Probab. Theory Relat. Fields 94, 83–90 (1992)

Probability Theory and Related Fields

Limit distributions of U-statistics resampled by symmetric stable laws*

Jerzy Szulga
Division of Mathematics, ACA, 120 Mathematics Annex, Auburn University, Auburn, AL 36849-5307, USA

Received June 15, 1990; in revised form January 1, 1992

Summary. If (Y_i) and (V_i) are independent random sequences such that Y_i are i.i.d. random variables belonging to the normal domain of attraction of a symmetric α-stable law, $0 < \alpha < 2$, and V_i are i.i.d. random variables, then the limit distributions of U-statistics $n^{-1/\alpha} \sum_{1 \leqq i_1, \ldots, i_d \leqq n} Y_{i_1} \ldots Y_{i_d} f(V_{i_1}, \ldots, V_{i_d})$, coincide with the probability laws of multiple stochastic integrals $X^d f = \int \ldots \int f(t_1, \ldots, t_d)\, dX(t_1) \ldots dX(t_d)$, with respect to a symmetric α-stable process $X(t)$.

Mathematics Subject Classification (1980): 60 H 05, 10 C 10, 62 G 99

1 Introduction

There is a vast literature devoted to study the limit behavior of distributions of U-statistics, which are suitably normalized multiple sums

$$\sum_{i_1 \leqq \ldots i_d \leqq n} f(X_{i_1}, \ldots, X_{i_d}), \tag{1}$$

subject to additional restrictions concerning independence and existence of moments of the underlying random variables (usually assumed to be independent), integrability and symmetry of the function f, etc., which allows one to use orthogonality techniques, conditional expectations, etc. A resampled U-statistic, by means of another random sequence (Y_i), independent of (X_i), is a particular case of the statistic (1). That is, up to a suitable normalization, it is a random quantity of the type

$$\sum_{i_1 \leqq, \ldots, i_d \leqq n} Y_{i_1} \ldots Y_{i_d} f(X_{i_1}, \ldots, X_{i_d}). \tag{2}$$

When the resampling variables Y_i have finite variances, the limit distributions of such statistics can be described by means of multiple Wiener-type stochastic

* The research was originated during author's visit at ORIE, Cornell University

integrals (more specifically, by multiple Kiefer integrals (cf., e.g., [1]), or in the terms of, so called, von Mises' statistics (see, e.g. [3]). A study of U-statistics "resampled" by stable laws was initiated by Dehling et al. [2]. It was shown that, under quite strong moment conditions allowing the use of martingale methods, multiple stable integrals are exactly the processes whose distributions appear in the limit of these statistics.

In [9] we weakened some of these conditions, replacing the requirement of the finitness of L^r-th norm of f (where $r > \alpha$, the index of stability) [2] by that of the finitness of a certain Orlicz norm. The new condition was sufficient but not necessary for the existence of a multiple stable integral. We also have conjectured that exactly the existence of the underlying multiple stable integral is a necessary and sufficient condition for the aforementioned weak convergence of resampled U-statistics.

This paper contains a proof of this conjecture, by reducing the problem to the investigation of properties of a standard Poisson process, in the spirit of [5]. This is also our general reference concerning details, a historical background, and the definition of a multiple stochastic integral.

It is quite elementary to obtain the desired limit theorem for a class of "simple" functions f. The nontrivial part of the limit theorem involves a uniform approximation, by means of a suitable metric, of functions from a larger class by "simple" functions. When second moments are available, one can apply the well established L^2-techniques. The use of L^r- or Orlicz norms had served exactly the same purpose. In this paper, in the absence of moments, we utilize an L^0-quasinorm which metrizes the convergence in probability:

$$\|X\|_0 = \mathsf{E}\psi(|X|) ,$$

where $\psi(x) = 1 - \exp\{-x\}$. This quasinorm is especially useful, when one deals with Poisson processes. It is worth to emphasize that the sought-for uniform approximation is based on a, so called, decoupling principle in L^0, for Poisson processes.

Let us point out that there is no known explicit and efficient quantitative description of a multiple stable integral, even in the symmetric case (except of the one-dimensional situation). The known descriptions are either of a qualitative nature (e.g., a recursive integrability criterion [5]), or are hardly applicable in the approximation (cf. [8], where a double stable integral is characterized by the finitness of a certain functional).

We shall use the symbol $X^d f$ for the multiple integral of f with respect to a process X, and write $Xf = X^1 f$. We note that the mentioned definition is the most general in the class of pure (symmetric or positive) jump Lévy processes, and contains all possible (reasonable) constructions [5]. We will identify $X^d(\mathbf{1}_A)$ and $X^d(A)$. Symbols $\overset{\mathscr{D}}{=}$ and $\overset{\mathscr{D}}{\to}$ denote the equality and convergence in distribution, respectively.

The multiparameter functions $f(t_1, \ldots, t_d)$ appearing in this paper are assumed to be real, symmetric, and vanishing on diagonals, i.e., whenever two or more of their arguments are equal.

We begin with the formulation of the main result. The next section provides the needed tools. However, Theorem 2.1 in this section may be of an intrinsic interest, and other limit theorems with distributions of Lévy multiple integrals in the limit may be derived from it. The last section contains the proof of the main result.

Let $X(\cdot)$ be an α-stable, $0<\alpha<2$, symmetric Lévy random measure on a separable finite measure space $(T, \mathscr{T}, \mu)$ with a Lévy measure $\nu(dx) = c_\alpha^{-1} dx/x^{-1-\alpha}$, where μ is a finite control measure and $c_\alpha = \int_{\mathbb{R}} (1-\cos x)\, dx/x^{1+\alpha}$. The nature of the problem, as we will see, is such that, without loss of generality, we may assume that $T = [0, 1]$ and μ is the Lebesgue measure. That is,

$$\begin{aligned} \mathsf{E} \exp\{uX(A)\} &= \exp\{-\mu(A)|u|^\alpha\} \\ &= \exp\left\{ -c_\alpha^{-1} \int_A \int_{\mathbb{R}\setminus\{0\}} (1-\cos ux) \frac{dx}{x^{1+\alpha}} \mu(ds) \right\}, \quad u \in \mathbb{R} . \end{aligned}$$

There follows the formulation of the main result of this paper.

Theorem 1 *Let (Y_i) and (V_i) be two independent sequences of* i.i.d. *random variables, Y_1 belong to the normal domain of attraction of a symmetric α-stable law, and V_1 be uniformly distributed on* $[0, 1]$. *The statistics*

$$\mathbb{D}_n = (c_\alpha \alpha)^{-d/\alpha} n^{-d/\alpha} \sum_{1 \leqq i_1, \ldots, i_d \leqq n} Y_{i_1} \ldots Y_{i_d} f(V_{i_1}, \ldots, V_{i_d}) \tag{3}$$

converge weakly to $X^d f$ if and only if the latter multiple integral exists. The analogous statement holds in the decoupled case, i.e., when X^d is replaced by the iterated integral $X_1 \ldots X_d$, and the quantities $Y_{i_1} \ldots Y_{i_d} f(V_{i_1}, \ldots, V_{i_d})$ in (3) *are replaced by $Y^1_{i_1} \ldots Y^d_{i_d} f(V^1_{i_1}, \ldots, V^d_{i_d})$, with all components being independent copies of each other.*

2 Poissonization

Let $(S, \mathscr{A}, \lambda)$ be a σ-finite separable atomless measure space. Let ζ be an abstract Poisson process with the intensity measure λ and atoms $\Gamma_i = \Gamma^{(\zeta)}$. Put

$$\zeta^d(f) = \sum_{1 \leqq i_1, \ldots, i_d < \infty} f(\Gamma_{i_1}, \ldots, \Gamma_{i_d}), \quad f: S^d \to \mathbb{R}_+ .$$

If $\varepsilon = (\varepsilon_i)$ is a Rademacher sequence independent of ζ, then a symmetrized version $(\zeta \circ \varepsilon)^d f$ is defined by "resampling" $\zeta^d f$

$$(\zeta \circ \varepsilon)^d f \overset{\mathrm{df}}{=} \sum_{i_1, \ldots, i_d} \varepsilon_{i_1} \ldots \varepsilon_{i_d} f(\Gamma_{i_1}, \ldots, \Gamma_{i_d}), \quad f: S^d \to \mathbb{R}_+ .$$

Consider a set $S_t \in \mathscr{A}$ such that $\lambda(S_t) = t, 0 \leqq t < \infty$, and an S-valued random variable V_t, uniformly distributed on S_t. Let $(V_{t,i})$ be a sequence of independent copies of V_t. Define a stochastic process, and its symmetrized version, as follows, for $f: S^d \to \mathbb{R}_+$, $1 \leqq n \leqq \infty$,

$$\begin{aligned} Q^d(t, n; f) &= \sum_{i_1, \ldots, i_d \leqq n} f(V_{t,i_1}, \ldots, V_{t,i_d}) \\ (Q \circ \varepsilon)^d(t, n; f) &= \sum_{i_1, \ldots, i_d \leqq n} \varepsilon_{i_1} \ldots \varepsilon_{i_d} f(V_{t,i_1}, \ldots, V_{t,i_d}) . \end{aligned}$$

It is convenient to refer to the above expressions as to multiple stochastic integrals of f (formally, they are pathwise multiple integrals with respect to suitable point processes).

The corresponding iterated integrals $\zeta_1 \ldots \zeta_d$, $(\zeta_1 \circ \varepsilon_1) \ldots (\zeta_d \circ \varepsilon_d)$, $Q_{1,t} \ldots Q_{d,t}$, and $Q_1 \circ \varepsilon_1, \ldots, Q_d \circ \varepsilon_d$ (called *decoupled* multiple integrals in the literature), where the components are independent copies of ζ, Q_t, and ε, respectively, are introduced analogously.

The proof of virtually any statement formulated in such an abstract language, which allows more flexibility in applications, can be reduced to a much simpler situation. In fact, one may consider a measure isomorphism $\Phi: \mathbb{R}_+ \to S$, transforming the Lebesgue measure to λ. Then ζ can be viewed as the image of the standard (unit rate) Poisson process ξ on the real line, which, in terms of random measures, means that $\zeta \overset{\mathscr{D}}{=} \xi \circ \Phi^{-1}$, $\Gamma^{(\zeta)} = \Phi \circ \Gamma^{(\xi)}$, where the latter gammas denote the arrival times of ξ, and $\zeta^d f \overset{\mathscr{D}}{=} \xi(f \circ \Phi)$. Similarly, $V_t \overset{\mathscr{D}}{=} \Phi \circ \pi \circ tU$, where U is a random variable, uniformly distributed on $[0, 1]$, and $\pi: \mathbb{R}_+ \to \mathbb{R}_t$ is a suitable measure isomorphism. Hence, on the real line

$$Q^d(t, n, f) \overset{\mathscr{D}}{=} \sum_{i_1, \ldots, i_d \leqq n} f(tU_{i_1}, \ldots, tU_{i_d}) . \tag{4}$$

In the sequel we will write simply $Q_n^d(f) \overset{\mathrm{df}}{=} Q^d(n, n, f)$, and $(Q \circ \varepsilon)_n^d(f) \overset{\mathrm{df}}{=} (Q \circ \varepsilon)^d(n, n, f)$.

Theorem 2 (i) *The distributions of $Q_n^d(f)$ converge weakly if and only if $\zeta^d f < \infty$. In this case*

$$(Q_n^d(f)) \overset{\mathscr{D}}{\to} (\zeta^d f) ,$$

as processes parametrized by the set $\{f: \zeta^d f < \infty\}$.

(ii) *The distributions of $(Q \circ \varepsilon)_n^d(f)$ converge weakly if and only if $(\zeta \circ \varepsilon)^d f$ exists. In this case*

$$((Q \circ \varepsilon)_n^d(f)) \overset{\mathscr{D}}{\to} ((\zeta \circ \varepsilon)^d f) ,$$

as processes whose parameter space is the set $\{f: \zeta^d f^2 < \infty\}$.

The similar statements hold for the decoupled integrals $Q_{n1} \ldots Q_{nd} f$ and $\zeta_1 \ldots \zeta_d f$, and their symmetrized analogs $(Q_{1n} \circ \varepsilon_1) \ldots (Q_{dn} \circ \varepsilon_d) f$ and $(\zeta_1 \circ \varepsilon_1) \ldots (\zeta_d \circ \varepsilon_d) f$.

We will need two auxiliary result concerning the tightness and symmetrization of the sequence $(Q_n^d(f))$.

Lemma 1 *Let $\mathbb{Q}_n$ denote one of two processes, Q_n^d or $Q_{n1} \ldots Q_{nd}$. Then there is a constant $C > 0$, depending only on d, such that*

$$\| \mathbb{Q}_n f \|_0 \leqq C \| \xi^d f \|_0 . \tag{5}$$

Proof. If G_t and G_t' are nondecreasing cadlag (right continuous with left limits) processes adapted to a filtration $(\mathscr{F}_t)$, and

$$\mathsf{P}[G_t \in \cdot \mid \mathscr{F}_{t-}] = \mathsf{P}[G_t' \in \cdot \mid \mathscr{F}_{t-}] \quad \text{a.s.,} \quad t > 0 , \tag{6}$$

then, for any non-negative $(\mathscr{F}_t)$-predictable process H_t on $\mathbb{R}_+$ there holds the inequality

$$\mathsf{E}(1 \wedge \int H \, \mathrm{d}G) \leqq 4 \mathsf{E}(1 \wedge \int H \, \mathrm{d}G') . \tag{7}$$

This property follows by a verbatim argument used in the proof of [5, Lemma 3.5] (the statement of the cited lemma is confined only to Poisson processes but this is not an essential restriction). Recall the reduced formula (4). Empirical processes based on identically distributed sequences of i.i.d. random variables have the property (6). That is, denote $G_n(s) = \sum_{i=1}^{n} \mathbf{1}_{\{U^{(i)} \leqq s\}}$, where $U^{(i)}$ are order statistics of the sequence U_i, and let $G_{j,n}$, $j = 1, \ldots, d$ be independent copies of G_n. Then

$$Q^d(t,n,f) = \int_{[0,1]^d} \ldots \int f(ts_1, \ldots, ts_d)\, \mathrm{d}G_n(\mathrm{d}s_1) \ldots \mathrm{d}G_n(\mathrm{d}s_d)\,,$$

and a similar integral formula holds for $Q_1 \ldots Q_d$. Notice that we may write $Q^d = Q^{d-1}Q$, where the inner operator acts on functions of $d-1$ variables. Then we apply (7) for $H = Q^{d-1}$, G as is, and $G' = G_d$, and use the inequalities $\psi(x) \leqq 1 \wedge x \leqq e/(e-1)\psi(x)$, and proceed by induction. Whence we infer that, for some constant c (of order $4^d d!$, cf. the proof of Lemma 3.6 in [5] whose line we are repeating almost literally), there holds a, so called, *decoupling principle*

$$c^{-1}\mathsf{E}\psi(Q_n^d f) \leqq \mathsf{E}\psi(Q_{n1} \ldots Q_{nd} f) \leqq c\mathsf{E}\psi(Q_n^d f)\,. \tag{8}$$

Hence, it is enough to prove (5) in the decoupled case, i.e., for $Q_{n1} \ldots Q_{nd}$. To this end, observe that $1 - (1 - x/n)^n \leqq e/(e-1)\psi(x)$, for $0 \leqq x \leqq n$. Then we check the inequality in the one-dimensional case:

$$\begin{aligned}\mathsf{E}\psi(Q_n(f)) &= 1 - \left(1 - \int_0^n (1 - \exp\{-f(t)\})\frac{\mathrm{d}x}{n}\right)^n \\ &\leqq \frac{e}{e-1}\psi\left(\int_0^n (1 - \exp\{-f(t)\})\right) \\ &\leqq \frac{e}{e-1}\,\mathsf{E}\psi(\xi f)\,,\end{aligned}$$

and iterate it, using the decoupling principles for empirical distributions (8) and Poisson processes, since they satisfy (6), too (cf. Lemma 3.6 in [5]. □

Lemma 2 *Let* $W\colon \mathbb{N}^d \to L_0$ *be a symmetric discrete time random field* (*i.e. invariant under permutations of its arguments, vanishing on diagonal hyperplanes. Let* $(W \circ \varepsilon)$ *denote its symmetrized version, i.e.* $(W \circ \varepsilon)(i_1, \ldots, i_d) = \varepsilon_{i_1} \ldots \varepsilon_{i_d} \cdot W(i_1, \ldots, i_d)$. *Denote* $S(W) = \sum_{\underline{i} \in \mathbb{N}^d} W(\underline{i})$. *Then there is a constant* $C = C(d)$ *such that*

$$\| S(W \circ \varepsilon) \|_0 \leqq \| S^{1/2}(W^2) \|_0 \leqq C \| S(W \circ \varepsilon) \|_0\,. \tag{9}$$

In particular,

(i) $S(W \circ \varepsilon)$ *converges* a.s (*or in probability*) iff $S(W^2) < \infty$,

(ii) $S(W \circ \varepsilon) \xrightarrow{\mathsf{P}} 0$ iff $S(W_n^2) \xrightarrow{\mathsf{P}} 0$,

(ii)′ $(S(W_n \circ \varepsilon))$ *is tight* iff $(S(W_n^2))$ *is tight.*

Proof. The left inequality in (9) follows by the concavity of ψ and Fubini's theorem. Also from Fubini's theorem, using properties of Rademacher multilinear forms (see, e.g., [6]) and conditioning, we infer that

$$\mathsf{P}(S^{1/2}(W^2) > t) \leqq \frac{3^{2d}}{(1-r)^2}\mathsf{P}(|S((W \circ \varepsilon))| > rt), \quad 0 < r < 1, \quad t > 0$$

(see Lemma 4.3 in [5]).

Since $\mathsf{E}\psi(X) = \int_0^\infty \exp\{-t\}\,\mathsf{P}(X > t)\,\mathrm{d}t$ for a $X \geqq 0$, and since $1 - \exp\{-x/r\} \leqq 1/r\psi(x)$ for $r < 1$, hence the right inequality (9) holds with a constant $C(r, d) \leqq 3^{2d}(r(1-r)^2)^{-1}$. The minimum of the latter expression is attained for $r = 2 - \sqrt{3}$ for which $(r(1-r)^2)^{-1} \leqq 7$, hence we may choose $C = 7 \cdot 9^d$.

The statements (i), (ii)′ and (ii) result immediately from (9). □

Proof of Theorem 2 As noted before, we can assume that $S = \mathbb{R}_+$, λ is the Lebesgue measure, and write ξ instead of ζ. For $t > 0$, write $f_t = f\mathbf{1}_{[0,t]^d}$, and let ξ_t be the Poisson process whose intensity measure has been restricted to $[0, t]$. It is enough to prove the theorem for the identical components. The decoupled case is similar.

(i) Assume that $\xi^d f < \infty$. By the monotone convergence theorem, for any $\varepsilon > 0$, we can find a number $t > 0$ such that $\|\xi^d(f - f_t)\|_0 < \varepsilon$. Hence, in view of the inequality (5), it suffices to prove the weak convergence for the function f_t.

Observe that

$$\xi_t^d f \stackrel{\mathcal{D}}{=} \xi^d f_t \stackrel{\mathcal{D}}{=} \sum_{1 \leqq i_1, \ldots, i_d \leqq \xi[0,t]} f_t(tU_{i_1}, \ldots, tU_{i_d}), \tag{10}$$

where ξ and (U_i) are mutually independent. Whence there follows immediately the formula for the Laplace transform

$$\mathsf{E}\exp\{-\xi_t^d f\}$$
$$= e^{-t} \sum_{k=0}^{\infty} \frac{1}{k!} \int \cdots \int_{[0,t]^k} \exp\left\{-\sum_{1 \leqq i_1, \ldots, i_d \leqq k} f(s_{i_1}, \ldots, s_{i_d})\right\} \mathrm{d}s_1 \ldots \mathrm{d}s_k . \tag{11}$$

Indeed, since f_t vanishes outside $[0, t]$, then, putting $N = \xi[0, t]$, the arrival times of ξ have the same distribution as the order statistics of a sequence $(tU_i : i \leqq N)$, where U_i are i.i.d. random variables uniformly distributed on [0, 1]'. Then we may write $\xi f_t \stackrel{\mathcal{D}}{=} \sum_{1 \leqq i \leqq N} f(tU_i)$. By the same token there follows formula (10). Let Q_n be the empirical measure process based on $U_1, \ldots, U_n$. Since

$$Q_n^d f = \int \cdots \int f \mathrm{d}Q_n \ldots \mathrm{d}Q_n \sum_{1 \leqq i_1, \ldots, i_d \leqq n} f(nU_{i_1}, \ldots, nU_{i_d}), \quad f : \mathbb{R}_+^d \to \mathbb{R}_+, \tag{12}$$

then

$$Q_n^d f_t \stackrel{\mathcal{D}}{=} \sum_{1 \leqq i_1, \ldots, i_d \leqq S_{t,n}} f_t(tU_{i_1}, \ldots, tU_{i_d}), \quad n \geqq t , \tag{13}$$

where $S_{n,t}$ is a binomial $b(n, t/n)$ random variable, independent of (U_i). Indeed, denoting $A_1 = [0, t]$ and $A_0 = (t, n]$ we have

$$\mathbf{1}_{[0,n]^n} = \sum_{\delta(i_1), \ldots, \delta(i_n) \in \{0,1\}} \mathbf{1}_{A_{\delta(i_1)} \times \cdots \times A_{\delta(i_n)}} .$$

Hence (13) follows:

$$\mathsf{E}\exp\{-Q_n^d f\} = n^{-n} \sum_{k=0}^{n} \binom{n}{k} (n-t)^{n-k} \int \cdots \int_{[0,t]^k} \exp\left\{-\sum_{1 \leqq i_1, \ldots, i_d \leqq k} f(t_{i_1}, \ldots, t_{i_d})\right\}.$$

By (10) and (13), the distributions of $Q_n^d f$ converge weakly to the distribution of $\xi^d f_t$.

Conversely, suppose that $\xi^d f = \infty$. Then $\sup_k \xi^d f_k = \infty$, where $f_k = f\mathbf{1}_{[0,k]^d}$. By the first part of the proof, for a fixed k, $\lim_n \mathsf{E}\psi(Q_n^d f_k) = \mathsf{E}\psi(\xi^d f_k)$. Therefore

$$1 \geqq \sup_n \mathsf{E}\psi(Q_n^d f) = \sup_n \sup_k \mathsf{E}\psi(X_n^d f_k) \geqq 1 .$$

In particular,

$$\inf_n \mathsf{E}\exp\{-Q_n^d f\} = 0 ,$$

hence, for every $c > 0$,

$$e^{-c} \inf_n \mathsf{P}(Q_n^d f \leqq c) = 0 ,$$

i.e., the sequence $(Q_n^d f)$ is not tight. □

(ii) By conditioning on ε, and using the same argument as for (10) and (13), we have

$$(\zeta \circ \varepsilon)_t^d f \stackrel{\mathscr{D}}{=} (\zeta \circ \varepsilon)^d f_t \stackrel{\mathscr{D}}{=} \sum_{1 \leqq i_1, \ldots, i_d \leqq \xi_t} \varepsilon_{i_1}, \ldots, \varepsilon_{i_d} f_t(V_{ti_1}, \ldots, V_{ti_d}) ,$$

and, for $n \geqq t$,

$$(Q \circ \varepsilon)_n(f_t) \stackrel{\mathscr{D}}{=} \sum_{1 \leqq i_1, \ldots, i_d \leqq S_{n,t}} \varepsilon_{i_1}, \ldots, \varepsilon_{i_d} f_t(V_{ti_1}, \ldots, V_{ti_d}) .$$

Whence, and from Lemma 2 combined with Lemma 1, which guarantee the tightness of the sequence $(Q \circ \varepsilon)_n^d(f - f_t))$, there follows the "if" part of (ii). Conversely, since the symmetrized integral $(\zeta \circ \varepsilon)^d f$ exists if and only if $\zeta^d f^2 < \infty$, then Lemma 2 yields the "only if" part of statement (ii) by a argument similar to that used in (i). □

3 Proof of the main result

The "only if" part of the theorem has been proved in [9] (it will follow immediately from Theorem 2). Thus, we will focus only on the proof of the "if" part, and for the identical components only. The decoupled case is similar.

Assume that a multiple stable integral $X^d f$ exists. First, by interpreting assertion (ii) of Theorem 2, we will show that the distributions of statistics

$$\mathbb{U}_n = (c_\alpha \alpha)^{-d/\alpha} n^{-d/\alpha} \sum_{1 \leqq i_1, \ldots, i_d \leqq n} \varepsilon_{i_1} \ldots \varepsilon_{i_d} \cdot U_{i_1}^{-1/\alpha} \ldots U_{i_d}^{-1/\alpha} f(W_{i_1}, \ldots, W_{i_d}) \tag{14}$$

converge weakly to $X^d f$, where (U_i) and (W_i) are two independent sequences of i.i.d. [0, 1]-uniform random variables. Note that $\varepsilon U^{-1/\alpha}$ has a, so called, Pareto distribution, which belongs to the normal domain of attraction of the symmetric α-stable law. Next, we will show that the desired weak convergence holds for any multiplier from the normal domain of attraction of the symmetric α-stable law.

It is well known that there is a Poisson process ζ on $(\mathbb{R}\backslash\{0\} \times [0, 1]$ and a Rademacher sequence ε, constructed from jumps of the process X_t, such that the following representation of a multiple integral holds

$$X^d f = (\zeta \circ \varepsilon)^d (L_d f) \quad \text{a.s.,} \qquad f : [0, 1]^d \to \mathbb{R} , \tag{15}$$

where $L_d f(x, t) = x_1 \ldots x_d \cdot f(t_1, \ldots, t_d)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d \backslash \{0\}$ (cf. [5]).

Let $S_n \nearrow \mathbb{R}\backslash\{0\}$ be such that $\nu(S_n) = n$, and let Z_n be random variables such that $\mathsf{P}(Z_n \in \cdot) = \nu(S_n \cap \cdot)/\nu(S_n)$. Let W be a random variable, uniformly distributed on [0, 1], and independent of (Z_n). Now, the pair (Z_n, W) plays the role of a uniformly distributed random variable V_n from Theorem 2.

In particular, we may choose $Z_n = (c_\alpha \alpha n U)^{-1/\alpha}$, where U is a random variable uniformly distributed on $[0, 1]$, and $S_n = [(\alpha c_\alpha n)^{-1/\alpha}, \infty)$ (so $\nu(S_n) = n$). Note that, for $t > (\alpha c_\alpha n)^{-1/\alpha}$,

$$\mathsf{P}(Z_n > t) = \frac{1}{c_\alpha n} \int_t \frac{dx}{x^{1+\alpha}} = \frac{1}{n\alpha c_\alpha t^\alpha}. \tag{16}$$

Thus, statistics (14) weakly converge to $X^d f$.

If Y is an arbitrary random variable whose distribution belongs to the domain of normal attraction of a symmetric α-stable law, then, obviously, the statistics $\mathbb{D}_n$ in (14) converge weakly to $X^d f$ if f is a simple function. Further,

$$\mathsf{P}(|Zn^{-1/\alpha}| > t) \leqq c\mathsf{P}(|\varepsilon Z_n| > t), \quad t > 0,$$

(Feller [4]) for some positive constant c. By combining (8) and Lemma 2, we infer that there are constants c' and c'' such that, for any X^d-integrable function f, one has

$$\|\mathbb{D}_n f\|_0 \leqq c' \|\mathbb{U}_n f\|_0 \leqq c'' \|X^d f\|_0 .$$

Since the subspace of multiple integrals X^d of simple functions is dense, by means of convergence in probability, in the space of all integrals (Theorem 6.2 in [5]), the proof is complete. □

References

1. Dehling, H., Denker, M., Philipp, W.: Invariance principles for sums of Banach space valued random elements and empirical processes. z. Wahrscheinlichkeitstheor. Verw. Geb. **62**, 509–552 (1984)
2. Dehling, D., Denker, M., Woyczyński, W.: Resampling U-statistics using p-stable laws. J. Multivariate Anal. **34**, 1–13 (1990)
3. Filippova, A.A.: Mises' theorem on asymptotic behavior of functionals of empirical distribution functions and its statistical applications. Theory Probab. Appl. 7, 24–57 (1962)
4. Feller, W. An introduction to probability theory and its applications, vol. 2, 2nd edn. New York: Wiley 1971
5. Kallenberg, O., Szulga, J.: Multiple integration with respect to Poisson and Lévy processes. Probab. Theory Relat. Fields **83**, 101–134 (1989)
6. Krakowiak, W., Szulga, J.: Multilinear random forms. Ann. Probab. **14**, 955–973 (1986)
7. Krakowiak, W., Szulga, J.: A multiple stochastic integral with respect to a strictly p-stable random measure. Ann. Probab. **16**, 764–777 (1988)
8. Rosiński, J., Woyczyński, W. A.: On Itô stochastic integration with respect to p-stable motion: Inner clock, integrability of sample paths, double and multiple integrals. Ann. Probab. **14**, 271–286 (1986)
9. Szulga, J.: On multiple stable integrals appearing as weak limits. In: Cambanis, S. et al. (eds.) Progress in probability, stable processes and related topics, pp. 305–316. Boston Basel Stuttgart: Birkhäuser 1991