Probab. Th. Rel. Fields 87, 121–128

Probability Theory and Related Fields

Transformations in functional iterated logarithm laws and regular variation

Wim Vervaat*

Mathematisch Instituut, Katholieke Universiteit, Toernooiveld 1, NL-6525 ED Nijmegen, The Netherlands

Received January 7, 1988; in revised form May 11, 1990

Summary. It is shown that functional iterated logarithm (log log) laws for geometric subsequences imply the corresponding laws for full sequences, and that the converse is not true. The implication is proved by simple algebraic arguments of regular variation type.

1 Introduction

In this paper we investigate the relations between functional iterated logarithm (log log) laws for collections of geometric subsequences and the corresponding law for the full sequence. The main conclusion is that in general the geometric subsequence law implies the full sequence law (for simple algebraic reasons), but that the converse implication need not hold true. Instances of the latter occur in cases of probabilistic interest.

Results have to be formulated in a rather abstract algebraic setting. Therefore we introduce the relevant notions in the context of Strassen's classical functional log log law for Brownian motion (Strassen (1964) and Freedman (1971)).

Let $X=(X(t))_{t\in[0,\infty)}$ be standard Brownian motion. It is a random variable with values in the metric space $F:=C[0,\infty)$ of real-valued continuous functions on $[0,\infty)$ with the topology of locally uniform convergence. Consider the transformations $\gamma(a,b):=F\to F$ defined by

$$\gamma(a,b)f:=af(b\cdot) \quad \text{for } a,b>0.$$

The $\gamma(a,b)$ form a group G, with composition of mappings as product. It becomes a metric group, isomorphic to $\mathbb{R}_+^2$, by declaring $\gamma(a,b)\mapsto(a,b)$ to be a homeomorphism. Here $\mathbb{R}_+$ denotes the group $(0,\infty)$ with multiplication. The group G acts

* Supported in part by a NATO grant for international collaboration in research

jointly continuously on F, i.e.,

(1.1) the mapping $G \times F \ni (\gamma, f) \mapsto \gamma f \in F$ is continuous.

If $(f_\alpha)_\alpha$ is a net in F and $K \subset F$, then

$$f_\alpha \rightsquigarrow K \quad \text{in } F$$

means that every subnet of (f_α) has a limit point and that K is the set of limit points of the full net. Necessarily K is compact.

Strassen's functional log log law now tells us that

(1.2) $$\gamma(a_s, s)X = a_s X(s\,\cdot) \rightsquigarrow K \quad \text{wp1 in } F \text{ as } s \to \infty$$

through $\mathbb{N}$ or $\mathbb{R}$, where

$$a_s := (2s \log\log s)^{-1/2} \quad \text{for } s > \mathrm{e}^{\mathrm{e}}$$

and

$$K := \{f \in F : f(0) = 0,\ f \text{ absolutely continuous, } \int_0^\infty (f'(t))^2\, \mathrm{d}t \leq 1\}.$$

Observe that for $u > 0$ and $s \to \infty$

(1.3) $$\gamma(a_{su}, su)\gamma(a_s, s)^{-1} = \gamma(a_{su}/a_s, u) \to \gamma(u^{-1/2}, u) \quad \text{in } G.$$

Now $(\gamma(u^{-1/2}, u))_{u>0}$ is a subgroup of G, isomorphic to $\mathbb{R}_+$ via $\gamma(u^{-1/2}, u) \mapsto u$. Moreover

(1.4) $$\gamma(u^{-1/2}, u)K = K \quad \text{for } u > 0.$$

Properties (1.3) and (1.4) will be crucial in our main result.

2 Regular variation

Let G be a metric group with unit element ι, not necessarily commutative. We say that a G-valued function φ on a neighborhood of ∞ in $\mathbb{R}$ is *regularly varying* if φ is measurable and

(2.1) $$\lim_{s \to \infty} \varphi(su)\varphi(s)^{-1} =: \chi(u)$$

exists in G for all $u > 0$. We had an example of this in (1.3) with $\varphi(s) = \gamma(a_s, s)$ and $\chi(u) = \gamma(u^{-1/2}, u)$.

Theorem 1 (cf. Balkema (1973, §9)). *If $\varphi : (a, \infty) \to G$ is regularly varying and χ is as in (2.1), then χ is a continuous homomorphism from $\mathbb{R}_+$ into G: $\chi(uv) = \chi(u)\chi(v)$, and the convergence in (2.1) holds locally uniformly in* $(0, \infty)$, *so*

(2.2) $$\lim_{s \to \infty} \varphi(su_s)\varphi(s)^{-1} = \chi(u)$$

whenever u_s is a function of s such that $u_s \to u$ in $(0, \infty)$ *as* $s \to \infty$.

We call φ *χ-varying*. If $\chi \equiv \iota$, we call φ *slowly varying*. The special case $G = \mathbb{R}_+$ (so $\chi(u) = u^\varrho$ for some $\varrho \in \mathbb{R}$) is classical regular variation (de Haan (1970), Seneta (1976), Bingham et al. (1987)).

3 The results

More generally than in Section 1, let F be a metric space and G a metric group that acts jointly continuously on F, i.e., (1.1) holds. We start with a nonprobabilistic result.

Lemma 1. *Let $\varphi:(a,\infty)\to G$ be χ-varying and fix $f\in F$.*
(a) *Fix $b>1$. If*

$$\varphi(b^n)f \rightsquigarrow K_{f,b} \quad \text{in } F \text{ as } n\to\infty, \tag{3.1}$$

for some compact $K_{f,b}\subset F$, then

$$\varphi(s)f \rightsquigarrow K_f := \bigcup_{u\geqq 1} \chi(u)K_{f,b} = \bigcup_{1\leqq u<b} \chi(u)K_{f,b} \quad \text{in } F \tag{3.2}$$

as $s\to\infty$ through $\mathbb{N}$ or $\mathbb{R}$.
(b) *If $c,d>1$, $\log c/\log d$ is irrational, (3.1) holds for $b=c$ and $b=d$ and $K_{f,c}=K_{f,d}$, then $K_f=K_{f,c}=K_{f,d}$.*

Remark. If (3.2) holds for different $b>1$, then K_f in (3.2) does not depend on b. Note that K_f is invariant for each $\chi(u)$.

Proof. (a) Straightforward by considerations around

$$\varphi(b^{k_\nu}r_\nu)f=\varphi(b^{k_\nu}r_\nu)\varphi(b^{k_\nu})^{-1}\varphi(b^{k_\nu})f\to\chi(r)g\in K_f$$

in case $k_\nu\to\infty$ through $\mathbb{N}$, $r_\nu\to r$ in $[1,b]$ and $\varphi(b^n)f\to g\in K_{f,b}$. In particular, (1.1) and Theorem 1 are essential.
(b) If $K_{f,c}=K_{f,d}=:K$, then $\chi(c^m d^n)K=\chi(c)^m\chi(d)^n K=K$ for all integers m and n. If $\log c/\log d$ is irrational, then $\{c^m d^n: m,n\in\mathbb{Z}\}$ is dense in $\mathbb{R}_+$, so $\chi(u)K=K$ for all $u>0$ by continuity of χ and joint continuity of the actions. Hence $K_f=K$. □

We will apply Lemma 1 in the following form to functional log log laws. Note that $K_{f,b}$ in Lemma 1 does not depend on f here, at least for almost all f in the set of possible values of the F-valued random variable X.

Theorem 2. *Let $\varphi:(a,\infty)\to G$ be χ-varying. Let X be a random variable in F and K a compact subset of F such that*

$$\varphi(b^n)X \rightsquigarrow K \quad \text{in } F \text{ wp} 1 \text{ as } n\to\infty \tag{3.3}$$

for all large $b>1$. Then

$$\varphi(s)X \rightsquigarrow K \quad \text{in } F \text{ wp} 1 \tag{3.4}$$

as $s\to\infty$ through $\mathbb{N}$, $\mathbb{R}$ or the integer powers of b for all $b>1$, and K is invariant for each $\chi(u)$.

Proof. Let $b_0\geqq 1$ be such that (3.3) holds for $b>b_0$. Selecting c and $d>b_0$ with $\log c/\log d$ irrational we conclude (3.4) as $s\to\infty$ through $\mathbb{R}$ or $\mathbb{N}$, by Lemma 1(b). If $b>1$, then $b^m>b_0$ for some natural m, and we conclude $K\supset K_b\supset K_{b^m}=K$. □

Remark. If we assume (3.3) for one fixed $b>1$, and in addition that K is invariant under χ, then (3.4) follows as $s\to\infty$ through $\mathbb{N}$ or $\mathbb{R}$ (Lemma 1(a)), but not necessarily as $s\to\infty$ through the integer powers of b for other $b>1$ (cf. § 5). In practice one always obtains the conditions of Theorem 2 as they stand.

4 Applications

Example 1. Strassen's loglog law. With (1.3) the first condition of Theorem 2 is satisfied. Consequently, it suffices to prove (1.2) as $s \to \infty$ through the integer powers of $b > 1$, for all large b. Formula (1.4) follows as a corollary, but can easily be verified directly.

The same also applies if $F = D[0, \infty)$ with Skorohod's J_1 topology (Whitt (1980)) and $X(t) := \sum_{k=1}^{[t]} \xi_k$, where the ξ_k's are iid real-valued random variables with zero mean and unit variance.

Example 2. In the same way as in Example 1, Theorem 2 applies to functional loglog laws for extremal processes (Wichura (1974a)) and stable processes (Wichura (1974b), Pakshirajan and Vasudeva (1981)), and to Strassen-type results for stationary finite-variance sequences with dependence in the limit (cf. surveys of Taqqu and Czado (1985) and Bingham (1986)).

Quite different examples (from a transformation point of view) occur in O'Brien and Vervaat (1990). Here we quote one.

Example 3. Let F be the space of Radon measures on the Borel sets of $E := [0, \infty) \times (0, \infty]$ with the topology of vague convergence, the coarsest topology that makes the evaluations $\mu \mapsto \int_E g \, d\mu$ continuous for continuous $g: E \to \mathbb{R}$ with compact support. Then F is Polish (Kallenberg (1983), Norberg (1986)). Let $\gamma(a, b, c)$ for $a, b, c > 0$ be the transformations of F determined by

$$\int_E g(t, x)(\gamma(a, b, c)\mu)(dt, dx) = a \int_E g(bt, cx)\mu(dt, dx)$$

for Radon measures μ on E and continuous $g: E \to \mathbb{R}$ with compact support. So $\gamma(a, b, c)$ multiplies the values of μ by a and transforms the two coordinates of E by factors b and c. The collection G of all these $\gamma(a, b, c)$ is a metric group, isomorphic to $\mathbb{R}_+^3$ via $\gamma(a, b, c) \mapsto (a, b, c)$, and acts jointly continuously on F.

Let X be a Poisson process in E with intensity $\mathbb{E}X = \pi$, where

$$\pi(dt, dx) = dt\, x^{-2} dx \quad \text{on } [0, \infty) \times (0, \infty), \tag{4.1}$$

$$\pi([0, \infty) \times \{\infty\}) = 0.$$

Set

$$\varphi(s) := \gamma((\log\log s)^{-1}, s^{-1}, s^{-1} \log\log s) \quad \text{for } s > e^e,$$

and let

$$K := \left\{ \mu \in F : \mu \ll \pi, \int_E \left(\frac{d\mu}{d\pi} \log \frac{d\mu}{d\pi} - \frac{d\mu}{d\pi} + 1 \right) \pi(dt, dx) \leq 1 \right\}.$$

One can prove that

$$\varphi(s) X \rightsquigarrow K \quad \text{wp1 in } F \tag{4.2}$$

as $s\to\infty$ through the integer powers of b for all large b. From

$$\varphi(su)\varphi(s)^{-1}=\gamma\left(\frac{\log\log s}{\log\log su},\frac{1}{u},\frac{1}{u}\frac{\log\log su}{\log\log s}\right)$$

$$\to\gamma\left(1,\frac{1}{u},\frac{1}{u}\right)=:\chi(u)\quad\text{as } s\to\infty\,(u>0)$$

we see that φ is χ-varying. From Theorem 2 it follows that (4.2) also holds as $s\to\infty$ through $\mathbb{N}$ or $\mathbb{R}$, and that $\chi(u)K=K$ for all $u>0$ (which also can be verified directly: note that π is invariant for each $\chi(u)$).

5 Back from the full sequence to geometric subsequences

We now study the possible patterns in (3.1) given (3.2). If

$$\text{(5.1)}\qquad \varphi(s)f\rightsquigarrow K_f\quad\text{as } s\to\infty\quad\text{through } \mathbb{R},$$

then the sequence $(\varphi(b^n)f)$ is relatively compact for each $b>1$, so there is some compact $K_{f,b}\subset K_f$ such that $\varphi(b^n)f\rightsquigarrow K_{f,b}$. Let $N_f:=\{b>1:K_{x,b}\neq K_f\}$. It is possible that N_f is not empty, as shows the following example.

Example 4. Let $\mathbb{T}:=[0,1)$ with addition modulo 1, and let $(c_k)_{k=1}^{\infty}$ be a sequence in $(0,1)$. Let $F:=\mathbb{T}^{\mathbb{N}}$ with the product topology, and G the automorphism group of F. Let $\varphi(s)\in G$ for $s\in\mathbb{R}_+$ be defined by

$$\varphi(s)f=(f(k)+c_k\log s)_{k=1}^{\infty}\quad\text{for } f=(f(k))_{k=1}^{\infty}\in F.$$

Then φ is φ-varying, being itself already a homomorphism from $\mathbb{R}_+$ into G. For all $f\in F$ we have $\varphi(s)f\rightsquigarrow F$ as $s\to\infty$ through $\mathbb{N}$ or $\mathbb{R}$, while $N_f=\left\{b>1:\log b\in\bigcup_{k=1}^{\infty}(c_k+\mathbb{Q})\right\}$.

If N_f is not empty, then it is unbounded, as it contains with each b also b^n for $n=2,3,\ldots$. We see that the converse of Theorem 2 is not true. If (3.4) holds as $s\to\infty$ through $\mathbb{N}$ or $\mathbb{R}$ (take $X=f$ wp1 in Example 4), then (3.3) may fail for certain b, and the set of such b is unbounded. We conclude that most functional log log laws in the literature have been weakened unnecessarily by stating them for s tending to ∞ through $\mathbb{N}$ or $\mathbb{R}$. A better formulation is: "$\varphi(s)X\rightsquigarrow K$ wp1 in F as $s\to\infty$ through the integer powers of b for each fixed $b>1$, and *consequently* also as $s\to\infty$ through $\mathbb{N}$ or $\mathbb{R}$".

Example 4 may look rather artificial and nonprobabilistic, but it is typical for what can happen in cases of probabilistic interest. In view of the theory of self-similar jump processes with stationary increments as developed in O'Brien and Vervaat (1985) it is interesting to generalize results like Example 3 to the case where X is a *Poincaré process* in $E:=\mathbb{R}\times(0,\infty)$, i.e., a point process invariant in distribution for the transformations $(t,x)\mapsto(at+b,ax)\,(a,b\in\mathbb{R},a>0)$ of E. Note that the Poisson process in Example 3 is just a restriction to smaller E of a Poincaré Poisson process. Consider the g-adic lattice process X in Section 3 of O'Brien and Vervaat (1985). It is obvious that N_X contains all rational powers of g. Moreover, it is not hard to think out variations on the g-adic lattice process such that N_χ contains all rational powers of all naturals.

So it has some interest to see how large N_f can be. We do not know whether N_f must be countable, as it is in all previous cases, but we can prove the following.

Theorem 3. *If (5.1) holds with regularly varying* φ, *then* $\operatorname{Leb} N_f = 0$.

6 Proof of Theorem 3

An additive notation is more convenient here. After the substitutions $\varphi \circ \log \to \varphi$, $\chi \circ \log \to \chi$, $\log b \to b$, $\log s \to s$ and $\log u \to u$ we have $\varphi(s+u)\varphi(s)^{-1} \to \chi(u)$, $\chi(u_1+u_2) = \chi(u_1)\chi(u_2)$, $\varphi(nb) f \rightsquigarrow K_{f,b}$ and $N_f = \{b>0 : K_{f,b} \neq K_f\}$. Furthermore, the image of Lebesgue measure on $(1, \infty)$ under log and Lebesgue measure on $\mathbb{R}_+ = (0, \infty)$ are mutually absolutely continuous.

For open $U \subset F$ with $U \cap K_f \neq \phi$, let

$$N_{f,U} := \{b > 0 : U \cap K_{f,b} = \phi\}.$$

Then

$$N_f = \bigcup_{U : U \cap K_f \neq \phi} N_{f,U}. \tag{6.1}$$

Since K_f is a compact metric space in the trace topology, there is a countable collection of open U's such that their intersections with K_f form a base of this topology, and (6.1) remains true if U varies through this collection. Therefore it suffices to prove Theorem 3 with $N_{f,U}$ instead of N_f, where $U \cap K_f \neq \phi$.

From $K_f = \bigcup_{0 \leq u < b} \chi(u) K_{f,b}$ (Lemma 1(a) in additive form), $\chi(0) = \imath$ and the joint continuity of χ we see that $\bigcup_{b>0} K_{f,b}$ is dense in K_f. So there is a $c > 0$ such that $U \cap K_{f,c} \neq \phi$. Select a $g \in U \cap K_{f,c}$ and an increasing sequence of integers (n_k) such that $\varphi(n_k c) f \to g$. Since $\chi(u) g \to g$ as $u \to 0$, there is an $\varepsilon > 0$ such that $\chi(u) g \in U$ for $|u| < \varepsilon$. Let $\delta < \varepsilon$ and set

$$B_{k,m} := \left(\frac{n_k c - \delta}{m}, \frac{n_k c + \delta}{m} \right),$$

$$B_k := \bigcup_{m=1}^{\infty} B_{k,m},$$

$$B := \limsup_{k \to \infty} B_k.$$

We now show that $B \subset \mathbb{R}_+ \backslash N_{f,U}$. If $b \in B$, then we have for infinitely many k that $m_k b = n_k c + \delta_k$ for some $m_k \in \mathbb{N}$, where $m_k \to \infty$ and $|\delta_k| < \delta$. Restricting ourselves to these k, we find that

$$\varphi(m_k b) f = \varphi(n_k c + \delta_k) f = \varphi(n_k c + \delta_k) \varphi(n_k c)^{-1} \varphi(n_k c) f$$

has all its limit points in $\{\chi(u) g : |u| \leq \delta\} \subset U$. So $U \cap K_{f,b} \neq \phi$, i.e., $b \notin N_{f,U}$.

The rest of this proof serves to show that

$$(\beta - \alpha)^{-1} \operatorname{Leb}(B \cap (\alpha, \beta)) \to 1 \quad \text{as } \beta \downarrow \alpha \text{ in } \mathbb{R}_+, \tag{6.2}$$

from which it follows that B has full Lebesgue measure in $\mathbb{R}_+$.

The analysis for the proof of (6.2) is facilitated by the following restrictions.

(a) Fix $b_0 > 0$ and consider only $(\alpha, \beta) \subset [b_0, \infty)$.

(b) Let $\delta < \frac{1}{2} b_0$, so that all $B_{k,m}$ are disjoint in $[b_0, \infty)$ for fixed k and varying m.

(c) Let $\delta < \frac{1}{2} c$ to obtain more uniformity for the next estimates.

(d) Assume that $n_{k+1}/n_k \to \infty$ as $k \to \infty$ (select a subsequence if necessary).

By standard estimates we obtain

$$(6.3) \qquad \operatorname{Leb}(B_k \cap (\alpha, \beta)) = \sum_{m: B_{k,m} \subset (\alpha, \beta)} \frac{2\delta}{m} + \operatorname{Leb} \bigcup_{m: B_{k,m} \cap \{\alpha, \beta\} \neq \phi} (B_{k,m} \cap (\alpha, \beta))$$
$$= 2\delta(\log \beta - \log \alpha) + R_{1,k} \frac{\beta\delta}{n_k},$$

where $|R_{1,k}|$ can be majorized by a bound independent of α, β, δ and n_k.

Let $j < k$. Applying (6.3) first with $B_{j,m}$ in place of (α, β) and adding up the result for $B_{j,m} \subset (\alpha, \beta)$ we obtain

$$(6.4) \qquad \operatorname{Leb}(B_j \cap B_k \cap (\alpha, \beta)) \leq \frac{\beta - \alpha}{\alpha\beta} \cdot 4\delta^2 + R_{2,k} \frac{1}{n_j} + R_{3,k} \frac{n_j}{n_k},$$

where $R_{2,k}$ and $R_{3,k}$ can be chosen independent of n_j and n_k (they still depend on α, β and δ, but this does not matter for the sequel). We now use the following Borel-Cantelli lemma, which follows from (iii) in the theorem of Kochen and Stone (1964).

Lemma 2. *If (A_k) is a sequence of events in the probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then*

$$\mathbb{P}(\limsup_{n \to \infty} A_n) \geq \limsup_{n \to \infty} \frac{\left(\sum_{k=1}^{n} \mathbb{P}(A_k)\right)^2}{\sum_{j=1}^{n} \sum_{k=1}^{n} \mathbb{P}(A_j \cap A_k)}.$$

Applying the lemma with $\Omega = (\alpha, \beta)$, $\mathbb{P} = (\beta - \alpha)^{-1}$ Leb and $A_k = B_k \cap (\alpha, \beta)$ we find (recall that $n_{k+1}/n_k \to \infty$)

$$\frac{1}{\beta - \alpha} \operatorname{Leb}(B \cap (\alpha, \beta)) \geq \alpha\beta \left(\frac{\log \beta - \log \alpha}{\beta - \alpha}\right)^2.$$

Letting $\beta \downarrow \alpha$ we obtain (6.2).

Acknowledgement. The paper has improved by comments of Andries Lenstra and George O'Brien. Some ideas already occur in the report Vervaat (1975), albeit with much less algebraic sophistication.

References

1. Balkema, A.A.: Monotone transformations and limit laws. Mathematical Centre Tracts **45**, Mathematisch Centrum, Amsterdam 1973
2. Bingham, N.H.: Variants on the law of the iterated logarithm. Bull. Lond. Math. Soc. **18**, 433–467 (1986)
3. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. Cambridge: Cambridge University Press 1987

4. De Haan, L.: On regular variation and its application to the weak convergence of sample extremes. Mathematical Centre Tracts **32**, Mathematisch Centrum, Amsterdam 1970
5. Freedman, D.: Brownian motion and diffusion. San Francisco: Holden-Day 1971
6. Kallenberg, O.: Random measures. New York London: Akademie-Verlag and Academic Press 1983
7. Kochen, S., Stone, C.: A note on the Borel-Cantelli lemma. Ill. J. Math. **8**, 248–251 (1964)
8. Matheron, G.: Random sets and integral geometry. New York: Wiley 1975
9. Mori, T., Oodaira, H.: A functional law of the iterated logarithm for sample sequences. Yokohama Math. J. **24**, 35–49 (1976)
10. Norberg, T.: Random capacities and their distributions. Probab. Th. Rel. Fields **73**, 281-297 (1986)
11. O'Brien, G.L., Vervaat, W.: Self-similar processes with stationary increments generated by point processes. Ann. Probab. **13**, 28–52 (1985)
12. O'Brien, G.L., Vervaat, W.: Loglog laws for point processes. (In preparation)
13. Pakshirajan, R.P., Vasudeva, R.: A functional law of the iterated logarithm for a class of subordinators. Ann. Probab. **9**, 1012–1018 (1981)
14. Seneta, E.: Regularly varying functions. (Lect. Notes Math., vol. 508), Berlin Heidelberg New York: Springer 1976
15. Strassen, V.: An invariance principle for the law of the iterated logarithm. Z. Wahrscheinlichkeitstheor. verw. Geb. **3**, 211–226 (1964)
16. Taqqu, M.S., Czado, C.: A survey of functional laws of the iterated logarithm for self-similar processes. Commun. Stat. Stochastic Models **1**, 77–115 (1985)
17. Vervaat, W.: Two papers on functional iterated logarithm laws. Report 75-05, Dept. of Math., University of Amsterdam 1975
18. Whitt, W.: Some useful functions for functional limit theorems. Math. Oper. Res. **5**, 67–85 (1980)
19. Wichura, M.J.: On the functional form of the law of the iterated logarithm for the partial maxima of independent identically distributed random variables. Ann. Probab. **2**, 202–230 (1974a)
20. Wichura, M.J.: Functional laws of the iterated logarithm for the partial sums of i.i.d. random variables in the domain of attraction of a completely asymmetric stable law. Ann. Probab. **2**, 1108–1138 (1974b)