Probab. Theory Relat. Fields 94, 53–68 (1992)

Probability Theory and Related Fields

Large deviations and the propagation of chaos for Schrödinger processes

Robert Aebi [1,*] **and Masao Nagasawa** [2]

[1] Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606, Japan
[2] Institut für Angewandte Mathematik der Universität Zürich, Rämistrasse 74, CH-8001 Zürich, Switzerland

Received December 21, 1990; in revised form January 1, 1992

Summary. Schrödinger processes due to Schrödinger (1931) (the definition of which is given in Sect. 4) are uniquely characterized by a large deviation principle, in terms of the relative entropy with respect to a reference process, which is a renormalized diffusion process with creation and killing in applications. An *approximate* Sanov property of a subset $A_{a,b}$ is shown, where $A_{a,b}$ denotes the set of all probability measures on a path space with prescribed marginal distributions $\{q_a, q_b\}$ at finite initial and terminal times a and b, respectively. It is shown that there exists the unique Markovian modification of n-independent copies of renormalized processes conditioned by the empirical distribution, and that the propagation of chaos holds for the system of interacting particles with the Schrödinger process as the limiting distribution.

1 Introduction

The aim of this article is to give a characterization of Schrödinger processes in terms of large deviations in the case of *unbounded or singular creation and killing*, and to prove a propagation of chaos result for Schrödinger processes. Since the initial and terminal distributions q_a and q_b, at $-\infty < a < b < \infty$, are fixed, when we discuss large deviations for Schrödinger processes, we take a set $A_{a,b}$ of probability measures on a path space $C([a,b], \mathbf{R}^d)$, with the pair $\{q_a, q_b\}$ as marginal distributions at a and b, respectively. To discuss the Sanov property of the subset $A_{a,b}$, there are two obstacles; namely, the empirical distribution does not belong to the set $A_{a,b}$ and moreover its interior $A^\circ_{a,b}$ is empty. To overcome these points we will introduce a sequence of enlarged subsets of probability measures, which approximates the subset $A_{a,b}$ (cf. Theorem 2.1). The third assertion of Theorem 4.1 states an approximate Sanov property of the set $A_{a,b}$ in the case of Schrödinger processes with of unbounded or singular creation and killing. Large deviations for Schrödinger processes have been dis-

* *Present address:* Department of Mathematics and Statistics, James Clark Maxwell Building, the King's Buildings, Mayfield Road, Edinburgh EH93JZ, UK

cussed by Föllmer (1988) for Brownian motions, and by Dawson et al. (1990) for a case of bounded creation and killing. However, in both papers the Sanov property of the set $A_{a,b}$ is not treated. The fifth assertion of Theorem 4.1 is on the propagation of chaos for systems of interacting diffusion processes converging to Schrödinger processes, for which the existence of a Markovian modification will be shown.

2 Approximate Sanov property

Let $\Omega = \mathrm{C}([a,b], \mathbf{R}^d)$, $-\infty < a < b < \infty$, be the space of continuous paths taking values in $\mathbf{R}^d$, $d \geqq 1$, with the Borel σ-field $\sigma(\Omega)$. We denote $X_t(\omega) = X(t,\omega) = \omega(t)$, for $\omega \in \Omega$. Let $\mathbf{M}_1(\Omega)$ be the space of probability measures on Ω endowed with Csiszar's τ_0-*topology:* For $\varepsilon > 0$ and finite measurable partitions $\mathscr{P}_k = \mathscr{P}_k(\Omega) = \{\Omega_1, \ldots, \Omega_k\}$, $k = 1, 2, \ldots$, of Ω the basic neighbourhoods of an element $P \in \mathbf{M}_1(\Omega)$ are defined by

$$(2.1) \qquad U(P, \varepsilon, \mathscr{P}_k) = \{R \in \mathbf{M}_1(\Omega) \colon |R(\Omega_i) - P(\Omega_i)| < \varepsilon, \textit{ for } i = 1, 2, \ldots, k, \\ \text{and } R \ll P \text{ on } \sigma(\mathscr{P}_k), \text{ i.e., } R(\Omega_i) = 0, \text{ if } P(\Omega_i) = 0\},$$

where $\sigma(\mathscr{P}_k)$ is the σ-field generated by the partition $\mathscr{P}_k$ (cf. Csiszar 1984).

For a pair of probability measures $\{q_a, q_b\}$ on $\mathbf{R}^d$, which will be fixed from now on, we consider a *subset* $A_{a,b}$ of $\mathbf{M}_1(\Omega)$ defined by

$$(2.2) \qquad A_{a,b} = \{P \in \mathbf{M}_1(\Omega) \colon P \circ X_r^{-1} = q_r, \text{ for } r = a, b\},$$

which is the class of continuous stochastic processes on $\mathbf{R}^d$ with the prescribed marginal distributions q_a and q_b at the fixed initial and terminal times a and b, respectively.

In the following we fix a probability measure $\bar{P} \in \mathbf{M}_1(\Omega)$, which will be specified in a practical application to Schrödinger processes in Sect. 4, and we always *assume* that

(2.3) *the set* $A_{a,b}$ *contains at least one element* P *with finite relative entropy* $\mathrm{H}(P|\bar{P}) < \infty$,

where the *relative entropy* $\mathrm{H}(P|\bar{P})$ of P with respect to $\bar{P}$ is defined by

$$\mathrm{H}(P|\bar{P}) = \int \left(\log \frac{dP}{d\bar{P}}\right) dP, \qquad \text{if } P \ll \bar{P} (= \infty, \text{otherwise}).$$

The measure $\bar{P}$ itself is not an element of the set $A_{a,b}$.

In order to define subsets which approximate the subset $A_{a,b}$ we consider a sequence of finite measurable partitions $\mathscr{P}_k(\mathbf{R}^d) = \{B_1, \ldots, B_k\}$ of $\mathbf{R}^d$, $k = 1, 2, \ldots$, where $\mathscr{P}_{k+1}(\mathbf{R}^d)$ is a refinement of $\mathscr{P}_k(\mathbf{R}^d)$ such that

$$(2.4) \qquad \sigma(\mathscr{P}_k(\mathbf{R}^d)) \subset \sigma(\mathscr{P}_{k+1}(\mathbf{R}^d)) \quad \text{and} \quad \sigma(\mathscr{P}_k(\mathbf{R}^d)) \uparrow \sigma(\mathbf{R}^d), \quad \text{as} \quad k \uparrow \infty.$$

In terms of the partitions $\mathscr{P}_k(\mathbf{R}^d)$ we define *a family of subsets* $A(\varepsilon, k)$ of $\mathbf{M}_1(\Omega)$, for $\varepsilon>0$ and $k=1, 2, \ldots$, by

$$(2.5)\qquad A(\varepsilon, k)=\{P\in\mathbf{M}_1(\Omega)\colon |P[X_r\in B_i]-q_r(B_i)|\leqq\frac{\varepsilon}{2^k},\quad \text{for } \forall B_i\in\mathscr{P}_k(\mathbf{R}^d),$$
$$\text{and } P\circ X_r^{-1}\ll\bar{P}\circ X_r^{-1} \text{ on } \sigma(\mathscr{P}_k(\mathbf{R}^d)), r=a, b\}.$$

The dependence of the set $A(\varepsilon, k)$ on $\varepsilon>0$ being not substantial but technical, which will be needed in Sect. 3, especially in Lemma 3.2 and Lemma 3.5, we will write $A(k)$ instead of $A(\varepsilon, k)$ for an arbitrary but fixed $\varepsilon>0$, when the explicit dependence on $\varepsilon>0$ will not be needed.

Let $\{\Omega^n, \bar{\mathbf{P}}\}$ be n-independent copies of $(\Omega, \bar{P})$, where $\bar{\mathbf{P}}=\bar{P}^n$ is the *n-product of the probability measure* $\bar{P}$. By L_n we denote the *empirical distribution*

$$(2.6)\qquad L_n(\omega)=\frac{1}{n}\sum_{i=1}^{n}\delta_{\omega_i},$$

where $\omega=(\omega_1, \ldots, \omega_n)\in\Omega^n$. We define the *conditional probability* $\mathbf{P}^{(n,k)}$ *of* $\bar{\mathbf{P}}$ *on the set* $\{\omega\in\Omega^n\colon L_n(\omega)\in A(k)\}$ in terms of the empirical distribution L_n by

$$(2.7)\qquad \mathbf{P}^{(n,k)}[\cdot]=\mathbf{P}^{(n)}_{A(k)}[\cdot]=\bar{\mathbf{P}}[\cdot\mid L_n\in A(k)].$$

Although the conditional probability depends on $\varepsilon>0$, we will not indicate this dependence explicitly following the convention stated after (2.5). Each marginal distribution on Ω of the conditional probability $\mathbf{P}^{(n,k)}$ belongs to the set $A(k)$ by lemma 4.2 of Csiszar (1984).

Instead of giving the definition of Csiszar projection, we quote Theorems 2.1 and 2.2 of Csiszar (1975) as a lemma for later reference.

Lemma 2.1 (Csiszar 1975) *Let Ω be a measurable space and $\bar{P}\in\mathbf{M}_1(\Omega)$ be fixed. If a subset A of $\mathbf{M}_1(\Omega)$ is convex and variation closed, and the subset A contains at least one element P with* $\mathrm{H}(P|\bar{P})<\infty$, *then there exists the unique* I-*projection $Q\in A$ of $\bar{P}$ on the set A such that*

$$(2.8)\qquad \inf_{P\in A}\mathrm{H}(P|\bar{P})=\mathrm{H}(Q|\bar{P}),$$

and it satisfies the inequality

$$(2.9)\qquad \mathrm{H}(P|\bar{P})\geqq\mathrm{H}(P|Q)+\mathrm{H}(Q|\bar{P}),\quad \textit{for } \forall P\in A.$$

From now on we will call the I-projection *Csiszar's projection.*

With the notations introduced above our basic theorem states

Theorem 2.1 (i) *Let $A_{a,b}$ and $A(k)$ be the subsets of* $\mathbf{M}_1(\Omega)$ *defined at* (2.2) *and* (2.5), *respectively. Then*

$$(2.10)\qquad A_{a,b}=\bigcap_{k\in\mathbf{N}}A(k),$$

and an approximate Sanov property *holds for the set* $A_{a,b}$ *in the following form*

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\log\bar{\mathbf{P}}[L_n\in A(k)]=-\inf_{P\in A_{a,b}}\mathrm{H}(P|\bar{P}), \tag{2.11}$$

where the infimum is attained by Csiszar's projection Q *of* $\bar{P}$ *on the set* $A_{a,b}$, *namely,*

$$\inf_{P\in A_{a,b}}\mathrm{H}(P|\bar{P})=\mathrm{H}(Q|\bar{P}). \tag{2.12}$$

(ii) *The process* $(X_1,\ldots,X_n)$ *with respect to the conditional probability* $\mathbf{P}^{(n,k)}$ *defined at* (2.7) *is asymptotically quasi-independent with the limiting distribution* Q, namely,

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\mathrm{H}(\mathbf{P}^{(n,k)}|Q^n)=0, \tag{2.13}$$

where Q^n *denotes the n-product of Csiszar's projection* Q.

3 Lemmas and proof of Theorem 2.1

We will apply a Csiszar's theorem (1984) on large deviations to the approximating subsets $A(\varepsilon,k)$, and then let k tend to infinity. For this we need simple Lemmas, and a proof of Theorem 2.1 will be given at the end of the section.

Lemma 3.1 *Let* $A(\varepsilon,k)$ *be the subset of* $\mathbf{M}_1(\Omega)$ *defined at* (2.5). *Then*

(i) $A(\varepsilon,k)$ *decreases as* $k\uparrow\infty$,
(ii) $A(\varepsilon,k)$ *decreases as* $\varepsilon\downarrow 0$,
(iii) $A_{a,b}=\bigcap_{k\in\mathbf{N}}A(\varepsilon,k)$, *for* $\forall\varepsilon>0$,

where $A_{a,b}$ *is the subset of* $\mathbf{M}_1(\Omega)$ *defined at* (2.2).

Proof. (i) Since $\mathscr{P}_{k+1}(\mathbf{R}^d)$ is a refinement of $\mathscr{P}_k(\mathbf{R}^d)$, if P is an element of $A(\varepsilon,k+1)$, then clearly $P\circ X_r^{-1}\ll\bar{P}\circ X_r^{-1}$ on $\sigma(\mathscr{P}_k(\mathbf{R}^d))$ for $r=a,b$. By the definition any $B\in\mathscr{P}_k(\mathbf{R}^d)$ is of the form $B=B_1\cup B_2$ with B_1, $B_2\in\mathscr{P}_{k+1}(\mathbf{R}^d)$, $B_1\cap B_2=\emptyset$, and hence we have

$$\begin{aligned}|P[X_r\in B_1\cup B_2]-q_r(B_1\cup B_2)|&\le|P[X_r\in B_1]-q_r(B_1)|+|P[X_r\in B_2]-q_r(B_2)|\\&\le\frac{\varepsilon}{2^{k+1}}+\frac{\varepsilon}{2^{k+1}}=\frac{\varepsilon}{2^k},\end{aligned}$$

for $r=a$ and b. Therefore, $P\in A(\varepsilon,k)$ and hence $A(\varepsilon,k)\supset A(\varepsilon,k+1)$.
(ii) is clear by the definition of $A(\varepsilon,k)$.

To show (iii) the inclusion "$\subset$" is obvious, since

$$A_{a,b}\subset A(\varepsilon,k),\qquad\text{for }\forall\varepsilon>0,\forall k\in\mathbf{N}.$$

In fact, by the definition (2.2) and assumption (2.3) on $A_{a,b}$, we have $q_r=P\circ X_r^{-1}\ll\bar{P}\circ X_r^{-1}$ on $\sigma(\mathscr{P}_k(\mathbf{R}^d))$, for any $P\in A_{a,b}$, and for $r=a,b$.

To show the converse inclusion " $\supset$ " let us take $P\in\bigcap_{k\in\mathbf{N}} A(\varepsilon,k)$, for $\varepsilon>0$. Suppose $P\notin A_{a,b}$, i.e., there are $\varepsilon_0>0$ and a subset $B\in\sigma(\mathbf{R}^d)$ satisfying

$$|P[X_r\in B]-q_r(B)|>\varepsilon_0, \quad \text{for } r=a \quad \text{or} \quad b.$$

Because $\sigma(\mathscr{P}_k(\mathbf{R}^d))\uparrow\sigma(\mathbf{R}^d)$, we can find $k\in\mathbf{N}$ and $B_i\in\mathscr{P}_k(\mathbf{R}^d)$ such that

$$|P[X_r\in B_i]-q_r(B_i)|>\frac{\varepsilon}{2^k}, \quad \text{for } r=a \quad \text{or} \quad b,$$

which is a contradiction. Thus " $\supset$ " holds.

Lemma 3.2 *Let $A^\circ(\varepsilon,k)$ denote the interior of the set $A(\varepsilon,k)$ with respect to the τ_0-topology. Then, for any $\varepsilon_0>0$*

(3.1) $$A^\circ(\varepsilon_0,k)\supset\bigcup_{0<\varepsilon<\varepsilon_0} A(\varepsilon,k).$$

Proof. Let $P_0\in\bigcup_{0<\varepsilon<\varepsilon_0} A(\varepsilon,k)$, i.e., $P_0\in A(\varepsilon_1,k)$ for $0<\varepsilon_1<\varepsilon_0$. For each $B_i\in\mathscr{P}_k(\mathbf{R}^d)$ define Ω_i^r by

$$\Omega_i^r=\{\omega\colon X_r(\omega)\in B_i\}, \quad \text{for } r=a,b,$$

and a finite measurable partition $\mathscr{P}$ of Ω by

$$\mathscr{P}=\mathscr{P}(\Omega)=\{\Omega_i^a\cap\Omega_j^b\colon i,j=1,2,\ldots,k\}.$$

Let $P\in U(P_0,\ \varepsilon_2,\ \mathscr{P})$. Then, $P\ll P_0$ on $\sigma(\mathscr{P})$, and moreover $P_0\circ X_r^{-1}\ll\bar{P}\circ X_r^{-1}$ and hence $P\circ X_r^{-1}\ll\bar{P}\circ X_r^{-1}$ on $\sigma(\mathscr{P}_k(\mathbf{R}^d))$, for $r=a,b$. Furthermore, for $\forall B_i\in\mathscr{P}_k(\mathbf{R}^d)$

$$\begin{aligned}|P[X_r\in B_i]-q_r(B_i)|&\leqq|P[\Omega_i^r]-P_0[\Omega_i^r]|+|P_0[X_r\in B_i]-q_r(B_i)|\\&\leqq k\varepsilon_2+\frac{\varepsilon_1}{2^k}, \quad \text{for } r=a,b.\end{aligned}$$

Therefore, if we choose a constant $\varepsilon_2>0$ so that $k\varepsilon_2+\frac{\varepsilon_1}{2^k}\leqq\frac{\varepsilon_0}{2^k}$, then $P\in A(\varepsilon_0,k)$, i.e., $U(P_0,\varepsilon_2,\mathscr{P})\subset A(\varepsilon_0,k)$, which implies (3.1).

Lemma 3.3 *The set $A(\varepsilon,k)$ is completely convex in the sense of Csiszar* (1984) *(hence convex) and variation closed.*

Proof. Let $(\Lambda,\mathscr{A},\mu)$ be an arbitrary probability space and $\eta(\lambda,B)$ be a probability kernel on $\Lambda\times\sigma(\Omega)$ such that $\eta(\lambda,\cdot)\in A(\varepsilon,k)$, for $\forall\lambda\in\Lambda$. Then, for any $B_i\in\mathscr{P}_k(\mathbf{R}^d)$,

$$\begin{aligned}|\mu\eta[X_r\in B_i]-q_r(B_i)|&\leqq\int|\eta(\lambda,\{X_r\in B_i\})-q_r(B_i)|\,\mu(d\lambda)\\&\leqq\frac{\varepsilon}{2^k}, \quad \text{for } r=a,b.\end{aligned}$$

Moreover, it is clear that $\mu\eta\circ X_r^{-1}\ll\bar{P}\circ X_r^{-1}$ on $\sigma(\mathscr{P}_k(\mathbf{R}^d))$, for $r=a$ and b. Therefore, $\mu\eta\in A(\varepsilon,k)$, i.e. the set $A(\varepsilon,k)$ is completely convex.

To show that the set $A(\varepsilon, k)$ is variation closed, let $\{P_n\} \subset A(\varepsilon, k)$ be a sequence which converges to $P \in \mathbf{M}_1(\Omega)$ in variation with respect to $\bar{P}$, namely,

$$|P_n - P|_{\mathrm{Var}.\bar{P}} = \int \left| \frac{dP_n}{d\bar{P}} - \frac{dP}{d\bar{P}} \right| d\bar{P} \to 0, \quad \text{as} \quad n \to \infty.$$

Then, for any $B_i \in \mathscr{P}_k(\mathbf{R}^d)$ and $r = a, b$,

$$|P[X_r \in B_i] - q_r(B_i)| \leqq |P[X_r \in B_i] - P_n[X_r \in B_i]| + |P_n[X_r \in B_i] - q_r(B_i)|,$$

where the second term on the right-hand side

$$|P_n[X_r \in B_i] - q_r(B_i)| \leqq \frac{\varepsilon}{2^k}$$

and the first term

$$|P[X_r \in B_i] - P_n[X_r \in B_i]| \leqq |P_n - P|_{\mathrm{Var}.\bar{P}} \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore,

$$|P[X_r \in B_i] - q_r(B_i)| \leqq \frac{\varepsilon}{2^k},$$

and hence $P \in A(\varepsilon, k)$, i.e., the set $A(\varepsilon, k)$ is variation closed.

In the following let us denote for simplicity

$$\text{(3.2)} \qquad \mathrm{H}(A \mid \bar{P}) = \inf_{P \in A} \mathrm{H}(P \mid \bar{P}),$$

for a subset $A \subset \mathbf{M}_1(\Omega)$.

Lemma 3.4 *Let* $\varepsilon > 0$ *and* $k \in \mathbf{N}$. *Then*

$$\text{(3.3)} \qquad -\mathrm{H}(A^\circ(\varepsilon, k) \mid \bar{P}) \leqq \liminf_{n \to \infty} \frac{1}{n} \log \bar{\mathbf{P}}[L_n \in A(\varepsilon, k)]$$

$$\leqq \limsup_{n \to \infty} \frac{1}{n} \log \bar{\mathbf{P}}[L_n \in A(\varepsilon, k)] \leqq -\mathrm{H}(A(\varepsilon, k) \mid \bar{P}),$$

where $\mathrm{H}(A(\varepsilon, k) \mid \bar{P})$ *is attained by Csiszar's projection* $P_{\varepsilon,k}$ *of* $\bar{P}$ *on the subset* $A(\varepsilon, k)$, *i.e.,*

$$\mathrm{H}(A(\varepsilon, k) \mid \bar{P}) = \mathrm{H}(P_{\varepsilon,k} \mid \bar{P}).$$

Proof. Since the set $A(\varepsilon, k)$ is completely convex by Lemma 3.3, we have

$$\frac{1}{n} \log \bar{\mathbf{P}}[L_n \in A(\varepsilon, k)] \leqq -\mathrm{H}(A(\varepsilon, k) \mid \bar{P}), \qquad \text{for } \forall n \in \mathbf{N},$$

by theorem 1 of Csiszar (1984), and the lower bound holds by lemma 4.1 in the same paper. Moreover, since the set $A(\varepsilon, k)$ is variation-closed by Lemma 3.3, and there is at least one $P \in A(\varepsilon, k)$ with $\mathrm{H}(P \mid \bar{P}) < \infty$ by the assumption (2.3) imposed on the set $A_{a,b}$, there exists Csiszar's projection $P_{\varepsilon,k}$ on the set $A(\varepsilon, k)$ by Lemma 2.1.

Lemma 3.5 (i) *There exists Csiszar's projection Q of $\bar{P}$ on the subset $A_{a,b}$ of $\mathbf{M}_1(\Omega)$ defined at* (2.2), *namely,*

$$\mathrm{H}(A_{a,b}|\bar{P})=\mathrm{H}(Q|\bar{P}). \tag{3.4}$$

(ii) *Moreover*

$$\lim_{k\to\infty}\mathrm{H}(A^\circ(\varepsilon,k)|\bar{P})=\lim_{k\to\infty}\mathrm{H}(A(\varepsilon,k)|\bar{P})=\mathrm{H}(Q|\bar{P}), \quad \textit{for } \forall\varepsilon>0. \tag{3.5}$$

(iii) *Csiszar's projection $P_{\varepsilon,k}$ of $\bar{P}$ on $A(\varepsilon,k)$ converges to Csiszar's projection Q of $\bar{P}$ on $A_{a,b}$ in entropy, and hence weakly as $k\to\infty$.*

Proof. Since the subset $A_{a,b}$ is also convex and variation closed because of (iii) of Lemma 3.1 and Lemma 3.3, there exists Csiszar's projection Q of $\bar{P}$ on $A_{a,b}$ by Lemma 2.1 under the entropy condition (2.3) imposed on the set $A_{a,b}$. Let $0<\varepsilon_1<\varepsilon_2$. Then by combining Lemma 3.1 with Lemma 3.2 we have

$$A^\circ(\varepsilon_1,k)\subset A(\varepsilon_1,k)\subset A^\circ(\varepsilon_2,k)\subset A(\varepsilon_2,k), \tag{3.6}$$

and hence

$$\mathrm{H}(A^\circ(\varepsilon_1,k)|\bar{P})\geqq\mathrm{H}(A(\varepsilon_1,k)|\bar{P})\geqq\mathrm{H}(A^\circ(\varepsilon_2,k)|\bar{P})\geqq\mathrm{H}(A(\varepsilon_2,k)|\bar{P}). \tag{3.7}$$

On the other hand,

$$\mathrm{H}(Q|\bar{P})=\mathrm{H}(A_{a,b}|\bar{P})\geqq\mathrm{H}(A(\varepsilon,k)|\bar{P})=\mathrm{H}(P_{\varepsilon,k}|\bar{P}). \tag{3.8}$$

Therefore, the lower semi-continuity of the relative entropy combined with Lemma 3.1 yields

$$\lim_{k\to\infty}\mathrm{H}(A(\varepsilon,k)|\bar{P})=\mathrm{H}(A_{a,b}|\bar{P}), \quad \text{for } \forall\varepsilon>0, \tag{3.9}$$

from which we can also conclude, because of (3.7),

$$\lim_{k\to\infty}\mathrm{H}(A^\circ(\varepsilon,k)|\bar{P})=\mathrm{H}(A_{a,b}|\bar{P}), \quad \text{for } \forall\varepsilon>0. \tag{3.10}$$

Thus, (3.5) is shown. The third statement follows from Csiszar's inequality (2.9)

$$\mathrm{H}(Q|\bar{P})-\mathrm{H}(P_{\varepsilon,k}|\bar{P})\geqq\mathrm{H}(Q|P_{\varepsilon,k}), \tag{3.11}$$

because $Q\in A_{a,b}\subset A(\varepsilon,k)$. The left-hand side of (3.11) vanishes as k tends to infinity, because of (3.8) and (3.9). Therefore,

$$\lim_{k\to\infty}\mathrm{H}(Q|P_{\varepsilon,k})=0, \quad \text{for } \forall\varepsilon>0,$$

that is, $P_{\varepsilon,k}$ converges to Q in entropy, and hence weakly (cf., e.g. lemma 3.1 in Csiszar 1975), which completes the proof.

Proof of Theorem 2.1 To show the equality (2.11) we let $n \to \infty$ and then $k \to \infty$ in

$$
\begin{aligned}
(3.12) \qquad & \left| \frac{1}{n} \log \bar{\mathbf{P}}[L_n \in A(\varepsilon, k)] + \mathrm{H}(Q \mid \bar{P}) \right| \\
& \leqq \left| \frac{1}{n} \log \bar{\mathbf{P}}[L_n \in A(\varepsilon, k)] + \mathrm{H}(A(\varepsilon, k) \mid \bar{P}) \right| \\
& \quad + \left| -\mathrm{H}(A(\varepsilon, k) \mid \bar{P}) + \mathrm{H}(Q \mid \bar{P}) \right|.
\end{aligned}
$$

Then, the first term on the right-hand side converges to zero by Lemma 3.4 together with Lemma 3.5, and the second term vanishes by Lemma 3.5. The second assertion follows from Csiszar's inequality

$$
(3.13) \qquad 0 \leqq \frac{1}{n} \mathrm{H}(\mathbf{P}^{(n,k)} \mid Q^n) \leqq -\mathrm{H}(A(\varepsilon, k) \mid \bar{P}) - \frac{1}{n} \log \bar{\mathbf{P}}[L_n \in A(\varepsilon, k)],
$$

(cf. (2.17) in Csiszar 1984) which connects the speed of convergence of the conditional probability $\mathbf{P}^{(n,k)}$ to the speed of convergence in the approximate Sanov property. The right-hand side of (3.13) vanishes as $n \to \infty$ and $k \to \infty$, because of the first assertion of Theorem 2.1 combined with Lemma 3.5, and hence we have (2.13), completing the proof.

Remark. We have stated Theorem 2.1 for probability measures on the space of continuous paths, since we intend to apply it to diffusion processes on $\mathbf{R}^d$. However, as we have seen, it is clear that the continuity of paths plays no role and can be avoided. Let $(S, \sigma(S))$ be a measurable space with a countably generated σ-field $\sigma(S)$, and Ω be the space of measurable functions on $[a, b]$ taking values in S. Consider the space $\mathbf{M}_1(\Omega)$ of probability measures on this space Ω. Then, Theorem 2.1 and all Lemmas in Sect. 3 remain valid on this setting.

4 Large deviations for Schrödinger processes and the propagation of chaos

Let $\{(t, X_t), P_{(s,x)}; (s, x) \in [a, b] \times \mathbf{R}^d\}$ be a space-time diffusion process determined by a diffusion equation

$$
(4.1) \qquad \frac{\partial p}{\partial t} + \frac{1}{2} \Delta p + \mathbf{a} \nabla p = 0,
$$

where Δ denotes the Laplace-Beltrami operator

$$
\Delta = \frac{1}{\sqrt{A}} \frac{\partial}{\partial x^i} \left(\sqrt{A}\, A^{ij} \frac{\partial}{\partial x^j} \right),
$$

with a bounded positive definite symmetric diffusion matrix $A^{jk}(t, x) = (\sigma^{jk}(t, x))^2$, $A = A(t, x) = |A_{ij}(t, x)|$. We assume that a vector potential $\mathbf{a}(t, x)$ satisfies a gauge condition $\nabla \mathbf{a} = 0$. We require necessary conditions which guarantee, by any means (cf., e.g. Stroock and Varadhan 1970), the existence of diffusion processes

$\{X_t, \mathrm{P}_{(s,x)}\}$ corresponding to the diffusion equation (4.1) (assuming e.g. the Novikov condition on $\mathbf{a}(t,x)$).

Let $c(s,x)$ be a measurable function on $[a,b]\times\mathbf{R}^d$ which may be unbounded or singular with the positive and negative parts denoted by $c^+(s,x)$ and $c^-(s,x)$, respectively. We consider diffusion processes with creation and killing $c(s,x) = c^+(s,x) - c^-(s,x)$. In terms of the killing part $c^-(s,x)$ setting

$$T_s = \inf\left\{t > s\colon \int_s^t c^-(r, X_r)\,dr = \infty\right\},$$

where $T_s = \infty$, if there is no such t, we adopt as the *state space* a subset $D \subset [a,b]\times\mathbf{R}^d$ defined by

$$D = \{(s,x)\colon \mathrm{P}_{(s,x)}[b < T_s] > 0\}.$$

Let us define, first of all, the *measure* $\mathrm{P}^-_{(s,x)}$ *with killing* $c^-(s,x)$ by

$$\mathrm{P}^-_{(s,x)}[F] = \mathrm{P}_{(s,x)}\left[\exp\left(-\int_s^b c^-(r,X_r)\,dr\right) F(\cdot)\,1_{\{b<T_s\}}\right], \tag{4.2}$$

for any non-negative measurable function F, where $\mathrm{P}[F]$ denotes the integral of F with respect to a measure P. Notice that the exponential function on the right-hand side in (4.2) is positive on the set $\{b < T_s\}$ and hence $\mathrm{P}^-_{(s,x)}[1] > 0$ for $\forall (s,x) \in D$. This means that with a positive probability a particle starting at a point in D survives until the terminal time b.

Then we require an *integrability condition* of the *creation* part $c^+(s,x)$

$$\mathrm{P}^-_{(s,x)}\left[\exp\left(\int_s^b c^+(r,X_r)\,dr\right)\right] < \infty, \text{ for } \forall s \text{ and a.e. } x \text{ in } (s,x)\in D, \tag{4.3}$$

where $\mathrm{P}^-_{(s,x)}$ is the measure defined at (4.2).

Now we define the *measure* $\mathrm{P}^c_{(s,x)}$ *with creation and killing* $c(s,x)$ by

$$\begin{aligned}\mathrm{P}^c_{(s,x)}[F] &= \mathrm{P}^-_{(s,x)}\left[\exp\left(\int_s^b c^+(r,X_r)\,dr\right) F(\cdot)\right] \\ &= \mathrm{P}_{(s,x)}\left[\exp\left(\int_s^b c(r,X_r)\,dr\right) F(\cdot)\,1_{\{b<T_s\}}\right].\end{aligned} \tag{4.4}$$

In terms of the measure $\mathrm{P}^c_{(s,x)}$ we define the function $\xi(s,x)$ by

$$\xi(s,x) = \mathrm{P}^c_{(s,x)}[1],$$

which is positive and finite in D because of the definition of D and (4.3).

The *renormalized measure* $\bar{P}$ of $\mathrm{P}^c_{(s,x)}$ is defined by

$$\bar{P}[F] = \int_{D_a} k(dx)\,\frac{1}{\xi(a,x)}\,\mathrm{P}^c_{(a,x)}[F], \tag{4.5}$$

where $k(dx)=k(x)dx$ with a probability density $k(x)>0$ and $D_a=\{x\colon (a,x)\in D\}$. We substitute this renormalized measure in place of the reference probability measure $\bar{P}$ which was fixed in Sect. 2.

In order to formulate the entropy requirement (2.3) on the set $A_{a,b}$ defined at (2.2) in terms of marginal distributions at the initial and terminal times, we define a probability measure $\bar{p}$ on $\mathbf{R}^d\times\mathbf{R}^d$ by

$$\bar{p}(A\times B)=\int_{D_a} k(dx)\frac{1}{\xi(a,x)}1_A(x)\,\mathrm{P}^{\mathrm{c}}_{(a,x)}[1_B(X_b)], \tag{4.6}$$

where we choose $k(x)>0$ so that $\log(k(x)/\xi(a,x))\in L^1(q_a)$, and consider the set $\mathscr{E}(a,b)$ of all probability measures on $\mathbf{R}^d\times\mathbf{R}^d$ with the given pair $\{q_a, q_b\}$ of probability measures on $\mathbf{R}^d$ as their marginal distributions at a and b. Then we assume:

(4.7) *There exists* $p\in\mathscr{E}(a,b)$ *with* $\mathrm{H}(p\,|\,\bar{p})<\infty$.

This condition requires that the distribution q_a must be absolutely continuous with respect to the distribution $k(dx)$, and the process with creation and killing must reach the support of the distribution q_b at the terminal time b with positive probability. Then we have

Lemma 4.1 (Föllmer 1988, Nagasawa 1990b) *Assume the conditions* (4.3) *and* (4.7). *Then there exists the unique*[1] *non-negative solution* $\{\hat{\phi}_a, \phi_b\}$ *for the* Schrödinger system:

$$q_a(A)=\int_{D_a}\hat{\phi}_a(x)\,1_A(x)\,dx\,\mathrm{P}^{\mathrm{c}}_{(a,x)}[\phi_b(X_b)],$$

$$q_b(B)=\int_{D_a}\hat{\phi}_a(x)\,dx\,\mathrm{P}^{\mathrm{c}}_{(a,x)}[\phi_b(X_b)\,1_B(X_b)],$$

with $\log\hat{\phi}_a\in L^1(q_a)$ *and* $\log\phi_b\in L^1(q_b)$, *where* $\mathrm{P}^{\mathrm{c}}_{(a,x)}$ *is defined at* (4.4).

In terms of the solution $\{\hat{\phi}_a, \phi_b\}$ of the Schrödinger system and the measure $\mathrm{P}^{\mathrm{c}}_{(a,x)}$ with creation and killing given at (4.4) we *define* a probability measure Q on $(\Omega, \sigma(\Omega))$ by

$$Q[F]=\int_{D_a}\hat{\phi}_a(x)\,dx\,\mathrm{P}^{\mathrm{c}}_{(a,x)}[F(\cdot)\,\phi_b(X_b)]. \tag{4.8}$$

Then we have (cf. Nagasawa 1990b)

Proposition 4.1 *The Csiszar projection Q of the renormalized measure $\bar{P}$ on the set $A_{a,b}$ coincides with the one defined at* (4.8).

We call $\{X_t, Q\}$ the *Schrödinger process for a given triplet* $\{c, q_a, q_b\}$. This is a diffusion process with the initial distribution q_a and with the drift $\nabla\log\phi_t$, which is singular if $\phi_t(x)$ vanishes on a subset, where

$$\phi_t(x)=\mathrm{P}_{(t,x)}\left[\exp\left(\int_t^b c(r,X_r)\,dr\right)\phi_b(X_b)\,1_{\{b<T_t\}}\right]$$

(cf. Nagasawa (1989, 1990b), Aebi (1989, Preprint)).

[1] Up to multiplicative constants depending on regions separated by the zero set of the solution

Then, the second assertion of Theorem 2.1 claims the asymptotic quasi-independence

(4.9) $$\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\mathrm{H}(\mathbf{P}^{(n,k)}|Q^n)=0,$$

i.e., the conditional process $\{(X_1,\ldots,X_n),\mathbf{P}^{(n,k)}\}$ converges in entropy and hence weakly to the infinite product of the Schrödinger process Q as $n\to\infty$ and $k\to\infty$. However, the conditional process $\{(X_1,\ldots,X_n),\mathbf{P}^{(n,k)}\}$ is not Markovian. Hence, we will need a *Markovian modification* $\mathbf{Q}^{(n,k)}$ of $\mathbf{P}^{(n,k)}$ which satisfies the entropy condition (4.10) below.

Let $\mathbf{P}$ and $\mathbf{Q}$ be probability measures on the space of right-continuous paths with the time parameter t running in $[a,b]$. If $\mathbf{Q}$ is Markovian and the marginal distributions at time t of $\mathbf{P}$ and $\mathbf{Q}$ coincide for $\forall t\in[a,b]$, then $\mathbf{Q}$ is called a *Markovian modification* of $\mathbf{P}$.

Lemma 4.2 *Let* $\mathbf{P}$ and $\mathbf{P}_0$ *be probability measures on the space of right-continuous (resp. continuous) paths satisfying* $\mathrm{H}(\mathbf{P}|\mathbf{P}_0)<\infty$, *where* $\mathbf{P}_0$ *is a Markov (resp. diffusion) process. Then there exists the unique Markovian (resp. diffusion) modification* $\mathbf{Q}$ *of* $\mathbf{P}$ *such that*

(4.10) $$\mathrm{H}(\mathbf{Q}|\mathbf{P}_0)\leqq\mathrm{H}(\mathbf{P}|\mathbf{P}_0).$$

Proof. Define a set A of probability measures $\mathbf{R}$ on the path space by

(4.11) $$A=\{\mathbf{R}:\mathbf{R}\circ X_t^{-1}=\mathbf{P}\circ X_t^{-1},\text{for }\forall t\in[a,b]\},$$

where X_t denotes the path function. Then the set A is convex and variation closed. Therefore, there exists a unique Csiszar's projection $\mathbf{Q}$ of $\mathbf{P}_0$ on the set A by Lemma 2.1, such that

$$\mathrm{H}(\mathbf{Q}|\mathbf{P}_0)=\mathrm{H}(A|\mathbf{P}_0).$$

Let us prove that $\mathbf{Q}$ is Markovian. Setting

$$t_j^{(m)}=\left(a+\frac{j}{2^m}\right)\wedge b,\qquad\text{for } m\in\mathbf{N},$$

we define a sequence of subsets $A^{(m)}$ of probability measures $\mathbf{R}$ by

(4.12) $$A^{(m)}=\{\mathbf{R}:\mathbf{R}\circ X_t^{-1}=\mathbf{P}\circ X_t^{-1},\text{for } t=t_j^{(m)},\forall j=0,1,2,\ldots\}.$$

Then the sets $A^{(m)}$ are convex and variation closed, and satisfy

(4.13) $$A^{(m)}\supset A^{(m+1)}\supset A,\quad\text{and}\quad\bigcap_m A^{(m)}=A,$$

because of the right-continuity of paths. Consequently,

(4.14) $$\mathrm{H}(A|\mathbf{P}_0)\geqq\mathrm{H}(A^{(m+1)}|\mathbf{P}_0)\geqq\mathrm{H}(A^{(m)}|\mathbf{P}_0),$$

and, furthermore, there exist Csiszar's projections $\mathbf{Q}^{(m)}$ such that

(4.15) $$\mathrm{H}(\mathbf{Q}^{(m)}|\mathbf{P}_0)=\mathrm{H}(A^{(m)}|\mathbf{P}_0),$$

respectively, by Lemma 2.1. It is clear that $\mathbf{Q}$ and $\mathbf{Q}^{(m)}$ are absolutely continuous with respect to $\mathbf{P}_0$, because of the assumption of the lemma combined with

(4.16) $$\mathrm{H}(\mathbf{Q}^{(m)}|\mathbf{P}_0) \leqq \mathrm{H}(\mathbf{Q}|\mathbf{P}_0) = \mathrm{H}(A|\mathbf{P}_0) \leqq \mathrm{H}(\mathbf{P}|\mathbf{P}_0) < \infty.$$

Then, the lower semi-continuity of the relative entropy, together with (4.13), (4.14) and (4.15), yields

(4.17) $$\lim_{m \to \infty} \mathrm{H}(\mathbf{Q}^{(m)}|\mathbf{P}_0) = \mathrm{H}(\mathbf{Q}|\mathbf{P}_0).$$

Because of Csiszar's inequality (2.9), we have

$$\mathrm{H}(\mathbf{Q}|\mathbf{P}_0) - \mathrm{H}(\mathbf{Q}^{(m)}|\mathbf{P}_0) \geqq \mathrm{H}(\mathbf{Q}|\mathbf{Q}^{(m)}),$$

since $\mathbf{Q} \in A^{(m)}$. Therefore,

(4.18) $$\lim_{m \to \infty} \mathrm{H}(\mathbf{Q}|\mathbf{Q}^{(m)}) = 0,$$

i.e., $\mathbf{Q}^{(m)}$ converges to $\mathbf{Q}$ in entropy, and hence weakly as m tends to infinity (cf., e.g. lemma 3.1 of Csiszar 1975). The process $\mathbf{Q}^{(m)}$ is Markovian, since it is a Schrödinger process on each subinterval $[t_j^{(m)}, t_{j+1}^{(m)}]$, $j = 0, 1, 2, \ldots$, with the marginal distributions $\mathbf{P} \circ X_t^{-1}$, for $t = t_j^{(m)}$ and $t_{j+1}^{(m)}$. Consequently, the process $\mathbf{Q}$ is Markovian as the limit of the sequence of the Markovian processes $\mathbf{Q}^{(m)}$. In fact, adding t and $t+r$, if necessary, to the set $\{t_j^{(m)}\}$ of the end points of subdivisions of the time interval in (4.12) and setting

$$P_r^{(m)} f(X_t) = \mathbf{Q}^{(m)}[f(X_{t+r})|X_t], \quad \text{and} \quad P_r^{(\infty)} f(X_t) = \mathbf{Q}[f(X_{t+r})|X_t],$$

we have for bounded measurable f

$$\lim_{m \to \infty} P_r^{(m)} f(X_t) = P_r^{(\infty)} f(X_t), \ \mathbf{P}\text{-a.e.},$$

and, with $n \leqq m$, $t_j = t_j^{(n)} \leqq t$, $t_k = t$, and bounded measurable g_j,

$$\begin{aligned}
&\mathbf{Q}\left[\prod_{j=1}^{k} g_j(X_{t_j}) f(X_{t+r})\right] \\
&= \lim_{m \to \infty} \mathbf{Q}^{(m)}\left[\prod_{j=1}^{k} g_j(X_{t_j}) f(X_{t+r})\right] = \lim_{m \to \infty} \mathbf{Q}^{(m)}\left[\prod_{j=1}^{k} g_j(X_{t_j}) P_r^{(m)} f(X_t)\right] \\
&= \lim_{m \to \infty} \mathbf{P}\left[\prod_{j=1}^{k} g_j(X_{t_j}) P_r^{(m)} f(X_t)\right] = \mathbf{P}\left[\prod_{j=1}^{k} g_j(X_{t_j}) P_r^{(\infty)} f(X_t)\right] \\
&= \mathbf{Q}\left[\prod_{j=1}^{k} g_j(X_{t_j}) P_r^{(\infty)} f(X_t)\right].
\end{aligned}$$

If $\mathbf{P}_0$ is a diffusion process, and $\mathbf{b}(t, \mathbf{x})$ (resp. $\mathbf{b}^{(m)}(t, \mathbf{x})$) denotes the drift coefficient of $\mathbf{Q}$ (resp. $\mathbf{Q}^{(m)}$), then the Maruyama-Girsanov theorem (cf. e.g., Liptser and Shiryayev 1977, Ikeda and Watanabe 1981, 1989) yields

$$\mathrm{H}(\mathbf{Q}\,|\,\mathbf{Q}^{(m)}) = \tfrac{1}{2}\mathbf{Q}\left[\int_a^b |\mathbf{b}(t, X_t) - \mathbf{b}^{(m)}(t, X_t)|^2\, dt\right], \tag{4.19}$$

which vanishes as $m \to \infty$, because of (4.18). Therefore, $\mathbf{b}^{(m)}(t, \mathbf{x})$ converges to $\mathbf{b}(t, \mathbf{x})$ in L^2 in the sense of the right-hand side of (4.19).

Remark. Notice that the (weak) limit of a sequence of Markov proceses is, in general, not a Markov process.

Let us now discuss the propagation of chaos for the Markovian modification $\mathbf{Q}^{(n,k)}$ with the Schrödinger process Q as limiting distribution.

In this paper we will formulate the propagation of chaos in a slightly different form to be fit for Schrödinger processes. Let Q be a Markov process on a path space Ω and $\mathbf{Q}^{(n,k)}$ be a Markov process on Ω^n, where we allow a family of doubly indexed probability measures with (n, k), as is typical in the case of Schrödinger processes.

Remark. The propagation of chaos for a system of interacting diffusion processes was introduced by McKean (1966, 1967), and is discussed in Nagasawa and Tanaka (1986, 1987a, b) in connection with Schrödinger processes. Cf. Tanaka (1984), Kusuoka and Tamura (1984), Ölschläger (1989), Sznitman (1989) and Dawson and Gärtner (1989) for the propagation of chaos.

We say the *propagation of chaos* holds for the Markov process $\mathbf{Q}^{(n,k)}$ with the limiting distribution Q as $n \to \infty$ and $k \to \infty$, if it has the following two properties

(4.20) $\mathbf{Q}^{(n,k)}$ *is asymptotically quasi-independent with the limiting distribution* Q,

(4.21) $\mathbf{Q}^{(n,k)}[L_n]$ *converges to the* Q *in entropy*

as $n \to \infty$ and $k \to \infty$, where L_n is the empirical distribution of $(X_1, \ldots, X_n)$.

Then, we have, applying Theorem 2.1 and Lemma 4.2,

Theorem 4.1 *Assume the conditions* (4.3) *and* (4.7). *Then:*

(i) *There exists the unique Markovian modification* $\mathbf{Q}^{(n,k)}$ *of* $\mathbf{P}^{(n,k)}$ *such that*

$$\mathrm{H}(\mathbf{Q}^{(n,k)}\,|\,Q^n) \leqq \mathrm{H}(\mathbf{P}^{(n,k)}\,|\,Q^n). \tag{4.22}$$

(ii) *The Markovian modification* $\{(X_1, \ldots, X_n), \mathbf{Q}^{(n,k)}\}$ *is a system of interacting diffusion processes with the Markovian drift coefficient* $\mathbf{b}^{n,k}(t, \mathbf{x})$ *(=interaction),* $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathbf{R}^d)^n$, *such that*

$$\mathbf{b}^{n,k}(t, \mathbf{x}) = \{\mathbf{b}_i^{n,k}(t, \mathbf{x}, L_n(\mathbf{x})) : i = 1, 2, \ldots, n\}, \tag{4.23}$$

where $\mathbf{b}_i^{n,k}$ *is the drift vector of* X_i.

(iii) *The approximate Sanov property*

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \bar{\mathbf{P}}[L_n \in A(k)] = -\inf_{P \in A_{a,b}} \mathrm{H}(P\,|\,\bar{P}) = -\mathrm{H}(Q\,|\,\bar{P}),$$

holds, where $A(k)\downarrow A_{a,b}$ *as* $k\to\infty$ *(cf. Lemma* 3.1*). In other words, the probability that the rare event* $\{L_n\in A(k)\}$ *occurs is given by*

$$\bar{\mathbf{P}}[L_n\in A(k)]\approx e^{-n\mathrm{H}(Q|\bar{P})}, \quad as \quad n\to\infty \quad and \quad k\to\infty.$$

(iv) *The Markovian modification* $\mathbf{Q}^{(n,k)}$ *is asymptotically quasi-independent with the Schrödinger process* Q *as limiting distribution;*

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{n}\mathrm{H}(\mathbf{Q}^{(n,k)}|Q^n)=0. \tag{4.24}$$

(v) *Moreover, the propagation of chaos holds for the Markovian modification* $\{(X_1,\ldots,X_n), \mathbf{Q}^{(n,k)}\}$ *with the Schrödinger process* Q *as limiting distribution, when* $n\to\infty$ *and* $k\to\infty$*. In fact,* $\mathbf{Q}^{(n,k)}$ *converges in entropy to the infinite product of the Schrödinger process* Q*, and* $\mathbf{Q}^{(n,k)}[L_n]$ *to* Q *also in entropy.*

Proof. We apply Lemma 4.2 to $\mathbf{P}=\mathbf{P}^{(n,k)}$ and $\mathbf{P}_0=Q^n$, where the assumption of the lemma is satisfied, because of (4.9). Therefore, there exists the unique Markovian modification $\mathbf{Q}^{(n,k)}$ of $\mathbf{P}^{(n,k)}$ with the Markovian drift coefficient $\mathbf{b}^{n,k}(t,\mathbf{x})$. Moreover, $\mathbf{P}^{(n,k)}$ and hence $\mathbf{Q}^{(n,k)}$ both depend on the empirical distribution L_n of $(X_1,\ldots,X_n)$ through the conditioning (cf. the definition at (2.7)). Therefore, the drift coefficient of $\mathbf{Q}^{(n,k)}$ has the form of (4.23). The asymptotic quasi-independence of $\mathbf{Q}^{(n,k)}$, i.e. (4.24), holds, because of (4.9) and (4.22). For the fifth statement of the propagation of chaos we formulate

Lemma 4.3 *The asymptotic quasi-independence of* $\mathbf{Q}^{(n,k)}$ *i.e.* (4.24) *implies* (4.21).

Proof. Since $\mathbf{Q}^{(n,k)}$ is the Markovian modification of $\mathbf{P}^{(n,k)}$, it is sufficient to show that $\mathbf{P}^{(n,k)}[L_n]$ converges in entropy to Q. Let us denote by $P^{(n,k)}$ the marginal distribution of $\mathbf{P}^{(n,k)}$ on Ω. Then we have

$$\mathbf{P}^{(n,k)}[L_n]=\frac{1}{n}\sum_{i=1}^{n}\mathbf{P}^{(n,k)}[\delta_{\omega_i}]=P^{(n,k)}. \tag{4.25}$$

This combined with (4.24) and

$$\mathrm{H}(P^{(n,k)}|Q)\leqq\frac{1}{n}\mathrm{H}(\mathbf{P}^{(n,k)}|Q^n), \tag{4.26}$$

(cf. (2.10) in Csiszar 1984) yields

$$\lim_{k\to\infty}\lim_{n\to\infty}\mathrm{H}(P^{(n,k)}|Q)=0, \tag{4.27}$$

i.e., $\mathbf{P}^{(n,k)}[L_n]$ converges to Q in entropy by (4.25).
Thus the proof of Theorem 4.1 is completed.

In connection with Theorem 4.1, see Nagasawa (1990b; Preprint) concerning the question what the Schrödinger equation is. As is known, it is difficult to treat the propagation of chaos for singular diffusion processes (cf. Nagasawa and Tanaka 1986, 1987a, b). Theorem 4.1 provides a way to discuss this difficult problem, although it treats an *inverse problem* of the propagation of chaos.

Remark. We have formulated assertions in this section for diffusion processes on $\mathbf{R}^d$, but the continuity of paths is not essential and can be avoided. Let S be a Polish space and $\{(t, X_t), P_{(s,x)}; (s,x) \in [a,b] \times S\}$, be a (strong) Markov process of right-continuous paths with left limits. Then, all statements in this section remain valid for this process, except the statements on drift coefficients.

References

Aebi, R.: MN-transformed α-diffusion with singular drift. Doctoral dissertation at the University of Zürich, 1989

Aebi, R.: Diffusions with singular drift related to wave functions. (Preprint)

Csiszar, I.: I-divergence geometry of probability distributions and minimization problems. Ann. Probab. **3**, 146–158 (1975)

Csiszar, I.: Sanov property, generalized I-projection and a conditional limit theorem. Ann. Probab. **12**, 768–793 (1984)

Dawson, D.A., Gärtner, J.: Long-time fluctuations of weakly interacting diffusions. Technical report series of the laboratory for research in statistics and probability. Carleton University 1984

Dawson, D.A., Gärtner, J.: Large deviations, free energy functional and quasi-potential for a mean field model of interacting diffusions. Mem. Am. Math. Soc. **78**, n° 398 1989

Dawson, D., Gorostiza, L., Wakolbinger, A: Schrödinger processes and large deviations. J. Math. Phys. **31**, 2385–2388 (1990)

Föllmer, H.: Random fields and diffusion processes. In: Hennequin, P.L. (ed.) Ecole d'été de Saint Flour XV–XVII, 1985–1987. (Lect. Notes Math., vol. 1362, pp. 101–203) Berlin Heidelberg New York: Springer 1988

Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Tokyo: Kodansha and Amsterdam New York: North-Holland 1981/1989

Kusuoka, S., Tamura, Y.: Gibbs measures for mean field potentials. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31**, 223–245 (1984)

Liptser, R.S., Shiryayev, A.N.: Statistics of random processes I. General theory. Berlin Heidelberg New York: Springer 1977

McKean, H.P.: A class of Markov processes associated with non-linear parabolic equations. Proc. Natl. Acad. Sci. **56**, 1907–1911 (1966)

McKean, H.P.: Propagation of chaos for a class of nonlinear parabolic equations. (Lecture Series in Differential Equations, Catholic University, pp. 41–57) Arlington, VA: Air Force Office Sci. Res. 1967

Nagasawa, M.: Transformations of diffusion and Schrödinger processes. Probab. Theory Relat. Fields **82**, 109–136 (1989)

Nagasawa, M.: Stochastic variational principle of Schrödinger processes. In: Cinlar, E. et al. (eds.) Seminar on stochastic processes. Boston Basel Stuttgart: Birkhäuser 1990a

Nagasawa, M.: Can the Schrödinger equation be a Boltzmann equation? In: Pinsky, M. (ed.) Diffusion processes and related problems in analysis. Boston Basel Stuttgart: Birkhäuser 1990b

Nagasawa, M.: The equivalence of diffusion and Schrödinger equations: A solution to Schrödinger's conjecture. Proc. Localno Conference 1991, Albeverio, S. (ed.)

Nagasawa, M., Tanaka, H.: Propagation of chaos for diffusing particles of two types with singular mean field interaction. Probab. Theory Relat. Fields **71**, 69–83 (1986)

Nagasawa, M., Tanaka, H.: Diffusion with interactions and collisions between coloured particles and the propagation of chaos. Probab. Theory Relat. Fields **74**, 161–198 (1987a)

Nagasawa, M., Tanaka, H.: A proof of the propagation of chaos for diffusion processes with drift coefficients not of average form. Tokyo J. Math. **10**, 403–418 (1987b)

Oelschläger, K.: Many-particle systems and the continuum description of their dynamics. University Heidelberg, 1989

Schrödinger, E.: Ueber die Umkehrung der Naturgesetze. Sitzungsber. Preuss. Akad. Wiss., Phys. Math. Kl. 144–153 (1931)

Schrödinger, E.: Théorie relativiste de l'électron et l'interprétation de la mécanique quantique. Ann. Inst. H. Poincaré **2**, 269–310 (1932)
Stroock, D.W., Varadhan, S.R.S.: Multidimensional diffusion processes. Berlin Heidelberg New York: Springer 1970
Sznitman, A.S.: Topics in propagation of chaos. Ecole d'été de probabilités de Saint Flour (1989)
Tanaka, H.: Limit theorems for certain diffusion processes with interaction. In: Itô, K. (ed.) Stochastic analysis, pp. 469–488. Toyko: Kinokuniya and Amsterdam: North-Holland 1984