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Probability Theory and Related Fields

On polynomial variance functions

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Summary. Let $\mathscr{F}$ be a natural exponential family on $\mathbb{R}$ and (V, Ω) be its variance function. Here, Ω is the mean domain of $\mathscr{F}$ and V, defined on Ω, is the variance of $\mathscr{F}$. A problem of increasing interest in the literature is the following: Given an open interval $\Omega \subset \mathbb{R}$ and a function V defined on Ω, is the pair (V, Ω) a variance function of some natural exponential family? Here, we consider the case where V is a polynomial. We develop a complex-analytic approach to this problem and provide necessary conditions for (V, Ω) to be such a variance function. These conditions are also sufficient for the class of third degree polynomials and certain subclasses of polynomials of higher degree.

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1 Introduction

Let $\mathscr{F}$ be a natural exponential family on $\mathbb{R}$ and (V, Ω) be its variance function. Here, Ω is the mean domain of $\mathscr{F}$ and V is the variance of $\mathscr{F}$ expressed in terms of the mean $\mu \in \Omega$. The pair (V, Ω) characterizes $\mathscr{F}$ within the class of natural exponential families. The problem of determining when such a pair constitutes the variance function of a natural exponential family has been studied by, amongst others, Morris [9], Mora [8], Bar-Lev and Enis [4], Letac and Mora [7], Jørgensen [5, 6], Bar-Lev and Bshouty [2], and Bar-Lev et al. [3]. Some statistical applications of such a determination, in the context of exponential dispersion models and generalized linear models, can be found in Jørgensen [6].

In this paper, we consider the situation where V is a polynomial and provide necessary conditions for (V, Ω) to be a variance function. One of the reasons for focusing on such variance functions is that, in the study of generalized linear models, a variance-mean relation (V, Ω) can be used to construct a statistical model for a data set, and an empirical variance-mean relation of a data set in the form of a polynomial of some degree can always be established. It should be noted that most of the natural exponential families with polynomial variance functions correspond to statistical models that apparently have not been considered before. Such natural exponential families, although having simple variance-mean relations, do not have tractable Laplace transforms (with the exception of quadratic variance functions or variance functions of the form $V(\mu)=\alpha\mu^n$, $\mu\in\mathbb{R}$). This fact also motivates the study of polynomial variance functions.

Let p_n denote a polynomial of degree n. Morris [9] identified all variance functions where $V=p_n$, $n\leqq 2$. Mora [8] identified all variance functions where $V=p_3$. (Details can be found in Letac and Mora [7].) Bar-Lev and Bshouty [2] showed that, except for the variance function of the binomial family, where $V(\mu)=j^{-1}\mu(j-\mu)$, $\Omega=(0,j)$, $j\in\mathbb{N}$, no polynomial variance function with bounded mean domain exists. It thus follows that the mean domain of any other polynomial variance function is either $\mathbb{R}$ or a semi-infinite interval. In this paper, we shall consider both cases. In the case where the mean domain is a semi-infinite interval, by affinity, one can assume that $\Omega=\mathbb{R}^+$. We so assume in what follows. Note that if (V,Ω) is a variance function and if $V=p_{2k+1}$, then Ω must be $\mathbb{R}^+$, since Ω must be the largest open interval on which V is positive real-analytic (see Letac and Mora [7]). Bar-Lev [1] showed that polynomials on $\mathbb{R}^+$ having nonnegative coefficients are variance functions of infinitely divisible natural exponential families.

It would appear that at least part of the difficulty in identifying variance functions amongst polynomials is that real-analytic techniques may not be sufficiently powerful for this general problem. Motivated by this consideration, we here develop a complex-analytic technique which permits us to obtain necessary conditions for polynomial variance functions on $\mathbb{R}^+$ and $\mathbb{R}$. These conditions are particularly powerful for polynomials of low degree.

In Sect. 2, we review some basic analytic properties of variance functions which are needed for the subsequent developments. In Sect. 3, we describe the idea underlying the technique used to obtain the main results. In Sect. 4, we apply this idea to the classes of polynomials of degree 3 and 4, and to subclasses of polynomials of degree 5 and 7. It will be shown that, in the case of third degree polynomials, the necessary conditions obtained here will imply that all coefficients must be nonnegative. This, together with the result of Bar-Lev [1], provides a characterization of those third degree polynomials which are variance functions. Although Mora [8] (see also Letac and Mora [7]) obtained such a characterization by a different approach, we, nevertheless, consider this case here to demonstrate the utility of our approach. Subclasses of polynomials of degree 5 and 7 are also characterized.

2 Preliminary notions and basic properties of variance functions

We first recall some definitions and properties of natural exponential families and their variance functions.

Let ν be a positive Radon measure on $\mathbb{R}$, which is not concentrated on one point. The Laplace transform and effective domain of ν are given, respectively, by $T(\theta)=\int_{-\infty}^{\infty}\exp(\theta x)\,\nu(dx)$ and $D=\{\theta\in\mathbb{R}\colon\ T(\theta)<\infty\}$. Let $\Theta=\operatorname{int}D$ and assume that Θ is nonempty. For $\theta\in\Theta$, define

$$F_\theta(dx)=[T(\theta)]^{-1}\exp(\theta x)\,\nu(dx).$$

The family of probability distributions $\mathscr{F}=\{F_\theta\colon \theta\in\Theta\}$ is called a natural exponential family generated by ν. The mean function of $\mathscr{F}$ is the mapping defined on Θ by

$$\mu(\theta)=\int_{-\infty}^{\infty} x\,F_\theta(d\,x).$$

The mean domain of $\mathscr{F}$ is $\Omega=\mu(\Theta)$, and μ is a one-to-one continuously differentiable mapping so that Ω is an open interval. Denote by $\theta=\theta(\mu)$ the inverse function of μ, and let V on Ω be defined by

$$V(\mu)=\int_{-\infty}^{\infty} (x-\mu)^2\,F_{\theta(\mu)}(d\,x).$$

The pair (V, Ω) is called the variance function of $\mathscr{F}$. Without loss of generality, we henceforth assume that $0\in\Theta$ and $T(0)=1$, so that ν is a probability measure. Define $\mu_0=\mu(0)$.

Let $\Omega=(a, b)$, $-\infty\leqq a<b\leqq\infty$. Then, θ, $T(\theta)$, and Θ can be expressed, respectively, by

$$\theta=\int_{\mu_0}^{\mu} d\,t/V(t), \tag{2.1}$$

$$T(\theta)=\exp\left\{\int_{\mu_0}^{\mu} t\,d\,t/V(t)\right\}, \tag{2.2}$$

and

$$\Theta=\left(\lim_{\mu\to a+}\int_{\mu_0}^{\mu} d\,t/V(t),\ \lim_{\mu\to b-}\int_{\mu_0}^{\mu} d\,t/V(t)\right). \tag{2.3}$$

The following lemma provides some analytic properties of T and μ (see Bar-Lev et al. [3]).

Lemma 1 *Let (V, Ω) be a variance function of a natural exponential family. Then*
(i) *the Laplace transform T, given by* (2.2), *is the restriction to Θ of a unique analytic function (which will also be denoted by T) on $\Theta\times\mathbb{R}\equiv S_\Theta\subseteq\mathbb{C}$;*
(ii) *μ is the restriction to Θ of a meromorphic function (which will also be denoted by μ) on S_Θ, with at most first order poles, where these poles are the zeroes of T;*
(iii) *V is the restriction to Ω of an analytic function (which will also be denoted by V) on some domain Q, $\Omega\subset Q\subseteq\mathbb{C}$. If V is a polynomial, then Q can be taken to be $\mathbb{C}$.*

It follows from Lemma 1 and the uniqueness of analytic continuation that

$$T(z)=\int_{\mu_0}^{\mu(z)} t\,d\,t/V(t), \qquad z=\theta+i\eta\in S_\Theta. \tag{2.4}$$

Moreover, the inverse form $m(z)$ of

$$(2.5)\qquad z=\int_{\mu_0}^{m} dt/V(t)$$

has a meromorphic branch which coincides with μ on S_Θ. In the sequel, for notational convenience, we shall write S for S_Θ, and, in order to avoid ambiguity, we shall refer to T as analytic on S, to μ as meromorphic on S, and to V as analytic on Q.

3 Necessary conditions for polynomial variance functions

We shall first discuss the method and derive results for the case $\Omega=\mathbb{R}^+$.

Let $(V,\mathbb{R}^+)$ be a variance function of a natural exponential family, and fix $\mu_0\in\mathbb{R}^+$. Then, by (2.3), with $a=0$ and $b=\infty$, $\Theta=(\theta_1,\theta_0)$, where $-\infty\leqq\theta_1<\theta_0\leqq\infty$. (Note that in the case where $(V,\mathbb{R}^+)$ is a variance function and V is a polynomial of degree $\geqq 2$, then $\theta_1=-\infty$ and θ_0 is finite.) The monotonic function $\mu(\theta)$, $\theta_1<\theta<\theta_0$, as defined implicitly by (2.1), admits a meromorphic continuation to $S=\{z\colon z=\theta+i\eta,\ \theta\in\Theta\}$, with at most first order poles there. Our plan is to determine for which polynomial V's such an extension is not possible, thereby excluding these polynomials from being variance functions.

We start by introducing the multivalued form

$$(3.1)\qquad z(m)=\int_{\mu_0}^{m} dt/V(t),\qquad m\in\mathbb{C}\setminus Z(V),$$

where $Z(V)$ denotes the zero set of V. This form is indeed multivalued, since $\operatorname{Res}(1/V, m_i)\neq 0$, for some $m_i\in Z(V)$, and thus its value at any point depends on the path of integration. For the case where V has a simple root at zero, this is particularly useful since Lemma 2, below, assures us that in this case the inverse function, $m(z)$, is periodic. The period of m is, in fact, imaginary, and shall be denoted by $i\,d$, $d>0$.

Considering this case (i.e., where m has a simple zero at the origin), we focus on the mapping m restricted to $S_1=S\cap\{z\colon |\Im(z)|<d/2\}$. We shall see that m is univalent in S_1 and that $m(S_1)$ misses $Z(V)$ and infinity. Since $m(S_1)$ is simply connected and symmetric with respect to the real axis, it then misses symmetric slits joining each $m_i\in Z(V)$ with infinity. We then consider an arbitrary simply connected domain M in the m-plane, which is $\mathbb{C}$ except for $\mathbb{R}^-$ and symmetric slits connecting each $m_i\in Z(V)$ with infinity (this is a "maximal" simply connected domain in $\mathbb{C}$, where $z(m)$ is one-valued). We then have: If $S_1\nsubseteq z(M)$, for all possible M, then V is not a variance function.

For the proofs of our theorems, we shall need some lemmas. Lemma 2 is due to Letac and Mora [7]. Here we provide a different formulation along with a short proof.

Lemma 2 *Let $(V,\mathbb{R}^+)$ be a variance function such that V is analytic at the origin. Then V has a first order zero at the origin* iff *$m(z)$ is periodic with period $2\pi i\alpha_0\equiv i\,d$, where $\alpha_0=\operatorname{Res}(1/V,0)>0$.*

Proof. Using partial fractions, it follows that the local behavior of the integral in (3.1), near $m=0$, is

(a) $z(m)=\alpha_0 \log m+H(m)$, $\alpha_0>0$, if V has a simple zero at the origin, or

(b) $z(m)=\sum_{k=1}^{r-1} \gamma_k/m^k+\alpha_0 \log m+H(m)$, $\gamma_{r-1}\neq 0$, if V has a zero of order $r\geqq 2$ at the origin.

Here, H denotes a function that is analytic in a neighborhood of the origin. The local behavior of $m(z)$ on the left half plane near infinity is thus

(a) $m(z)=f(e^{z/\alpha_0})$, in the first case, or

(b) $m(z)=f(z^{-1/r})$, in the second case,

where f is analytic in a neighborhood of infinity. By the uniqueness of the meromorphic continuation, m is periodic with period $2\pi i\alpha_0$ iff V has a simple zero at the origin. Also, $\alpha_0=1/V'(0)>0$, since $V>0$ on $\mathbb{R}^+$ and $V(0)=0$. □

We now address our original problem. Let $(p, \mathbb{R}^+)$ be a variance function, where p is a polynomial of degree $n>2$. Then $p(0)=0$ and $\Theta=(-\infty, \theta_0)$, where $0<\theta_0<\infty$. Hence, S is a half plane. Let r denote the order of the zero of p at the origin and let

$$S_0=\begin{cases} S_1, & \text{if } r=1,\\ S, & \text{if } r=2.\end{cases}$$

Lemma 3 *Let $(p, \mathbb{R}^+)$ be a variance function, where p is a polynomial of degree $n>2$. Then, for each $s\in Z(p)\backslash\{0\}$, there exists a slit connecting s to infinity that does not intersect $m(S)$. These slits can be taken to be symmetric with respect to the real axis.*

Proof. We first show that each $s\in Z(p)$ is not in $m(S)$. Indeed, if $s\in Z(p)$, then $\int_{\mu_0}^{s} dt/p(t)$ diverges for every path that joins μ_0 and s. On the other hand, if $s\in m(S)$, then there exists a path Γ, joining μ_0 and s, and a $z\in S$ such that $\int_\Gamma dt/p(t)=z$. Since z is finite, this is a contradiction. Hence, $Z(p)\cap m(S)=\phi$.

We next show that $\infty\notin m(S)$. Assume, to the contrary, that $\infty\in m(S)$. Then m has a pole in S and therefore T has a zero there. By (2.4), there exists a path joining μ_0 to infinity such that $\int_{\mu_0}^{\infty} t\,dt/p(t)=-\infty$. However, this integral converges for all paths joining μ_0 to infinity. (Here, we have used the fact that $|p(t)|\sim|t|^n$, $n\geqq 2$, as t tends to infinity.) Hence, $\infty\notin m(S)$.

We now proceed to show that m is univalent in S_0. Assume to the contrary that m is not univalent in S_0. Then there exist $z_1, z_2\in S_0$, $z_1\neq z_2$, such that $m(z_1)=m(z_2)$. Let Λ denote the straight line segment that joins z_1 and z_2. Since S_0 is convex, $\Lambda\subset S_0$. Now, $m(\Lambda)$ is a closed curve in the m-plane and, therefore,

$$z_1-z_2=\int_\Lambda dz=\oint_{m(\Lambda)} dm/p(m)\neq 0.$$

We conclude that $m(\Lambda)$ is a curve that surrounds at least one of the zeroes of p. But, by the residue theorem, the integral on the right is independent of

z_1-z_2 in a small neighborhood of z_1-z_2. Hence, m is periodic with period $\kappa=z_1-z_2$. If κ is imaginary, then $|\kappa|<d$, which contradicts Lemma 2. If κ is not imaginary, then, by the Schwartz reflection principle, $\bar{\kappa}$ is also a period of m. Since κ and $\bar{\kappa}$ are independent periods, m is doubly periodic and is, thus, meromorphic in $\mathbb{C}$. This is possible only if p is a second degree polynomial (see Theorem 4.2 of Bar-Lev et al. [3]), which contradicts a premise of the lemma. Accordingly, m is univalent and, since S_0 is simply connected, so is $m(S_0)$. Thus, there exist slits joining each $s\in Z(p)$ with infinity that do not intersect $m(S)$. Since m is real on the real axis and S_0 is symmetric with respect to the real axis, so is $m(S)$. This makes it possible to choose these slits to be symmetric with respect to the real axis. □

As a result of Lemma 3, we need focus on only the upper half plane. For convenience, we adopt the following notation in the sequel. Let M denote a simply connected domain, symmetric with respect to the real axis, consisting of $\mathbb{C}$ except for $\mathbb{R}^-$ and symmetric slits joining each $s\in Z(p)\backslash\{0\}$ to infinity. Also, if A is a set and b a number, we write $A+b$ to denote the set of all numbers of the form $a+b$, where $a\in A$, and (for $A\neq\mathbb{R}$) we write A^+ to denote the set $A\cap\{z\colon \Im(z)>0\}$.

Lemma 4 *Let p be a polynomial of degree >2, with a zero of order r, $r=1$, 2, at the origin. Then, the mapping m, as defined in* (2.5) *where $V=p$, is meromorphic in S, with $m(\theta)=\mu(\theta)$ for all $\theta\in\Theta$,* iff *there exists a symmetric simply connected domain $L\subset M$, $\partial L\supset\mathbb{R}^-$, such that, for $r=1$, 2, $z(\partial L^+)\cap S_0^+=\phi$ and, for $r=1$, $z(\mathbb{R}^-)\supset\Theta+i\,d/2$, also. Here, for $r=1$, $S_0^+=S_1^+$ and, for $r=2$, $S_0^+=S^+$.*

Proof. For $r=1$, 2, by Lemma 3, m is univalent in S_0 and symmetric with respect to the real axis. Hence, "necessity" follows by choosing $L=m(S_0)$.

For the "sufficiency" part of the proof, by the open mapping theorem, the condition that $z(\partial L^+)\cap S_0^+=\phi$ implies that $S_0^+\subset z(L^+)$ and, by the inverse mapping theorem, m is meromorphic on S_0^+. Since Θ is mapped by m onto $\mathbb{R}^+$, by the Schwartz reflection principle, m is meromorphic on $\mathbb{R}^+$ and is extendable as a meromorphic function to S_0. If $r=2$, $S_0=S$ and we are done. If $r=1$, then, using Lemma 2 and the condition that $z(\mathbb{R}^-)\supset\Theta+i\,d/2$, we have

$$m(\theta+i\,d/2)=m(\theta-i\,d/2)=\overline{m(\overline{\theta-i\,d/2})}=\overline{m(\theta+i\,d/2)},$$

for $\theta\in\Theta$, so that m is real on $\Theta+i\,d/2$ and, by periodicity, on $\Theta-i\,d/2$ also. By the Schwartz reflection principle, m is meromorphic on these lines and thus on $\bigcup_{k\in\mathbb{Z}}(S_0+i\,d\,k)=S$. This concludes the proof of the lemma. □

Before stating our basic theorem, we shall introduce some additional necessary notation. Let $\mathscr{M}_p$ denote the set of nonreal zeroes of (a polynomial) p, whose imaginary parts are positive, and let $\mathscr{A}_p$ denote the set of real zeroes of p, except for zero. Let l denote the number of elements in $\mathscr{M}_p\cup\mathscr{A}_p$. An ordering $\{s_j\}_{1 \leqq j \leqq l}$ of $\mathscr{M}_p \cup \mathscr{A}_p$ is an order on the set such that $m \prec c$, for all $m \in \mathscr{M}_p$ and $c \in \mathscr{A}_p$, and $c \prec d$, if $c < d$ and $c, d \in \mathscr{A}_p$. We also let

$$\mathscr{P}_j = \begin{cases} -2\pi i \operatorname{Res}(1/p, s_j), & \text{if } s_j \in \mathscr{M}_p \\ -\pi i \operatorname{Res}(1/p, s_j), & \text{if } s_j \in \mathscr{A}_p. \end{cases}$$

We are now ready to formulate our first theorem.

Theorem 1 *Let p be a polynomial of degree $n > 2$ and let the multiplicity, r, of the zero of p at the origin not exceed 2. Then, if $(p, \mathbb{R}^+)$ is a variance function of some natural exponential family, it is necessary that there exists an ordering of $\mathscr{M}_p \cup \mathscr{A}_p$ such that all partial sums of $\mathscr{P}_j$ satisfy*

$$\text{(3.2)} \qquad \text{(i) } \theta_0 + \sum_{j=1}^{k} \mathscr{P}_j \notin S_0^+ \quad \textit{and} \quad \text{(ii) } \theta_0 + \sum_{j=1}^{k} \mathscr{P}_j \notin \Theta + i d/2, \quad \textit{for } r = 1,$$

and

$$\text{(3.3)} \qquad \theta_0 + \sum_{j=1}^{k} \mathscr{P}_j \notin S_0^+, \quad \textit{for } r = 2,$$

for $k = 1, 2, \ldots, l$.

Proof. We shall see that (3.2) and (3.3) provide necessary and sufficient conditions for $\mu(\theta)$ to admit a meromorphic continuation to S. For brevity and simplicity, we shall restrict the proof to the case where each element of $\mathscr{M}_p$ is a simple zero of p. (Proofs for the other cases can be given similarly.) By Lemma 4, this is the case iff there exists an M and $L \subset M$ such that $z(\partial L^+) \cap S_0^+$ is empty. If M^+ is given a positive orientation (i.e., counterclockwise) and its boundary ∂M^+ is considered in terms of prime ends (i.e., a boundary point on a slit that is not a root of p is considered as two different points from the two sides of the slit that passes through it), then an L with the above-mentioned property exists iff the element of $z(M^+)$ that contains the positive real axis has no singularities in S_0^+. To check this, we shall therefore trace ∂M^+ and discuss its image in the z-plane. A convenient way to do so is to consider $M_R = M \cap \{z: |z| < R\}$, instead of M, and let R tend to infinity. Note that the circular arcs of ∂M_R^+ (such as $\widehat{CD}$ and $\widehat{HI}$) are mapped onto very small arcs (see Fig. 1). Indeed,

$$|z(F) - z(G)| = \left| \int_{\mu_0}^{F} dt/p(t) - \int_{\mu_0}^{G} dt/p(t) \right| = \left| \int_{G}^{F} dt/p(t) \right| \leqq \pi R^{1-n} \beta,$$

where $\beta \in \mathbb{R}^+$. Hence, the right hand side of the above inequality tends to zero, as R tends to infinity. Starting with A in tracing the curve, and letting $R \to \infty$, the image is seen to look as in Fig. 2, if $r = 1$, or as in Fig. 3, if $r = 2$, where the lowercase letters are the images of the corresponding capital letters appearing in Fig. 1. Furthermore,

$$s - a = -\pi i \operatorname{Res}(1/p, 0), \qquad g - e = -2\pi i \operatorname{Res}(1/p, s_1),$$

$$l - g = -2\pi i \operatorname{Res}(1/p, s_2) \quad \text{and} \quad r - q = -\pi i \operatorname{Res}(1/p, s_3).$$

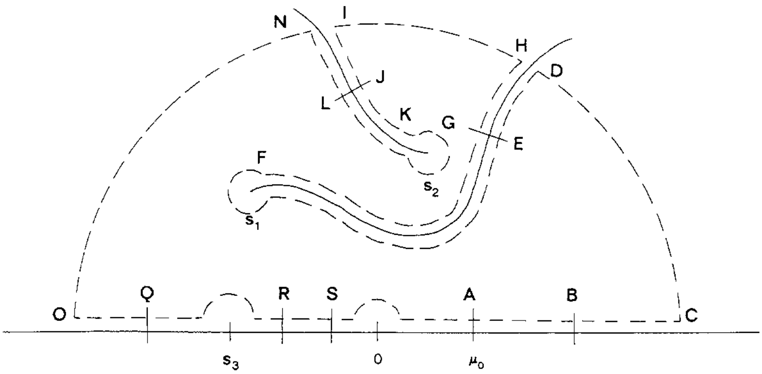

Fig. 1. ∂M_R^+ as a closed curve

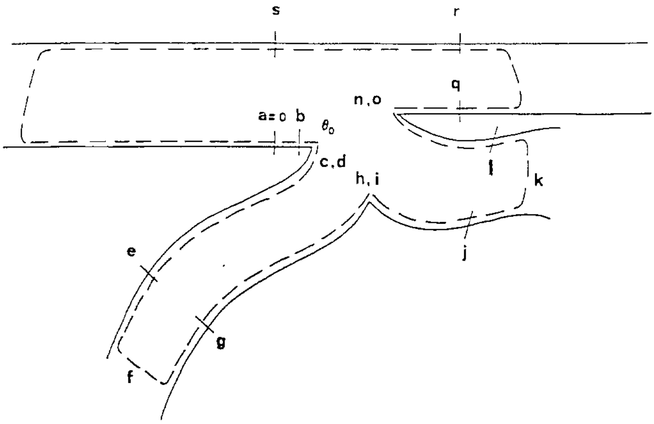

Fig. 2. $z(\partial M_R^+)$ for $r=1$

Note that the singularities of $m(z)$ (none of which are first order poles) are exactly

$$\theta_0 + \sum_{j=1}^{k} \mathscr{P}_j, \qquad k=1, 2, \ldots, l,$$

which, by Lemma 4, must satisfy (3.2), for $r=1$, or (3.3), for $r=2$. This concludes the proof. □

Remark 1 Theorem 1 provides necessary conditions for $(p, \mathbb{R}^+)$ to be a variance function, where r, the multiplicity of the zero of p at the origin, does not exceed 2. By employing similar techniques, one can also derive necessary conditions for the case where $r>2$. Such conditions, however, are not sufficiently restrictive to be useful for polynomials of degree $\geqq 5$, so this case is not considered here.

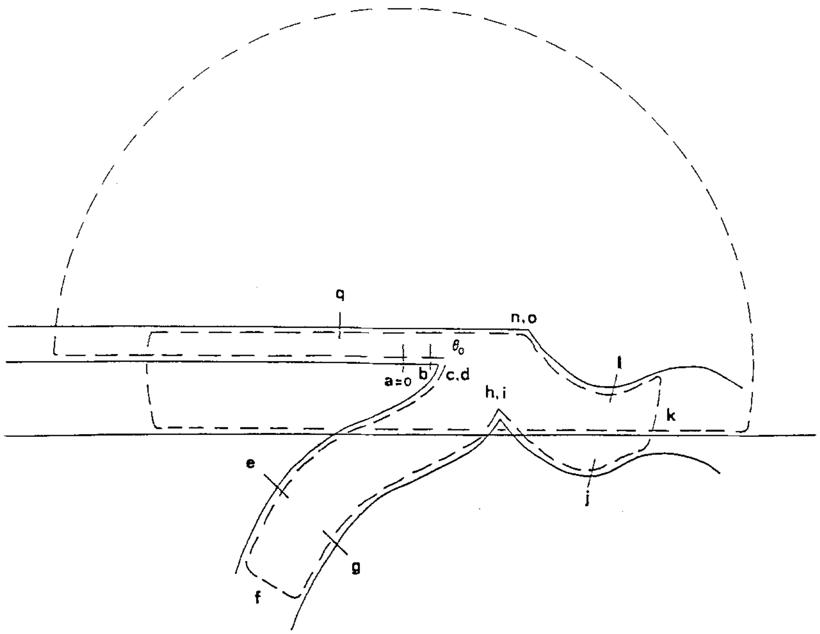

Fig. 3. $z(\partial M_R^+)$ for $r=2$

Remark 2 Actually, p has real coefficients, which implies that any $\mathscr{P}_j$, corresponding to an $s_j \in \mathscr{A}_p$, is imaginary. Hence, if (3.2) or (3.3) is satisfied for $\mathscr{M}_p$ it is automatically satisfied for $\mathscr{A}_p \cup \mathscr{M}_p$.

We now consider the case where $\Omega = \mathbb{R}$. Here, if $(p, \mathbb{R})$ is a variance function, where p is a polynomial of degree >2, then the corresponding space $\Theta = (\theta_1, \theta_0)$, as defined by (2.3), is a finite interval. The mean function μ, as defined on Θ by (2.1), admits a meromorphic continuation to S, with at most first order poles there. We continue to use the notation introduced in the paragraph preceding Theorem 1. However, note that, here, $\mathscr{A}_p$ is empty so that $\mathscr{P}_j = -2\pi i \operatorname{Res}(1/p, s_j), j=1, \ldots, l$. Theorem 2, which follows, provides necessary conditions for $(p, \mathbb{R})$ to be a variance function. Its proof is similar to that of Theorem 1 and, thus, is omitted for brevity. Instead, we sketch the proof of Lemma 5, which is the analogue of Lemma 3.

Lemma 5 *Let* $(p, \mathbb{R})$ *be a variance function, where* p *is a polynomial of degree* >2. *Assume that* (I) *for an arbitrary ordering of* $\mathscr{M}_p$, $\sum_{j=1}^{k} \mathscr{P}_j$, $k=1, \ldots, l$, *is not imaginary.*

Then, for each $s \in Z(p)$, *there exists a slit connecting* s *to infinity that does not intersect* $m(S)$. *These slits can be taken to be symmetric with respect to the real axis.*

Proof. As in Lemma 3, one shows that $(\{\infty\} \cup Z(p)) \cap m(S) = \phi$. One then proceeds to show that m is univalent in S. This can be done by negation, as follows. If, for some $z_1 \neq z_2$, $z_1, z_2 \in S$, $m(z_1) = m(z_2)$, then the same argument as in Lemma 3 shows that $\kappa = z_1 - z_2$ is a period of m. (I), above, implies that κ is not imaginary and, therefore, it is either real or complex. If κ is real, then $|\kappa|<\theta_0-\theta_1$ and, therefore, m is meromorphic in $\mathbb{C}$. Also, if κ is complex (but *not* imaginary), then m is meromorphic in $\mathbb{C}$. Indeed, $\bar{\kappa}$ is also a period of m and, thus, so is $\kappa-\bar{\kappa}=2i\Im(\kappa)$. Now, κ and $2i\Im(\kappa)$ are independent periods of m and, since the fundamental parallelogram domain of m, defined by κ and $2i\Im(\kappa)$, fits in the strip of regularity of m, m is meromorphic in $\mathbb{C}$. Since this can happen only if p is of degree 2 (see Theorem 4.2 of Bar-Lev et al. [3]), we have a contradiction. The remainder of the proof follows as in Lemma 3. □

Theorem 2 *Let p be a polynomial of degree >2. Then, if $(p, \mathbb{R})$ is a variance function of some natural exponential family and condition* (I) *of Lemma* 5 *is satisfied, it is necessary that there exists an ordering of $\mathcal{M}_p$ such that*

$$\theta_0+\sum_{j=1}^{k}\mathcal{P}_j\notin S^+, \tag{3.4}$$

for all $k=1, \ldots, l$.

4 Applications

We shall now apply the results of Sect. 3 and note that they provide satisfactory necessary conditions for polynomial variance functions of small degree. By these conditions, along with the sufficient conditions of Bar-Lev [1], we will be able to completely characterize some subclasses of polynomials. We state Bar-Lev's conditions in the following lemma.

Lemma 6 *Let p be a polynomial of degree $\geqq 1$ having nonnegative coefficients. Then $(p, \mathbb{R}^+)$ is a variance function of an infinitely divisible natural exponential family.*

Our applications in the sequel will refer to different classes of polynomials with one complex parameter; viz., all zeroes of p will be fixed, except for two complex conjugate zeroes m_1 and $\bar{m}_1$. The region of variability of m_1 such that the coefficients of p are nonnegative will be referred to as the $\mathfrak{B}$-region.

In Theorem 3, we characterize the class of third degree polynomials which are variance functions of natural exponential families. Theorem 4 is devoted to the study of fourth degree polynomials. It will reduce substantially the characterization problem for this class. Theorem 5 and Theorem 6 are devoted to the study of one parameter families of fifth and seventh degree polynomials, respectively.

The following theorem is due to Mora [8]. A complete characterization of all third degree polynomials which are variance functions of natural exponential families is found in Letac and Mora [7]. Here, by employing the necessary conditions of Sect. 3 and Lemma 6, we present a shorter proof.

Theorem 3 *Let p be a third degree polynomial. Then $(p, \mathbb{R}^+)$ is a variance function* iff *the coefficients of p are nonnegative.*

Proof. The "sufficiency" part follows immediately from Lemma 6. We now prove "necessity". Put $p(m)=\alpha m(m^2+\beta m+\gamma)$ and assume that $(p, \mathbb{R}^+)$ is a variance function. Then, it is clear that $\alpha>0$ and $\gamma\geqq 0$, so that it is only necessary to show that $\beta \geqq 0$. If the roots of p were real, they should be nonpositive (since p does not vanish on $\mathbb{R}^+$) and therefore its coefficients are nonnegative. The only other nontrivial case is when $m_1 = a + ib$, $b > 0$, and $\bar{m}_1$ are the complex conjugate roots of p. In this case, we apply Theorem 1 with $\mathscr{M}_p = \{m_1\}$, $\mathscr{A}_p = \phi$, $r = 1$, and $l = 1$. Here, we have $m_1 = s_1$ and

$$\mathscr{P}_1 = -2\pi i \operatorname{Res}(1/p, s_1) = \frac{-2\pi i}{\alpha m_1 (m_1 - \bar{m}_1)} = \frac{-\pi \bar{m}_1}{\alpha |m_1|^2 \Im(m_1)}.$$

Thus, $\Im(\mathscr{P}_1) = \pi/[\alpha(a^2+b^2)]$ and $\Re(\mathscr{P}_1) = -\pi a/[\alpha b(a^2+b^2)]$. Since $\Im(\mathscr{P}_1) = d/2$, condition (3.2) implies that $\Re(\mathscr{P}_1) \geqq 0$ or, equivalently, that $a \leqq 0$. Since $\beta = -2a$, we have $\beta \geqq 0$. □

We next address the problem of fourth degree polynomial variance functions. Let (p, Ω) be a variance function, where p is a fourth degree polynomial and Ω is either $\mathbb{R}$ or $\mathbb{R}^+$. If $\Omega = \mathbb{R}$, then p must have two nonreal zeroes and their conjugates; viz., m_1, $\bar{m}_1$, m_2, and $\bar{m}_2$, where, without loss of generality, $\Im(m_i) > 0$, $i = 1, 2$. By using affinity, we can assume, without loss of generality, that the zeroes of p are i, $-i$, m_1, and $\bar{m}_1$, where $\Re(m_1) \geqq 0$ and $\Im(m_1) > 0$; i.e., p has the form

(4.1) $p(m) = \alpha(m^2+1)(m - m_1)(m - \bar{m}_1), \quad \alpha > 0, \quad \Re(m_1) \geqq 0, \quad \Im(m_1) > 0.$

If $\Omega = \mathbb{R}^+$, then p has one zero at the origin and another real zero, say a. If $a \neq 0$, then, by using affinity, we may assume that $a = -1$. The other two zeroes of p are either real, in which case they must be nonpositive since p does not vanish on $\mathbb{R}^+$ (and thus by Lemma 6, $(p, \mathbb{R}^+)$ is a variance function of an infinitely divisible natural exponential family), or they are two complex conjugate roots, m_1 and $\bar{m}_1$, with $\Im(m_1) > 0$. Consequently, for $\Omega = \mathbb{R}^+$, two forms of p should be analyzed:

(4.2) $\qquad p(m) = \alpha m(m+1)(m - m_1)(m - \bar{m}_1), \quad \alpha > 0, \ \Im(m_1) > 0$

and

(4.3) $\qquad p(m) = \alpha m^2 (m - m_1)(m - \bar{m}_1), \quad \alpha > 0, \ \Im(m_1) > 0.$

The following theorem provides necessary conditions for (p, Ω) to be a variance function, where p is of the form (4.1), (4.2), or (4.3).

Theorem 4 (i) *Let p be given by* (4.1). *Then, for* $(p, \mathbb{R})$ *to be a variance function, it is necessary that either* $|m_1| \leqq 1$ *or* $\Re(m_1^2 + 1) \leqq 0$.
(ii) *Let p be given by* (4.2). *Then, for* $(p, \mathbb{R}^+)$ *to be a variance function, it is necessary that* $\Re(m_1^2 + m_1) < 0$ *or* $\Re(m_1) < -1/2$.
(iii) *Let p be given by* (4.3). *Then, for* $(p, \mathbb{R}^+)$ *to be a variance function, it is necessary that* $|\arg(m_1)| > \pi/4$.

Proof. (i) We apply Theorem 2 with $l = 2$ and $\mathscr{M}_p = \{i, m_1\}$. Two orderings of $\mathscr{M}_p$ are possible: (a) $s_1 = i$ and $s_2 = m_1$, or (b) $s_2 = i$ and $s_1 = m_1$. In case (a), we have

$$\mathscr{P}_1 = -2\pi i \operatorname{Res}(1/p, m_1) = \frac{-\pi[|m_1|^2 - 1 + 2i\Re(m_1)]}{\alpha |m_1^2 + 1|^2} \equiv x_1$$

and

$$\mathscr{P}_2 = -2\pi i \operatorname{Res}(1/p, i) = \frac{-\pi(\bar{m}_1^2 + 1)}{\alpha \Im(m_1) |m_1^2 + 1|^2} \equiv x_2.$$

Whereas in case (b), $\mathscr{P}_1 = x_2$ and $\mathscr{P}_2 = x_1$. Note that condition (I), of Lemma 5, is satisfied if $|m_1| \neq 1$ and $\bar{m}_1^2 + 1$ is not imaginary. In which case, at least one of the two orderings should satisfy condition (3.4); i.e.,

(4.4) $$\theta_0 + \mathscr{P}_1 \notin S^+$$

and

(4.5) $$\theta_0 + \mathscr{P}_1 + \mathscr{P}_2 \notin S^+.$$

Clearly, for both orderings $\mathscr{P}_1 + \mathscr{P}_2 = -\int_{-\infty}^{\infty} dt/p(t)$. But, since $\int_{-\infty}^{\infty} dt/p(t) = \theta_0 - \theta_1$, it follows that $\theta_0 + \mathscr{P}_1 + \mathscr{P}_2 = \theta_1 \notin S^+$, and thus condition (4.5) holds for both orderings. Therefore, condition (4.4) holds, for either ordering, if and only if $\Re(\mathscr{P}_1) \geqq 0$ or $\Re(\mathscr{P}_2) \geqq 0$. This implies the desired result.

(ii) We apply Theorem 1 with $r=1$, $l=2$, $\mathscr{M}_p = \{m_1\}$, and $\mathscr{A}_p = \{-1\}$. Here, there is only one ordering, namely $s_1 = m_1$ and $s_2 = -1$. We have

$$\mathscr{P}_1 = -2\pi i \operatorname{Res}(1/p, m_1) = \frac{-\pi \bar{m}_1 (\bar{m}_1 + 1)}{\alpha \Im(m_1) |m_1|^2 |m_1 + 1|^2}$$

and

$$\mathscr{P}_2 = -i\pi \operatorname{Res}(1/p, -1) = \frac{i\pi}{\alpha |m_1 + 1|^2}.$$

Since $-2\pi i \operatorname{Res}(1/p, 0) = -\pi i/(\alpha |m_1|^2)$, the period of m (see Lemma 2) is $id = \pi i/(\alpha |m_1|^2)$. Now, if $\Im(\mathscr{P}_1) > 0$, then $\Im(\mathscr{P}_1) = d/2 - \Im(\mathscr{P}_2) < d/2$ and, therefore, by Remark 2, it is necessary that $\Re(\mathscr{P}_1) \geqq 0$ for (3.2) to hold; i.e., (3.2) holds iff $\{\Im(\mathscr{P}_1) > 0$ and $\Re(\mathscr{P}_1) \geqq 0\}$ or $\Im(\mathscr{P}_1) \leqq 0$. The desired result then follows.

(iii) We apply Theorem 1 with $r=2$, $l=1$, $\mathscr{M}_p = \{m_1\}$, and $\mathscr{A}_p = \phi$. Therefore $s_1 = m_1$ and

$$\mathscr{P}_1 = -2\pi i \operatorname{Res}(1/p, m_1) = \frac{-\pi \bar{m}_1^2}{\alpha \Im(m_1) |m_1|^4}.$$

For (3.3) to be satisfied, it is necessary that $\Re(\mathscr{P}_1) \geqq 0$ or $\Im(\mathscr{P}_1) \leqq 0$. This yields the desired result. □

It is interesting to graphically illustrate the "necessary" region of m_1 for the polynomial in (4.2). This "necessary" region is specified by the necessary conditions in part (ii) of Theorem 4. For this polynomial, in which $\Omega = \mathbb{R}^+$, note that the "necessary" region and the $\mathfrak{B}$-region (i.e., the region of variability of m_1 such that the coefficients of p are nonnegative) touch in three distinct points. Since the $\mathfrak{B}$-region belongs to the region of infinitely divisible natural exponential families, it would be interesting to characterize the remaining region (viz., the region between the boundaries of the "necessary" region and the $\mathfrak{B}$-region).

The possible types of fifth degree polynomial variance functions can be obtained by an analysis similar to that employed for fourth degree polynomial variance functions (see the paragraph preceding Theorem 4). Our next application concerns two such types.

Theorem 5 (i) *Let* $p(m)=\alpha m(m-m_1)^2(m-\bar{m}_1)^2$, *where* $\alpha>0$ *and* $\Im(m_1)>0$. *Then* $(p, \mathbb{R}^+)$ *is a variance function* iff $\Re(m_1)\leqq 0$.
(ii) *Let* $p(m)=\alpha m^2(m+1)(m-m_1)(m-\bar{m}_1)$, *where* $\alpha>0$ *and* $\Im(m_1)>0$, *and put* $m_1=a+ib$. *Then a necessary condition for* $(p, \mathbb{R}^+)$ *to be a variance function is that*

$$b^2>a^2(a+1)/(3a+1) \quad \text{or} \quad b^2<3a^2+2a. \tag{4.6}$$

Proof. (i) The "sufficiency" part follows immediately, since if $\Re(m_1)\leqq 0$, then p has nonnegative coefficients, so that by Lemma 6, $(p, \mathbb{R}^+)$ is a variance function of an infinitely divisible natural exponential family. For "necessity", we apply Theorem 1 with $r=1$, $l=1$, $\mathscr{M}_p=\{m_1\}$, and $\mathscr{A}_p=\phi$. Here, only one ordering exists for which $s_1=m_1$. Therefore

$$\mathscr{P}_1=-2\pi i\,\mathrm{Res}(1/p, m_1)=-2\pi i\left(\frac{1}{\alpha m(m-\bar{m}_1)^2}\right)'\bigg|_{m=m_1}=\frac{-\pi(3m_1-\bar{m}_1)\,\bar{m}_1^2}{4\alpha\,|m_1|^4(\Im(m_1))^3}.$$

Here, $\Im(\mathscr{P}_1)=\pi/(2\alpha\,|m_1|^4)=d/2$ and, thus, for (3.2) to be satisfied, it is necessary that $\Re(\mathscr{P}_1)\geqq 0$. Put $m_1=a+ib$, then

$$\Re(\mathscr{P}_1)=\frac{-\pi a(2a^2+6b^2)}{4\alpha(a^2+b^2)\,b^3}.$$

Since $\Re(\mathscr{P}_1)\geqq 0$ iff $a=\Re(m_1)\leqq 0$, the desired result follows.
(ii) We apply Theorem 1 with $r=2$, $l=2$, $\mathscr{M}_p=\{m_1\}$, and $\mathscr{A}_p=\{-1\}$. There is only one ordering with $s_1=m_1$ and $s_2=-1$, and

$$\mathscr{P}_1=-2\pi i\,\mathrm{Res}(1/p, m_1)=\frac{-\pi(\bar{m}_1+1)\,\bar{m}_1^2}{\alpha\,|m_1+1|^2\,|m_1|^4\,\Im(m_1)}.$$

Now, by Remark 2, (3.3) is satisfied iff $\Re(\mathscr{P}_1)\geqq 0$ or $\Im(m_1)\leqq 0$, and this implies (4.6). □

Our last application generalizes the result in Theorem 5(i).

Theorem 6 *Let* $p(m)=\alpha m(m-m_1)^3(m-\bar{m}_1)^3$, *where* $\alpha>0$ *and* $\Im(m_1)>0$. *Then* $(p, \mathbb{R}^+)$ *is a variance function* iff $\Re(m_1)\leqq 0$.

Proof. The "sufficiency" part is immediate by Lemma 6. We prove "necessity" by applying Theorem 1 with $r=1$, $l=1$, $\mathscr{M}_p=\{m_1\}$, and $\mathscr{A}_p=\phi$. Only one ordering is possible with $s_1=m_1$ and

$$\mathscr{P}_1=-2\pi i\,\mathrm{Res}(1/p, m_1)=-\pi i\left(\frac{1}{\alpha m(m-\bar{m}_1)^3}\right)''\bigg|_{m=m_1}$$
$$=\frac{-\pi i\{8i\bar{m}_1^3[\Im(m_1)]^3+12\bar{m}_1^2[\Im(m_1)]^2\,|m_1|^2-12i\bar{m}_1\,|m_1|^4\,\Im(m_1)\}}{\alpha\,|m_1|^6\,|2\Im(m_1)|^6}.$$

As in Theorem 5(i), $\Im(\mathscr{P}_1)=d/2$ and it is, therefore, necessary that $\Re(\mathscr{P}_1)\geqq 0$. This is easily seen to give, for $m_1=a+ib$, that

$$-b\,a(40\,b^2\,a^2+60\,b^4+12\,a^4)\geqq 0$$

and, therefore, $\Re(m_1)=a\leqq 0$. This concludes the proof of the theorem. □

Finally, let $p(m)=\alpha\,m(m-m_1)^n(m-\bar{m}_1)^n$, where $\alpha>0$ and $\Im(m_1)>0$. We conjecture that $(p,\mathbb{R}^+)$ is a variance function, for all $n\in\mathbb{N}$, iff $\Re(m_1)\leqq 0$. (Note that this conjecture has been proved for $n=1, 2$, and 3 in Theorems 3, 5(i), and 6, respectively.)

References

1. Bar-Lev, S.K.: Contribution to discussion of paper by B. Jørgensen: Exponential dispersion models. J. R. Stat. Soc., Ser. B **49**, 153–154 (1987)
2. Bar-Lev, S.K., Bshouty, D.: Rational variance functions. Ann. Stat. **17**, 741–748 (1989)
3. Bar-Lev, S.K., Bshouty, D., Enis, P.: Variance functions with meromorphic means. Ann. Probab. **19**, 1349–1366 (1991)
4. Bar-Lev, S.K., Enis, P.: Reproducibility and natural exponential families with power variance functions. Ann. Stat. **14**, 1507–1522 (1986)
5. Jørgensen, B.: Some properties of exponential dispersion models. Scand. J. Stat. **13**, 187–198 (1986)
6. Jørgensen, B.: Exponential dispersion models (with discussion). J. R. Stat. Soc., Ser. B **49**, 127–162 (1987)
7. Letac, G., Mora, M.: Natural real exponential families with cubic variances. Ann. Stat. **18**, 1–37 (1990)
8. Mora, M.: A classification of cubic variance functions of exponential families on **R**. C.R. Acad. Sci., Paris, Ser. I **302**, 587–590 (1986)
9. Morris, C.N.: Natural exponential families with quadratic variance functions. Ann. Stat. **10**, 65–80 (1982)