \cdot \log_2 t} \geq \frac{\xi(t_k) - t_{k+1}}{t_k \log^{-\hat{\alpha}} t_k \cdot \log_2 t_k}$$

$$= \frac{\xi(t_k) - t_k}{t_k \log^{-\hat{\alpha}} t_k \cdot \log_2 t_k} - \frac{t_{k+1} - t_k}{t_k \log^{-\hat{\alpha}} t_k \cdot \log_2 t_k}. \tag{2.26}$$

An elementary calculation implies

$$\lim_{k\to\infty} \frac{t_{k+1} - t_k}{t_k \log^{-\hat{\alpha}} t_k \cdot \log_2 t_k} = 0 \tag{2.27}$$

which together with (2.26) yields

$$\liminf_{t\to\infty} \frac{\xi(t) - t}{t\log^{-\hat{\alpha}} t \cdot \log_2 t} = \liminf_{k\to\infty} \frac{\xi(t_k) - t_k}{t_k \log^{-\hat{\alpha}} t_k \cdot \log_2 t_k} \quad \text{a.s.} \tag{2.28}$$

This proves (2.23), by (2.25) and (2.28).

Step 2 Assume $0 < \alpha < 2$. Then

$$\liminf_{t\to\infty} \frac{\xi(t) - t}{t\log^{-\hat{\alpha}} t \cdot \log_2 t} \leq -(1-\varepsilon) C_1 \quad \text{a.s.} \tag{2.29}$$

for every $0 < \varepsilon < (2-\alpha)/8 < \frac{1}{4}$.

Proof. Let

$$b_k = \exp(k^{(1+\varepsilon^2)/\tilde{\alpha}}), \quad a_k = b_k - (1-2\varepsilon) C_1 b_k \log^{-\hat{\alpha}} b_k \cdot \log_2 b_k,$$

$$y_{k,i} = (2\log(a_k + i))^{1/2}, \quad \theta_{k,i} = \log^{-8/\alpha}(a_k + i),$$

$$\bar{y}_{k,i} = y_{k,i} - \theta_{k,i}^{\alpha/4}/y_{k,i}, \quad \hat{y}_{k,i} = y_{k,i}^{2/\alpha}/\theta_{k,i},$$

$$E_k = \{\xi(b_k) \leq a_k\} = \left\{ \sup_{a_k < s \leq b_k} \frac{X(s)}{(2\log s)^{1/2}} < 1 \right\},$$

$$A_k = \bigcap_{0 \leq i \leq [b_k - a_k]} \left\{ \max_{0 \leq j \leq \hat{y}_{k,i}} X(a_k + i + j\theta_{k,i} y_{k,i}^{-2/\alpha}) \leq \bar{y}_{k,i} \right\}.$$

It suffices to show that

$$P(E_n \text{ i.o.}) = 1 . \tag{2.30}$$

Clearly, for $m \geqq 1$

$$P\left(\bigcup_{k=m}^{\infty} A_k\right) \leqq P\left(\bigcup_{k=m}^{\infty} E_k\right) + \sum_{k=m}^{\infty} P(A_k E_k^c) \leqq P\left(\bigcup_{k=m}^{\infty} E_k\right)$$

$$+ \sum_{k=m}^{\infty} \sum_{i=0}^{b_k - a_k} P\left(\max_{0 \leqq j \leqq \hat{y}_{k,i}} X(a_k + i + j\theta_{k,i} y_{k,i}^{-2/\alpha}) \leqq \bar{y}_{k,i}, \sup_{0 \leqq s \leqq 1} X(a_k + i + s) \geqq y_{k,i}\right)$$

$$= P\left(\bigcup_{k=m}^{\infty} E_k\right) + \sum_{k=m}^{\infty} \sum_{i=0}^{b_k - a_k} P\left(\max_{0 \leqq j \leqq \hat{y}_{k,i}} X(j\theta_{k,i} y_{k,i}^{-2/\alpha}) \leqq \bar{y}_{k,i}, \sup_{0 \leqq s \leqq 1} X(s) \geqq y_{k,i}\right)$$

$$:= P\left(\bigcup_{k=m}^{\infty} E_k\right) + L_m . \tag{2.31}$$

Using Lemma 2.3, we have, for some $K_0 > 0$

$$L_m \leqq K_0 \sum_{k=m}^{\infty} \sum_{i=0}^{b_k - a_k} y_{k,i}^{2/\alpha} \psi(y_{k,i}) \theta_{k,i}^{\alpha/2 - 1} \exp\left(-\frac{\theta_{k,i}^{-\alpha/4}}{K_0}\right)$$

$$\leqq K \sum_{k=m}^{\infty} \sum_{i=0}^{b_k - a_k} \log^{9\hat{\alpha}}(a_k + i) \cdot (a_k + i)^{-1} \exp\left(-\frac{\log^2(a_k + i)}{K_0}\right)$$

$$\leqq K \sum_{k=m}^{\infty} \sum_{i=0}^{\infty} \log^{9\hat{\alpha}}(a_k + i) \cdot (a_k + i)^{-3}$$

$$\leqq K \sum_{k=m}^{\infty} a_k^{-1}$$

$$\leqq K m^{-4}$$

provided m is large enough. Therefore

$$\lim_{m \to \infty} L_m = 0$$

and

$$\lim_{m \to \infty} P\left(\bigcup_{k=m}^{\infty} E_k\right) \geqq \lim_{m \to \infty} P\left(\bigcup_{k=m}^{\infty} A_k\right) .$$

To finish the proof of (2.30), we only need to show that

$$P(A_n \text{ i.o.}) = 1 . \tag{2.32}$$

Similarly to (2.24), we have

$$\log^{\tilde{\alpha}} b_k - \log^{\tilde{\alpha}} a_k \sim \frac{(1 - 2\varepsilon)(2 + \alpha) C_1}{2\alpha} \log_2 b_k .$$

Now from Lemma 2.5 it follows that

$$P(A_k) \geqq \frac{1}{4}\exp\left(- \frac{(1+\varepsilon)\alpha(2C)^{1/\alpha} H_\alpha}{(2+\alpha)\sqrt{\pi}}(\log^{\tilde{\alpha}} b_k - \log^{\tilde{\alpha}} a_k)\right) - N a_k^{-\tau}$$
$$\geqq \frac{1}{4}\exp\left(- \frac{(1+2\varepsilon)(2C)^{1/\alpha}(1-2\varepsilon)C_1}{2\sqrt{\pi}}\log_2 b_k\right) - 2N b_k^{-\tau}$$
$$\geqq \frac{1}{8} k^{-(1-\varepsilon^4)}$$

for every k sufficiently large. Hence

$$\sum_{k=1}^{\infty} P(A_k) = \infty \tag{2.33}$$

and (2.32) will be implied by

$$\liminf_{n\to\infty} \frac{\sum_{1 \leqq k \neq l \leqq n} P(A_k A_l)}{(\sum_{k=1}^{n} P(A_k))^2} \leqq 1 \tag{2.34}$$

by the general form of the Borel-Cantelli lemma (cf. [9]). We below verify that (2.34) is satisfied. It is easy to see that

$$\frac{a_{k+1} - b_k}{b_{k+1} - b_k} \sim 1 . \tag{2.35}$$

Applying Lemma 2.1, we get that for $k < l$

$$P(A_k A_l) \leqq P(A_k)P(A_l)$$
$$+ \sum_{i=0}^{b_k - a_k} \sum_{j=0}^{\hat{y}_{k,i}} \sum_{u=0}^{b_l - a_l} \sum_{v=0}^{\hat{y}_{l,u}} \frac{\bar{\tau}(i,j,u,v)}{\sqrt{1-\bar{\tau}(i,j,u,v)}} \exp\left(- \frac{\frac{1}{2}(\bar{y}_{k,i}^2 + \bar{y}_{l,u}^2)}{1+\bar{\tau}(i,j,u,v)}\right)$$
$$:= P(A_k)P(A_l) + C_{k,l} \tag{2.36}$$

where

$$\bar{\tau}(i,j,u,v) = |r(a_l + u + v\theta_{l,u} y_{l,u}^{-2/\alpha} - a_k - i - j\theta_{k,i} y_{k,i}^{-2/\alpha})|$$
$$\leqq r^*(a_l - b_k - 1)$$
$$\leqq r^*(\tfrac{1}{2}(b_l - b_{l-1}))$$
$$\leqq (b_l - b_{l-1})^{-\gamma} \leqq \min(1,\gamma)/4$$

by (2.35) and (1.6), for every k, l ($k < l$) sufficiently large. Therefore

$$C_{k,l} \leqq K \sum_{i=0}^{b_k - a_k} \sum_{u=0}^{b_l - a_l} \hat{y}_{k,i}\hat{y}_{l,u}(b_l - b_{l-1})^{-\gamma} \exp\left(- \frac{\log(a_k + i) + \log(a_l + u)}{1+\gamma/4}\right)$$
$$\leqq K b_k^{\gamma/4} \log^{9/\alpha} b_k \cdot b_l^{\gamma/4} \log^{9/\alpha} b_l \cdot (b_l - b_{l-1})^{-\gamma}$$
$$\leqq K \exp(-\gamma l^{\alpha/4}/8)$$

for some constant K not depending on k and l, and for every $k < l$ sufficiently large. Hence we have

$$\sum_{0 \leq k < l < \infty} C_{k,l} < \infty . \tag{2.37}$$

Now (2.34) follows from (2.36), (2.37) and (2.33). This proves (2.30) and so does (2.29).

Step 3 If $\alpha = 2$, then

$$\liminf_{t \to \infty} \frac{\log(\xi(t)/t)}{\log_2 t} \geqq -(1 + 2\varepsilon)^2 C_2 \quad \text{a.s.} \tag{2.38}$$

and

$$\liminf_{t \to \infty} \frac{\log(\xi(t)/t)}{\log_2 t} \leqq -(1 - 2\varepsilon) C_2 \quad \text{a.s.} \tag{2.39}$$

for every $0 < \varepsilon < \frac{1}{4}$, where $C_2 = 2\sqrt{\pi}/(H_2\sqrt{2C}) = 2\pi/\sqrt{2C}$.

Proof. Put

$$t_k = e^k, \quad s_k = t_k \exp(-(1 + 2\varepsilon)^2 C_2 \log_2 t_k), \qquad k = 1, 2, \ldots .$$

Proceeding the same way as that of the proof of (2.25), one can obtain that

$$\liminf_{k \to \infty} \frac{\log(\xi(t_k)/t_k)}{\log_2 t_k} \geqq -(1 + 2\varepsilon)^2 C_2 \quad \text{a.s.} \tag{2.40}$$

On the other hand, it is clear that

$$\liminf_{t \to \infty} \frac{\log(\xi(t)/t)}{\log_2 t} = \liminf_{k \to \infty} \frac{\log(\xi(t_k)/t_k)}{\log_2 t_k} \quad \text{a.s.} \tag{2.41}$$

since $\lim_{k \to \infty} \log(t_{k+1}/t_k)/\log_2 t_k = 0$. This proves (2.38), by (2.40) and (2.41).

Let

$$b_k = \exp(k^{1+\varepsilon^2}), \quad a_k = b_k \exp(-(1 - 2\varepsilon) C_2 \log_2 b_k), \qquad k = 1, 2, \ldots .$$

Noting that

$$\frac{a_{k+1} - b_k}{a_{k+1}} \sim 1 ,$$

along the same line of the proof of (2.30), we also have

$$\liminf_{k \to \infty} \frac{\log(\xi(b_k)/b_k)}{\log_2 b_k} \leqq -(1 - 2\varepsilon) C_2 \quad \text{a.s.}$$

This proves (2.39).

Now the proof of Theorem 1.1 is completed.

3 Infinite series of independent Ornstein-Uhlenbeck processes

In this section we consider a special stationary Gaussian process, infinite series of independent Ornstein-Uhlenbeck processes. A real valued stationary Gaussian process $\{X(t), -\infty < t < \infty\}$ is called an Ornstein-Uhlenbeck process with coefficients γ and λ $(\gamma \geqq 0, \lambda > 0)$ if $EX(t) = 0$ and $EX(s)X(t) = (\gamma/\lambda)\exp(-\lambda|t-s|)$. Let $Y(t) = (X_1(t), X_2(t), \ldots)$, where $\{X_i(t), -\infty < t < \infty\}$ are independent Ornstein-Uhlenbeck processes with coefficients γ_i and λ_i $(i = 1, 2, \ldots)$. The process $Y(\cdot)$ was first studied by Dawson [2] as the stationary solution of the infinite array of stochastic differential equations

$$\mathrm{d}X_i(t) = -\lambda_i X_i(t)\mathrm{d}t + \sqrt{2\gamma_i}\,\mathrm{d}W_i(t), \quad i = 1, 2, \ldots,$$

where $\{W_i(t), -\infty < t < \infty\}$ are independent standard Wiener processes. The path properties of $Y(\cdot)$ have been extensively investigated by various authors during the past two decades. We refer to Csáki et al. [1] and references therein.

We concern here with the infinite sums of $Y(\cdot)$. Assume

$$0 < \Gamma_0^2 = \sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i} < \infty . \tag{3.1}$$

Define

$$X(t) = \frac{1}{\Gamma_0} \sum_{i=1}^{\infty} X_i(t), \tag{3.2}$$

$$\sigma^2(t) = \frac{1}{\Gamma_0^2} \sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i}(1 - e^{-\lambda_i t}), \quad t \geqq 0 . \tag{3.3}$$

It is clear that $\{X(t), t \geqq 0\}$ is a stationary Gaussian process with $EX(t) = 0$, $EX^2(t) = 1$ and covariance function

$$r(t) = EX(t+s)X(s) = 1 - \sigma^2(t) \quad \text{for } s, t \geqq 0 . \tag{3.4}$$

Theorem 3.1 *Let $\{X(t), t \geqq 0\}$ be the infinite sums of independent Ornstein-Uhlenbeck processes, defined as in* (3.2). *Put $\xi(t) = \sup\{s : 0 \leqq s \leqq t, X(s) \geqq (2\log s)^{1/2}\}$, $t \geqq 0$. Let $\sigma^2(t)$ be as in* (3.3). *Assume that there exist $0 < \alpha \leqq 1$, $C > 0$ and $\delta > 0$ such that*

$$\sum_{i=1}^{\infty} \frac{\gamma_i}{\min(\lambda_i, \lambda_i^{1+\delta})} < \infty , \tag{3.5}$$

$$\lim_{t \downarrow 0} \frac{\sigma^2(t)}{t^\alpha} = C . \tag{3.6}$$

Then (1.8) *holds.*

Proof. It is easy to find that (1.7) is satisfied since $\sigma^2(t)$ is a positive non-decreasing function for $t > 0$. By Theorem 1.1, it suffices to verify that (1.6) is satisfied. Notice that

$$\begin{aligned} r(t) &= \frac{1}{\Gamma_0^2} \sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i} e^{-\lambda_i t} \\ &= \frac{1}{\Gamma_0^2} \left(\sum_{i:\lambda_i \geqq t^{-1/2}} + \sum_{i:\lambda_i < t^{-1/2}} \right) \frac{\gamma_i}{\lambda_i} e^{-\lambda_i t} \\ &\leqq \frac{1}{\Gamma_0^2} \left(\sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i} \exp(-t^{1/2}) + \sum_{i:\frac{1}{\lambda_i} > t^{1/2}} \frac{\gamma_i}{\lambda_i} \right) \\ &\leqq \exp(-t^{1/2}) + \frac{1}{\Gamma_0^2} \sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i^{1+\delta}} \cdot t^{-\delta/2} \\ &\leqq K t^{-\delta/2} \end{aligned}$$

for some constant K and for every $t \geqq 2$. This proves (1.6), as desired.

Corollary 3.1 *Let* $\{X(t), t \geqq 0\}$ *and* $\{\xi(t), t \geqq 0\}$ *be defined as in Theorem* 3.1. *Assume for some* $\delta > 0$

$$\sum_{i=1}^{\infty} \frac{\gamma_i}{\min(1, \lambda_i^{1+\delta})} < \infty . \tag{3.7}$$

Then

$$\liminf_{t \to \infty} \frac{\xi(t) - t}{t \log^{-1/2} t \log_2 t} = -\frac{3\sqrt{\pi}}{2\Gamma_1} \quad \text{a.s.} \tag{3.8}$$

where $\Gamma_1 = \Gamma_0^{-2} \sum_{i=1}^{\infty} \gamma_i$.

Proof. Clearly, (3.7) implies (3.5) and

$$\sum_{i=1}^{\infty} \gamma_i < \infty .$$

The latter yields

$$\lim_{t \downarrow 0} \frac{\sigma^2(t)}{t} = \Gamma_1 .$$

Now (3.8) follows from Theorem 3.1 with $\alpha = 1$, $C = \Gamma_1$ and $H_1 = 1$.

4 Fractional Wiener process

Let $\{Z(t), t \geqq 0\}$ be a fractional Wiener process of order α, i.e., a centred Gaussian process with stationary increments and variance $EZ^2(t) = t^\alpha$, where $0 < \alpha < 2$. Consider

$$\eta(t) = \sup\{s : 0 \leqq s \leqq t, Z(s) \geqq (2s^\alpha \log_2 s)^{1/2}\}, \quad t \geqq 0 . \tag{4.1}$$

By the upper class of increments for $Z(t)$ (cf. [4]), one has

$$\lim_{t \to \infty} \eta(t) = \infty \quad \text{a.s.}$$

and

$$\limsup_{t \to \infty} (\eta(t) - t) = 0 \quad \text{a.s.}$$

The following theorem presents the lower bound of $\eta(\cdot)$.

Theorem 4.1 *We have*

$$\liminf_{t \to \infty} \frac{(\log_2 t)^{(2-\alpha)/(2\alpha)} \cdot \log(\eta(t)/t)}{\log t \cdot \log_3 t} = -\frac{(2+\alpha)\sqrt{\pi}}{\alpha H_\alpha} \quad \text{a.s.} \tag{4.2}$$

Proof. Let

$$X(t) = Z(e^t)/e^{\alpha t/2}, \quad t \geqq 0 .$$

Then, $EX(t) = 0$, $EX^2(t) = 1$ and

$$EX(s)X(t) = \tfrac{1}{2}(e^{\alpha(t-s)/2} + e^{\alpha(s-t)/2} - |e^{(t-s)/2} - e^{(s-t)/2}|^{\alpha}) .$$

Hence, $\{X(t), t \geqq 0\}$ is a stationary Gaussian process with correlation function

$$r(t) = \tfrac{1}{2}(e^{\alpha t/2} + e^{-\alpha t/2} - (e^{t/2} - e^{-t/2})^{\alpha}) \quad \text{for } t \geqq 0 . \tag{4.3}$$

It is not difficult to find that

$$r(t) - 1 \sim \tfrac{1}{2}t^{\alpha} \quad \text{as} \quad t \downarrow 0 , \tag{4.4}$$

$$r(t) \sim e^{-\frac{\alpha}{2}t} + \alpha e^{-(1-\frac{\alpha}{2})t} \quad \text{as } t \to \infty \tag{4.5}$$

and

$$\sup_{t \geqq s} |r(t)| < 1 \quad \text{for every } s > 0 . \tag{4.6}$$

Therefore, by Theorem 1.1

$$\liminf_{t \to \infty} \frac{(\log t)^{(2-\alpha)/(2\alpha)}(\xi(t) - t)}{t \cdot \log_2 t} = -\frac{(2+\alpha)\sqrt{\pi}}{\alpha H_{\alpha}} \quad \text{a.s.} \tag{4.7}$$

where $\xi(t)$ is defined as in (1.4). Let

$$\bar{\eta}(t) = \sup\{s : 1 \leqq s \leqq t : Z(s) \geqq (2s^{\alpha}\log_2 s)^{1/2}\} \quad \text{for } s \geqq 1 .$$

Then

$$\xi(t) = \log\bar{\eta}(e^t) \quad \text{a.s. for every } t \geqq 0 . \tag{4.8}$$

Consequently, we have, by (4.7)

$$\liminf_{t \to \infty} \frac{(\log_2 t)^{(2-\alpha)/(2\alpha)} \cdot \log(\bar{\eta}(t)/t)}{\log t \cdot \log_3 t} = -\frac{(2+\alpha)\sqrt{\pi}}{\alpha H_{\alpha}} \quad \text{a.s.} \tag{4.9}$$

This proves (4.2) by (4.9) and the fact that $|\bar{\eta}(t) - \eta(t)| \leqq 1$ for every $t \geqq 1$.

Acknowledgements. The author thanks an associate editor and the editor for their helpful comments.

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