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The scaling limit for a stochastic PDE and the separation of phases

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Summary. We investigate the problem of singular perturbation for a reaction-diffusion equation with additive noise (or a stochastic partial differential equation of Ginzburg-Landau type) under the situation that the reaction term is determined by a potential with double-wells of equal depth. As the parameter ϵ (the temperature of the system) tends to 0, the solution converges to one of the two stable phases and consequently the phase separation is formed in the limit. We derive a stochastic differential equation which describes the random movement of the phase separation point. The proof consists of two main steps. We show that the solution stays near a manifold M^{ϵ} of minimal energy configurations based on a Lyapunov type argument. Then, the limit equation is identified by introducing a nice coordinate system in a neighborhood of M^{ϵ}.

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1 Introduction

Phenomena like dynamic phase transition, pattern formation, generation of an interface between two coexisting phases and propagation of wave fronts or pulses in excitable media can be described by reaction-diffusion equations; see [3, 22, 23, 26]. In various physical situations, however, it becomes necessary to take an external random force into account as an additional effect in the equation; see [5, 27, 31, 39, 43, 49]. The scaling limit for such reaction-diffusion equations has been investigated under several different circumstances. The aim of the present paper is to extend these attempts for such equations with a random additional term. We discuss in 1-dimension the case where the reaction term is bistable and creates a standing wave front.

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1.1 Problem and main result

We consider a stochastic partial differential equation (SPDE)

$$(1.1)\qquad \frac{\partial u}{\partial t} = \Delta u + \epsilon^{-1} f(u) + \kappa_\epsilon a(x)\dot{w}_t(x), \quad t > 0,\ x \in \mathbf{R},$$

where $\Delta = d^2/dx^2$, $\epsilon > 0$ is a small parameter, κ_ϵ is a positive constant which depends only on ϵ and represents the strength of the noise, $a(x) \in C_0^2(\mathbf{R})$ describes how the strength of the noise varies with x and $\dot{w}_t(x)$ is a 2-parameter Gaussian white noise. The reaction term f is smooth: $f \in C^\infty(\mathbf{R})$, and satisfies the following conditions:

$$(1.2)\qquad \begin{cases} (a)\ f \text{ has exactly three zeros } \pm 1, 0\ (: f(\pm 1) = f(0) = 0) \\ \qquad \text{and } f'(\pm 1) < 0, f'(0) > 0, \\ (b)\ f \text{ is odd} : f(u) = -f(-u), \\ (c)\ \text{there exist } \bar{c}_1, \bar{c}_2, \bar{p} > 0 \text{ such that} \\ \qquad |f(u)| \le \bar{c}_1(1 + |u|^{\bar{p}}) \text{ and } f'(u) \le \bar{c}_2 \text{ for every } u \in \mathbf{R}. \end{cases}$$

A typical example is $f(u) = u - u^3$. The condition (a) means that the underlying reaction dynamics $du/dt = f(u)$ are bistable with stable states $u = \pm 1$. It is well-known under this condition that the equation $\partial v/\partial t = \Delta v + f(v)$ on $\mathbf{R}$ requiring $v(\pm\infty) = \pm 1$ admits a travelling wave solution $v(t,y) = \tilde{m}(y - ct), y \in \mathbf{R}$. The condition $\int_{-1}^{1} f(u)\,du = 0$ which follows from (b) implies that the speed of the travelling wave vanishes: $c \equiv c(f) = 0$. These kinds of properties were investigated first by Kanel' and then by [4, 21, 38]. Throughout the paper we denote by $m = m(y), y \in \mathbf{R}$, the shape of the standing wave front, that is, the solution of the steady state problem (with $c = 0$):

$$(1.3)\qquad \Delta m + f(m) = 0,\ y \in \mathbf{R} \quad \text{and} \quad m(\pm\infty) = \pm 1.$$

The solution m is increasing in y and unique up to translation, so we can normalize it as $m(0) = 0$. The potential F is defined from f by $F(u) = -\int_1^u f(u')\,du'$, $u \in \mathbf{R}$, and hence $f = -F'$ and F is of double-well type with equal global minima at $u = \pm 1$. For instance, $m(y) = \tanh(y/\sqrt{2})$ and $F(u) = (u^2 - 1)^2/4$ when $f(u) = u - u^3$. The condition (b), which is much stronger than $\int_{-1}^{1} f(u)\,du = 0$, will be necessary, in particular, to derive a kind of centering condition (in Corollary 7.1 below). The condition (c) guarantees the existence and uniqueness of the global solutions u to (1.1), see Sect. 2.

The goal of this paper is to study the asymptotic behavior as the parameter $\epsilon \downarrow 0$ of the solution $u = u^\epsilon(t,x)$ of (1.1) subject to the boundary conditions $u^\epsilon(t, \pm\infty) = \pm 1$ at infinity with a properly scaled constant κ_ϵ. Indeed, we discuss the problem under the situation that the noise becomes small as the reaction term gets large and therefore we take

$$(1.4)\qquad \kappa_\epsilon = \epsilon^\gamma,$$

where γ is a positive constant; we shall assume a technical condition $\gamma > 19/4$.

Since $u = \pm 1$ are the two stable states (or stable phases in physical terminology) of f, one would expect that u^ϵ converges to one of these two phases as $\epsilon \downarrow 0$. In fact, we obtain $\lim_{\epsilon\downarrow 0} f(u^\epsilon) = 0$ at least formally by multiplying both sides of (1.1) by ϵ and this suggests that u^ϵ converges to 1 or -1. From the boundary conditions which u^ϵ satisfies, a phase separation point $\xi \in \mathbf{R}$ may be formed and $u^\epsilon(t,x)$ might converge to χ_{ξ_t}, where χ_ξ is a function defined by $\chi_\xi(x) = 1$ for $x > \xi$ and -1 for $x < \xi$. Our main result claims that this type of statement is actually true if the time parameter is properly scaled (see subsection 1.2 below) and can roughly be formulated as follows: The position ξ_t of phase separation moves according to a stochastic differential equation (SDE) on $\mathbf{R}$

$$d\xi_t = \alpha_1 a(\xi_t)dB_t + \alpha_2 a(\xi_t)a'(\xi_t)dt \tag{1.5}$$

where α_1, α_2 are certain constants defined only through the function f and B_t is a standard Brownian motion; see Theorem 8.1 for a precise formulation of the main result. We shall assume for simplicity that the initial data $u^\epsilon(0,x)$ of (1.1) is of the form $m_\xi^\epsilon(x) := m(\epsilon^{-1/2}(x-\xi))$ for some $\xi \in \mathbf{R}$; in other words, it is a minimizer (minimal configuration) of the Ginzburg-Landau-Wilson free energy functional $\mathscr{H}^\epsilon$ defined by

$$\mathscr{H}^\epsilon(u) = \int_{\mathbf{R}} \left\{ \frac{1}{2}|\nabla u|^2(x) + \epsilon^{-1}F(u(x)) \right\} dx, \quad \nabla = \frac{d}{dx}, \tag{1.6}$$

in the class of configurations u satisfying boundary conditions $u(\pm\infty) = \pm 1$. This means that we avoid discussing the problem of initial generation of a layer (cf. [24]) and investigate the single transition layer case only.

1.2 Relevant time scale for (1.1), cut-off for $a(x)$ and the constant α_1

The small factor $\kappa_\epsilon = \epsilon^\gamma$ appearing in (1.1) makes the drift term dominate the noise term and, as a result, the solution u is attracted toward the set M^ϵ of minimizers of the energy functional $\mathscr{H}^\epsilon$ after a long time. Therefore $u = u^\epsilon(t,x)$ behaves like $m_{\tilde{\xi}_t^\epsilon}^\epsilon(x)$ for some $\tilde{\xi}_t^\epsilon \in \mathbf{R}$. The relevant time change for the equation (1.1) is $t \longmapsto \epsilon^{-1/2-2\gamma}t$; the effect of the noise survives in this time scale. In fact, under this time change, the drift term is multiplied by $\epsilon^{-1/2-2\gamma}$ while the noise term is multiplied by $(\epsilon^{-1/2-2\gamma})^{1/2}$ and consequently we obtain an SPDE

$$\frac{\partial \bar{u}}{\partial t} = \epsilon^{-1/2-2\gamma}\left\{\Delta\bar{u} + \epsilon^{-1}f(\bar{u})\right\} + \epsilon^{-1/4}a(x)\dot{w}_t(x), \tag{1.7}$$

for $\bar{u} \equiv \bar{u}^\epsilon(t,x) := u^\epsilon(\epsilon^{-1/2-2\gamma}t, x)$. However, the principal effect of the noise comes from the region near the interface (see the proof of Lemma 8.2), and therefore the main contribution from the white noise becomes $O(\epsilon^{1/4})$ in size since the width of the interface in a typical configuration $m_{\tilde{\xi}_t^\epsilon}^\epsilon(x)$ is $O(\epsilon^{1/2})$. This balances with the diverging factor $\epsilon^{-1/4}$ and $\epsilon^{-1/4}a(x)\dot{w}_t(x)$ becomes a quantity of order one.

As we shall point out later (subsection 1.3 (a)), Carr and Pego [10] found out that the relevant time scale for the equation (1.1) without noise is $\exp(C\epsilon^{-1/2})$ where C is order one. This means that there is no apparent effect from the drift term in the time scale $\epsilon^{-1/2-2\gamma}t$; it only pushes the solution toward M^ϵ and the motion along M^ϵ can not be observed in this time span (if there is no external force). Only the effect of the noise term survives in this scaling limit and pushes the wave front randomly. The condition $\gamma > 19/4$ is rather technical, but is introduced to obtain an adequate speed of convergence of u^ϵ to M^ϵ (although a similar result is expected for all $\gamma \geq 0$). The speed of convergence can be computed if one can employ $\mathscr{H}^\epsilon$ as a Lyapunov function. However, to do this, some modification (see Sects. 4, 5) is required in practice because of the very singular nature of the white noise. This is the main reason for our assumption that the noise term be comparatively weak.

We assume in addition the technical condition that the support of the noise term of the SPDE (1.1) is compact. This assumption is necessary to impose the boundary conditions $u(\pm\infty) = \pm 1$ at infinity to the SPDE (1.1). There is always a single random wave front determined by these boundary conditions. This cut-off assumption merely localizes the problem and, as far as we are concerned with the interface motion, it never changes the essential physical behavior of the system, since the limit dynamics of the interface are determined in a quite local way (namely, it depends only on the noise near the interface) as we explained above.

The diffusion constant α_1^2 (at x such that $a(x) = 1$) given by the formula (8.5) below coincides with that conjectured in the physical literature (see (3.47), (3.48) in [43]). The constant α_1^2 has both kinetic and thermodynamic meanings; namely, α_1^2 is the so-called mobility (see the formula (4.7) in [50]) and $1/\alpha_1^2$ is proportional to the surface tension (see (4.8) in [50] or [16, 43]). This kind of relation among these quantities is called an Einstein relation and a variant of the Green-Kubo formula.

1.3 Motivation and bibliographical notes

(a) Scaling limit for reaction-diffusion equations. The scaling limit for a reaction-diffusion equation without random term:

$$\frac{\partial u}{\partial t} = \Delta u + \epsilon^{-1} f(u), \quad t > 0,\ x \in \mathbf{R}^d \tag{1.8}$$

has been investigated by several authors. The reaction term f may depend on the space-variable x, in such case $f = f(x,u)$, and the d-dimensional Laplacian Δ may be replaced with a second order elliptic operator.

Freidlin [26, 28] discussed the case where the reaction term $f(x,u)$ is of KPP (Kolmogorov-Petrovskii-Piskunov) type for every x, i.e., as a function of u, $f(\pm 1) = 0, f(u) > 0$ in $(-1,1), f(u) < 0$ in $\mathbf{R} \setminus [-1,1]$; in other words, the reaction dynamics have only two equilibrium states, the stable state 1 and unstable state -1. The bistable case (i.e., f satisfies the condition (1.2)-(a)) was studied by Gärtner [36] in higher dimensions and by Fife and Hsiao [24] in 1-dimension. The relevant time scale is $O(\epsilon^{1/2})$ in both the KPP and the bistable cases. This can be easily understood, assuming $d = 1$ and $f = f(u)$ for simplicity, from the fact that (1.8) has a solution of the form $u^\epsilon(t,x) = \tilde{m}(\epsilon^{-1/2}(x - c\epsilon^{-1/2}t))$ which propagates with speed $c\epsilon^{-1/2}$, where $c = c(f)$ and $\tilde{m} = \tilde{m}(y)$ are the speed and the shape, respectively, of the travelling front: $\Delta\tilde{m} + c\nabla\tilde{m} + f(\tilde{m}) = 0$ on $\mathbf{R}$ satisfying $\tilde{m}(\pm\infty) = \pm 1$. Note that $c(f)$ is not uniquely determined in the KPP case although it is unique in the bistable case. The rescaled function $\bar{u} \equiv \bar{u}^\epsilon(t,x) := u^\epsilon(\epsilon^{1/2}t, x)$ of the solution $u = u^\epsilon(t,x)$ of (1.8) satisfies

$$\frac{\partial \bar{u}}{\partial t} = \epsilon^{1/2}\Delta\bar{u} + \epsilon^{-1/2} f(\bar{u})$$

and it was proved that $\bar{u}^\epsilon(t,x)$ converges to $\chi_{G_t}(x)$ as $\epsilon \downarrow 0$, where $\chi_{G_t}(x) = 1$ for $x \in G_t$ and -1 for $x \notin G_t$. The evolution law for the region $G_t \subset \mathbf{R}^d$ was derived in both the KPP case [28] and bistable case [36]. In particular, in the KPP case, G_t expands as the time t grows, since $u = 1$ is the only stable state. In the 1-dimensional bistable and single layer case, G_t is an interval (ξ_t, ∞) and the motion of the phase separation point $\xi_t \in \mathbf{R}$ is governed by an ODE $\dot{\xi}_t = c(\xi_t)$, where $c(\xi) = c(f(\xi, \cdot))$, see [24].

If the bistable reaction term $f = f(u)$ fulfills an additional condition $\int_{-1}^{1} f(u)\,du = 0$, then $c(f) = 0$ so that the interface doesn't move and accordingly we need to introduce a much longer time scale to observe a considerable propagation effect. In fact, under this vanishing condition, the relevant time scale becomes $O(1)$. Recently several authors including [44], [18] proved that the solution $u = u^\epsilon(t,x)$ of (1.8) converges to $\chi_{G_t}(x)$ as $\epsilon \downarrow 0$ and derived the motion by mean curvature as an evolution law for the boundary ∂G_t of G_t. In 1-dimension, however, the relevant time scale turns out to be $\exp(C\epsilon^{-1/2})$. This fact was found by [10]. Indeed, it is apparent that we have to observe a longer time span than $O(1)$ since the curvature of the plane wave vanishes, but the actual propagation speed is extremely slow, see also [35].

(b) Drumhead model. Part of the motivation of the present paper comes from the investigation of the so-called drumhead model, which is a model for the interface between two coexisting thermodynamic phases. Diehl et al. [16] developed a static theory. They started from the Ginzburg-Landau-Wilson free energy functional $\mathcal{H}^\epsilon(u)$ for scalar order parameter $u = \{u(x), x \in \mathbf{R}^d\}$ defined by a d-dimensional analog of (1.6) (especially, $\nabla u = (\partial u/\partial x_1, \dots, \partial u/\partial x_d)$ in d dimensions) and then derived a free energy for the interfaces by investigating the low temperature limit $\epsilon \downarrow 0$ of $\mathcal{H}^\epsilon(u)$. The parameter ϵ indicates the temperature of the system in this model. The corresponding kinetic theory was discussed by Kawasaki and Ohta [43]. The dynamics corresponding to the energy functional (1.6) can be described by the so-called time-dependent Ginzburg-Landau stochastic equation

$$\frac{\partial u}{\partial t} = -D\mathcal{H}^\epsilon(x, u) + \dot{w}_t(x), \quad t > 0,\ x \in \mathbf{R}^d, \tag{1.9}$$

where $D\mathscr{H}^\epsilon(x,u) = -\Delta u(x) + \epsilon^{-1}F'(u(x))$ denotes the functional derivative of $\mathscr{H}^\epsilon$. In general, the time evolution $u(t,\cdot)$ is naturally regarded as the dynamics corresponding to $\mathscr{H}^\epsilon$ if it has the (formal) Gibbs state $Z_\epsilon^{-1}e^{-\mathscr{H}^\epsilon(u)}\mathscr{D}(u)$ associated with $\mathscr{H}^\epsilon$ as an equilibrium measure, where $\mathscr{D}(u) = \prod_{x\in\mathbf{R}^d} du(x)$ denotes a "flat measure" (Feynman measure) on the space of all order parameters and Z_ϵ is a "normalizing constant". It can be seen, at least heuristically, that the stochastic evolution determined by (1.9) has such property. Kawasaki and Ohta investigated the asymptotic behavior of the solution $u = u^\epsilon(t,x)$ of (1.9) as the temperature $\epsilon \downarrow 0$ and derived a stochastic equation describing the random movement of the interface between two phases, namely, they derived randomly perturbed motion by mean curvature but based on rather heuristic arguments, see also [46]. The mathematical theory for SPDE's of the type (1.9), including their relation to the Gibbs states associated with the energy functional $\mathscr{H}^\epsilon$, has been developed in some detail in [32].

Unfortunately, it is a well-known mathematical fact that the SPDE (1.9) has a meaningful solution only when the space dimension $d = 1$ or if the noise term $\dot{w}_t(x)$ is sufficiently smooth in the variable x (cf. [33]). Therefore, in this paper, we restrict ourselves to the 1-dimensional case.

(c) Other related stochastic models. There are some attempts to derive wave front propagation from the underlying microscopic particle systems. The reaction-diffusion equation itself can be derived from the so-called Glauber-Kawasaki dynamics, a stochastic system of interacting particles on a cubic lattice, by taking the hydrodynamic limit, see De Masi et al. [12]. Therefore, it would be quite desirable to show the formation of interfaces and derive the motion by mean curvature directly from the particle system. In 1-dimension, De Masi et al. [15] proved the formation of multiple interfaces and investigated their spatial patterns for the Glauber-Kawasaki dynamics starting off from an unstable distribution.

The fluctuation theory [49] at rather heuristic level suggests that the second correction to the reaction-diffusion equation in the hydrodynamic limit for the Glauber-Kawasaki dynamics is given by $\epsilon^{1/2}\dot{w}_t(x)$ and this leads us to the SPDE (1.1) (with 1 rather than ϵ^{-1} in front of $f(u)$ and $\kappa_\epsilon = \epsilon^{1/2}$). Therefore, it is natural to think of the SPDE (1.1) as describing the dynamics at an intermediate level between microscopic and macroscopic levels.

The motion by mean curvature has been derived from the Glauber dynamics with Kac potentials, long range interactions of mean field type, by a series of papers by an Italian group after taking two stages of scaling limits: First, a certain integro-differential equation is obtained from the microscopic dynamics by taking the so-called mesoscopic scaling limit (i.e., the time is kept finite while the space scale is chosen so that the ranges of Kac potentials become $O(1)$), see [13]. Then, the motion by mean curvature is derived from this integro-differential equation in the phase transition regime by taking the macroscopic scaling limit (i.e., diffusion type space-time scaling limit), see [14]. See also [50] for the derivation of the motion by mean curvature from Ginzburg-Landau stochastic lattice model or from Glauber dynamics. The SPDE (1.1) (with $\epsilon = 1$) has been derived from the Glauber dynamics with Kac potentials at the critical temperature, see [6].

Mueller and Sowers [45] investigated the random travelling waves arising from the SPDE of the form (1.1) (with $\epsilon = 1$) with reaction term f of KPP type and noise term $\lambda\sqrt{u(1-u)}\dot{w}_t(x)$, where $\lambda > 0$ is small but fixed.

(d) Inertial manifold. The concept of the inertial manifold (or center manifold) was introduced to describe the long time behavior of infinite-dimensional dissipative dynamical systems in terms of finite-dimensional systems, cf. Temam [52]. See Flandori [25] for an extension to stochastic dynamics. In our case, the 1-dimensional space M^ϵ consisting of all minimizers of $\mathcal{H}^\epsilon$ plays the role of the inertial manifold for the infinite-dimensional stochastic dynamical system determined by the SPDE (1.1). Indeed, the solution is attracted toward M^ϵ after long time (since the drift is the gradient of $\mathcal{H}^\epsilon$ so that the flow converges to M^ϵ) and only the motion of the stochastic dynamics along M^ϵ can be observed in the limit. The corresponding finite-dimensional problem was studied by [34, 42] systematically.

The asymptotic behavior as $\epsilon \downarrow 0$ of the equilibrium measure of the SPDE (1.9) for $x \in [0,1]$ with Dirichlet boundary conditions at both edges $x = 0, 1$ was discussed in [30] where continuum of minima appears like in the present paper. The case of finitely many minima was discussed by Freidlin [27] and by Faris and Jona-Lasinio [20] where the problem of large deviation was investigated. The large deviation problem (for sums of independent random vectors) under the situation that the set of minima forms a manifold was discussed by Bolthausen [7] and then by Chiyonobu [11].

1.4 Organization of the paper

In Sect. 2, by applying the maximum principle for PDE, we shall prove that the solution $u = u^\epsilon(t,x)$ of the SPDE (1.1) takes values only in a neighborhood of $[-1,1]$, the region containing two stable states $\{\pm 1\}$ of the reaction term f, with high probability for small $\epsilon > 0$. This is an effect of the strong drift term $\epsilon^{-1}f(u)$ in (1.1), which forces u toward the stable states of f. In Sect. 3, we investigate the property of the functional $\mathcal{H}(v)$ which is obtained from $\mathcal{H}^\epsilon$ by introducing the spatial scaling $x \longmapsto y = \epsilon^{-1/2}x$. We study in particular the asymptotical structure of $\mathcal{H}$ near M, which denotes the space consisting of all minimizers $\{m_\eta; \eta \in \mathbf{R}\}$ of $\mathcal{H}(v)$ restricted to such v's that satisfy the boundary conditions $v(\pm\infty) = \pm 1$ at infinity; namely $m_\eta(y) = m(y-\eta)$ with m determined by (1.3). In Sect. 4, we introduce a proper space-time scaling for the SPDE (1.1) and derive a new SPDE for the scaled process $v_t = v^\epsilon(t,y)$. We shall prove that v_t stays near M. The idea for completing this is to employ the functional $\mathcal{H}$ as a Lyapunov function. However, since $v^\epsilon(t,y)$ is not C^1 in the variable y, this idea does not work directly and consequently it becomes necessary to introduce a smooth approximation v_t^δ of v_t. The process v_t^δ is defined by applying Friedrichs' mollifier to v_t. In Sect. 5, we calculate the time derivative of $\mathscr{H}(v_t^\delta)$ and prove that v_t stays near M. Section 6 supplies the calculations in the previous section and provides some error estimates. The limiting SDE (1.5) for the position ξ_t of the phase separation is derived in Sect. 8 based on the method used by Katzenberger [42] for a finite dimensional problem. Namely, we shall introduce a nice coordinate $\zeta = \zeta(v) \in \mathbf{R}$ defined in a neighborhood of M from the limit map of the classical flow (i.e., a solution of PDE) associated with the SPDE (1.1). This coordinate is useful since it eliminates the diverging drift term which appears in the scaled SPDE for $v^\epsilon(t,y)$. Some necessary properties of the classical flow are summarized in Sect. 7 and proved in Sect. 9.

After completing the work, the author has received the paper [8] which discusses a similar problem. They considered an SPDE $\partial u/\partial t = \epsilon^{-1}\{\Delta u + f(u)\} + \dot{w}_t(y)$ for $y \in [-\epsilon^{-1}, \epsilon^{-1}]$ with $f(u) = u - u^3$ by imposing Neumann boundary conditions at both edges (instead of introducing cut-off function a in our case) and proved that $u = u^\epsilon(t,y)$ converges to $m_{\eta_t}(y)$ as $\epsilon \downarrow 0$ in a distributional sense. Here, $m_\eta(y)$ is the function introduced above, $\eta_t = \eta_0 + \alpha_1 B_t$ with a standard Brownian motion B_t and the constant α_1 is exactly the same as that obtained in the present paper. The limit keeps the shape m of the wave front and the sharp transition does not arise. The reason is that the basic scale for u^ϵ here stays at the microscopic (or mesoscopic) level; however, their result seems to be closely related to ours since the macroscopic scaling limit (i.e., diffusion type scaling limit) for $m_{\eta_t}(y)$ results in $\chi_{\xi_t}(x)$ with Brownian motion ξ_t having diffusion constant α_1^2.

2 Uniform bound

Throughout the paper, we suppose that $a(x) = 0$ if $|x| \geq 1$ and $|a(x)| \leq 1$ without loss of generality. Here we derive a uniform bound on the solution $u = u^\epsilon(t,x)$ of the SPDE (1.1). The main tool is the maximum principle [29, 47] for the PDE of the form (1.1) without random term. The argument in this section depends heavily on the assumption that the support of a be compact.

Before giving the uniform bound, we quickly refer to the existence and uniqueness result known for the SPDE (1.1). This SPDE with $\kappa_\epsilon = \epsilon^\gamma$ is sometimes written in the form

$$(1.1') \qquad du = \{\Delta u + \epsilon^{-1} f(u)\}dt + \epsilon^\gamma a(x) dw_t(x), \quad x \in \mathbf{R},$$

where $w_t(x)$ is the so-called cylindrical Brownian motion on $L^2(\mathbf{R})$, in other words, $dw_t(x)/dt = \dot{w}_t(x)$ is the 2-parameter Gaussian white noise. The mathematical meaning to the SPDE (1.1) is given by rewriting it in an integral form (solution in this sense is called mild solution) or in a weak form (generalized solution), i.e., (1.1′) is interpreted by multiplying test functions $\varphi(x)$ to its both sides. Then, the SPDE (1.1) has a unique solution satisfying $u_t^\epsilon = u^\epsilon(t,\cdot) \in C([0,\infty), \mathscr{E})$ a.s. if $u_0^\epsilon \in \mathscr{E}$. Here, $\mathscr{E} = \{u \in C(\mathbf{R}); \|u\|_{-\lambda} < \infty$ for every $\lambda > 0\}$ is a Fréchet space equipped with a family of norms $\{\|\cdot\|_{-\lambda}\}_{\lambda>0}$ defined by $\|u\|_{-\lambda} = \sup_{x\in\mathbf{R}} |u(x)| e^{-\lambda|x|}$. This result follows from the assumption (1.2)-(c) by applying Theorems 5.1 and 5.2 in Iwata [40] (with slight modification because of $a(x)$ introduced in our equation).

Now, we start the task to derive a uniform bound. Let $u_2 = u_2^\epsilon(t,x)$ be a solution of the SPDE

$$\begin{cases} \frac{\partial u_2}{\partial t} = \Delta u_2 + \epsilon^\gamma a(x)\dot{w}_t(x), & t > 0,\ x \in \mathbf{R}, \\ u_2(0,x) = 0, \end{cases} \tag{2.1}$$

namely $u_2^\epsilon(t,x) = \epsilon^\gamma Y(t,x)$, where

$$Y(t,x) = \int_0^t \int_{\mathbf{R}} q_{t-s}(x,x')a(x')dw_s(x')dx', \tag{2.2}$$

and $q_t(x,x') = (4\pi t)^{-1/2}\exp\{-(x-x')^2/4t\}$ is the heat kernel.

Lemma 2.1. *There exists $Y(\omega) \in \cap_{p\geq 1}L^p(\Omega)$ such that*

$$|u_2^\epsilon(t,x)| \leq \epsilon^\gamma Y(\omega), \quad t \in [0,1], x \in \mathbf{R}, 0 < \epsilon < 1.$$

Proof. From Lemma 2.3 of [33, p.499], the following moment estimate on $Y(t,x)$ holds: For arbitrary $\delta > 0$, there exists $C > 0$ such that

$$E[|Y(t,x) - Y(t',x')|^2] \leq C\{|t-t'|^{1/2} + |x-x'|^{1-\delta}\}, \tag{2.3}$$

for every $t, t' \in [0,1], x, x' \in \mathbf{R}$. Noting that $\{Y(t,x)\}$ is a Gaussian system and applying Kolmogorov-Totoki's theorem (cf. [33, p.501], [53, p.273]), this estimate shows that, for every $p > 1$ and $\kappa > 0$, there exists $Z(\omega) \in L^p(\Omega)$ such that

$$|Y(t,x) - Y(t',x')| \leq Z(\omega)\{|t-t'|^{1/4-2/p-\kappa} + |x-x'|^{1/4-2/p-\kappa}\}, \tag{2.4}$$

for every $t, t' \in [0,1]$, $x, x' \in [-2,2]$. Since $Y(0,x) \equiv 0$, this implies that $Y := \sup_{t\in[0,1],x\in[-2,2]} |Y(t,x)| \in \cap_{p\geq 1}L^p(\Omega)$. For $|x| \geq 2$, $Y(t,x)$ satisfies the heat equation with initial data 0 and boundary data $|Y(t,\pm 2)| \leq Y, t \in [0,1]$. Therefore, the maximum principle for the heat equation implies that $|Y(t,x)| \leq Y$ also for $|x| \geq 2$ and $t \in [0,1]$. □

Let $\bar{u}^\epsilon(t) \equiv \bar{u}^\epsilon(t;K,\delta), K,\delta > 0$, be the solution of an ODE

$$\frac{d\bar{u}^\epsilon}{dt} = \epsilon^{-1}\bar{f}^\delta(\bar{u}^\epsilon), \quad t > 0; \quad \bar{u}^\epsilon(0) = K, \tag{2.5}$$

where $\bar{f}^\delta(u) \in C([0,\infty))$ is chosen as $\bar{f}^\delta(u) \geq \sup_{|u_2|\leq\delta} f(u+u_2), u \geq 0$.

Lemma 2.2. *Assume that $|u^\epsilon(0,x)| \leq K, K > 0$ and $|u_2^\epsilon(t,x)| \leq \bar{\delta}, \bar{\delta} > 0$, holds for every $t \in [0,1], x \in \mathbf{R}$. Then, we have*

$$|u^\epsilon(t,x)| \leq \bar{u}^\epsilon(t;K,\bar{\delta}) + \bar{\delta}, \quad t \in [0,1], x \in \mathbf{R}.$$

Proof. The function $u_1^\epsilon(t,x) := u^\epsilon(t,x) - u_2^\epsilon(t,x)$ satisfies a PDE

$$\begin{cases} \frac{\partial u_1^\epsilon}{\partial t} = \Delta u_1^\epsilon + \epsilon^{-1} f(u_1^\epsilon + u_2^\epsilon), \\ u_1^\epsilon(0,x) = u^\epsilon(0,x). \end{cases} \tag{2.6}$$

Therefore, noting an inequality $d\bar{u}^\epsilon/dt \geq \epsilon^{-1} f(\bar{u}^\epsilon + u_2^\epsilon)$ for $\bar{u}^\epsilon(t) = \bar{u}^\epsilon(t; K, \bar{\delta})$, we see that $v(t,x) := \bar{u}^\epsilon(t) - u_1^\epsilon(t,x)$ satisfies

$$Lv \equiv \Delta v + c(t,x)v(t,x) - \frac{\partial v}{\partial t} \leq 0$$

where

$$c(t,x) = \epsilon^{-1} \frac{f(\bar{u}^\epsilon + u_2^\epsilon) - f(u_1^\epsilon + u_2^\epsilon)}{v}.$$

However, the condition $f'(u) \leq \bar{c}_2$ in (1.2)-(c) implies that $c(t,x) \leq \bar{c}_2 \epsilon^{-1}$; in particular, $c(t,x)$ is bounded from above. Furthermore, the initial data $v(0,x) = K - u^\epsilon(0,x) \geq 0$ and one can easily show that $v(t,x) \geq -Ce^{-c|x|^2}, t \in [0,1], x \in \mathbf{R}$, with some $C, c > 0$. Therefore, the maximum principle (see Theorem 9 of [29, p.43]) proves $v(t,x) \geq 0$ and this gives the upper bound on $u^\epsilon : u^\epsilon(t,x) \leq \bar{u}^\epsilon(t) + \bar{\delta}, t \in [0,1]$. The lower bound is shown analogously, recall the condition (1.2)-(b). □

Theorem 2.1. *If* $|u^\epsilon(0,x)| \leq K, K > 0$, *then*

$$\lim_{\epsilon \downarrow 0} P\left\{ |u^\epsilon(t,x)| \leq \max\{K, 1\} + \delta, t \in [0, \epsilon^{-1/2-2\gamma}T], x \in \mathbf{R} \right\} = 1, \quad T, \delta > 0.$$

Proof. We may assume $K \geq 1$ without loss of generality. Set $c_1 := \inf_{1 \leq u \leq 4}\{f(1) - f(u)\}/(u-1)$; note that $c_1 > 0$, since $f'(1) < 0$ and $f(u) < f(1) = 0$ for $u > 1$ from the condition (1.2)-(a). Then, we have $f(u) \leq -c_1(u-1)$ for $1 \leq u \leq 4$ and consequently, for $0 < \bar{\delta} < 1$, we can choose

$$\bar{f}^{\bar{\delta}}(u) = -c_1\{u - (1+\bar{\delta})\}, \quad u \in [1+\bar{\delta}, 3],$$

in (2.5) with $\delta = \bar{\delta}$. Therefore, if $1 + \bar{\delta} \leq K \leq 3$, we obtain by solving the ODE (2.5)

$$\bar{u}^\epsilon(t; K, \bar{\delta}) = \{K - (1+\bar{\delta})\}e^{-c_1\epsilon^{-1}t} + (1+\bar{\delta}), \quad t \geq 0. \tag{2.7}$$

Let us temporarily assume $|u^\epsilon(0,x)| \leq 1 + \epsilon^{\gamma/2}$ and $0 < \epsilon < 1$. Then, applying Lemma 2.2 with $\bar{\delta} = \epsilon^\gamma Y(\omega)$ and $K = 1 + \epsilon^{\gamma/2}$ and noting Lemma 2.1 and (2.7), we see that

$$|u^\epsilon(t,x)| \leq \epsilon^{\gamma/2}e^{-c_1\epsilon^{-1}t} + 1 + \epsilon^\gamma Y(\omega), \quad t \in [0,1], x \in \mathbf{R},$$

if $\bar{\delta} < 1$ and $1 + \bar{\delta} \leq K$ (i.e., if $\epsilon^\gamma Y(\omega) \leq \epsilon^{\gamma/2}$). Therefore, using Chebyshev's inequality, we have

$$P\left\{ \begin{array}{l} |u^\epsilon(t,x)| \leq 1 + 2\epsilon^{\gamma/2} \text{ for } t \in [0,1/2] \\ |u^\epsilon(t,x)| \leq 1 + \epsilon^{\gamma/2} \text{ for } t \in [1/2,1] \end{array} \right\}$$
$$\geq P\{\epsilon^\gamma Y \leq (1 - e^{-c_1/2})\epsilon^{\gamma/2}\} \geq 1 - (1 - e^{-c_1/2})^{-p}\epsilon^{\gamma p/2}E[Y^p], \quad p > 1.$$

Hence, by the Markov property of $u^\epsilon(t,x)$,

$$P\left\{|u^\epsilon(t,x)| \le 1 + 2\epsilon^{\gamma/2} \text{ for } t \in [0, \epsilon^{-1/2-2\gamma}T]\right\}$$
$$\ge \left\{1 - C\epsilon^{\gamma p/2}\right\}^{\epsilon^{-1/2-2\gamma}T+1}, \quad C > 0,$$

which converges to 1 as $\epsilon \downarrow 0$ if p is taken sufficiently large: $p > 4 + 1/\gamma$.

To treat the general case, divide the time interval $[0, \epsilon^{-1/2-2\gamma}T]$ into $[0,1]$ and $[1, \epsilon^{-1/2-2\gamma}T]$. If $K > 3$, we determine $\bar{u}^\epsilon(t)$ by solving the ODE (2.5) with $\bar{f}^\delta(u) \equiv -c_2 := \sup_{2 \le u \le K+1} f(u) < 0$ (i.e., $\bar{u}^\epsilon(t) = K - c_2\epsilon^{-1}t$) until the time when it reaches 3 and afterwards we continue it as in (2.7). (If $K \le 3$, we determine $\bar{u}^\epsilon(t)$ simply by (2.7).) Then, on the time interval $[0,1]$, we apply the results of Lemmas 2.1 and 2.2 with $\bar{u}^\epsilon(t)$ constructed as above and $\bar{\delta} = \epsilon^\gamma Y(\omega)$. In particular, this shows $\lim_{\epsilon \downarrow 0} P\{|u^\epsilon(1,x)| \le 1 + \epsilon^{\gamma/2},\ x \in \mathbf{R}\} = 1$. Therefore, we can use the result established above on the extra interval $[1, \epsilon^{-1/2-2\gamma}T]$. The conclusion follows by combining these results by the Markov property of $u^\epsilon(t,x)$. □

Remark 2.1. We can remove the condition (1.2)-(c) to derive our main result (Theorem 8.1) under the convention that we consider the stopped process $u^\epsilon(t \wedge \tau^\epsilon_{K'}, x)$ instead of the solution $u^\epsilon(t,x)$ of (1.1) itself, where $\tau^\epsilon_{K'} = \inf\{t > 0; \sup_x |u^\epsilon(t,x)| > K'\}$ for $K' > \max\{1, \sup_x |u^\epsilon(0,x)|\}$.

3 Energy functional

We shall denote by $H^n \equiv H^n(\mathbf{R}), n \in \mathbb{Z}_+ = \{0,1,2,\cdots\}$, the Sobolev spaces equipped with the usual norms $||\cdot||_{H^n}$ defined by

(3.1) $$||s||^2_{H^n} = \sum_{k=0}^{n} ||\nabla^k s||^2_{L^2}, \quad \nabla^k s = d^k s/dy^k,$$

for $s = s(y), y \in \mathbf{R}$, where $||\cdot||_{L^2}$ stands for the norm of the space $L^2 \equiv L^2(\mathbf{R})$. Let $H^n + m$ be the classes of all functions $v = v(y), y \in \mathbf{R}$, such that $v - m \in H^n$, where $m = m(y)$ is the function uniquely determined by the ODE (1.3) satisfying $m(0) = 0$. We associate an energy functional with $v \in \mathrm{Dom}(\mathscr{H}) := H^1 + m$ by

(3.2) $$\mathscr{H}(v) = \int_{\mathbf{R}} \left\{\frac{1}{2}|\nabla v|^2(y) + F(v(y))\right\} dy - C_*,$$

where the constant $C_* > 0$ is chosen as $\min_{v \in \mathrm{Dom}(\mathscr{H})} \mathscr{H}(v) = 0$; for instance, $C_* = 2\sqrt{2}/3$ when $f(u) = u - u^3$. This functional is translation-invariant. Since every critical point of $\mathscr{H}$ automatically satisfies (1.3), the minimum of $\mathscr{H}$ is attained on the set $M = \{m_\eta; \eta \in \mathbf{R}\}$, where m_η is a function defined by shifting m:

(3.3) $$m_\eta(y) = m(y - \eta), \quad y \in \mathbf{R}.$$

The (formal) functional derivative of $\mathcal{H}$ is given by

$$D\mathcal{H}(y,v) = -\Delta v(y) - f(v(y)), \tag{3.4}$$

so that we adopt (3.4) as a definition of $D\mathcal{H}(\cdot,v) \in L^2$ for $v \in H^2 + m$; recall that $F' = -f$. The aim of this section is to investigate the structure of the functional $\mathcal{H}(v)$ and its derivative $\|D\mathcal{H}(\cdot,v)\|_{L^2}^2$ near M. To this end, we need the non-degeneracy of the second derivative (Hessian) of $\mathcal{H}$ at $v = m \in M$ to the normal direction to M:

Lemma 3.1. *Let $\mathcal{A}$ be the Sturm-Liouville (or 1-dimensional Schrödinger) operator:*

$$\mathcal{A} = -\Delta - f'(m(y)), \tag{3.5}$$

in L^2 with $\mathrm{Dom}(\mathcal{A}) = H^2$. Then $\mathcal{A}$ is selfadjoint with discrete spectrum in $(-\infty, f_*)$, where $f_* = -f'(1) > 0$. The principal eigenvalue is $\mu = 0$ and simple. The corresponding eigenspace is spaned by ∇m (called the Goldstone mode in physical literature). In particular, let $\mu_* > 0$ denote the second eigenvalue of $\mathcal{A}$ less than f_* if one exists, or $\mu_* = f_*$ otherwise. Then $\mu = 0$ is the only eigenvalue in $(-\infty, \mu_*)$.

Proof. See [10, p. 536] or [44, 2]. □

Now we introduce the so-called Fermi coordinates of v in a neighborhood of M: Let $\eta(v) \in \mathbf{R}$ be such η that $\mathrm{dist}(v,M) := \min_{\eta\in\mathbf{R}} \|v - m_\eta\|_{L^2}$ is attained; see [37, 34] for the definition of Fermi coordinates in finite dimensional spaces. Notice that η exists uniquely for v sufficiently close to M, i.e., for $v \in L^2 + m$ satisfying $\mathrm{dist}(v,M) \le \beta_0$ with some $\beta_0 > 0$. We call the pair $(\eta(v), s(v) := v - m_{\eta(v)}) \in \mathbf{R} \times L^2$ the Fermi coordinates of v. It follows that

$$\langle v, \nabla m_{\eta(v)}\rangle = \langle s(v), \nabla m_{\eta(v)}\rangle = 0, \tag{3.6}$$

where $\langle \cdot,\cdot \rangle$ denotes the usual inner product of the space L^2.

Lemma 3.2. *There exist $c_1, c_2 > 0$ such that*

$$c_1\|s\|_{H^1}^2 \le \langle \mathcal{A}s, s\rangle \le c_2\|s\|_{H^1}^2, \tag{3.7}$$

for all $s \in C^\infty(\mathbf{R}) \cap H^1$ satisfying $\langle s, \nabla m\rangle = 0$ and

$$c_1\|s\|_{H^2}^2 \le \|\mathcal{A}s\|_{L^2}^2 \le c_2\|s\|_{H^2}^2, \tag{3.8}$$

for all $s \in C^\infty(\mathbf{R}) \cap H^2$ satisfying $\langle s, \nabla m\rangle = 0$.

Proof. The upper bounds in (3.7) and (3.8) are derived quite easily. For the lower bounds, we first notice that Lemma 3.1 immediately implies

$$\langle \mathcal{A}s, s\rangle \ge \mu_*\|s\|_{L^2}^2, \tag{3.9}$$

$$\|\mathcal{A}s\|_{L^2}^2 \ge \mu_*^2\|s\|_{L^2}^2. \tag{3.10}$$

for $s \in L^2$ satisfying $\langle s, \nabla m\rangle = 0$. On the other hand, writing $c_3 = \max_{|u|\le 1} f'(u)$,

$$\begin{aligned}\langle \mathscr{A}s, s\rangle &= \|\nabla s\|_{L^2}^2 - \int_{\mathbf{R}} f'(m(y))s^2(y)\,dy \\ &\ge \|s\|_{H^1}^2 - (c_3+1)\|s\|_{L^2}^2,\end{aligned} \tag{3.11}$$

and, by a Gårding type inequality (which is shown by simple computation in our case),

$$\|\mathscr{A}s\|_{L^2}^2 \ge c_4\|s\|_{H^2}^2 - c_5\|s\|_{L^2}^2, \tag{3.12}$$

with some $c_4, c_5 > 0$. Then, the lower bound in (3.7) with $c_1 = \mu_*/(\mu_*+c_3+1)$ is verified from (3.9) and (3.11). Indeed, we may use (3.11) if $\|s\|_{L^2}^2 \le \|s\|_{H^1}^2/(\mu_*+c_3+1)$ and (3.9) if $\|s\|_{L^2}^2 \ge \|s\|_{H^1}^2/(\mu_*+c_3+1)$. The lower bound in (3.8) follows from (3.10) and (3.12) similarly. □

Theorem 3.1. *There exist $c_1, c_2 > 0$ and $\beta_1, 0 < \beta_1 \le \beta_0$, such that*

$$c_1\|s\|_{H^1}^2 \le \mathscr{H}(v) \le c_2\|s\|_{H^1}^2, \tag{3.13}$$

holds for all $v \in H^1 + m : \|s\|_{H^1} \le \beta_1$ and

$$c_1\|s\|_{H^2}^2 \le \|D\mathscr{H}(\cdot, v)\|_{L^2}^2 \le c_2\|s\|_{H^2}^2, \tag{3.14}$$

holds for all $v \in H^2 + m : \|s\|_{H^1} \le \beta_1$, where $s = s(v)$ and β_0 is the constant appearing just after Lemma 3.1.

Proof. Using the ODE (1.3) for m_η and recalling $\mathscr{H}(m_\eta) = 0$ and $f = -F'$, we have

$$\mathscr{H}(v) = \frac{1}{2}\langle \mathscr{A}_\eta s, s\rangle + \int_{\mathbf{R}} U(y;\eta,s)\,dy \tag{3.15}$$

$$\|D\mathscr{H}(\cdot, v)\|_{L^2}^2 = \|\mathscr{A}_\eta s + V\|_{L^2}^2, \tag{3.16}$$

where

$$\mathscr{A}_\eta = -\Delta - f'(m_\eta(y)), \quad \eta \in \mathbf{R}, \tag{3.17}$$

and

$$\begin{aligned} U &\equiv U(y;\eta,s) \\ &:= F(m_\eta(y)+s(y)) - F(m_\eta(y)) - F'(m_\eta(y))s(y) - \frac{1}{2}F''(m_\eta(y))s(y)^2 \\ V &\equiv V(y;\eta,s) \\ &:= F'(m_\eta(y)+s(y)) - F'(m_\eta(y)) - F''(m_\eta(y))s(y). \end{aligned}$$

Since Sobolev's imbedding theorem shows $\|s\|_{L^\infty} \le C_1\|s\|_{H^1}$, $\|s\|_{H^1} \le \beta_1$ implies $\|s\|_{L^\infty} \le C_1\beta_1$. Therefore, by using Taylor's formula,

$$\left|\int_{\mathbf{R}} U\,dy\right| \le \frac{1}{6} \sup_{|v|\le C_1\beta_1+1} |F'''(v)| \times ||s||_{L^3}^3$$
$$\le C_2||s||_{L^\infty}||s||_{L^2}^2 \le C_3||s||_{H^1}^3.$$

Hence, (3.13) follows from (3.15) and (3.7) shifted by $\eta = \eta(v)$; notice (3.6). Similarly, since

$$||V||_{L^2} \le C_4||s^2||_{L^2} = C_4||s||_{L^4}^2 \le C_5||s||_{H^1}||s||_{H^2},$$

we have from (3.16)

$$\left|||D\mathscr{H}(\cdot,v)||_{L^2}^2 - ||\mathscr{A}_\eta s||_{L^2}^2\right| \le 2||\mathscr{A}_\eta s||_{L^2}||V||_{L^2} + ||V||_{L^2}^2$$
$$\le C_6\{||s||_{H^1} + ||s||_{H^1}^2\}||s||_{H^2}^2.$$

Therefore (3.14) follows from (3.8). □

4 Scaled SPDE and smooth approximation

Let us introduce a scaling in space and time for the solution $u^\epsilon(t,x)$ of (1.1):

(4.1) $$v \equiv v^\epsilon(t,y) := u^\epsilon(\epsilon^{-1/2-2\gamma}t, \epsilon^{1/2}y), \quad t > 0,\ y \in \mathbf{R}.$$

Then, since $\dot{w}_{\epsilon^{-1/2-2\gamma}t}(\epsilon^{1/2}y) = \epsilon^{-1/2-\gamma}\dot{w}_t(y)$ (generally $\dot{w}_{at}(bx) = (a/b)^{1/2}\dot{w}_t(x)$ for $a, b > 0$) in the sense of distribution, the SPDE (1.1) with $\kappa_\epsilon = \epsilon^\gamma$ can be rewritten as

(4.2) $$\frac{\partial v}{\partial t} = \epsilon^{-3/2-2\gamma}\{\Delta v + f(v)\} + \epsilon^{-1/2}a(\epsilon^{1/2}y)\dot{w}_t(y), \quad t > 0,\ y \in \mathbf{R},$$

which amounts to the same as

(4.2′) $$dv = \epsilon^{-3/2-2\gamma}\{\Delta v + f(v)\}dt + \epsilon^{-1/2}a(\epsilon^{1/2}y)dw_t(y), \quad t > 0,\ y \in \mathbf{R}.$$

In order to show that the solution $v_t = v^\epsilon(t,\cdot)$ of (4.2) stays near the minimizer M of the energy functional $\mathscr{H}$ at least if it starts near M, we shall employ $\mathscr{H}$ as a Lyapunov function; however, this idea does not work directly, since $v_t(y)$ is not differentiable in y (P-a.s.) so that $\mathscr{H}(v_t)$ has no meaning. To avoid this inconvenience we consider its smooth approximation.

Let $\rho \in C_0^\infty(\mathbf{R})$ be a non-negative symmetric function on $\mathbf{R}$ satisfying $\rho(y) = 0$ if $|y| \ge 1$ and $\int_{\mathbf{R}} \rho(y)\,dy = 1$ and let $\psi \in C^\infty(\mathbf{R})$ be a function such that $\psi(y) = 1$ for $y \le 0, \psi(y) = 0$ for $y \ge 1$ and $0 < \psi(y) \le 1$ for $0 \le y < 1$. These two functions ρ and ψ will be fixed to Sect. 6. For $0 < \delta < 1$ and $v \in L^\infty_{loc}(\mathbf{R}) \cap C^\infty(\mathbf{R} \setminus [-\epsilon^{-1/2}, \epsilon^{-1/2}])$, we define $\Phi(v) = \Phi^\delta(v) \in C^\infty(\mathbf{R})$ by

(4.3) $$\Phi(v)(z) = \{v * \rho^{\delta(z)}\}(z)$$
$$= \int_{|y'|\le 1} \rho(y')v(z - \delta(z)y')\,dy',$$

where $\rho^\delta(\cdot) = \rho(\cdot/\delta)/\delta$ and $\delta(z) = \delta\psi(|z| - \epsilon^{-1/2} - 1)$; notice that $\delta(z) = \delta$ for $|z| \le \epsilon^{-1/2} + 1$ and $\delta(z) = 0$ for $|z| \ge \epsilon^{-1/2} + 2$. We use the convention: $v * \rho^0(z) = v(z)$. Now a smooth approximation of $v_t = v_t^\epsilon$ is introduced by

$$v^\delta(t,z) \equiv v^{\epsilon,\delta}(t,z) := \Phi^\delta(v^\epsilon(t,\cdot))(z) \in C^\infty(\mathbf{R}). \tag{4.4}$$

In other words, $v^\delta(t,z)$ is the smoothed version of $v(t,z)$ for $|z| \le \epsilon^{-1/2} + 1$, not smoothed for $|z| \ge \epsilon^{-1/2} + 2$ and weakly smoothed on an intermediate region $\epsilon^{-1/2} + 1 \le |z| \le \epsilon^{-1/2} + 2$.

In the rest of this section, we prepare some bounds on the solution $v(t,y) = v^\epsilon(t,y)$ of the SPDE (4.2). We always assume

$$0 < \epsilon < 1,$$

in the following. Let $v_2(t,y) = v_2^\epsilon(t,y)$ be a solution of the SPDE

$$\frac{\partial v_2}{\partial t} = \epsilon^{-3/2-2\gamma}\Delta v_2 + \epsilon^{-1/2} a(\epsilon^{1/2}y)\dot{w}_t(y), \quad t > 0,\ y \in \mathbf{R}, \tag{4.5}$$

satisfying $v_2(0,y) = 0$ and set $v_1(t,y) = v(t,y) - v_2(t,y)$; namely, $v_1(t,y) = u_1^\epsilon(\epsilon^{-1/2-2\gamma}t, \epsilon^{1/2}y)$ and $v_2(t,y) = u_2^\epsilon(\epsilon^{-1/2-2\gamma}t, \epsilon^{1/2}y)$, where $u_1^\epsilon, u_2^\epsilon$ are defined in Sect. 2. Notice that $v_1(t,y)$ solves a nonlinear PDE

$$\frac{\partial v_1}{\partial t} = \epsilon^{-3/2-2\gamma}\{\Delta v_1 + f(v_1 + v_2)\} \tag{4.6}$$

having initial data $v_1(0,y) = v(0,y)$.

Lemma 4.1. *(Estimates on v_2) For every $T > 0, p > 1$ and $\kappa > 0$,*
(i) there exists a positive random variable $Z_\epsilon(\omega) \in L^p(\Omega)$ such that

$$|v_2(t,y) - v_2(t',y')| \le Z_\epsilon(\omega)\{\epsilon^{-1/8+\gamma/2}|t-t'|^{1/4-2/p-\kappa} + \epsilon^{1/16+3\gamma/4-(2/p+\kappa)(3/4+\gamma)}|y-y'|^{1/4-2/p-\kappa}\},$$

for every $0 \le t \le T$, $|y|, |y'| \le \epsilon^{-1/2} + 3$ and $\sup_{0<\epsilon<1} E[Z_\epsilon^p] < \infty$,
(ii) there exists a family of positive random variables $\{Z_{t,\epsilon}(\omega) \in L^p(\Omega)\}$ which are jointly measurable in (t,ω) such that

$$|v_2(t,y) - v_2(t,y')| \le Z_{t,\epsilon}(\omega)\epsilon^{1/4+\gamma-(1/2p+\kappa)(3/4+\gamma)}|y-y'|^{1/2-1/2p-\kappa},$$

for every $0 \le t \le T$, $|y|, |y'| \le \epsilon^{-1/2} + 3$ and $\sup_{0\le t\le T, 0<\epsilon<1} E[Z_{t,\epsilon}^p] < \infty$.

Remark 4.1. The estimate (ii) is better than (i) for both Hölder exponent in y and decay rate in ϵ, but holds only for fixed t and will be used only for the proof of Proposition 5.3 below.

Proof of Lemma 4.1 Consider a rescaled process $\tilde{v}_2(t,x) := v_2(t, \epsilon^{-3/4-\gamma}x)$ of $v_2(t,x)$. Then, since $\dot{w}_t(\epsilon^{-3/4-\gamma}x) = \epsilon^{3/8+\gamma/2}\dot{w}_t(x)$ in the sense of distribution, $\tilde{v}_2$ satisfies the SPDE

$$\frac{\partial \tilde{v}_2}{\partial t} = \Delta \tilde{v}_2 + \epsilon^{-1/8+\gamma/2} a(\epsilon^{-1/4-\gamma} x)\dot{w}_t(x),$$

and this implies $\tilde{v}_2(t,x) = \epsilon^{-1/8+\gamma/2}\tilde{Y}(t,x)$, where $\tilde{Y}(t,x) \equiv \tilde{Y}_\epsilon(t,x)$ is defined by the formula (2.2) with $a(x')$ replaced by $a(\epsilon^{-1/4-\gamma}x')$. Since a is bounded, the estimate (2.3) still holds for $\tilde{Y}(t,x)$ and for $t,t' \in [0,T], x,x' \in \mathbf{R}$. Therefore, the estimate (2.4) holds for $\tilde{Y}(t,x)$ in place of $Y(t,x)$ and for $t,t' \in [0,T], x,x' \in [-4,4]$; notice that $Z(\omega) = Z_\epsilon(\omega)$ may depend on ϵ, but satisfies $\sup_{0<\epsilon<1} E[Z_\epsilon^p] < \infty$. This estimate on $\tilde{Y}(t,x)$ shows the assertion (i) by noting that $v_2(t,y) = \epsilon^{-1/8+\gamma/2}\tilde{Y}(t,\epsilon^{3/4+\gamma}y)$ and $|y| \le \epsilon^{-1/2}+3$ implies $|\epsilon^{3/4+\gamma}y| \le 4$. To prove (ii), taking $t = t'$ in the estimate (2.3) for $\tilde{Y}(t,x)$, we have

$$(2.3') \qquad E[|\tilde{Y}(t,x) - \tilde{Y}(t,x')|^2] \le C|x-x'|^{1-\delta}, \quad 0 \le t \le T, x,x' \in \mathbf{R},$$

for every $\delta > 0$. Then, the similar argument as above implies the assertion (ii). □

Lemma 4.2. *(Estimate on v_1) Assume $\gamma \ge 1/4$ and $v(0,\cdot) \in C_b^1(\mathbf{R})$; in particular, both $v(0,\cdot)$ and its derivative $v'(0,\cdot)$ in y are bounded. Then, we have*

$$\sup_{0<\epsilon<1} E[Y_\epsilon^p] < \infty, \quad T > 0, p > 1,$$

where $Y_\epsilon(\omega) = \sup_{0\le t\le \tau_K \wedge T, y\in\mathbf{R}} |(v_1^\epsilon)'(t,y)|$, $v_1^\epsilon(t,y) = v_1(t,y)$ *and* $\tau_K = \inf\{t > 0; \sup_{y\in\mathbf{R}} |v(t,y)| > K\}$, $K > 0$.

Proof. First we notice

$$(4.7) \qquad \sup_{0<\epsilon<1} E[\sup_{0\le t\le T}\sup_{y\in\mathbf{R}} |v_2(t,y)|^p] < \infty, \quad T > 0, p > 1.$$

In fact, since $v_2(0,y) = 0$ and $\gamma \ge 1/4$, Lemma 4.1-(i) with $(t',y') = (0,y)$ proves (4.7) with $\sup_{|y|\le\epsilon^{-1/2}+3}$ in place of $\sup_{y\in\mathbf{R}}$. However, this shows (4.7) with the help of the maximum principle for the heat equation; similar argument was used in the proof of Lemma 2.1. Now, set $\tilde{v}_1(t,y) = v_1(\epsilon^{3/2+2\gamma}t,y)$ and $\tilde{v}(t,y) = v(\epsilon^{3/2+2\gamma}t,y)$. Then, the equation (4.6) is rewritten in $\partial\tilde{v}_1/\partial t = \Delta\tilde{v}_1 + f(\tilde{v})$, which can be rewritten further in an integral form

$$(4.8) \qquad \tilde{v}_1(t,y) = \int_{\mathbf{R}} e^{-ct} q_t(y,y')v_0(y')\,dy' + \int_0^t ds \int_{\mathbf{R}} e^{-c(t-s)} q_{t-s}(y,y')\{c\tilde{v}_1(s,y') + f(\tilde{v}(s,y'))\}\,dy',$$

where $c \in \mathbf{R}$ and q_t is the heat kernel. The proof of lemma is easily completed by taking the derivative of the both sides of (4.8) in y; we take $c > 0$ (the decay factor e^{-ct} plays an important role since we consider on the time scale $\epsilon^{-(3/2+2\gamma)}$ for $\tilde{v}_1$) and note that (4.7) implies

$$\sup_{0<\epsilon<1} E[\sup_{0\le t\le \epsilon^{-(3/2+2\gamma)}\{\tau_K\wedge T\}} \sup_{y\in\mathbf{R}} |c\tilde{v}_1(t,y) + f(\tilde{v}(t,y))|^p] < \infty. \quad \square$$

5 Energy estimate for scaled SPDE

Let $v_t^\delta = v^\delta(t,\cdot)$ be the smooth approximation, defined by (4.4), of the solution $v_t = v_t^\epsilon$ of the SPDE (4.2). Then, its stochastic differential (in t) is given by

$$(5.1)\qquad dv_t^\delta(z) = \epsilon^{-3/2-2\gamma} b_t^\delta(z)\,dt + d\mu_t^\delta(z), \quad z \in \mathbf{R},$$

where $b_t^\delta(z) \equiv b_t^{\epsilon,\delta}(z)$ and $\mu_t^\delta(z) \equiv \mu_t^{\epsilon,\delta}(z)$ are defined by

$$b_t^\delta(z) = \begin{cases} \langle v_t(\cdot), \Delta_y \rho^{\delta(z)}(z-\cdot)\rangle + \langle f(v_t(\cdot)), \rho^{\delta(z)}(z-\cdot)\rangle, & |z| < \epsilon^{-1/2}+2, \\ \Delta_z v_t^\delta(z) + f(v_t^\delta(z)), & |z| \ge \epsilon^{-1/2}+2, \end{cases}$$

$$\mu_t^\delta(z) = \begin{cases} \epsilon^{-1/2}\int_0^t \langle \rho^\delta(z-\cdot)a(\epsilon^{1/2}\cdot), dw_s\rangle, & |z| < \epsilon^{-1/2}+1, \\ 0, & |z| \ge \epsilon^{-1/2}+1. \end{cases}$$

In fact, when $|z| \ge \epsilon^{-1/2}+2$, we see $v_t^\delta(z) = v_t(z)$ and $a(\epsilon^{1/2}z) = 0$ so that (5.1) is the same as (4.2′), while (5.1) is the SPDE (4.2′) multiplied by a test function $\varphi(y) = \rho^{\delta(z)}(z-y)$ when $|z| < \epsilon^{-1/2}+2$. Note that $\rho^{\delta(z)}(z-y)a(\epsilon^{1/2}y) \neq 0$ implies "$|y| \le \epsilon^{-1/2}$ and $|z-y| \le 1$" and especially $|z| \le \epsilon^{-1/2}+1$, for which we have $\delta(z) = \delta$. For every $p \ge 1$, let us calculate $d\mathscr{H}^p(v_t^\delta)$ based on Itô's formula; remind that $\mathscr{H}^p(v_t^\delta)$ denotes the p-th power of $\mathscr{H}(v_t^\delta)$.

Lemma 5.1.

$$(5.2)\qquad \begin{aligned} d\mathscr{H}^p(v_t^\delta) &= p\mathscr{H}^{p-1}(v_t^\delta)\langle dv_t^\delta, D\mathscr{H}(\cdot, v_t^\delta)\rangle \\ &\quad + \frac{1}{2}\epsilon^{-1}\int_{\mathbf{R}} V^{p,\delta}(y, v_t^\delta)a^2(\epsilon^{1/2}y)\,dy\cdot dt, \end{aligned}$$

where

$$V^{p,\delta}(y,v) = p(p-1)\mathscr{H}^{p-2}(v)\{V_1^\delta(y,v)\}^2 + p\mathscr{H}^{p-1}(v)V_2^\delta(y,v)$$

and

$$\begin{aligned} V_1^\delta(y,v) &= \{D\mathscr{H}(\cdot,v) * \rho^\delta\}(y), \\ V_2^\delta(y,v) &= \langle\{-\Delta - f'(v(\cdot))\}\rho^\delta(\cdot-y), \rho^\delta(\cdot-y)\rangle. \end{aligned}$$

Proof. First, we notice a general fact that the quadratic variational process $[\mu^1,\mu^2]_t$ of two martingales $\mu_t^i = \int_0^t \langle h_s^i, dw_s\rangle, i = 1,2$, is given by

$$[\mu^1,\mu^2]_t = \int_0^t \langle h_s^1, h_s^2\rangle\,ds,$$

where h_t^i are L^2-valued $\sigma\{w_s; s \le t\}$-adapted measurable processes. Then, computing $d\{\Delta v_t^\delta(z)\cdot v_t^\delta(z)\}$ by (usual) Itô's formula for each $z \in \mathbf{R}$ and integrating it in z, we have

$$\begin{aligned} d\langle \Delta v_t^\delta, v_t^\delta\rangle &= 2\langle \Delta v_t^\delta, dv_t^\delta\rangle + \int_{\mathbf{R}} dz\cdot d[\Delta\mu_\cdot^\delta(z), \mu_\cdot^\delta(z)]_t \\ &= 2\langle \Delta v_t^\delta, dv_t^\delta\rangle + \epsilon^{-1}\int_{\mathbf{R}} \langle \Delta\rho^\delta(\cdot-y), \rho^\delta(\cdot-y)\rangle a^2(\epsilon^{1/2}y)dy\cdot dt. \end{aligned}$$

On the other hand,

$$dF(v_t^\delta(z)) = F'(v_t^\delta(z))dv_t^\delta(z) + \frac{1}{2}F''(v_t^\delta(z))d[\mu_\cdot^\delta(z), \mu_\cdot^\delta(z)]_t$$
$$= F'(v_t^\delta(z))dv_t^\delta(z)$$
$$+\frac{1}{2}\epsilon^{-1}F''(v_t^\delta(z)) \int_{\mathbf{R}} \{\rho^\delta(z-y)\}^2 a^2(\epsilon^{1/2}y)dy \cdot dt.$$

Therefore, recalling $f = -F'$ and the definitions (3.2) and (3.4) of $\mathcal{H}(v)$ and $D\mathcal{H}(y,v)$, respectively, we obtain

$$\text{(5.3)} \qquad d\mathcal{H}(v_t^\delta) = d\left\{-\frac{1}{2}\langle \Delta v_t^\delta, v_t^\delta\rangle + \int_{\mathbf{R}} F(v_t^\delta(z))\,dz\right\}$$
$$= \langle dv_t^\delta, D\mathcal{H}(\cdot, v_t^\delta)\rangle + \frac{1}{2}\epsilon^{-1}\int_{\mathbf{R}} V_2^\delta(y, v_t^\delta)a^2(\epsilon^{1/2}y)\,dy \cdot dt.$$

Now, the conclusion follows since we have

$$d\mathcal{H}^p(v_t^\delta) = p\mathcal{H}^{p-1}(v_t^\delta)d\mathcal{H}(v_t^\delta) + \frac{1}{2}p(p-1)\mathcal{H}^{p-2}(v_t^\delta)d[\mathcal{H}(v_\cdot^\delta), \mathcal{H}(v_\cdot^\delta)]_t.$$

Note that (5.3) implies, in particular, that the martingale part of $d\mathcal{H}(v_t^\delta)$ is

$$\langle d\mu_t^\delta, D\mathcal{H}(\cdot, v_t^\delta)\rangle = \epsilon^{-1/2}\langle V_1^\delta(\cdot, v_t^\delta)a(\epsilon^{1/2}\cdot), dw_t\rangle$$

which shows

$$d[\mathcal{H}v_\cdot^\delta), \mathcal{H}(v_\cdot^\delta)]_t = \epsilon^{-1}\int_{\mathbf{R}} \{V_1^\delta(y, v_t^\delta)\}^2 a^2(\epsilon^{1/2}y)\,dy \cdot dt. \quad \square$$

To estimate the second term of (5.2) we prepare

Lemma 5.2. *For every $K > 0$, there exists $C_K > 0$ such that*

$$\int_{\mathbf{R}} V^{p,\delta}(y,v)a^2(\epsilon^{1/2}y)\,dy$$
$$\le p(p-1)\mathcal{H}^{p-2}(v)\|D\mathcal{H}(\cdot,v)\|_{L^2}^2 + C_K\delta^{-3}\epsilon^{-1/2}p\mathcal{H}^{p-1}(v)$$

holds if $\sup_{|y|\le\epsilon^{-1/2}+\delta}|v(y)| \le K$.

Proof. Using the Hausdorff-Young inequality and recalling that $|a(x)| \le 1$, we have

$$\int_{\mathbf{R}} \{V_1^\delta(y,v)\}^2 a^2(\epsilon^{1/2}y)\,dy \le \|D\mathcal{H}(\cdot,v)\|_{L^2}^2.$$

On the other hand,

$$V_2^\delta(y,v) \le \langle -\Delta\rho^\delta, \rho^\delta\rangle + \sup_{|v|\le K}|f'(v)|\,\|\rho^\delta\|_{L^2}^2 \le C_K'\delta^{-3}.$$

Therefore, we have the conclusion noting that $a(\epsilon^{1/2}y) = 0$ for $|y| \ge \epsilon^{-1/2}$. $\square$

Next we compute the first term in the right hand side of (5.2). To this end, we rewrite the term $b_t^\delta(z)$ appearing in (5.1).

Lemma 5.3.

$$\langle v_t(\cdot), \Delta_y \rho^{\delta(z)}(z-\cdot)\rangle = \Delta v_t^\delta(z) + R_t^{\delta,1}(z),$$

where

$$\begin{aligned} R_t^{\delta,1}(z) &= \delta''(z)\int_{|y'|\le 1} \rho(y')v_t'(z-\delta(z)y')y'\,dy' \\ &+ \int_{|y'|\le 1} \rho(y')v_t''(z-\delta(z)y')\left[\{1-\delta'(z)y'\}^2 - 1\right] dy'. \end{aligned}$$

Especially $R_t^{\delta,1}(z) = 0$ *if* $|z| \notin [\epsilon^{-1/2}+1, \epsilon^{-1/2}+2]$.

Proof. The final assertion is true because $\delta'(z) = \delta''(z) = 0$ for $|z| \notin [\epsilon^{-1/2} + 1, \epsilon^{-1/2}+2]$. When $|z| \le \epsilon^{-1/2}+1$, $\delta(z) = \delta$ and this implies $v_t^\delta(z) = v_t * \rho^\delta(z)$. Therefore, the conclusion holds with $R_t^{\delta,1}(z) = 0$. When $|z| \ge \epsilon^{-1/2}+1$, noting that $v_t(z-\delta(z)y')$, $|y'| \le 1$, is differentiable in z, we have

$$\begin{aligned} \Delta v_t^\delta(z) &= \int_{|y'|\le 1} \rho(y')\Delta_z\{v_t(z-\delta(z)y')\}\,dy' \\ &= \int_{|y'|\le 1} \rho(y')\left[v_t''(z-\delta(z)y')\{1-\delta'(z)y'\}^2 - v_t'(z-\delta(z)y')\delta''(z)y'\right] dy'. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \langle v_t(\cdot), \Delta_y\rho^{\delta(z)}(z-\cdot)\rangle &= \int_{\mathbf{R}} v_t''(y)\rho^{\delta(z)}(z-y)\,dy \\ &= \int_{|y'|\le 1} \rho(y')v_t''(z-\delta(z)y')\,dy'. \end{aligned}$$

Therefore, the conclusion follows also when $|z| \ge \epsilon^{-1/2}+1$. □

Now, we get

$$\begin{aligned} b_t^\delta(z) &= \{\Delta v_t^\delta(z) + R_t^{\delta,1}(z)\} + \{f(v_t^\delta(z)) + R_t^{\delta,2}(z)\} \\ &= -D\mathscr{H}(z, v_t^\delta) + R_t^\delta(z), \end{aligned}$$

where

$$R_t^{\delta,2}(z) := \begin{cases} \langle f(v_t(\cdot)), \rho^{\delta(z)}(z-\cdot)\rangle - f(v_t^\delta(z)), & \text{if } |z| \le \epsilon^{-1/2}+2, \\ 0, & \text{if } |z| \ge \epsilon^{-1/2}+2, \end{cases}$$

and $R_t^\delta(z) = R_t^{\delta,1}(z) + R_t^{\delta,2}(z)$. Inserting this into (5.1), we have

$$\begin{aligned} &p.\mathscr{H}^{p-1}(v_t^\delta)\langle dv_t^\delta, D\mathscr{H}(\cdot, v_t^\delta)\rangle \\ &= p.\mathscr{H}^{p-1}(v_t^\delta)\epsilon^{-3/2-2\gamma}\langle D\mathscr{H}(\cdot, v_t^\delta), -D\mathscr{H}(\cdot, v_t^\delta) + R_t^\delta(\cdot)\rangle\,dt + d\mu_t^\delta \end{aligned}$$

where

$$\mu_t^\delta = p\int_0^t \mathcal{H}^{p-1}(v_s^\delta)\langle D\mathcal{H}(\cdot, v_s^\delta), d\mu_s^\delta(\cdot)\rangle$$

is a martingale. We have proved the following proposition by virtue of Lemmas 5.1, 5.2 and by using a simple estimate: $\langle D\mathcal{H}(\cdot, v_t^\delta), R_t^\delta(\cdot)\rangle \le \{||D\mathcal{H}(\cdot, v_t^\delta)||_{L^2}^2 + ||R_t^\delta||_{L^2}^2\}/2$.

Proposition 5.1. *Whenever* $\sup_{0\le s\le t} ||v_s||_{L^\infty} \le K$, *we have*

$$\begin{aligned}\mathcal{H}^p(v_t^\delta) &+ \frac{p}{2}\epsilon^{-3/2-2\gamma}\int_0^t \mathcal{H}^{p-1}(v_s^\delta)||D\mathcal{H}(\cdot, v_s^\delta)||_{L^2}^2\, ds \\ &\le \mathcal{H}^p(v_0^\delta) + \mu_t^\delta + \frac{p}{2}\epsilon^{-3/2-2\gamma}\int_0^t \mathcal{H}^{p-1}(v_s^\delta)||R_s^\delta||_{L^2}^2\, ds \\ &+ \frac{p(p-1)}{2}\epsilon^{-1}\int_0^t \mathcal{H}^{p-2}(v_s^\delta)||D\mathcal{H}(\cdot, v_s^\delta)||_{L^2}^2\, ds \\ &+ C_K p\epsilon^{-3/2}\delta^{-3}\int_0^t \mathcal{H}^{p-1}(v_s^\delta)\, ds.\end{aligned}$$

Let us introduce a stopping time:

$$\tau_K := \inf\{t > 0; \sup_{y\in\mathbf{R}} |v(t,y)| > K\}, \quad K > 0. \tag{5.4}$$

We shall prove the following two estimates on the error terms $R_t^{\delta,1}$ and $R_t^{\delta,2}$, respectively, in the next section.

Proposition 5.2. *Assume that the initial data* $v(0,y)$ *of the SPDE (4.2) satisfies*

$$\sup_{|y|\ge\epsilon^{-1/2}+\frac{1}{2}} \{|v'(0,y)| + |v''(0,y)|\} < \infty.$$

Then, there exists $C > 0$ *such that*

$$|R_t^{\delta,1}(z)| \le C\delta, \quad |z| \in [\epsilon^{-1/2}+1, \epsilon^{-1/2}+2],\ 0 < \delta < 1/2,\ t \le \tau_K.$$

Proposition 5.3. *Assume* $\gamma \ge 1/4$ *and* $v(0,\cdot) \in C_b^1(\mathbf{R})$. *Then, for every* $T > 0, \kappa > 0$ *and* $p > 1$, *there exists a family of positive random variables* $\{Y_\epsilon(\omega), Z_{t,\epsilon}(\omega) \in L^p(\Omega)\}$, *the latter of which are jointly measurable in* (t,ω) *such that*

$$|R_t^{\delta,2}(z)| \le Y_\epsilon(\omega)\delta + Z_{t,\epsilon}(\omega)\epsilon^{1/4+\gamma-\kappa}\delta^{1/2-\kappa},\ |z| \le \epsilon^{-1/2}+2,\ t \le \tau_K \wedge T,$$

and

$$\sup_{0<\epsilon<1} E[Y_\epsilon^p] < \infty, \qquad \sup_{0\le t\le T, 0<\epsilon<1} E[Z_{t,\epsilon}^p] < \infty.$$

Combining these two estimates, under the assumptions of these two propositions, we immediately obtain

$$(5.5)\qquad ||R_t^\delta||_{L^2}^2 \le C_1\{(Y_\epsilon^2+1)\delta^2 + Z_{t,\epsilon}^2\epsilon^{1/2+2\gamma-2\kappa}\delta^{1-2\kappa}\}\epsilon^{-1/2}, t\le \tau_K\wedge T,$$

for $T>0$, $0<\epsilon<1$ and $0<\delta<1/2$. Set

$$\sigma_\delta(\beta)\equiv\sigma_{\epsilon,\delta}(\beta) := \inf\{t>0; ||s(v_t^\delta)||_{H^1}>\beta\},\quad \beta>0,$$
$$\sigma=\sigma_\delta(\beta)\wedge\tau_K\wedge T,$$

where $\beta=\beta_\epsilon$ will be determined later in such a manner that $\beta\downarrow 0$ as $\epsilon\downarrow 0$. Then, Proposition 5.1 shows

$$(5.6)\qquad E[\mathscr{H}^p(v_{t\wedge\sigma}^\delta)] + \frac{p}{2}\epsilon^{-3/2-2\gamma}E[\int_0^{t\wedge\sigma}\mathscr{H}^{p-1}(v_s^\delta)||D\mathscr{H}(\cdot,v_s^\delta)||_{L^2}^2\,ds]$$
$$\le \mathscr{H}^p(v_0^\delta) + \frac{p}{2}\epsilon^{-3/2-2\gamma}E[\int_0^{t\wedge\sigma}\mathscr{H}^{p-1}(v_s^\delta)||R_s^\delta||_{L^2}^2\,ds]$$
$$+\frac{p(p-1)}{2}\epsilon^{-1}E[\int_0^{t\wedge\sigma}\mathscr{H}^{p-2}(v_s^\delta)||D\mathscr{H}(\cdot,v_s^\delta)||_{L^2}^2\,ds]$$
$$+C_Kp\epsilon^{-3/2}\delta^{-3}E[\int_0^{t\wedge\sigma}\mathscr{H}^{p-1}(v_s^\delta)\,ds].$$

For simplicity, from now on in this paper, we shall assume $v_0\in M$ (i.e., $v_0=m_\eta$ with some $\eta\in\mathbf{R}$) for the initial data v_0 of the SPDE (4.2); in particular, v_0 fulfills the assumptions of Propositions 5.2 and 5.3. Then, a bound on the term $\mathscr{H}(v_0^\delta)$ appearing in the right hand side of (5.6) can be easily derived:

Lemma 5.4. *$\mathscr{H}(m_\eta^\delta)\le C_2\delta$, where C_2 is a constant independent of $\eta\in\mathbf{R}$.*

Proof. Since $\mathscr{H}(m_\eta)=0$, we see

$$\mathscr{H}(m_\eta^\delta) = \frac{1}{2}\int_{\mathbf{R}}\{|\nabla m_\eta^\delta(z)|^2-|\nabla m_\eta(z)|^2\}\,dz$$
$$+\int_{\mathbf{R}}\{F(m_\eta^\delta(z))-F(m_\eta(z))\}\,dz.$$

The conclusion follows by showing

$$|\nabla m_\eta^\delta(z)-\nabla m_\eta(z)|\le C\delta\{|\nabla m_\eta(z)|+|\nabla^2 m_\eta(z)|\},$$
$$|m_\eta^\delta(z)-m_\eta(z)|\le C\delta|\nabla m_\eta(z)|.\quad\square$$

Especially, noting $\mathscr{H}(v)\ge 0$, we have from (5.6) with $p=1$ and Lemma 5.4

$$
\begin{aligned}
&E\left[\int_0^{t\wedge\sigma} \|D\mathcal{H}(\cdot, v_s^\delta)\|_{L^2}^2\, ds\right] \\
&\le 2C_2\epsilon^{3/2+2\gamma}\delta + E[\int_0^{t\wedge\sigma} \|R_s^\delta\|_{L^2}^2\, ds] + 2C_K\epsilon^{2\gamma}\delta^{-3}T \\
&\le 2C_2\epsilon^{3/2+2\gamma}\delta + C_1\epsilon^{-1/2}E[\int_0^t \{(Y_\epsilon^2+1)\delta^2 + Z_{s,\epsilon}^2\epsilon^{1/2+2\gamma-2\kappa}\delta^{1-2\kappa}\}\, ds] \\
&\quad +2C_K\epsilon^{2\gamma}\delta^{-3}T \\
&\le 2C_2\epsilon^{3/2+2\gamma}\delta + C_3\left[\epsilon^{-1/2}\{\delta^2+\epsilon^{1/2+2\gamma-2\kappa}\delta^{1-2\kappa}\} + \epsilon^{2\gamma}\delta^{-3}\right]
\end{aligned}
$$

for arbitrary $\kappa > 0$. Taking $\delta = \epsilon^{1/10+2\gamma/5}$ (so that $\epsilon^{-1/2}\delta^2 = \epsilon^{2\gamma}\delta^{-3} = \epsilon^{-3/10+4\gamma/5} \gg \epsilon^{2\gamma}\delta$ from which the second term in braces in the last line becomes negligible), we obtain

$$
(5.7) \qquad E\left[\int_0^{t\wedge\sigma} \|D\mathcal{H}(\cdot, v_s^\delta)\|_{L^2}^2\, ds\right] \le C_4\epsilon^{-3/10+4\gamma/5}.
$$

We return to the estimate (5.6) and set

$$
A_p := E\left[\int_0^{t\wedge\sigma} \mathcal{H}^{p-1}(v_s^\delta)\|D\mathcal{H}(\cdot, v_s^\delta)\|_{L^2}^2\, ds\right].
$$

Because of (5.5) and recalling $\delta = \epsilon^{1/10+2\gamma/5}$, we have

$$
\|R_s^\delta\|_{L^2}^2 \le C_1\{Y_\epsilon^2 + Z_{s,\epsilon}^2 + 1\}\epsilon^{-3/10+4\gamma/5},
$$

by taking sufficiently small $\kappa > 0$ in (5.5). Therefore, (5.6) implies

$$
\begin{aligned}
(5.8) \qquad A_p &\le C_5\epsilon^{3/2+2\gamma+p(1/10+2\gamma/5)} + (p-1)\epsilon^{1/2+2\gamma}A_{p-1} \\
&\quad +\epsilon^{-3/10+4\gamma/5}E[\int_0^{t\wedge\sigma} \mathcal{H}^{p-1}(v_s^\delta)X_{s,\epsilon}\, ds],
\end{aligned}
$$

where $C_5 = C_{5,p} = 2C_2^p/p$ and $X_{s,\epsilon} = C_1\{Y_\epsilon^2+Z_{s,\epsilon}^2+1\}+2C_K$. However, applying Hölder's inequality and then using (5.7),

$$
\begin{aligned}
(5.9) \qquad A_{p-1} &= E\left[\int_0^{t\wedge\sigma} \mathcal{H}^{p-2}(v_s^\delta)\|D\mathcal{H}(\cdot, v_s^\delta)\|_{L^2}^2\, ds\right] \\
&\le \{A_p\}^{(p-2)/(p-1)}E\left[\int_0^{t\wedge\sigma} \|D\mathcal{H}(\cdot, v_s^\delta)\|_{L^2}^2\, ds\right]^{1/(p-1)} \\
&\le C_6\epsilon^{(-3/10+4\gamma/5)/(p-1)}\{A_p\}^{(p-2)/(p-1)}, \quad p \ge 2.
\end{aligned}
$$

To give a bound on the third term in the right hand side of (5.8), we use Theorem 3.1. In fact, since $\beta = \beta_\epsilon$ will be taken such that $\beta \downarrow 0$ as $\epsilon \downarrow 0$,

$$
(5.10) \qquad \mathcal{H}(v_t^\delta) \le C_7\|D\mathcal{H}(\cdot, v_t^\delta)\|_{L^2}^2, \quad 0 \le t \le \sigma,
$$

holds with some $C_7 > 0$ for all sufficiently small $\epsilon > 0$ (such that $\beta_\epsilon \le \beta_1$ =the constant appearing in Theorem 3.1). Therefore, using Hölder's inequality,

$$(5.11)\qquad E\left[\int_0^{t\wedge\sigma} \mathscr{H}^{p-1}(v_s^\delta)X_{s,\epsilon}\,ds\right]$$
$$= E\left[\int_0^{t\wedge\sigma} \mathscr{H}^{p-1-1/q'}(v_s^\delta)\mathscr{H}^{1/q'}(v_s^\delta)X_{s,\epsilon}\,ds\right]$$
$$\leq E\left[\int_0^{t\wedge\sigma} \mathscr{H}^{p}(v_s^\delta)\,ds\right]^{1/p'}\cdot E\left[\int_0^{t\wedge\sigma} \mathscr{H}(v_s^\delta)\,ds\right]^{1/q'}\cdot E\left[\int_0^{t\wedge\sigma} X_{s,\epsilon}^{r'}\,ds\right]^{1/r'}$$
$$\leq C_8\epsilon^{(-3/10+4\gamma/5)/q'}\{A_p\}^{1/p'},$$

for $1 < p', q', r' < \infty$ satisfying $1/p' + 1/q' + 1/r' = 1$ and $p'(p-1-1/q') = p$, namely,

$$(5.12)\qquad \frac{1}{p'} = \frac{p-2+\frac{1}{r'}}{p-1},\quad \frac{1}{q'} = \frac{1-\frac{p}{r'}}{p-1}.$$

We take r' to be sufficiently large. The last inequality in (5.11) is shown by using (5.10) and (5.7). Summarizing (5.8), (5.9) and (5.11), we obtain

$$(5.13)\qquad A_p \leq C_5\epsilon^{3/2+2\gamma+p(1/10+2\gamma/5)}$$
$$+C_6(p-1)\epsilon^{1/2+2\gamma+(-3/10+4\gamma/5)/(p-1)}\{A_p\}^{(p-2)/(p-1)}$$
$$+C_8\epsilon^{(-3/10+4\gamma/5)(1+1/q')}\{A_p\}^{1/p'},$$

which proves

$$(5.14)\qquad \tilde{A}_p \leq C_5\epsilon^{3/2+2\gamma+2p(1-\gamma)/5} + C_6(p-1)\{\tilde{A}_p\}^{(p-2)/(p-1)} + C_8\{\tilde{A}_p\}^{1/p'},$$

for $\tilde{A}_p = \epsilon^{-p(-3/10+4\gamma/5)}A_p$; note that $1/2+2\gamma > -3/10+4\gamma/5$ and also that (5.12) implies $(1+1/q') = p(1-1/p')$. However, a bound $x \leq c_1 + c_2x^\alpha + c_3x^\beta$ for $x \geq 0$ with $0 < \alpha, \beta < 1$ implies $x \leq C\max\{c_1, 1\}$ with some constant C independent of c_1. Therefore, we obtain from (5.14)

$$(5.15)\qquad A_p \leq C_9\epsilon^{p(-3/10+4\gamma/5)} \times \max\{\epsilon^{3/2+2\gamma+2p(1-\gamma)/5}, 1\},\quad p > 2\,.$$

Let us return to (5.6). We have

$$\epsilon^{3/2+2\gamma}\; E[\mathscr{H}^p(v_{t\wedge\sigma}^\delta)] \leq \epsilon^{3/2+2\gamma}\mathscr{H}^p(v_0^\delta) + \frac{p}{2}E\left[\int_0^{t\wedge\sigma} \mathscr{H}^{p-1}(v_s^\delta)\|R_s^\delta\|_{L^2}^2\,ds\right]$$
$$+\frac{p(p-1)}{2}\epsilon^{1/2+2\gamma}A_{p-1} + C_K p\epsilon^{2\gamma}\delta^{-3}E\left[\int_0^{t\wedge\sigma} \mathscr{H}^{p-1}(v_s^\delta)\,ds\right],$$

which is bounded by the same quantity ($\times p/2$) in the right hand side of (5.13). In fact, we may follow the same lines through (5.8) to (5.13). Therefore, assuming that $v_0 \in M$ and using (5.15), we have

$$\epsilon^{3/2+2\gamma}E[\mathscr{H}^p(v_{t\wedge\sigma}^\delta)] \leq C_{10}\epsilon^{p(-3/10+4\gamma/5)} \times \max\{\epsilon^{3/2+2\gamma+2p(1-\gamma)/5}, 1\},$$

which verifies

$$\begin{aligned}(c_1\beta^2)^p P\{\sigma \le t\} &\le E[\mathscr{H}^p(v^\delta_{t\wedge\sigma})]\\ &\le C_{10}\epsilon^{p(-3/10+4\gamma/5)} \times \max\{\epsilon^{2p(1-\gamma)/5}, \epsilon^{-3/2-2\gamma}\}.\end{aligned}$$

We have used Theorem 3.1 to get the first inequality. Now, first assuming $3/8 < \gamma \le 1$, we choose $\beta : \beta^2 = \epsilon^{-3/10+4\gamma/5-\kappa}$ for arbitrary but sufficiently small $\kappa > 0$; notice that $\beta \downarrow 0$ as $\epsilon \downarrow 0$ so that the condition $\beta \le \beta_1$ is fulfilled. Then

$$P\{\sigma \le t\} \le c_1^{-p} C_{10}\epsilon^{p\kappa-3/2-2\gamma}$$

which converges to 0 as $\epsilon \downarrow 0$; take $p = (3/2+2\gamma)/\kappa+1$. Secondly when $\gamma > 1$, we choose $\beta : \beta^2 = \epsilon^{1/10+2\gamma/5-\kappa}$ and then similar argument works. Noting Theorem 2.1 we have proved

Proposition 5.4. *Assume $\gamma > 3/8$ and $v_0 \in M$ for the SPDE (4.2). Take $\beta = \epsilon^{-3/20+2\gamma/5-\kappa}$ when $3/8 < \gamma \le 1$ and $\beta = \epsilon^{1/20+\gamma/5-\kappa}$ when $\gamma \ge 1$ for arbitrary small $\kappa > 0$. Then,*

$$\lim_{\epsilon\downarrow 0} P\{\sigma_{\epsilon,\delta}(\beta) \le t\} = 0, \quad t > 0,$$

where $\delta = \epsilon^{1/10+2\gamma/5}$.

Finally, we rewrite this estimate on the smooth approximation v^δ_t of $v_t = v^\epsilon_t$ in an estimate on v_t itself. We prepare

Lemma 5.5.

(i) $||m_{\eta_1} - m_{\eta_2}||_{L^2} \le ||\nabla m||_{L^2} \cdot |\eta_1 - \eta_2|, \quad \eta_1, \eta_2 \in \mathbf{R}$,
(ii) $|\eta(v_1) - \eta(v_2)| \le C||v_1 - v_2||_{L^2}, \quad$ *for* $v_i : \mathrm{dist}(v_i, M) \le \beta_1$.

Proof. The first assertion (i) is straightforward. To show (ii),

$$|\eta(v_1) - \eta(v_2)| = \left| \int_0^1 \langle D\eta(\cdot, v_a), v_1 - v_2\rangle \, da \right| \le C||v_1 - v_2||_{L^2},$$

where $v_a = (1-a)v_1 + av_2, a \in [0,1]$, and $C = \sup_{a\in[0,1]} ||D\eta(\cdot, v_a)||_{L^2} < \infty$ from Lemma 9.5 below. □

Theorem 5.1. *Assume $\gamma > 3/8$ and $v_0 \in M$ for the SPDE (4.2). Take $\beta = \epsilon^{-3/20+2\gamma/5-\kappa}$ when $3/8 < \gamma \le 1$ and $\beta = \epsilon^{1/20+\gamma/5-\kappa}$ when $\gamma \ge 1$ for arbitrary small $\kappa > 0$. Then,*

$$\lim_{\epsilon\downarrow 0} P\{\mathrm{dist}(v^\epsilon_t, M) \le \beta \text{ for every } 0 \le t \le T\} = 1, \quad T > 0.$$

Proof. Since $s(v) = s(v^\delta) + (v - v^\delta) - (m_{\eta(v)} - m_{\eta(v^\delta)})$, we have from Lemma 5.5

$$||s(v_t)||_{L^2} \le ||s(v^\delta_t)||_{L^2} + C_1||v_t - v^\delta_t||_{L^2}.$$

However, using Lemmas 4.1-(i) and 4.2,

$$
\begin{aligned}
|v_t(z) - v_t^\delta(z)| &\le \int_{|y'|\le 1} \rho(y')|v_t(z) - v_t(z-\delta(z)y')|\,dy' \\
&\le Y_\epsilon \delta + Z_\epsilon \epsilon^{1/16+3\gamma/4-(2/p+\kappa)(3/4+\gamma)}\delta^{1/4-2/p-\kappa}, \\
&\qquad\qquad t \le \tau_K \wedge T,
\end{aligned}
$$

for $|z| \le \epsilon^{-1/2}+2$ and $v_t(z) = v_t^\delta(z)$ for $|z| \ge \epsilon^{-1/2}+2$. Therefore, taking $\delta = \epsilon^{1/10+2\gamma/5}$

$$
\begin{aligned}
||v_t - v_t^\delta||_{L^2} &\le C_2\epsilon^{-1/4} \sup_{|z|\le\epsilon^{-1/2}+2} |v_t(z) - v_t^\delta(z)| \\
&\le C_2\{Y_\epsilon \epsilon^{-3/20+2\gamma/5} + Z_\epsilon \epsilon^{-13/80+17\gamma/20-\kappa'}\},
\end{aligned}
$$

where $\kappa' > 0$ can be taken arbitrarily small. Since $-13/80 + 17\gamma/20$ is larger than $-3/20 + 2\gamma/5$ when $\gamma > 3/8$ and also it is larger than $1/20 + \gamma/5$ when $\gamma \ge 1$, we obtain the conclusion from Proposition 5.4 □

Before closing this section, we prepare a lemma which will become necessary in Sect. 8. Let $H^\alpha = H^\alpha(\mathbf{R}), \alpha \ge 0$, be the Sobolev spaces defined by interpolating $\{H^n\}_{n\in\mathbb{Z}_+}$ as usual; see Sect. 9.

Lemma 5.6. *Under the same conditions as in Theorem 5.1, we have*

$$
\lim_{\epsilon\downarrow 0} P\{v_t = v_t^\epsilon \in H^\alpha + m \textit{ for every } 0 \le t \le T\} = 1, \quad T > 0,\ \alpha < 1/4.
$$

Proof. First we note $v_t = (v_t - v_t^\delta) + s(v_t^\delta) + m_{\eta(v_t^\delta)}$. Since $m_\eta - m \in H^1 \subset H^\alpha$ for every $\eta \in \mathbf{R}$ and $s(v_t^\delta) \in H^1 \subset H^\alpha$ for $t \le \sigma_{\epsilon,\delta}(\beta)$, the conclusion follows from Proposition 5.4 if one can prove

$$
(5.16) \quad \lim_{\epsilon\downarrow 0} P\{v_t - v_t^\delta \in H^\alpha \text{ for every } 0 \le t \le T\} = 1, \quad T > 0,\ \alpha < 1/4.
$$

To this end, we prove that the norm $|v_t - v_t^\delta|_{H^\alpha}$ is finite for $t \le \tau_2$; see the formula (9.97) for the definition of the norm $|\cdot|_{H^\alpha}$ and Lemma 4.2 or (5.4) for τ_2 (we take $K = 2$). However, this is easy based on the estimate on the Hölder continuity of $v_t(z)$ (Lemmas 4.1-(i) and 4.2) by noting $v_t(z) - v_t^\delta(z) = 0$ for $|z| \ge \epsilon^{-1/2} + 2$ and $v_t^\delta \in C^\infty(\mathbf{R})$. Therefore, since Theorem 2.1 implies $\lim_{\epsilon\downarrow 0} P\{\tau_2 > T\} = 1,\ T > 0$, (5.16) is concluded. □

Remark 5.1. A stronger result $P\{v_t^\epsilon \in C([0,T], H^\alpha + m)\} = 1, \alpha < 1/4, \epsilon > 0$, can be shown by rewriting (4.2) in an equivalent integral form. The proof is omitted since Lemma 5.6 is shown easily and sufficient for our purpose.

6 Error estimates

In this section the proofs of Propositions 5.2 and 5.3 will be given. To show the estimate on $R_t^{\delta,1}(z)$, let us consider an auxiliary PDE for $\tilde{v} = \tilde{v}(t,y)$ on the half line $[0,\infty)$

$$(6.1)\qquad \frac{\partial \tilde{v}}{\partial t} = \Delta\tilde{v} + f(\tilde{v}), \quad t > 0,\ y \geq 0,$$

where initial data $\tilde{v}(0,y)$ and Dirichlet boundary data $\tilde{v}(t,0)$ at $y = 0$ are given.

Lemma 6.1. *For every $T > 0$,*

$$\sup_{0\leq t\leq T, y\geq 0} |\tilde{v}(t,y)| \leq \max\{\sup_{y\geq 0}|\tilde{v}(0,y)|,\ \sup_{0\leq t\leq T}|\tilde{v}(t,0)|, 1\}.$$

Proof. Use the maximum principle [29, 47] noting that $f(v) \leq 0$ for $v \geq 1$ and $f(v) \geq 0$ for $v \leq -1$. □

We give an estimate of Schauder's type on $\tilde{v}$. What we need is a mixture of the so-called boundary estimate in t and the interior estimate in x, so that the estimates stated in [29, Chapters 3 and 7] are not directly applicable. However, the extension is easy.

Lemma 6.2. *Assume*

$$(6.2)\qquad \sup_{y\geq 0}\sum_{k=0}^{2}|\nabla^k\tilde{v}(0,y)| + \sup_{0\leq t\leq T}|\tilde{v}(t,0)| \leq K < \infty, \nabla^k\tilde{v} = d^k\tilde{v}/dy^k.$$

Then, for every $\delta > 0$, there exists $C = C_{K,\delta}$ (which does not depend on T) such that

$$(6.3)\qquad \sup_{0\leq t\leq T, y\geq\delta}|\nabla^k\tilde{v}(t,y)| \leq C, \quad k = 1,2.$$

Proof. First we prove the following assertion: Let $\bar{v}(t,y)$ and $\bar{f}(t,y)$ be functions satisfying

$$(6.4)\qquad \frac{\partial \bar{v}}{\partial t} = \Delta\bar{v} + \bar{f}(t,y), \quad t > 0,\ y \geq \bar{\delta},$$

for $\bar{\delta} \geq 0$ and

$$(6.5)\qquad \sup_{y\geq\bar{\delta}}|\nabla\bar{v}(0,y)| + \sup_{0\leq t\leq T, y\geq\bar{\delta}}\{|\bar{v}(t,y)| + |\bar{f}(t,y)|\} \leq K.$$

Then, for every $\delta > 0$, there exists $C = C_{K,\delta}$ (independent of T) such that

$$(6.6)\qquad \sup_{0\leq t\leq T, y\geq\bar{\delta}+\delta}|\nabla\bar{v}(t,y)| \leq C.$$

Indeed, assuming $\bar{\delta} = 0$ without loss of generality, we rewrite (6.4) in an equivalent integral form: $\bar{v}(t,y) = \bar{v}_1(t,y) + \bar{v}_2(t,y)$, where

$$\bar{v}_1 = \int_0^\infty e^{-ct} p_t(y,y')\bar{v}(0,y')\,dy' - \int_0^t e^{-c(t-s)} \frac{\partial p_{t-s}}{\partial y'}(y,0)\bar{v}(s,0)\,ds,$$
$$\bar{v}_2 = \int_0^t ds \int_0^\infty e^{-c(t-s)} p_{t-s}(y,y')\{c\bar{v}(s,y') + \bar{f}(s,y')\}\,dy',$$

for every $c > 0$ (this c is introduced to obtain a global estimate in t). Here, $p_t(y,y') = q_t(y,y') - q_t(y,-y')$, $y,y' \geq 0$, is the fundamental solution of the heat equation on $[0,\infty)$ having Dirichlet 0-boundary condition at $y = 0$; q_t is the heat kernel on $\mathbf{R}$. The desired estimate (6.6) is shown for $\bar{v}_1$ (in place of $\bar{v}$) by a straightforward calculation and for $\bar{v}_2$ by noting that $c\bar{v}(s,y') + \bar{f}(s,y')$ is bounded from the condition (6.5).

Now we return to the proof of (6.3). First, we apply the result mentioned above by taking $\bar{\delta} = 0, \bar{v}(t,y) = \tilde{v}(t,y)$ and $\bar{f}(t,y) = f(\tilde{v}(t,y))$. The condition (6.5) is fulfilled from (6.2) and Lemma 6.1. Hence, we obtain (6.3) for $k = 1$. Next, we take $\bar{\delta} > 0, \bar{v}(t,y) = \nabla\tilde{v}(t,y)$ and $\bar{f}(t,y) = f'(\tilde{v}(t,y))\nabla\tilde{v}(t,y)$. Then, the result mentioned above is again applicable and we get (6.3) for $k = 2$. □

Proof of Proposition 5.2. For $|z| > \epsilon^{-1/2}$, $v(t,z) = v^\epsilon(t,z)$ satisfies the PDE

$$\frac{\partial v}{\partial t} = \epsilon^{-3/2-2\gamma}\{\Delta v + f(v)\}.$$

Therefore, under the assumption of Proposition 5.2, Lemma 6.2 (choose $T = \epsilon^{-3/2-2\gamma}\tau_K$) shows

$$\sup_{0<\epsilon<1} \sup_{0\leq t\leq\tau_K, |y|\geq\epsilon^{-1/2}+1/2} \{|v'(t,y)| + |v''(t,y)|\} < \infty.$$

Hence, $v_t'(z - \delta(z)y')$ and $v_t''(z - \delta(z)y')$ appearing in the expression for $R_t^{\delta,1}(z)$ given in Lemma 5.3 are both bounded for $t \leq \tau_K$, $|z| \in [\epsilon^{-1/2} + 1, \epsilon^{-1/2} + 2]$, $|y'| \leq 1, 0 < \delta < 1/2$ and $0 < \epsilon < 1$. Furthermore, we have $|\delta''(z)| \leq C\delta$ and $|\{1 - \delta'(z)y'\}^2 - 1| \leq C\delta$. This completes the proof. □

The next task is to prove the estimate on $R_t^{\delta,2}(z)$.

Proof of Proposition 5.3. Decompose $R_t^{\delta,2}(z)$ into a sum of $r_t^1(z)$ and $r_t^2(z)$, which are defined by

$$r_t^1(z) = f(v_t(z)) - f(v_t^\delta(z))$$
$$r_t^2(z) = \langle f(v_t(\cdot)), \rho^{\delta(z)}(z - \cdot)\rangle - f(v_t(z)).$$

Using Lemmas 4.1-(ii) and 4.2, $r_t^1(z)$ is bounded by

$$|r_t^1(z)| \leq \sup_{|v|\leq K} |f'(v)| \times \left| \int_{|y'|\leq 1} \rho(y')\{v_t(z - \delta(z)y') - v_t(z)\}\,dy' \right|$$
$$\leq C_1\{Y_\epsilon\delta + Z_{t,\epsilon}\epsilon^{1/4+\gamma-(1/2p+\kappa)(3/4+\gamma)}\delta^{1/2-1/2p-\kappa}\}, \quad t \leq \tau_K \wedge T.$$

The second term $r_t^2(z)$ is similarly bounded by

$$\begin{aligned} |r_t^2(z)| &= \left| \int_{|y'|\le 1} \rho(y')\{f(v_t(z-\delta(z)y')) - f(v_t(z))\}\, dy' \right| \\ &\le C_2\{Y_\epsilon \delta + Z_{t,\epsilon}\epsilon^{1/4+\gamma-(1/2p+\kappa)(3/4+\gamma)}\delta^{1/2-1/2p-\kappa}\}, \quad t \le \tau_K \wedge T. \end{aligned}$$

We may rewrite $\max\{(1/2p+\kappa)(3/4+\gamma), 1/2p+\kappa\}$ as κ again, since p and κ can be taken sufficiently large and small, respectively. □

7 Analysis on the classical flow

The result of Sect. 2 (Theorem 2.1) shows that the solution $v_t^\epsilon(y)$ of the SPDE (4.2) stays in the region $[-1-\delta, 1+\delta], \delta > 0$, with high probability as $\epsilon \downarrow 0$ if its initial data satisfies $|v_0^\epsilon(y)| \le 1$. From this observation, for the investigation of the asymptotic behavior of v_t^ϵ, it does not harm our argument if one introduce a cut-off to the potential $F(v)$ (or, to the reaction term f) for large $|v|$. We therefore consider a modified mild potential function $\tilde{F} \in C^\infty(\mathbf{R})$ satisfying the condition

$$(7.1) \qquad \begin{cases} (a) & \tilde{F}(v) = F(v) \text{ for } |v| \le 2 \text{ and } \tilde{F} \text{ is symmetric} \\ & \text{(i.e., } \tilde{F}(-v) = \tilde{F}(v), v \in \mathbf{R}), \\ (b) & \text{the } k\text{-th derivatives } \tilde{F}^{(k)} \text{ of } \tilde{F} \\ & \text{are bounded on } \mathbf{R} \text{ for } k = 2,3,4; \\ & \text{in particular, } \tilde{F}''(v) \ge -\bar{c}_2 \text{ for all } v \in \mathbf{R}, \end{cases}$$

recall that the condition (1.2) implies $F(-v) = F(v)$ and $F''(v) \ge -\bar{c}_2$.

Let $\{v_t = v_t(\cdot)\}$ be the dynamical system determined by solving the PDE

$$(7.2) \qquad \begin{cases} \frac{\partial v_t}{\partial t} = \Delta v_t + \tilde{f}(v_t), & t > 0,\ y \in \mathbf{R}, \\ v_0 = v, \end{cases}$$

where $\tilde{f}(v) = -\tilde{F}'(v)$. We denote v_t by $v_t(\cdot; v)$ to elucidate its initial data. We shall state four theorems (Theorems 7.1-7.4) concerning this dynamical system without proof; the proof will be given in Sect. 9. These theorems will be applied in the next section to establish the main result of the present paper. The first theorem establishes the limit map $\zeta = \zeta(v)$ for v close to M. A constant β_2 appearing in Theorems 7.1-7.4 are common and can be taken properly such that $0 < \beta_2 \le \beta_1$, where β_1 is the constant given in Theorem 3.1. We employ the notations introduced in previous sections consecutively; in particular, recall $\mathrm{dist}(v, M) = \min_{\eta \in \mathbf{R}} \|v - m_\eta\|_{L^2}$ and $L^2 + m = \{v; v - m \in L^2 = L^2(\mathbf{R})\}$.

Theorem 7.1. *There exists a limit in the space $H^1 = H^1(\mathbf{R})$*

$$\lim_{t\to\infty} v_t(\cdot; v) =: m_\zeta \in M$$

if the initial data $v \in L^2 + m$ satisfies $\mathrm{dist}(v, M) \le \beta_2$. We denote this ζ by $\zeta(v)$.

We use the following notations:

$$\mathscr{V}_\beta = \{v \in L^2 + m;\ \mathrm{dist}(v, M) \le \beta\},$$
$$\mathscr{V}_{\beta,0} = \{v \in \mathscr{V}_\beta; \zeta(v) = 0\}, \quad \beta > 0.$$

Let H be a Hilbert space and let $\mathscr{L}(L^2, H)$ be the class of all bounded linear operators from L^2 to H. As usual, we call a map $v \in \mathscr{V}_{\beta_2} \longmapsto \Phi(v) \in H$ Fréchet differentiable if there exists $D\Phi(\cdot; v) \in \mathscr{L}(L^2, H)$ such that

$$\|\Phi(v+h) - \Phi(v) - D\Phi(\cdot; v)h\|_H = o(\|h\|_{L^2}), \tag{7.3}$$
$$\text{as } \|h\|_{L^2} \downarrow 0.$$

Especially when $H = \mathbf{R}$, we regard $D\Phi(\cdot, v) \in \mathscr{L}(L^2, \mathbf{R}) \cong L^2$ and, furthermore, if $D\Phi(y, v) \in L^2$ is Fréchet differentiable in v and its derivative is an operator of Hilbert-Schmidt type from L^2 to L^2 having an integral representation

$$D\{D\Phi(y, \cdot; v)\}h = \int_{\mathbf{R}} D^2\Phi(y, y_2, v)h(y_2)\,dy_2, \quad h \in L^2, \tag{7.4}$$

with a kernel function $D^2\Phi(y_1, y_2, v) \in L^2(\mathbf{R}^2)$, then Φ is called twice Fréchet differentiable with derivatives $D\Phi(y, v)$ and $D^2\Phi(y_1, y_2, v)$.

Theorem 7.2. *The map $\zeta(v) \in \mathbf{R}$ is Fréchet differentiable in $v \in \mathscr{V}_{\beta_2}$ and its derivative has an estimate:*

$$\|D\zeta(\cdot, v) - D\zeta(\cdot, m)\|_{L^2} \le C\sqrt{\mathrm{dist}(v, M)}, \quad C > 0, \tag{7.5}$$

for every $v \in \mathscr{V}_{\beta_2,0}$. Moreover, for every $\bar\eta \in \mathbf{R}$

$$D\zeta(y, m_{\bar\eta}) = D\eta(y, m_{\bar\eta}) = -\frac{\nabla m_{\bar\eta}(y)}{\|\nabla m\|_{L^2}^2}, \tag{7.6}$$

recall that $\eta = \eta(v)$ is defined through the Fermi coordinates.

Theorem 7.3. *The map $\zeta(v) \in \mathbf{R}$ is twice Fréchet differentiable in $v \in \mathscr{V}_{\beta_2}$ and its second derivative has the following properties:*

$$\left|\int_{\mathbf{R}} y^p\{D^2\zeta(y, y, v) - D^2\zeta(y, y, m)\}\,dy\right| \le C\sqrt{\mathrm{dist}(v, M)}, \quad C > 0, \tag{7.7}$$

for every $v \in \mathscr{V}_{\beta_2,0}$ and $p = 0, 1$, and

$$\sup_{v \in \mathscr{V}_{\beta_2,0}} \int_{\mathbf{R}} y^2|D^2\zeta(y, y, v)|\,dy < \infty. \tag{7.8}$$

Moreover, for every $\eta \in \mathbf{R}$

$$\begin{aligned} &D^2\zeta(y, y, m_\eta) \\ &= -\frac{1}{\|\nabla m\|_{L^2}^2}\int_0^\infty dt \int_{\mathbf{R}} p_{0,t}(y, z; m_\eta)^2 f''(m_\eta(z))\nabla m_\eta(z)\,dz, \end{aligned} \tag{7.9}$$

where $p_{0,t}(y, z; m_\eta)$ denotes the fundamental solution of $\partial/\partial t - \Delta - f'(m_\eta)$; see Paragraph 9.1 in Sect. 9 recalling that $F(v) = \tilde F(v)$ for $|v| \le 1$ from the condition (7.1)-(a).

It will be useful to notice the shift and reflection-invariance of the functional $\zeta = \zeta(v)$. Consider mappings $\{\mathscr{S}_z\}_{z\in\mathbf{R}}$ and $\mathscr{R}$ defined by $\mathscr{S}_z v(y) = v(y-z)$, $\mathscr{R}v(y) = -v(-y)$, $y \in \mathbf{R}$, respectively. Then, we have

Lemma 7.1. *For every $v \in \mathscr{V}_{\beta_2}$ and $y, y_1, y_2, z \in \mathbf{R}$,*

(i) $\zeta(\mathscr{S}_z v) = \zeta(v) + z, \quad \zeta(\mathscr{R}v) = -\zeta(v),$

(ii) $D\zeta(y, \mathscr{S}_z v) = D\zeta(y - z, v),$

$D^2\zeta(y_1, y_2, \mathscr{S}_z v) = D^2\zeta(y_1 - z, y_2 - z, v),$

(iii) $D\zeta(y, \mathscr{R}v) = D\zeta(-y, v)$

$D^2\zeta(y_1, y_2, \mathscr{R}v) = -D^2\zeta(-y_1, -y_2, v).$

Proof. If $v_t = v_t(\cdot; v)$ is a solution of the PDE (7.2), then $\mathscr{S}_z v_t$ and $\mathscr{R}v_t$ are also; we use the symmetricity of $\tilde{F}$, see (7.1)-(a), for $\mathscr{R}v_t$. This implies $\mathscr{S}_z\{v_t(\cdot; v)\} = v_t(\cdot; \mathscr{S}_z v)$ and $\mathscr{R}\{v_t(\cdot; v)\} = v_t(\cdot; \mathscr{R}v)$. Letting $t \to \infty$, we obtain the assertion (i). The assertions (ii) and (iii) follow from (i) without difficulty. □

Corollary 7.1. *For all $\eta \in \mathbf{R}$,*

$$\int_{\mathbf{R}} D^2\zeta(y, y, m_\eta)\, dy = 0.$$

Proof. We may assume $\eta = 0$ without loss of generality by the shift-invariance; see Lemma 7.1-(ii). Then, this can be proven from the concrete expression (7.9); note that $\int_{\mathbf{R}} p_{0,t}(y, z; m)^2 dy = p_{0,2t}(z, z; m)$ is an even function in z, while $f''(m(z))\nabla m(z)$ is odd from the assumption (1.2)-(b). Or, the proof can be completed by noting $D^2\zeta(y, y, m) = -D^2\zeta(-y, -y, m)$, which follows from Lemma 7.1-(iii). □

The next theorem gives a basic identity related to the classical flow v_t. It will be derived from $0 = d\zeta(v_t)/dt = \langle D\zeta(\cdot, v_t), \Delta v_t + \tilde{f}(v_t)\rangle$, $t > 0$, by letting $t \downarrow 0$.

Theorem 7.4. *For every $v \in \mathscr{V}_{\beta_2}$, we have $D\zeta(\cdot, v) \in \cap_{\delta>0} H^{2-\delta}$. In addition, for every $v \in \mathscr{V}_{\beta_2} \cap (H^\delta + m)$ with some $\delta > 0$ and satisfying $\|v\|_{L^\infty} \le 2$,*

$$\langle D\zeta(\cdot, v), \Delta v + f(v)\rangle = 0,$$

where $H^\alpha, \alpha \ge 0$, denotes the Sobolev space; see Paragraph 9.8 below.

Remark 7.1. We use a cut-off potential $\tilde{F}$, however this is not actually necessary. In fact, when we consider the PDE (7.2) with original potential F instead of $\tilde{F}$, the maximum principle proves that the solution v_t satisfies $\|v_t\|_{L^\infty} \le \max\{\|v_0\|_{L^\infty}, 1\}$ if the initial data v_0 is bounded, cf. Lemma 6.1. One can prove that Theorems 7.1–7.4 hold by replacing $L^2 + m$ with $(L^2 + m) \cap L^\infty$.

8 Identification of the limit

We rely on the argument which was used by Katzenberger [42] in a finite-dimensional situation. Let $u^\epsilon(t,x)$ be the solution of the SPDE (1.1) with $\kappa_\epsilon = \epsilon^\gamma$ having initial data: $u^\epsilon(0,x) = m_\xi^\epsilon(x)$ with some $\xi \in \mathbf{R}$. Here, m_ξ^ϵ is a function defined by $m_\xi^\epsilon(x) = m(\epsilon^{-1/2}(x-\xi))$. We introduce its time-change:

$$\bar{u}^\epsilon(t,x) := u^\epsilon(\epsilon^{-1/2-2\gamma}t, x), \quad t > 0,\ x \in \mathbf{R}, \tag{8.1}$$

and investigate the asymptotic behavior of $\bar{u}^\epsilon(t,x)$ as $\epsilon \downarrow 0$. Our main result is formulated as follows.

Theorem 8.1. *Assume $\gamma > 19/4$ for the SPDE (1.1) with $\kappa_\epsilon = \epsilon^\gamma$ and consider the time change $\bar{u}^\epsilon(t,x)$, defined by (8.1), of its solution $u^\epsilon(t,x)$ with initial data m_ξ^ϵ. Then,*
(i) there exists an $\mathbf{R}$-valued process $\xi_t^\epsilon(\omega)$ such that

$$\lim_{\epsilon\downarrow 0} P\left\{ \sup_{0\le t\le T} \|\bar{u}^\epsilon(t,\cdot) - \chi_{\xi_t^\epsilon}\|_{L^2} > \delta \right\} = 0$$

for every $\delta > 0$ and $T > 0$. Recall that the function $\chi_\xi(x) = 1$ for $x > \xi$ and -1 for $x < \xi$.
(ii) The distribution on the space $C([0,T],\mathbf{R})$ of ξ_t^ϵ converges weakly to that of a solution ξ_t of the SDE (1.5) starting at ξ as $\epsilon \downarrow 0$. Two constants α_1 and α_2 appearing in (1.5) are defined by the formula (8.5) below.

The proof will be completed after several steps. Notice that $v_t^\epsilon(y) \equiv v^\epsilon(t,y) := \bar{u}^\epsilon(t,\epsilon^{1/2}y)$ satisfies the SPDE (4.2) and consider a stopping time

$$\tilde{\sigma}_\epsilon = \inf\{t > 0; \mathrm{dist}(v_t^\epsilon, M) > \beta \text{ or } \|v_t^\epsilon\|_{L^\infty} > 2 \text{ or } v_t^\epsilon \notin H^\delta + m\},$$

where $\beta = \epsilon^{1/20+\gamma/5-\kappa}$ with sufficiently small $\kappa > 0$ and $0 < \delta < 1/4$. Then, from Theorem 2.1, Theorem 5.1 and Lemma 5.6 (note that the initial data $v_0^\epsilon = m_{\epsilon^{-1/2}\xi} \in M$ satisfies all required conditions), we see

$$\lim_{\epsilon\downarrow 0} P\{\tilde{\sigma}_\epsilon \le T\} = 0, \quad T > 0. \tag{8.2}$$

Using the map $\zeta = \zeta(v)$ constructed in the previous section, we set

$$\xi_t^\epsilon := \epsilon^{1/2}\zeta(v_{t\wedge\tilde{\sigma}_\epsilon}^\epsilon), \quad t \ge 0. \tag{8.3}$$

Then, applying Itô's formula, we have

$$\xi_t^\epsilon = \xi_0^\epsilon + \mu_{t\wedge\tilde{\sigma}_\epsilon}^\epsilon + \frac{1}{2}\int_0^{t\wedge\tilde{\sigma}_\epsilon} b_s^\epsilon\, ds, \tag{8.4}$$

where

$$\mu_t^\epsilon = \int_0^t \langle a(\epsilon^{1/2}\cdot) D\zeta(\cdot, v_s^\epsilon), dw_s\rangle, \quad t \le \tilde\sigma_\epsilon,$$

$$b_t^\epsilon = \epsilon^{-1/2} \int_{\mathbf{R}} a^2(\epsilon^{1/2} y) D^2\zeta(y, y, v_t^\epsilon)\, dy, \quad t \le \tilde\sigma_\epsilon.$$

In fact, an extra bad term $\epsilon^{-1-2\gamma}\int_0^t \langle D\zeta(\cdot, v_s^\epsilon), \Delta v_s^\epsilon + f(v_s^\epsilon)\rangle\, ds$ appears in the right hand side of (8.4) but it vanishes fortunately because of Theorem 7.4; note that $v_s^\epsilon \in \mathscr{V}_{\beta_2} \cap (H^\delta + m)$ and $\|v_s^\epsilon\|_{L^\infty} \le 2$ for $s \le \tilde\sigma_\epsilon$. This is the trick of Katzenberger. We define two constants α_1 and α_2 by

$$(8.5) \qquad \begin{cases} \alpha_1 = \frac{1}{\|\nabla m\|_{L^2}}, \\ \alpha_2 = -\frac{1}{\|\nabla m\|_{L^2}^2} \int_0^\infty dt \int_{\mathbf{R}^2} y p_{0,t}(y, z; m)^2 f''(m(z)) \nabla m(z)\, dydz, \end{cases}$$

where $p_{0,t}(y, z; m)$ denotes the fundamental solution of $\partial/\partial t - \Delta - f'(m)$.

Lemma 8.1. *For every sufficiently small* $\epsilon > 0$,

$$\sup_{t \ge 0} |b_{t\wedge\tilde\sigma_\epsilon}^\epsilon - 2\alpha_2 a(\xi_t^\epsilon) a'(\xi_t^\epsilon)| \le C \max\{\epsilon^{\gamma/10 - 19/40 - \kappa/2}, \epsilon^{1/2}\}.$$

Proof. We always assume $0 \le t \le \tilde\sigma_\epsilon$. By Taylor expansion, we see

$$a^2(\epsilon^{1/2} y) = a^2(\xi_t^\epsilon) + (a^2)'(\xi_t^\epsilon)\cdot \epsilon^{1/2}(y - \zeta(v_t^\epsilon)) + r_t^\epsilon(y),$$

where

$$|r_t^\epsilon(y)| \le \frac{1}{2} \sup_{x\in\mathbf{R}} |(a^2)''(x)| \cdot \{\epsilon^{1/2}(y - \zeta(v_t^\epsilon))\}^2.$$

Therefore,

$$(8.6) \qquad \begin{aligned} b_t^\epsilon =& \epsilon^{-1/2} a^2(\xi_t^\epsilon) \int_{\mathbf{R}} D^2\zeta(y, y, v_t^\epsilon)\, dy \\ &+ (a^2)'(\xi_t^\epsilon) \int_{\mathbf{R}} (y - \zeta(v_t^\epsilon)) D^2\zeta(y, y, v_t^\epsilon)\, dy + R_t^\epsilon, \end{aligned}$$

where

$$\begin{aligned} |R_t^\epsilon| =& \left| \epsilon^{-1/2} \int_{\mathbf{R}} D^2\zeta(y, y, v_t^\epsilon) r_t^\epsilon(y)\, dy \right| \\ \le& C_1 \epsilon^{1/2} \int_{\mathbf{R}} \{y - \zeta(v_t^\epsilon)\}^2 |D^2\zeta(y, y, v_t^\epsilon)|\, dy. \end{aligned}$$

However, using Lemma 7.1-(ii), we have

$$\int_{\mathbf{R}} \{y - \bar\zeta\}^2 |D^2\zeta(y, y, v)|\, dy = \int_{\mathbf{R}} y^2 |D^2\zeta(y, y, \mathscr{S}_{-\bar\zeta} v)|\, dy, \quad \bar\zeta \in \mathbf{R},$$

and, in addition, $v \in \mathscr{V}_{\beta_2}$ implies $\mathscr{S}_{-\zeta(v)} v \in \mathscr{V}_{\beta_2, 0}$. Hence, noting that $v_t^\epsilon \in \mathscr{V}_{\beta_2}$ for small enough $\epsilon > 0$ since $\beta \le \beta_2$ for such ϵ, we obtain from (7.8):

$$|R_t^\epsilon| \le C_1\epsilon^{1/2} \sup_{v\in \mathcal{T}_{\beta_2,0}} \int_{\mathbf{R}} y^2 |D^2\zeta(y,y,v)|\,dy \le C_2\epsilon^{1/2}.$$

For the first term in the right hand side of (8.6), we observe

$$\begin{aligned}
&\left|\epsilon^{-1/2}\int_{\mathbf{R}} D^2\zeta(y,y,v_t^\epsilon)\,dy\right| \\
&= \left|\epsilon^{-1/2}\int_{\mathbf{R}} \{D^2\zeta(y,y,\mathcal{S}^\epsilon_{-\zeta(v_t)}v_t^\epsilon) - D^2\zeta(y,y,m)\}\,dy\right| \\
&\le C_3\epsilon^{-1/2}\beta^{1/2} = C_3\epsilon^{\gamma/10-19/40-\kappa/2}.
\end{aligned}$$

We have applied Lemma 7.1-(ii) and Corollary 7.1 for the first equality, and then (7.7) with $p = 0$ for the second inequality. For the second term of (8.6), since the definition (8.5) of α_2 and the formula (7.9) imply $\alpha_2 = \int_{\mathbf{R}} yD^2\zeta(y,y,m)\,dy$,

$$\begin{aligned}
&\left|\int_{\mathbf{R}} (y-\zeta(v_t^\epsilon))D^2\zeta(y,y,v_t^\epsilon)\,dy - \alpha_2\right| \\
&= \left|\int_{\mathbf{R}} y\{D^2\zeta(y,y,\mathcal{S}^\epsilon_{-\zeta(v_t)}v_t^\epsilon) - D^2\zeta(y,y,m)\}\,dy\right| \le C_4\beta^{1/2}.
\end{aligned}$$

Here, we have used Lemma 7.1-(ii) and then (7.7) with $p = 1$. Now, the proof of the lemma is completed. □

The next task is to investigate the martingale term μ_t^ϵ of ξ_t^ϵ. Note that its quadratic variational process is given by

$$[\mu^\epsilon]_t \equiv [\mu^\epsilon,\mu^\epsilon]_t = \int_0^t \bar{b}_s^\epsilon\,ds, \quad t \le \tilde{\sigma}_\epsilon,$$

where

$$\bar{b}_t^\epsilon = \int_{\mathbf{R}} a^2(\epsilon^{1/2}y)D\zeta(y,v_t^\epsilon)^2\,dy, \quad t \le \tilde{\sigma}_\epsilon.$$

Lemma 8.2. *For all sufficiently small* $\epsilon > 0$,

$$\sup_{t\ge 0} |\bar{b}^\epsilon_{t\wedge\tilde{\sigma}_\epsilon} - \alpha_1^2 a^2(\xi_t^\epsilon)| \le C\max\{\epsilon^{\gamma/10+1/40-\kappa/2}, \epsilon^{1/2}\}.$$

Proof. We always assume $0 \le t \le \tilde{\sigma}_\epsilon$ again. Recalling the definition (8.5) of α_1 and the formula (7.6), we see

$$\bar{b}_t^\epsilon - \alpha_1^2 a^2(\xi_t^\epsilon) = \bar{b}_t^\epsilon - a^2(\xi_t^\epsilon)\int_{\mathbf{R}} D\zeta(y,m_{\zeta(v_t^\epsilon)})^2\,dy = I_t^{1,\epsilon} + I_t^{2,\epsilon},$$

where

$$\begin{aligned}
I_t^{1,\epsilon} &= \int_{\mathbf{R}} a^2(\epsilon^{1/2}y)\{D\zeta(y,v_t^\epsilon)^2 - D\zeta(y,m_{\zeta(v_t^\epsilon)})^2\}\,dy, \\
I_t^{2,\epsilon} &= \int_{\mathbf{R}} \{a^2(\epsilon^{1/2}y) - a^2(\xi_t^\epsilon)\}D\zeta(y,m_{\zeta(v_t^\epsilon)})^2\,dy.
\end{aligned}$$

However, if $\epsilon > 0$ is sufficiently small such that $\beta \le \beta_2$, we have from Theorem 7.2

$$|I_t^{1,\epsilon}| \le ||a^2||_{L^\infty}||D\zeta(\cdot,v_t^\epsilon) - D\zeta(\cdot,m_{\zeta(v_t^\epsilon)})||_{L^2}||D\zeta(\cdot,v_t^\epsilon) + D\zeta(\cdot,m_{\zeta(v_t^\epsilon)})||_{L^2} \le C_1\beta^{1/2}$$

and

$$|I_t^{2,\epsilon}| \le \epsilon^{1/2}||(a^2)'||_{L^\infty}\int_{\mathbf{R}}|y-\zeta(v_t^\epsilon)|D\zeta(y,m_{\zeta(v_t^\epsilon)})^2\,dy = C_2\epsilon^{1/2}\int_{\mathbf{R}}|y|D\zeta(y,m)^2\,dy = C_3\epsilon^{1/2}. \quad \square$$

Lemma 8.3. *The distribution of ξ_t^ϵ converges weakly on $C([0,T],\mathbf{R})$ to that of ξ_t, the solution of the SDE (1.5) starting at ξ, as $\epsilon \downarrow 0$ for every $T > 0$.*

Proof. The proof is quite standard. Let $\mathscr{L}$ be the generator of the process ξ_t; more precisely,

$$\mathscr{L} = \frac{\alpha_1^2}{2}a^2(\xi)\frac{\partial^2}{\partial\xi^2} + \alpha_2 a(\xi)a'(\xi)\frac{\partial}{\partial\xi}.$$

Applying Itô's formula by noting (8.4), we see

$$\text{(8.7)} \quad \varphi(\xi_t^\epsilon) = \varphi(\xi_s^\epsilon) + \frac{1}{2}\int_s^t\{\varphi'(\xi_r^\epsilon)b_r^\epsilon + \varphi''(\xi_r^\epsilon)\bar{b}_r^\epsilon\}\,dr + \bar{\mu}_t^\epsilon = \varphi(\xi_s^\epsilon) + \frac{1}{2}\int_s^t\{\mathscr{L}\varphi(\xi_r^\epsilon) + R_r^\epsilon\}\,dr + \bar{\mu}_t^\epsilon, \quad 0 \le s \le t \le \bar{\sigma}_\epsilon,$$

for every $\varphi \in C_b^2(\mathbf{R})$, where $\bar{\mu}_t^\epsilon$ is a martingale and R_t^ϵ is a process with bound

$$\text{(8.8)} \quad \sup_{0\le t\le\bar{\sigma}_\epsilon}|R_t^\epsilon| \le C_1\max\{\epsilon^{\gamma/10-19/40-\kappa/2},\epsilon^{1/2}\}.$$

Indeed, this bound immediately follows from Lemmas 8.1 and 8.2. On the other hand, these two lemmas also prove

$$E[|\xi_t^\epsilon - \xi_s^\epsilon|^4] \le C_2(t-s)^2, \quad 0 \le s \le t < \infty,$$

which implies the tightness of $\{P^\epsilon\}_{0<\epsilon<1}$, where P^ϵ denotes the distribution of $\xi_t^\epsilon, t \in [0,T]$, on the space $C([0,T],\mathbf{R})$. From (8.7), (8.8) and noting (8.2), we see that every limit $\bar{P}$ of P^ϵ as $\epsilon \downarrow 0$ is a solution of the $\mathscr{L}$-martingale problem and this completes the proof. $\square$

Proof of Theorem 8.1. First notice that

$$\text{(8.9)} \quad \sup_{v\in\mathscr{V}_{\beta_2}}||v - m_{\zeta(v)}||_{L^2} < \infty.$$

In fact, since $||v - m_{\zeta(v)}||_{L^2} \le ||s(v)||_{L^2} + ||m_{\eta(v)} - m_{\zeta(v)}||_{L^2}$ where $v = s(v) + m_{\eta(v)}$ denotes the Fermi coordinates of v, (8.9) is deduced from $||s(v)||_{L^2} \le \beta_2$ and the boundedness of $||m_{\eta(v)} - m_{\zeta(v)}||_{L^2}$ (use Lemma 5.5-(i) and (9.46) given below in Sect. 9). Therefore, we get

$$\begin{aligned}||\bar{u}_t^\epsilon - \chi_{\xi_t^\epsilon}||_{L^2} &\leq ||\bar{u}_t^\epsilon - m_{\xi_t^\epsilon}^\epsilon||_{L^2} + ||m_{\xi_t^\epsilon}^\epsilon - \chi_{\xi_t^\epsilon}||_{L^2} \\ &= \epsilon^{1/4}||v_t^\epsilon - m_{\zeta(v_t^\epsilon)}||_{L^2} + \epsilon^{1/4}||m - \chi_0||_{L^2} \leq C\epsilon^{1/4},\end{aligned}$$

for $0 \leq t \leq \tilde{\sigma}_\epsilon$. Noting (8.2), the proof of the theorem is concluded. □

Remark 8.1. Slight modification in the proof leads us to the assertion (i) of Theorem 8.1 with $||\bar{u}^\epsilon(t,\cdot) - m_{\xi_t^\epsilon}^\epsilon||_{L^\infty}$ in place of $||\bar{u}^\epsilon(t,\cdot) - \chi_{\xi_t^\epsilon}||_{L^2}$.

9 Proofs of Theorems 7.1–7.4

In this section we shall prove four theorems formulated in Sect. 7. The contents are divided into 8 paragraphs (*P. 9.1–9.8*). *P. 9.1* and *P. 9.3* have technical characters. We establish energy inequalities for the PDE (7.2) and its linearized PDE. Several estimates on the corresponding semigroup are also given. In *P. 9.2*, the Fréchet derivatives of the Fermi coordinates $\eta = \eta(v)$ are calculated. Theorem 7.1 is proved in *P. 9.3* by using the formula for $\eta_t(v) := \eta(v_t(\cdot; v))$ given in *P. 9.2*; recall that $v_t(\cdot; v)$ denotes the solution of the PDE (7.2) with initial data v. Then, after deriving concrete formulas for the Fréchet derivatives of $v_t(\cdot; v)$ in *P. 9.4*, the Fréchet derivatives of $\zeta(v) := \lim_{t\to\infty} \eta_t(v)$ are computed in *P. 9.5*. Based on these formulas, Theorems 7.2 and 7.3 are shown in *P. 9.6* and *P. 9.7*, respectively. Finally, the proof of Theorem 7.4 is given in *P. 9.8*. In this section we always use the potential $\tilde{F}$ rather than $\tilde{f}$ to avoid confusion; recall $\tilde{f} = -\tilde{F}'$.

9.1 Energy inequalities for the PDE (7.2) and its linearized equation

Here, we give energy estimates on the solution $v_t = v_t(\cdot; v)$ of the PDE (7.2) with initial data $v \in L^2 + m$. Similar energy estimates will be given also for the linearized equation of (7.2). We denote by $\tilde{\mathcal{H}}$ the energy functional $\mathcal{H}$ defined by (3.2) with F replaced by $\tilde{F}$ (but with the same constant C_*). We assume $\tilde{c}_2 = 1$ in the condition (7.1)-(b) for $\tilde{F}$ (i.e. $\tilde{F}''(v) \geq -1$) only for notational simplicity.

Lemma 9.1. *Assume $v, \bar{v} \in L^2 + m$ and denote $v_t = v_t(\cdot; v)$ and $\bar{v}_t = v_t(\cdot; \bar{v})$.*
(i) Then, we have

$$||v_t - \bar{v}_t||_{L^2} \leq ||v - \bar{v}||_{L^2} e^t, \tag{9.1}$$

$$\int_0^t ||\nabla(v_s - \bar{v}_s)||_{L^2}^2 \, ds \leq \frac{1}{2} ||v - \bar{v}||_{L^2}^2 e^{2t}, \tag{9.2}$$

and, in particular, these estimates hold for $\bar{v}_t = \bar{v} = m_\eta$ for every $\eta \in \mathbf{R}$.
(ii) For every $T > 0$, there exists $C = C_T > 0$ such that

$$\tilde{\mathscr{H}}(v_t) \le \frac{C}{t}\,\mathrm{dist}(v, M), \quad t \in (0,T], \quad \text{if} \quad \mathrm{dist}(v,M) \le 1.$$

(iii) There exist $c, \bar{\beta} > 0$ *such that*

$$\tilde{\mathscr{H}}(v_t) \le e^{-c(t-s)}\tilde{\mathscr{H}}(v_s), \quad t \ge s \ge 0, \quad \text{if} \quad \tilde{\mathscr{H}}(v_s) \le \bar{\beta}.$$

Remark 9.1. Since $||s||_{H^1} \le \beta_1$ implies $||s||_{L^\infty} \le C_1\beta_1$ with the help of Sobolev's imbedding theorem (see the proof of Theorem 3.1), if $\beta_1 \le 1/C_1$, we have $\tilde{\mathscr{H}}(v) = \mathscr{H}(v)$ for $v = m_{\eta(v)} + s(v)$ such that $||s(v)||_{H^1} \le \beta_1$; recall the assumption (7.1)-(a) for $\tilde{F}$. Therefore, Theorem 3.1 holds also for $\tilde{\mathscr{H}}$ by replacing β_1 with $\min\{\beta_1, 1/C_1\}$ if necessary.

Proof of Lemma 9.1. We divide the proof into four steps.

Step 1: Here, we prove the assertion (i). To this end, noting $\tilde{F}''(v) \ge -1$, we see

$$\begin{aligned}\frac{\partial}{\partial t}||v_t - \bar{v}_t||_{L^2}^2 &= 2\langle \Delta(v_t - \bar{v}_t), v_t - \bar{v}_t\rangle - 2\langle \tilde{F}'(v_t) - \tilde{F}'(\bar{v}_t), v_t - \bar{v}_t\rangle \\ &\le -2||\nabla(v_t - \bar{v}_t)||_{L^2}^2 + 2||v_t - \bar{v}_t||_{L^2}^2.\end{aligned}$$

Integrate the both sides to obtain

$$\begin{aligned}||v_t - \bar{v}_t||_{L^2}^2 + 2\int_0^t ||\nabla(v_s - \bar{v}_s)||_{L^2}^2\,ds \\ \le ||v - \bar{v}||_{L^2}^2 + 2\int_0^t ||v_s - \bar{v}_s||_{L^2}^2\,ds.\end{aligned} \tag{9.3}$$

The estimate (9.1) is shown from this inequality by neglecting the second term in the left hand side and by applying the Gronwall inequality. Afterwards, (9.2) follows by inserting (9.1) into the second term of the right hand side of (9.3). In particular, from (1.3) and recalling that $F(v) = \tilde{F}(v)$ for $|v| \le 1$, m_η is a stationary solution of the PDE (7.2) for every $\eta \in \mathbf{R}$ and therefore (9.1) and (9.2) hold for $\bar{v}_t = \bar{v} = m_\eta$.

Step 2: In this step, we show an elementary bound on the potential $\tilde{F}$

$$\tilde{F}(v) \le \frac{C_1}{\delta}(v-m)^2 + (1+\delta)\tilde{F}(m), \quad C_1 > 0, \tag{9.4}$$

for all $v \in \mathbf{R}, |m| \le 1$ and $0 < \delta \le 1$. Indeed, this is true for all $|v| \ge 2$ and $|m| \le 1$, since the boundedness of $\tilde{F}''(v)$ (see the assumption (7.1)) implies $|\tilde{F}(v)| \le C_2(v^2+1)$ and since we have $(v^2+1) \le 5(v-m)^2$ and $\tilde{F}(m) \ge 0$ on this region. Therefore, it is sufficient to show (9.4) for all $|v| \le 2, |m| \le 1, 0 < \delta \le 1$ with some $C_1 > 0$. To this end, we first see $\tilde{F}(m) \ge C_3(m-1)^2$ for all $0 \le m \le 1$ with some $C_3 > 0$. In fact, this is shown for m in a neighborhood of 1 from $\tilde{F}(1) = \tilde{F}'(1) = 0$ and $\tilde{F}''(1) > 0$, and then for other m from $\tilde{F}(v) > 0$ in $[0, 1)$. On the other hand, noting $\tilde{F}'(1) = 0$ again, we have

$$\begin{aligned}\tilde F(v) - \tilde F(m) &= \tilde F'(m)(v-m) + \frac{1}{2}\tilde F''(v_*)(v-m)^2 \\ &= \tilde F''(v_{**})(m-1)(v-m) + \frac{1}{2}\tilde F''(v_*)(v-m)^2 \\ &\le C_4\left\{|(m-1)(v-m)| + (v-m)^2\right\} \\ &\le \delta C_3(m-1)^2 + \left\{\frac{C_4^2}{4C_3\delta} + C_4\right\}(v-m)^2, \quad |v|\le 2,\ |m|\le 1,\end{aligned}$$

for some $|v_*| \le 2, |v_{**}| \le 1$ and $C_4 > 0$; a simple estimate $|ab| \le (a^2+b^2)/2$ has been used to derive the final line. Therefore, since $C_3(m-1)^2 \le \tilde F(m)$ if $0 \le m \le 1$, (9.4) is shown for $|v| \le 2$ and $0 \le m \le 1$. Replacing (v,m) by $(-v,-m)$, we obtain (9.4) also for $-1 \le m \le 0$.

Step 3: Let us prove

$$\int_0^t \tilde{\mathscr{H}}(v_s)\,ds \le \frac{C_5}{\delta}||v_0 - m_\eta||_{L^2}^2 e^{2t} + C_*\delta t, \tag{9.5}$$

for all $0 < \delta \le 1, t \ge 0$ and $\eta \in \mathbf{R}$, where C_* is the constant appearing in (3.2). Using (9.4) and (9.1) with $\bar v_t = m_\eta$, we have

$$\int_0^t ds \int_{\mathbf{R}} \tilde F(v_s(y))\,dy \le \frac{C_1}{2\delta}||v_0 - m_\eta||_{L^2}^2(e^{2t}-1) + t(1+\delta)\int_{\mathbf{R}} \tilde F(m_\eta(y))\,dy. \tag{9.6}$$

On the other hand, first using $(a+b)^2 \le (1+1/\delta)a^2 + (1+\delta)b^2$ and then from (9.2),

$$\begin{aligned}\int_0^t ||\nabla v_s||_{L^2}^2\,ds &\le \int_0^t \left\{\left(1+\frac{1}{\delta}\right)||\nabla(v_s - m_\eta)||_{L^2}^2 + (1+\delta)||\nabla m_\eta||_{L^2}^2\right\} ds \\ &\le \frac{1}{2}\left(1+\frac{1}{\delta}\right)||v_0 - m_\eta||_{L^2}^2 e^{2t} + t(1+\delta)||\nabla m_\eta||_{L^2}^2.\end{aligned} \tag{9.7}$$

Now, recall the definition of $\tilde{\mathscr{H}}(v)$ and the fact $\tilde{\mathscr{H}}(m_\eta) = \mathscr{H}(m_\eta) = 0$ and sum up the both sides of (9.6) and (9.7) multiplied by $1/2$. Then, we obtain (9.5).

Step 4: The estimate (9.5) is not an estimate on $\tilde{\mathscr{H}}(v_t)$ for each t but that on its integration in t. To get an estimation on $\tilde{\mathscr{H}}(v_t)$ itself, we observe $||\partial v_t/\partial t||_{L^2}^2 + \partial\tilde{\mathscr{H}}(v_t)/\partial t = 0$ from the PDE (7.2), which proves $t||\partial v_t/\partial t||_{L^2}^2 + \partial\{t\tilde{\mathscr{H}}(v_t)\}/\partial t = \tilde{\mathscr{H}}(v_t)$ and consequently

$$t\tilde{\mathscr{H}}(v_t) \le \int_0^t \tilde{\mathscr{H}}(v_s)\,ds.$$

Combine this estimate with (9.5), in which we take $\delta = ||v_0 - m_\eta||_{L^2}$. Then, we obtain

$$\tilde{\mathscr{H}}(v_t) \le \frac{C_T}{t}||v_0 - m_\eta||_{L^2}, \quad t \in (0,T], \quad \text{if } ||v_0 - m_\eta||_{L^2} \le 1,$$

with some $C_T > 0$. Taking infimum in η, we obtain the assertion (ii). Finally, the assertion (iii) is shown since Theorem 3.1 (see Remark 9.1 as well) implies that

$$\frac{d}{dt}\tilde{\mathscr{H}}(v_t) = -||D\tilde{\mathscr{H}}(\cdot, v_t)||_{L^2}^2 \le -c\tilde{\mathscr{H}}(v_t), \quad c = c_1/c_2 > 0,$$

holds if $||s(v_t)||_{H^1} \le \beta_1$ and therefore if $\tilde{\mathscr{H}}(v_t) \le \bar{\beta} := c_1\beta_1^2$. □

As an immediate consequence of Lemma 9.1, we obtain

Corollary 9.1. *There exist $C, c > 0$ and $\beta_2, 0 < \beta_2 \le \beta_1$, such that $v \in \mathscr{V}_{\beta_2}$ (i.e., $\mathrm{dist}(v, M) \le \beta_2$) implies $v_t(\cdot; v) \in \mathscr{V}_{\beta_1}$ for all $t \ge 0$ and*

(i) $||s_t||_{L^2} \le e^t \,\mathrm{dist}(v, M), \quad t \ge 0,$
(ii) $||s_t||_{H^1} \le C\sqrt{\mathrm{dist}(v, M)}e^{-ct}, \quad t \ge 1,$

where β_1 is the constant appearing in Theorem 3.1 and $s_t = s(v_t(\cdot; v))$ is defined in terms of the Fermi coordinates of $v_t(\cdot; v)$. Especially, we have

$$\text{(iii)} \qquad ||s_t||_{L^2} \le C\sqrt{\mathrm{dist}(v, M)}e^{-ct}, \quad t \ge 0.$$

Proof. The assertion (i) (as long as $v_t(\cdot; v) \in \mathscr{V}_{\beta_1}$, i.e., $||s_t||_{L^2} \le \beta_1$ and hence the Fermi coordinates are defined for v_t) follows from (9.1) with $\bar{v}_t = m_\eta$; take infimum in η. On the other hand, Lemma 9.1-(ii) shows that $\tilde{\mathscr{H}}(v_1) \le \bar{\beta}$ if $\mathrm{dist}(v, M) \le \beta_2 := \min\{\bar{\beta}/C_1, 1, \beta_1\}$, and therefore we have

$$c_1||s_t||_{H^1}^2 \le \tilde{\mathscr{H}}(v_t) \le e^{-c(t-1)}C_1 \,\mathrm{dist}(v, M), \quad t \ge 1,$$

if $||s_0||_{L^2} \le \beta_2$ from Lemma 9.1-(iii) with $s = 1$, Theorem 3.1 and Remark 9.1; recall that C_1 = "C_T with $T = 1$". This proves (ii). The assertion (iii) follows from (i) and (ii). Finally we remark that, by taking β_2 smaller if necessary, we have $||s_t||_{L^2} \le \beta_1$ for all $t \ge 0$ from (iii). □

Now we give bounds concerning the linearized equation of the PDE (7.2):

$$\text{(9.8)} \qquad \frac{\partial u_t}{\partial t} = \left(\Delta - \tilde{F}''(v_t(\cdot; v))\right) u_t + r_t, \quad t \ge 0,$$

having an inhomogeneous term $r_t(z) \in L^2([0, T], L^2), T > 0$.

Lemma 9.2. *We have*

$$\text{(9.9)} \qquad ||u_t||_{L^2}^2 \le e^{3t}\left\{||u_0||_{L^2}^2 + \int_0^t ||r_s||_{L^2}^2\, ds\right\},$$

$$\text{(9.10)} \qquad \int_0^t ||\nabla u_s||_{L^2}^2\, ds \le \frac{1}{2}e^{3t}\left\{||u_0||_{L^2}^2 + \int_0^t ||r_s||_{L^2}^2\, ds\right\}, \quad 0 \le t \le T.$$

Proof. Since we have

$$\frac{\partial}{\partial t}||u_t||_{L^2}^2 = 2\langle \Delta u_t, u_t\rangle - 2\int_{\mathbf{R}} \tilde{F}''(v_t(z))u_t^2(z)\, dz + 2\langle r_t, u_t\rangle,$$

by using $2\langle r_t, u_t\rangle \le ||r_t||_{L^2}^2 + ||u_t||_{L^2}^2$, the proof can be completed in a similar fashion to Step 1 of the proof of Lemma 9.1. □

Let $p_{s,t}(y,z) \equiv p_{s,t}(y,z;v), 0 \le s \le t < \infty$, be the fundamental solution for $\partial/\partial t - \{\Delta_z - \tilde{F}''(v_t(z;v))\}$. Namely, the solution $u(t,z), t \ge s$, of the PDE (9.8) without inhomogeneous term (i.e., $r_t = 0$) having initial data $u(s,z) = h(z)$ has an integral representation $u(t,z) = \int_{\mathbf{R}} p_{s,t}(y,z)h(y)\,dy,\ t \ge s$. Then, we have a Gaussian bound on the fundamental solution $p_{s,t}(y,z)$:

Lemma 9.3. *There exists $C = C_T > 0$ such that*

$$C^{-1}q_{s,t}(y,z) \le p_{s,t}(y,z;v) \le Cq_{s,t}(y,z)$$

for every $0 \le s < t : t - s \le T$, where $q_{s,t}(y,z) = q_{t-s}(y,z)$ is the heat kernel; see the formula below (2.2).

Proof. The conclusion follows from Feynman-Kac formula:

$$p_{s,t}(y,z;v) = E_{s,t;y,z}[e^{-\int_s^t c(r,\sqrt{2}B_r)dr}]q_{s,t}(y,z),$$

where $c(r,z) = \tilde{F}''(v_r(z;v))$ is bounded and $E_{s,t;y,z}[\cdot]$ denotes the expectation with respect to the pinned Brownian motion B_r starting from $y/\sqrt{2}$ at time s and reaching $z/\sqrt{2}$ at time t. □

9.2 Fermi coordinates $\eta_t = \eta_t(v)$ of $v_t(\cdot;v)$ and Fréchet derivatives of η_0

In the following, we always assume $v \in \mathscr{V}_{\beta_2}$, where $\beta_2, 0 < \beta_2 \le \beta_1$, will be chosen later in an appropriate manner; see Proposition 9.1 and also Corollary 9.1 above. For instance, β_2 is taken so small that the solution $v_t = v_t(\cdot;v)$ of the PDE (7.2) has a representation in terms of Fermi coordinates: $v_t = m_{\eta_t} + s_t$, where $\eta_t \equiv \eta_t(v) := \eta(v_t(\cdot;v)), s_t \equiv s_t(v) := s(v_t(\cdot;v))$ for all $t \ge 0$ and $v \in \mathscr{V}_{\beta_2}$. The next lemma computes the derivative of η_t in t.

Lemma 9.4.
$$\frac{d\eta_t}{dt} = \frac{\langle \Delta v_t - \tilde{F}'(v_t), \nabla m_{\eta_t}\rangle}{\langle v_t, \nabla^2 m_{\eta_t}\rangle}, \quad t > 0.$$

Proof. Since the property (3.6) of the Fermi coordinates shows $\langle v_t, \nabla m_{\eta_t}\rangle = 0$, the conclusion follows from

$$0 = \frac{d}{dt}\langle v_t, \nabla m_{\eta_t}\rangle = \langle \frac{dv_t}{dt}, \nabla m_{\eta_t}\rangle - \langle v_t, \nabla^2 m_{\eta_t}\rangle \frac{d\eta_t}{dt}. \quad \square$$

Our first goal will be to give concrete formulas for the Fréchet derivatives of $\eta_t = \eta_t(v)$ in v (Lemmas 9.8 and 9.11), which will be useful to derive bounds on the Fréchet derivatives of the limit map $\zeta(v) = \lim_{t\to\infty} \eta_t(v)$. To this end, we naturally start with the computation of those for $\eta_0 = \eta_0(v)$; recall that $\eta_0(v) = \eta(v)$ is the Fermi coordinate of v itself.

Lemma 9.5. *The map $\eta(v) \in \mathbf{R}$ is twice Fréchet differentiable in $v \in \mathscr{V}_{\beta_2}$ and its derivatives are given by*

$$(9.11)\qquad D\eta(y, v) = \frac{\nabla m_{\eta(v)}(y)}{\langle v, \nabla^2 m_{\eta(v)}\rangle},$$

$$(9.12)\qquad D^2\eta(y_1, y_2, v) = \frac{\nabla m_{\eta(v)}(y_1)\nabla m_{\eta(v)}(y_2)\langle v, \nabla^3 m_{\eta(v)}\rangle}{\langle v, \nabla^2 m_{\eta(v)}\rangle^3} - \frac{\nabla^2 m_{\eta(v)}(y_1)\nabla m_{\eta(v)}(y_2) + \nabla m_{\eta(v)}(y_1)\nabla^2 m_{\eta(v)}(y_2)}{\langle v, \nabla^2 m_{\eta(v)}\rangle^2}.$$

Proof. Since

$$\nabla m_{\eta(v+h)}(\cdot) - \nabla m_{\eta(v)}(\cdot) = -\{\eta(v+h) - \eta(v)\} \int_0^1 \nabla^2 m_{(1-a)\eta(v)+a\eta(v+h)}(\cdot)\, da,$$

noting that $\langle v, \nabla m_{\eta(v)}\rangle = 0$ from (3.6), we have

$$\eta(v+h) - \eta(v) = \frac{\langle h, \nabla m_{\eta(v+h)}\rangle}{\int_0^1 \langle v, \nabla^2 m_{(1-a)\eta(v)+a\eta(v+h)}\rangle da}, \qquad h \in L^2.$$

From this formula, one can easily prove that $\eta(v)$ is Fréchet differentiable and its first derivative is given by (9.11). The formula (9.12) for the second derivative is similarly shown by slightly more works; we omit the detail. □

The following four functions will play a role of basic ingredients for giving concrete expressions of $D\eta_t(y, v)$ and $D^2\eta_t(y_1, y_2, v)$; see Lemmas 9.8 and 9.11 below.

$$(9.13)\qquad \begin{cases} \varphi_k(t) \equiv \varphi_k(t, v) := \langle \Delta v_t - \tilde{F}'(v_t), \nabla^k m_{\eta_t}\rangle \\ \psi_k(t, y) \equiv \psi_k(t, y, v) := \langle p_{0,t}(y, \cdot), \{\Delta - \tilde{F}''(v_t)\}\nabla^k m_{\eta_t}\rangle \\ \tilde{\varphi}_k(t) \equiv \tilde{\varphi}_k(t, v) := \langle v_t, \nabla^{k+1} m_{\eta_t}\rangle \\ \tilde{\psi}_k(t, y) \equiv \tilde{\psi}_k(t, y, v) := \langle p_{0,t}(y, \cdot), \nabla^{k+1} m_{\eta_t}\rangle, \\ \quad k = 1, 2, \cdots. \end{cases}$$

In addition, we introduce two more functions.

$$(9.14)\qquad \begin{cases} \Lambda(t) \equiv \Lambda(t, v) = \frac{\varphi_1(t)\tilde{\varphi}_2(t) - \tilde{\varphi}_1(t)\varphi_2(t)}{\{\tilde{\varphi}_1(t)\}^2}, \\ \Xi(t, y) \equiv \Xi(t, y, v) = \frac{\tilde{\varphi}_1(t)\psi_1(t, y) - \varphi_1(t)\tilde{\psi}_1(t, y)}{\{\tilde{\varphi}_1(t)\}^2}, \\ \quad t > 0,\ y \in \mathbf{R},\ v \in \mathscr{V}_{\beta_2}. \end{cases}$$

9.3 Estimates on the basic ingredients and proof of Theorem 7.1

In order to give decay estimates on $d\eta_t/dt$ (necessary for showing the existence of $\zeta = \lim_{t\to\infty} \eta_t$) and also to give bounds on $D\eta_t$ and $D^2\eta_t$, therefore on $D\zeta$ and $D^2\zeta$, we need bounds on the ingredients $\varphi_k, \psi_k, \tilde{\varphi}_k, \tilde{\psi}_k$. Especially to establish the estimates like (7.7) and (7.8), we need decay estimates of $\psi_k(t,y)$ and $\tilde{\psi}_k(t,y)$ in y as $|y| \to \infty$. This forces us to introduce weighted L^p-spaces

$$L^p_\lambda = L^p(\mathbf{R}, e^{p\lambda\theta(y)}dy), \quad \lambda \in \mathbf{R},\ p \geq 1,$$

where a function $\theta \in C^\infty(\mathbf{R})$ satisfying $\theta(y) = |y|$ for $|y| \geq 1$ will be fixed throughout the paper. We also introduce

$$\mathscr{V}_{\beta_2,K} := \{v \in \mathscr{V}_{\beta_2};\ |\eta(v)| \leq K\} = \{v;\ \mathrm{dist}(v,M) \leq \beta_2,\ |\eta(v)| \leq K\},$$

for $K > 0$. Then, the basic estimates are summarized as follows:

Proposition 9.1. *There exists $\beta_2, 0 < \beta_2 \leq \beta_1$, such that the following estimates hold for all $v \in \mathscr{V}_{\beta_2,K}, K > 0$, with some $C, c, \bar{c}, \bar{\lambda} > 0$ dependent on K.*

(i) $|\varphi_k(t)| \leq C\sqrt{\mathrm{dist}(v,M)}e^{-ct}$,

(ii) $|\tilde{\varphi}_k(t)| \leq C$,

$|\tilde{\varphi}_k(t)| \geq c$, *for $k = 1,3,5,\cdots$ (odd),*

(iii) $\|\psi_1(t,\cdot)\|_{L^2_\lambda} \leq C\sqrt{\mathrm{dist}(v,M)}e^{-ct}$, *for every $\lambda : 0 \leq \lambda < \bar{\lambda}$,*

$\|\psi_k(t,\cdot)\|_{L^2_\lambda} \leq Ce^{\bar{c}\lambda t}$, $k = 2,3,\cdots$, *for every $\lambda : 0 \leq \lambda < \bar{\lambda}$,*

(iv) $\|\tilde{\psi}_k(t,\cdot)\|_{L^2_\lambda} \leq Ce^{\bar{c}\lambda t}$, *for every $\lambda : 0 \leq \lambda < \bar{\lambda}$.*

Remark 9.2. The condition $|\eta(v)| \leq K$ is necessary only for the estimates (iii) and (iv) to be uniform in v. The constants $C, c, \bar{c}, \bar{\lambda}$ are common for $1 \leq k \leq k_0$ for each k_0.

Proof of Proposition 9.1-(i) and (ii). Let β_2 be the constant determined by Corollary 9.1. Noting that (1.3) implies $\Delta m_{\eta_t} = \tilde{F}'(m_{\eta_t})$, we have

$$\begin{aligned}|\varphi_k(t)| &\leq |\langle v_t - m_{\eta_t}, \nabla^{k+2}m_{\eta_t}\rangle| + \|\tilde{F}'(v_t) - \tilde{F}'(m_{\eta_t})\|_{L^2}\|\nabla^k m_{\eta_t}\|_{L^2} \\ &\leq \|s_t\|_{L^2}\{\|\nabla^{k+2}m\|_{L^2} + \|\tilde{F}''\|_{L^\infty}\|\nabla^k m\|_{L^2}\}.\end{aligned}$$

This proves (i) with the help of Corollary 9.1-(iii). On the other hand, to prove (ii), we decompose $\tilde{\varphi}_k(t) = \langle v_t - m_{\eta_t}, \nabla^{k+1}m_{\eta_t}\rangle + \langle m_{\eta_t}, \nabla^{k+1}m_{\eta_t}\rangle$ and use Corollary 9.1-(iii); note that $|\langle m_{\eta_t}, \nabla^{k+1}m_{\eta_t}\rangle| = c_k > 0$ if k is odd. If necessary, we choose smaller β_2. □

Now, let us complete the proof of Theorem 7.1, since we shall need the fact that the limit $\zeta(v) = \lim_{t\to\infty}\eta_t(v)$ exists for $v \in \mathscr{V}_{\beta_2}$ to state Lemma 9.6 below. This lemma will be necessary for the proof of Proposition 9.1-(iii), (iv).

Proof of Theorem 7.1. Let β_2 be the same constant as above. From Lemma 9.4 and Proposition 9.1-(i), (ii), we see

$$\left|\frac{d\eta_t}{dt}\right| = \left|\frac{\varphi_1(t)}{\tilde{\varphi}_1(t)}\right| \leq C\sqrt{\mathrm{dist}(v,M)}e^{-ct}, \quad t > 0,\ v \in \mathscr{V}_{\beta_2}. \tag{9.15}$$

This proves that the limit $\zeta \equiv \zeta(v) := \lim_{t\to\infty}\eta_t(v)$ exists. Now, the conclusion of Theorem 7.1 readily follows from Corollary 9.1-(ii) by recalling that $v_t = m_{\eta_t} + s_t$. □

The next lemma gives estimates on the semigroup $\{T_{s,t;v}\}$ defined by

$$T_{s,t;v}\varphi(y) := \langle p_{s,t}(y,\cdot;v),\varphi\rangle, \quad \varphi \in L^2,\ 0 \le s \le t < \infty.$$

In particular, $T_{0,t;m_\zeta}$ is sometimes denoted by $e^{-t\mathscr{A}_\zeta}$, $\zeta \in \mathbf{R}$. Recall that $\mathscr{A}_\zeta = -\Delta + \tilde{F}''(m_\zeta(y))$ is the operator defined by (3.17); we have replaced $-f'$ with $\tilde{F}''$ noting that $-f' = F''$ and $F(v) = \tilde{F}(v)$ for $|v| \le 1$.

Lemma 9.6. *There exist $C, c, \bar{c}, \bar{\lambda} = \bar{\lambda}_K > 0$ for every $K > 0$ such that*

(i) $$\|e^{-t\mathscr{A}_\zeta}\|_{L^2_\lambda \to L^2_\lambda} \le e^{\bar{c}|\lambda| t}, \quad |\lambda| < \bar{\lambda},\ \zeta \in \mathbf{R},$$

(ii) $$\|T_{s,t;v}\|_{L^2_\lambda \to L^2_\lambda} \le Ce^{\bar{c}|\lambda|(t-s)}, \quad |\lambda| < \bar{\lambda},\ v \in \mathscr{V}_{\beta_2},$$

(iii) $$\|e^{-t\mathscr{A}_\zeta} - P_{\{\nabla m_\zeta\}}\|_{L^2_\lambda \to L^2_\lambda} \le e^{-ct}, \quad |\lambda| < \bar{\lambda},\ |\zeta| \le K,$$

(iv) $$\|\{T_{0,t;v} - e^{-t\mathscr{A}_{\zeta(v)}}\}P_{\{\nabla m_{\zeta(v)}\}^\perp}\|_{L^2 \to L^2} \le C\sqrt{\operatorname{dist}(v,M)}\,e^{-ct}, \quad v \in \mathscr{V}_{\beta_2},$$

where β_2 is the constant determined in the proof of Theorem 7.1, $\zeta(v) = \lim_{t\to\infty} \eta_t(v)$, $P_{\{\nabla m_\zeta\}}\cdot = \langle\cdot,\phi_\zeta\rangle\phi_\zeta$, $\phi_\zeta = \nabla m_\zeta/\|\nabla m\|_{L^2}$ and $P_{\{\nabla m_\zeta\}^\perp} = I - P_{\{\nabla m_\zeta\}}$.

Proof. We assume $|\lambda| \le 1$ in the proof. First, simply denoting by $\mathscr{A} = \mathscr{A}_\zeta$, $\bar{T}_t = e^{-t\mathscr{A}}$ and $\phi = \phi_\zeta$, we prove (i) and (iii). Introduce an isometry $\tau = \tau_\lambda : L^2_\lambda \to L^2$ defined by $\tau g = g \cdot e^{\lambda\theta}$ and consider an operator $\tilde{\mathscr{A}} = \tau\mathscr{A}\tau^{-1} : L^2 \to L^2$. Then, a simple computation shows that $\tilde{\mathscr{A}} = \mathscr{A} + \mathscr{B}$ with

$$\mathscr{B}\tilde{g} = 2\lambda\nabla\theta \cdot \nabla\tilde{g} + \{\lambda\nabla^2\theta - \lambda^2(\nabla\theta)^2\}\tilde{g}.$$

From an inequality of Gårding type, which can be easily shown by noting that $\nabla\theta$ and $\nabla^2\theta$ are bounded and $|\lambda| \le 1$:

$$|\langle\mathscr{B}\tilde{g},\tilde{g}\rangle_{L^2}| \le C_1|\lambda|\{\langle\mathscr{A}\tilde{g},\tilde{g}\rangle_{L^2} + \|\tilde{g}\|^2_{L^2}\},$$

with $C_1 > 0$ independent of $\zeta \in \mathbf{R}$, we obtain

$$\begin{aligned}\langle\mathscr{A}g,g\rangle_{L^2_\lambda} &= \langle\mathscr{A}\tilde{g},\tilde{g}\rangle_{L^2} + \langle\mathscr{B}\tilde{g},\tilde{g}\rangle_{L^2} \\ &\ge \{1 - C_1|\lambda|\}\langle\mathscr{A}\tilde{g},\tilde{g}\rangle_{L^2} - C_1|\lambda|\,\|\tilde{g}\|^2_{L^2},\end{aligned} \tag{9.16}$$

where $\tilde{g} = \tau g$. Now assume $|\lambda| \le 1/C_1$ in addition. Then,

$$\langle\mathscr{A}g,g\rangle_{L^2_\lambda} \ge -C_1|\lambda|\,\|\tilde{g}\|^2_{L^2} = -C_1|\lambda|\,\|g\|^2_{L^2_\lambda}. \tag{9.17}$$

Therefore, we have

$$\frac{d}{dt}\|u_t\|^2_{L^2_\lambda} = -2\langle\mathscr{A}u_t,u_t\rangle_{L^2_\lambda} \le 2C_1|\lambda|\,\|u_t\|^2_{L^2_\lambda}, \quad u_t = \bar{T}_t u_0,$$

which implies $||u_t||_{L^2_\lambda}^2 \le e^{2C_1|\lambda|t}||u_0||_{L^2_\lambda}^2$ and this proves (i). Noting that we have $\langle \mathcal{A}\tilde{g}, \tilde{g}\rangle_{L^2} \ge \mu_* ||\tilde{g} - \langle \tilde{g}, \phi\rangle\phi||_{L^2}^2$ from Lemma 3.1 (μ_* denotes the spectral gap of $\mathcal{A}$) and also noting elementary facts $||\tilde{g}||_{L^2}^2 = ||\tilde{g} - \langle \tilde{g}, \phi\rangle\phi||_{L^2}^2 + \langle \tilde{g}, \phi\rangle^2$ and $||\tilde{g} - \langle \tilde{g}, \phi\rangle\phi||_{L^2}^2 \ge \{||\tilde{g}||_{L^2} - |\langle \tilde{g}, \phi\rangle|\}^2$, we obtain from (9.16):

$$\text{(9.18)} \qquad \langle \mathcal{A}g, g\rangle_{L^2_\lambda} \ge \{\mu_* - C_1(\mu_* + 1)|\lambda|\}\{||\tilde{g}||_{L^2}^2 - 2||\tilde{g}||_{L^2}|\langle \tilde{g}, \phi\rangle|\}$$

if $C_1(\mu_* + 2)|\lambda| \le \mu_*$. However, recalling $\tilde{g} = \tau_\lambda g$, we have

$$\text{(9.19)} \qquad |\langle \tilde{g}, \phi\rangle| = |\langle \tilde{g}, (1 - e^{-\lambda\theta})\phi\rangle| \le C_\zeta(\lambda)||\tilde{g}||_{L^2}, \quad \text{if } \langle g, \phi\rangle = 0$$

with $C_\zeta(\lambda) = ||(1 - e^{-\lambda\theta})\phi_\zeta||_{L^2}$ which satisfies $\lim_{\lambda\to 0} C_\zeta(\lambda) = 0$ uniformly in $|\zeta| \le K$; use Lebesgue's dominated convergence theorem. Therefore, from (9.18) and (9.19), we obtain

$$\text{(9.20)} \qquad \langle \mathcal{A}g, g\rangle_{L^2_\lambda} \ge \mu_*||\tilde{g}||_{L^2}^2/2 = \mu_*||g||_{L^2_\lambda}^2/2, \quad \text{if } \langle g, \phi\rangle = 0$$

if $|\lambda|$ is sufficiently small that $\{\mu_* - C_1(\mu_*+1)|\lambda|\}\{1-2C_\zeta(\lambda)\} \ge \mu_*/2$. However, $\{\bar{T}_t\}$ is a semigroup on the space $\mathbb{H} := \{g \in L^2_\lambda; \langle g, \phi\rangle = 0\}$, i.e., $u_t = \bar{T}_t u_0 \in \mathbb{H}$ if $u_0 \in \mathbb{H}$. In fact, noting Lemma 9.3, $\bar{T}_t : L^2_\lambda \to L^2_\lambda$ holds for every $t \ge 0$, since heat semigroup has this property (cf. Lemma 2.1 of [33]), and moreover, $d\langle \bar{T}_t u_0, \phi\rangle_{L^2}/dt = 0$ implies $\langle \bar{T}_t u_0, \phi\rangle_{L^2} = \langle u_0, \phi\rangle_{L^2}$. Hence, for $u_0 \in \mathbb{H}$, from (9.20),

$$\frac{d}{dt}||u_t||_{L^2_\lambda}^2 = -2\langle \mathcal{A}u_t, u_t\rangle_{L^2_\lambda} \le -\mu_*||u_t||_{L^2_\lambda}^2,$$

which proves $||u_t||_{L^2_\lambda} \le e^{-\mu_* t/2}||u_0||_{L^2_\lambda}$. This concludes the assertion (iii).

To show (ii), we see

$$\text{(9.21)} \qquad T_{s,t} = e^{-(t-s)\mathcal{A}} + \int_s^t T_{s,r}\tilde{c}_r e^{-(t-r)\mathcal{A}}\,dr,$$

where $T_{s,t} = T_{s,t;v}$, $\mathcal{A} = \mathcal{A}_{\zeta(v)}$ and $\tilde{c}_r(z) = \tilde{c}(r,z) = \tilde{F}''(m_{\zeta(v)}(z)) - \tilde{F}''(v_r(z;v))$. In fact, since $T^*_{s,t}\psi(z) = \int_{\mathbf{R}} p_{s,t}(y,z)\psi(y)\,dy$ is a solution of the PDE (9.8) for $t \ge s$ with $r_t = 0$, decomposing $\{\Delta - \tilde{F}''(v_t(\cdot;v))\}T^*_{s,t}\psi = -\mathcal{A}T^*_{s,t}\psi + \tilde{c}_t(\cdot)T^*_{s,t}\psi$ and regarding $\tilde{c}_t T^*_{s,t}\psi$ as a perturbation term, we have

$$\text{(9.22)} \qquad T^*_{s,t} = e^{-(t-s)\mathcal{A}} + \int_s^t e^{-(t-r)\mathcal{A}}\tilde{c}_r T^*_{s,r}\,dr.$$

Taking the adjoint of the both sides of (9.22), we obtain (9.21). From (9.21) and (i), we have

$$\text{(9.23)} \qquad ||T_{s,t}||_{L^2_\lambda\to L^2_\lambda} \le e^{\bar{c}|\lambda|(t-s)} + \int_s^t ||T_{s,r}||_{L^2_\lambda\to L^2_\lambda}||\tilde{c}_r \cdot||_{L^2_\lambda\to L^2_\lambda} e^{\bar{c}|\lambda|(t-r)}\,dr.$$

Here, we see

$$(9.24)\qquad ||\tilde{c}_t \cdot ||_{L^2_\lambda \to L^2_\lambda} \le ||\tilde{c}_t||_{L^\infty} \le C_2 e^{-ct}, \quad t \ge 0.$$

In fact, by Sobolev's imbedding theorem and then from Corollary 9.1-(ii) and (9.15), we have

$$(9.25)\qquad \begin{aligned} ||\tilde{c}_t||_{L^\infty} &\le ||\tilde{F}'''||_{L^\infty} ||m_{\zeta(v)} - v_t||_{L^\infty} \\ &\le C_3\{||s_t||_{H^1} + ||m_{\zeta(v)} - m_{\eta_t}||_{H^1}\} \\ &\le C_4\sqrt{\operatorname{dist}(v, M)}c^{-ct}, \quad t \ge 1, \end{aligned}$$

and evidently $||\tilde{c}_t||_{L^\infty}(\le 2||\tilde{F}''||_{L^\infty})$ is bounded for $0 \le t \le 1$. From (9.23) and (9.24), the assertion (ii) is shown without difficulty.

Finally, we prove (iv). We simply denote ζ for $\zeta(v)$. From (9.21) and (ii) with $\lambda = 0$,

$$(9.26)\qquad \begin{aligned} &||\{T_{0,t;v} - e^{-t\mathcal{A}_\zeta}\}P_{\{\nabla m_\zeta\}^\perp}||_{L^2\to L^2} \\ &= \left\| \int_0^t T_{0,r;v}\tilde{c}_r e^{-(t-r)\mathcal{A}_\zeta} P_{\{\nabla m_\zeta\}^\perp}\, dr \right\|_{L^2\to L^2} \le C\int_0^t a(r,t)\, dr, \end{aligned}$$

where

$$(9.27)\qquad \begin{aligned} a(r,t) :=& ||\tilde{c}_r e^{-(t-r)\mathcal{A}_\zeta} P_{\{\nabla m_\zeta\}^\perp}||_{L^2\to L^2} \\ =& ||e^{-(t-r)\mathcal{A}_\zeta} P_{\{\nabla m_\zeta\}^\perp} \cdot \tilde{c}_r||_{L^2\to L^2} \\ \le& ||e^{-(t-r)\mathcal{A}_\zeta} P_{\{\nabla m_\zeta\}^\perp}||_{L^1\to L^2} ||\tilde{c}_r \cdot ||_{L^2\to L^1}. \end{aligned}$$

We notice that $||e^{-t\mathcal{A}_\zeta} P_{\{\nabla m_\zeta\}^\perp}||_{L^2\to L^2} \le e^{-\mu_* t}$ and $\sup_{0\le t\le 1} t^{1/4}||e^{-t\mathcal{A}_\zeta}||_{L^1\to L^2} < \infty$; use Lemma 9.3 by noting $||q_t * g||_{L^2} \le ||q_t||_{L^2}||g||_{L^1} \le C_5 t^{-1/4}||g||_{L^1}$, where q_t is the heat kernel. Therefore

$$(9.28)\qquad \begin{aligned} &||e^{-t\mathcal{A}_\zeta} P_{\{\nabla m_\zeta\}^\perp}||_{L^1\to L^2} \\ &\le \begin{cases} ||e^{-\frac{t}{2}\mathcal{A}_\zeta} P_{\{\nabla m_\zeta\}^\perp}||_{L^2\to L^2} ||e^{-\frac{t}{2}\mathcal{A}_\zeta}||_{L^1\to L^2} \le C_6 t^{-1/4}, & t \le 1, \\ ||e^{-(t-1)\mathcal{A}_\zeta} P_{\{\nabla m_\zeta\}^\perp}||_{L^2\to L^2} ||e^{-\mathcal{A}_\zeta}||_{L^1\to L^2} \le C_6 e^{-\mu_* t}, & t \ge 1, \end{cases} \end{aligned}$$

On the other hand, from Corollary 9.1-(iii) (similarly to the proof of (9.25)), we see

$$(9.29)\qquad ||\tilde{c}_r \cdot ||_{L^2\to L^1} = ||\tilde{c}_r||_{L^2} \le C_7\sqrt{\operatorname{dist}(v, M)}c^{-cr}, \quad r \ge 0.$$

From (9.27)-(9.29), we obtain an estimate on $a(r,t)$ and then, inserting it into the right hand side of (9.26), we see that the left hand side of (9.26) is bounded by

$$CC_6C_7\sqrt{\operatorname{dist}(v, M)} \left\{ \int_{t-1}^t (t-r)^{-\frac{1}{4}} e^{-cr}\, dr + \int_0^{t-1} e^{-\mu_*(t-r)} e^{-cr}\, dr \right\}$$

and therefore by $C_8\sqrt{\operatorname{dist}(v, M)}c^{-ct}$. This completes the proof of (iv). □

Remark 9.3. (i) The estimate (9.17) shows that $-\mathscr{A} - C_1|\lambda|$ is dissipative on the space L^2_λ; see [51, 54]. Similarly, (9.20) shows that $-\mathscr{A}+\mu_*/2$ is dissipative on the space $\mathbb{H}$.
(ii) Since the operator $\mathscr{B}$ is $\mathscr{A}$-compact, applying Weyl's theorem concerning the perturbation theory, we see that the essential spectrum of $\tilde{\mathscr{A}}$ is the same as that of $\mathscr{A}$, c.f. [41, 48].
(iii) The assertion (iv) of Lemma 9.6 will be used only in the proof of Lemma 9.12.

Now we are at the position to complete the proof of Proposition 9.1.

Proof of Proposition 9.1. The assertions (i) and (ii) are already shown. Using Lemma 9.6-(ii) and $\{\Delta - \tilde{F}''(m_{\eta_t})\}\nabla m_{\eta_t} = 0$ (Lemma 3.1 implies $\mathscr{A}\nabla m = 0$ and recall the assumption (7.1)-(a)), we have

$$\begin{aligned}
||\psi_1||_{L^2_\lambda} =& ||T_{0,t;v}\{\Delta - \tilde{F}''(v_t)\}\nabla m_{\eta_t}||_{L^2_\lambda} \\
\le& Ce^{\bar{c}\lambda t}||\{\tilde{F}''(m_{\eta_t}) - \tilde{F}''(v_t)\}\nabla m_{\eta_t}||_{L^2_\lambda} \\
\le& Ce^{\bar{c}\lambda t}||s_t||_{L^2}||\tilde{F}'''||_{L^\infty} \sup_{y\in\mathbf{R}}\left\{|\nabla m_{\eta_t}(y)|e^{\lambda\theta(y)}\right\},
\end{aligned}$$

for sufficiently small $\lambda > 0$. Therefore, the estimate (iii) for $k = 1$ follows from Corollary 9.1-(iii). Notice that the condition $|\eta(v)| \le K$ is necessary for the bound to be uniform in v; in fact, this condition implies $\sup_{t\ge 0, v\in\mathscr{V}_{\bar{\beta}_2}} |\eta_t(v)| < \infty$ from (9.15). Similarly as above, we have

$$||\psi_k||_{L^2_\lambda} \le Ce^{\bar{c}\lambda t}||\{\Delta - \tilde{F}''(v_t)\}\nabla^k m_{\eta_t}||_{L^2_\lambda}$$

and the norm in the right hand side is uniformly finite in $v \in \mathscr{V}_{\bar{\beta}_2,K}$ if $\lambda > 0$ is sufficiently small and this shows (iii) for $k \ge 2$. Finally, again using Lemma 9.6-(ii), we have

$$||\tilde{\psi}_k||_{L^2_\lambda} = ||T_{0,t;v}\nabla^{k+1}m_{\eta_t}||_{L^2_\lambda} \le Ce^{\bar{c}\lambda t}||\nabla^{k+1}m_{\eta_t}||_{L^2_\lambda},$$

which completes (iv). □

The next corollary immediately follows from Proposition 9.1; recall (9.14) for Λ and Ξ.

Corollary 9.2. *For all $v \in \mathscr{V}_{\bar{\beta}_2,K}$ and for sufficiently small $\lambda > 0$*

$$|\Lambda(t)| + ||\Xi(t,\cdot)||_{L^2_\lambda} \le C\sqrt{\mathrm{dist}(v,M)}e^{-ct}.$$

9.4 Fréchet derivatives of $v_t(\cdot;v)$ in v

Here, we study the Fréchet differentiability of the solution $v_t(\cdot;v)$ of the PDE (7.2) in its initial data v.

Lemma 9.7. *(i) The map $v \in L^2 + m \longmapsto v_t(\cdot; v) \in L^2 + m$ is Fréchet differentiable, namely, there exists $D\{v_t(\cdot; v)\} \in \mathscr{L}(L^2, L^2)$ such that (7.3) holds with $\Phi(v), D\Phi(\cdot; v)$ and H replaced by $v_t(\cdot; v), D\{v_t(\cdot; v)\}$ and L^2, respectively. Moreover, the derivative $D\{v_t(\cdot; v)\}$ has an integral representation*

$$D\{v_t(\cdot; v)\}h(z) = \int_{\mathbf{R}} D\{v_t(z; v)\}(y)h(y)\,dy, \quad h \in L^2,$$

with a kernel $D\{v_t(z; v)\}(y) = p_{0,t}(y, z; v) \in L^2(\mathbf{R}^2)$; see Paragraph 9.1 for $p_{0,t}(y, z; v)$.
(ii) The map $v \in L^2 + m \longmapsto D\{v_t(z; v)\}(y) \in L^2(\mathbf{R}^2)$ is Fréchet differentiable and the derivative $D[D\{v_t(z, \cdot; v)\}(y)] \in \mathscr{L}(L^2, L^2(\mathbf{R}^2))$ has an integral representation

$$D[D\{v_t(z, \cdot; v)\}(y)]h = \int_{\mathbf{R}} D^2\{v_t(z; v)\}(y, y_2)h(y_2)\,dy_2, \quad h \in L^2,$$

with a kernel

$$\begin{aligned} &D^2\{v_t(z; v)\}(y_1, y_2) \\ &= -\int_0^t \int_{\mathbf{R}} p_{s,t}(z', z; v)\tilde{F}'''(v_s(z'; v))p_{0,s}(y_1, z'; v)p_{0,s}(y_2, z'; v)\,ds dz'. \end{aligned}$$

Remark 9.4. (i) Fréchet derivative of $v_t(\cdot; v)$ in v is defined by that of $v_t(\cdot; v) - m \in L^2$.
(ii) Rather formal proof (in the sense of functional or Gâteaux derivatives) can be given quickly. In fact, differentiating the both sides of the PDE (7.2) for $v_t = v_t(z; v)$ in v, we get the PDE (9.8) with $r_t = 0$ and initial data $\delta_y(z)$ (delta function) for $D\{v_t(z; v)\}(y)$. This implies $D\{v_t(z; v)\}(y) = p_{0,t}(y, z; v)$. Taking further differentiation of (9.8), we obtain the PDE (9.8) with $r_t(z) = -\tilde{F}'''(v_t(z))D\{v_t(z; v)\}(y_1)D\{v_t(z; v)\}(y_2)$ and initial data 0 for $D^2\{v_t(z; v)\}(y_1, y_2)$. This leads us to the formula given in the assertion (ii).

Proof of Lemma 9.7. To complete the proof of (i), it is sufficient to show

$$(9.30) \qquad ||u_t^{(1)}(\cdot; v, h)||_{L^2} \le C_1 ||h||_{L^2}^2,$$

with $C_1 > 0$ (independent of v), where $u_t^{(1)} = u_t^{(1)}(\cdot; v, h) := v_t(\cdot; v + h) - v_t(\cdot; v) - u_t(\cdot; v, h)$ and $u_t = u_t(\cdot; v, h) = \int_{\mathbf{R}} p_{0,t}(y, \cdot; v)h(y)dy$ denotes a solution of the PDE (9.8) with $r_t = 0$ and $u_0 = h$. We see that $u_t^{(1)}$ satisfies the PDE (9.8) with

$$r_t(\cdot) = -\tilde{F}'(v_t(\cdot; v + h)) + \tilde{F}'(v_t(\cdot; v)) + \tilde{F}''(v_t(\cdot; v))v_t(\cdot; v + h; v)$$

where $v_t(\cdot; v + h; v) = v_t(\cdot; v + h) - v_t(\cdot; v)$. However, using Sobolev's imbedding theorem

$$\begin{aligned} ||r_t||_{L^2} &\le \frac{1}{2}||\tilde{F}'''||_{L^\infty} ||v_t(\cdot; v + h; v)||_{L^4}^2 \\ &\le C_2 ||v_t(\cdot; v + h; v)||_{H^1} ||v_t(\cdot; v + h; v)||_{L^2}. \end{aligned}$$

Therefore, since $u_0^{(1)} = 0$, we have from (9.9) and then from (9.1), (9.2)

$$||u_t^{(1)}(\cdot; v, h)||_{L^2}^2 \le e^{3t} \int_0^t ||r_s||_{L^2}^2 \, ds \le C_3 ||h||_{L^2}^4, \quad 0 \le t \le T,$$

which proves (9.30). Secondly, to prove (ii), we set

$$\begin{aligned} \bar{u}_t &= \bar{u}_t(\cdot; v, h_1, h_2) \\ &:= \int_{\mathbf{R}} D^2\{v_t(z; v)\}(y_1, y_2) h_1(y_1) h_2(y_2) dy_1 dy_2, \quad h_1, h_2 \in L^2, \end{aligned}$$

where the kernel $D^2\{v_t(z; v)\}(y_1, y_2)$ is the function defined in the assertion (ii). Note that $\bar{u}_t$ satisfies the PDE (9.8) with an inhomogeneous term $r_t(\cdot) = -\tilde{F}'''(v_t(\cdot; v)) u_t(\cdot; v, h_1) u_t(\cdot; v, h_2)$. To conclude (ii), it is sufficient to show

$$(9.31) \qquad ||u_t^{(2)}(\cdot; v, h_1, h_2)||_{L^2} \le C_4 ||h_2||_{L^2}^{3/2},$$

with $C_4 > 0$ independent of v and $h_1 : ||h_1||_{L^2} = 1$, where $u_t^{(2)} = u_t^{(2)}(\cdot; v, h_1, h_2) := u_t(\cdot; v + h_2, h_1) - u_t(\cdot; v, h_1) - \bar{u}_t(\cdot; v, h_1, h_2)$. However, we see that $u_t^{(2)}$ satisfies the PDE (9.8) with $r_t = r_t^{(1)} + r_t^{(2)}$, where

$$\begin{aligned} r_t^{(1)}(\cdot) &= -u_t(\cdot; v, h_1) \left\{ \tilde{F}''(v_t(\cdot; v + h_2)) - \tilde{F}''(v_t(\cdot; v)) - \tilde{F}'''(v_t(\cdot; v)) u_t(\cdot; v, h_2) \right\}, \\ r_t^{(2)}(\cdot) &= -\left\{ \tilde{F}''(v_t(\cdot; v + h_2)) - \tilde{F}''(v_t(\cdot; v)) \right\} \left\{ u_t(\cdot; v + h_2, h_1) - u_t(\cdot; v, h_1) \right\}. \end{aligned}$$

Therefore, noting $u_0^{(2)} = 0$ and applying (9.9) again, (9.31) follows if we can prove

$$(9.32) \qquad \int_0^t ||r_s^{(1)}||_{L^2}^2 \, ds \le C_5 ||h_2||_{L^2}^3,$$

$$(9.33) \qquad \int_0^t ||r_s^{(2)}||_{L^2}^2 \, ds \le C_6 ||h_2||_{L^2}^4.$$

To prove (9.32), we notice that the Gaussian bound (Lemma 9.3) shows

$$(9.34) \qquad ||u_t(\cdot; v, h_1)||_{L^\infty} \le C_T t^{-1/4}, \quad 0 < t \le T,$$

recall that $||h_1||_{L^2} = 1$. Hence, we have the following two types of bounds on $||r_t^{(1)}||_{L^2}$:

$$\begin{aligned} &||r_t^{(1)}||_{L^2} \\ &\le C_T t^{-1/4} \left\{ ||\tilde{F}'''||_{L^\infty} ||u_t^{(1)}(\cdot; v, h_2)||_{L^2} + \frac{1}{2} ||\tilde{F}''''||_{L^\infty} ||v_t(\cdot; v + h_2; v)||_{L^4}^2 \right\} \\ &\le C_7 t^{-1/4} \left\{ ||h_2||_{L^2}^2 + ||v_t(\cdot; v + h_2; v)||_{L^4}^2 \right\}, \end{aligned}$$

and

$$\begin{aligned} ||r_t^{(1)}||_{L^2} &\le C_T t^{-1/4} \left\{ ||\tilde{F}'''||_{L^\infty} ||v_t(\cdot; v + h_2; v)||_{L^2} + ||\tilde{F}'''||_{L^\infty} ||u_t(\cdot; v, h_2)||_{L^2} \right\} \\ &\le C_7 t^{-1/4} ||h_2||_{L^2}. \end{aligned}$$

In fact, the first bound follows by applying Taylor's formula for the term in parentheses and then by using (9.30), while the second one is derived by estimating the first two terms and the last term separately and then by applying (9.1) and (9.9). Therefore, we obtain (9.32) by dividing the interval $[0,t]$ of the integration in the left hand side into the union of $I_1 = [0, ||h_2||_{L^2}^2]$ and $I_2 = (||h_2||_{L^2}^2, t]$. We use the bound $||r_t^{(1)}||_{L^2} \le C_7 t^{-1/4}||h_2||_{L^2}$ on the interval I_1, while we use $||r_t^{(1)}||_{L^2} \le C_7||h_2||_{L^2}^{-1/2}\{||h_2||_{L^2}^2 + ||v_t(\cdot; v+h_2; v)||_{L^4}^2\}$ on I_2 and compute similarly to deriving the bound on $||u_t^{(1)}(\cdot; v, h)||_{L^2}^2$. Finally to prove (9.33), using (9.1), we see

$$||r_t^{(2)}||_{L^2} \le ||\tilde{F}'''||_{L^\infty}||v_t(\cdot; v+h_2; v)||_{L^2}||\tilde{u}_t||_{L^\infty} \le C_8 e^t ||h_2||_{L^2}||\tilde{u}_t||_{H^1},$$

where $\tilde{u}_t = \tilde{u}_t(\cdot; v, h_1, h_2) := u_t(\cdot; v+h_2, h_1) - u_t(\cdot; v, h_1)$. However, using the energy bounds (9.9) and (9.10) for $\tilde{u}_t$, one can prove $\int_0^t ||\tilde{u}_s||_{H^1}^2\, ds \le C_9||h_2||_{L^2}^2$ and this completes the proof of (9.33). □

9.5 Representations of $D\zeta$ and $D^2\zeta$

We begin with the computation of $D\eta_t$.

Lemma 9.8. *The map $\eta_t(v)$ is Fréchet differentiable in $v \in \mathscr{V}_{\beta_2}$ and its derivative is given by*

$$D\eta_t(y, v) = e^{\int_0^t \Lambda(s)ds} D\eta(y, v) + \int_0^t \Xi(s, y) e^{\int_s^t \Lambda(r)dr}\, ds, \tag{9.35}$$

where $\Lambda(t)$ and $\Xi(t, y)$ are the functions defined by (9.14).

Proof. Noting the formula $D\{v_t(z; v)\}(y) = p_{0,t}(y, z; v)$ shown in Lemma 9.7-(i) and also noting

$$D\{\nabla^k m_{\eta_t}(z)\}(y, v) = -\nabla^{k+1} m_{\eta_t}(z) D\eta_t(y, v), \tag{9.36}$$

we obtain

$$D\{\varphi_k(t)\}(y, v) = \psi_k(t, y) - \varphi_{k+1}(t) D\eta_t(y, v), \tag{9.37}$$

$$D\{\tilde{\varphi}_k(t)\}(y, v) = \tilde{\psi}_k(t, y) - \tilde{\varphi}_{k+1}(t) D\eta_t(y, v). \tag{9.38}$$

Since $d\eta_t/dt = \varphi_1(t)/\tilde{\varphi}_1(t)$ from Lemma 9.4, using (9.37) and (9.38),

$$\begin{aligned} \frac{d}{dt} D\eta_t(y, v) &= \{\tilde{\varphi}_1 D\varphi_1 - \varphi_1 D\tilde{\varphi}_1\}/\tilde{\varphi}_1^2 \\ &= \Lambda(t) D\eta_t(y, v) + \Xi(t, y). \end{aligned} \tag{9.39}$$

Solving this equation in $D\eta_t(y, v)$, we get the conclusion. □

Remark 9.5. In the proof, more precisely saying, we first obtain (9.36)-(9.39) with $D\{\nabla^k m_{\eta_t}(z)\}(y,v)$, $D\eta_t(y,v)$, $D\{\varphi_k(t)\}(y,v)$ and $D\{\tilde{\varphi}_k(t)\}(y,v)$ replaced by $\nabla^k m_{\eta_t(v+h)}(z) - \nabla^k m_{\eta_t(v)}(z)$, $\eta_t(v+h) - \eta_t(v)$, $\varphi_k(t,v+h) - \varphi_k(t,v)$ and $\tilde{\varphi}_k(t,v+h) - \tilde{\varphi}_k(t,v)$, respectively, and with small errors $o(\|h\|_{L^2})$. Notice that, as a function of y, the right hand side of (9.35) is in L^2_λ with some $\lambda > 0$ and therefore in L^2.

Before computing the second derivative of $\eta_t(v)$, we prepare two lemmas.

Lemma 9.9.

$$D\{\Lambda(t)\}(y,v) = \tilde{\Lambda}(t)D\eta_t(y,v) + \tilde{\Xi}(t,y),$$

where

$$\tilde{\Lambda}(t) = \tilde{\varphi}_1^{-3}\left[2\varphi_1\tilde{\varphi}_2^2 - 2\tilde{\varphi}_1\varphi_2\tilde{\varphi}_2 + \tilde{\varphi}_1^2\varphi_3 - \varphi_1\tilde{\varphi}_1\tilde{\varphi}_3\right],$$
$$\tilde{\Xi}(t,y) = \tilde{\varphi}_1^{-3}\left[-2\varphi_1\tilde{\varphi}_2\tilde{\psi}_1 + \tilde{\varphi}_1\{\tilde{\varphi}_2\psi_1 + \varphi_1\tilde{\psi}_2 + \varphi_2\tilde{\psi}_1\} - \tilde{\varphi}_1^2\psi_2\right],$$

and $\varphi_k = \varphi_k(t), \tilde{\varphi}_k = \tilde{\varphi}_k(t), \psi_k = \psi_k(t,y), \tilde{\psi}_k = \tilde{\psi}_k(t,y)$.

Proof. Take the Fréchet derivative of $\Lambda(t)$ given in (9.14) by noting (9.37) and (9.38). □

Lemma 9.10.

$$D\Xi(t,y_1,y_2,v) \equiv D\{\Xi(t,y_1)\}(y_2,v) = J_1 - J_2 + J_3 - J_4 - J_5,$$

where $J_i = J_i(t,y_1,y_2,v)$ *are given by*

$$\begin{aligned}
J_1 =& \{\tilde{\varphi}_1(t)\}^{-3}\{J_{1,1}(t,y_1,y_2,v) + J_{1,2}(t,y_1,v)D\eta_t(y_2,v)\},\\
&J_{1,1} = 2\varphi_1(t)\tilde{\psi}_1(t,y_1)\tilde{\psi}_1(t,y_2) - \tilde{\varphi}_1(t)\psi_1(t,y_1)\tilde{\psi}_1(t,y_2)\\
&\qquad -\tilde{\varphi}_1(t)\tilde{\psi}_1(t,y_1)\psi_1(t,y_2),\\
&J_{1,2} = \tilde{\varphi}_1(t)\tilde{\varphi}_2(t)\psi_1(t,y_1) + \tilde{\varphi}_1(t)\varphi_2(t)\tilde{\psi}_1(t,y_1)\\
&\qquad +\tilde{\varphi}_1(t)\varphi_1(t)\tilde{\psi}_2(t,y_1) - 2\varphi_1(t)\tilde{\varphi}_2(t)\tilde{\psi}_1(t,y_1),\\
&J_2 = \{\tilde{\varphi}_1(t)\}^{-2}\varphi_1(t)\langle D^2\{v_t(\cdot)\}(y_1,y_2), \nabla^2 m_{\eta_t}\rangle,\\
&J_3 = \{\tilde{\varphi}_1(t)\}^{-1}\langle D^2\{v_t(\cdot)\}(y_1,y_2), \{\Delta - \tilde{F}''(v_t)\}\nabla m_{\eta_t}\rangle,\\
&J_4 = \{\tilde{\varphi}_1(t)\}^{-1}\psi_2(t,y_1)D\eta_t(y_2,v),\\
&J_5 = \{\tilde{\varphi}_1(t)\}^{-1}\langle p_{0,t}(y_1,\cdot)p_{0,t}(y_2,\cdot), \tilde{F}'''(v_t)\nabla m_{\eta_t}\rangle.
\end{aligned}$$

Proof. Use the formulas (9.37), (9.38) and

$$\begin{aligned}
D\{\psi_1(t,y_1)\}(y_2,v) &= \langle D^2\{v_t(\cdot)\}(y_1,y_2), \{\Delta - \tilde{F}''(v_t)\}\nabla m_{\eta_t}\rangle\\
&\quad -\langle p_{0,t}(y_1,\cdot)p_{0,t}(y_2,\cdot), \tilde{F}'''(v_t)\nabla m_{\eta_t}\rangle - \psi_2(t,y_1)D\eta_t(y_2,v),\\
D\{\tilde{\psi}_1(t,y_1)\}(y_2,v) &= \langle D^2\{v_t(\cdot)\}(y_1,y_2), \nabla^2 m_{\eta_t}\rangle - \tilde{\psi}_2(t,y_1)D\eta_t(y_2,v),
\end{aligned}$$

which are shown by noting Lemma 9.7-(ii). □

Now, the following formula for $D^2\eta_t$ is immediate from Lemmas 9.9 and 9.10 by differentiating (9.35).

Lemma 9.11. *The map $\eta_t(v)$ is twice Fréchet differentiable in $v \in \mathscr{V}_{\beta_2}$ and its second derivative is given by*

$$D^2\eta_t(y_1, y_2, v) = \sum_{i=1}^{4} I_i(t, y_1, y_2, v), \tag{9.40}$$

where

$$\begin{aligned}
I_1(t, y_1, y_2, v) &= \int_0^t DA(s)(y_2, v)\, ds \cdot e^{\int_0^t A(s)ds} D\eta(y_1, v),\\
I_2(t, y_1, y_2, v) &= e^{\int_0^t A(s)ds} D^2\eta(y_1, y_2, v),\\
I_3(t, y_1, y_2, v) &= \int_0^t \Xi(s, y_1) \int_s^t DA(r)(y_2, v)\, dr \cdot e^{\int_s^t A(r)dr}\, ds,\\
I_4(t, y_1, y_2, v) &= \int_0^t D\Xi(s, y_1, y_2, v) e^{\int_s^t A(r)dr}\, ds.
\end{aligned}$$

We obtain the following representations of $D\zeta$ and $D^2\zeta$ by taking the limit $t \to \infty$ in the formulas for $D\eta_t$ and $D^2\eta_t$ given in Lemmas 9.8 and 9.11. In fact, using Proposition 9.1, Corollary 9.2 and Lemmas 9.9, 9.10, one can prove that $D\eta_t(y, v)$ and $D^2\eta_t(y_1, y_2, v)$ converge uniformly in $v \in \mathscr{V}_{\beta_2}$ to $D\zeta(y, v)$ in the space L^2 and to $D^2\zeta(y_1, y_2, v)$ in $L^2(\mathbf{R}^2)$, respectively, where $D\zeta$ and $D^2\zeta$ are given by the formulas (9.41) and (9.42) below; to establish the convergence of the term $I_4(t, y_1, y_2, v)$, we use some estimates established in Paragraph 9.7 below.

Proposition 9.2. *The map $\zeta(v)$ is twice Fréchet differentiable in $v \in \mathscr{V}_{\beta_2}$ and its derivatives are given by*

$$D\zeta(y, v) = e^{\int_0^\infty A(s)ds} D\eta(y, v) + \int_0^\infty \Xi(s, y) e^{\int_s^\infty A(r)dr}\, ds, \tag{9.41}$$

$$D^2\zeta(y_1, y_2, v) = \sum_{i=1}^{4} I_i(y_1, y_2, v), \tag{9.42}$$

where

$$\begin{aligned}
I_1(y_1, y_2, v) &= \int_0^\infty DA(t)(y_2, v)\, dt \cdot e^{\int_0^\infty A(s)ds} D\eta(y_1, v),\\
I_2(y_1, y_2, v) &= e^{\int_0^\infty A(s)ds} D^2\eta(y_1, y_2, v),\\
I_3(y_1, y_2, v) &= \int_0^\infty \Xi(t, y_1) \int_t^\infty DA(s)(y_2, v)\, ds \cdot e^{\int_t^\infty A(s)ds}\, dt,\\
I_4(y_1, y_2, v) &= \int_0^\infty D\Xi(t, y_1, y_2, v) e^{\int_t^\infty A(s)ds}\, dt.
\end{aligned}$$

9.6 Proof of Theorem 7.2

Here, we give the proof of Theorem 7.2. We notice that

$$(9.43)\qquad \Lambda(t, m_\eta) = 0 \quad \text{and} \quad \Xi(t, \cdot, m_\eta) = 0, \quad \eta \in \mathbf{R},$$

and this especially implies the formula (7.6) from (9.41); the second equality in (7.6) follows from (9.11). Now, assume $v \in \mathscr{V}_{\beta_2,0}$, i.e., $v \in \mathscr{V}_{\beta_2}$ and $\zeta(v) = 0$. Then, using (9.41) and (7.6)

$$(9.44)\qquad ||D\zeta(\cdot, v) - D\zeta(\cdot, m)||_{L^2} \le ||D\eta(\cdot, v)||_{L^2}\left|e^{\int_0^\infty \Lambda(s)ds} - 1\right|$$
$$+||D\eta(\cdot, v) - D\eta(\cdot, m)||_{L^2} + \int_0^\infty ||\Xi(s, \cdot)||_{L^2} e^{\int_s^\infty \Lambda(r)dr} ds.$$

Here, $||D\eta(\cdot, v)||_{L^2}$ is bounded from (9.11) and

$$(9.45)\qquad \left|e^{\int_0^\infty \Lambda(s)ds} - 1\right| \le C_1\sqrt{\text{dist}(v, M)}$$

by using $|e^x - 1| \le |x|e^{|x|}$ and Corollary 9.2. Furthermore, the second term in the right hand side of (9.44) is bounded by $C_2\sqrt{\text{dist}(v, M)}$ from (9.11) since (9.15) proves

$$(9.46)\quad |\eta(v)| = |\eta(v) - \zeta(v)| = |\int_0^\infty \frac{d\eta_t(v)}{dt}dt| \le C_3\sqrt{\text{dist}(v, M)}, \quad v \in \mathscr{V}_{\beta_2,0},$$

and the third term is bounded by $C_4\sqrt{\text{dist}(v, M)}$ from Corollary 9.2. Hence, the proof of Theorem 7.2 is completed.

Remark 9.6. We have actually a better estimate: The right hand side of (7.5) can be replaced by $C\,\text{dist}(v, M)$, since one can prove $\sup_{v \in \mathscr{V}_{\beta_2}} ||D^2\zeta(\cdot, \cdot, v)||_{L^2(\mathbf{R}^2)} < \infty$.

9.7 Proof of Theorem 7.3

Now, we move to the proof of Theorem 7.3. To show (7.7) and (7.8), it is sufficient to prove

$$(9.47;\text{i})\qquad ||I_i(y, y, v) - I_i(y, y, m)||_{L^1_\lambda(\mathbf{R}_y)} \le C\sqrt{\text{dist}(v, M)}, \quad v \in \mathscr{V}_{\beta_2,0},$$

for some $\lambda > 0$ and $i = 1, 2, 3, 4$, where $I_i(y, y, v)$ are functions defined in Proposition 9.2. In fact, note that $|\int_{\mathbf{R}} y^p I(y)dy| \le ||I||_{L^1_\lambda} \times \sup_{y\in\mathbf{R}} |y|^p e^{-\lambda\theta(y)}, p = 0, 1, 2$.

9.7.1. Estimate on $D\Lambda(t)$

Lemma 9.12. *There exists $C > 0$ such that*

$$||D\{\Lambda(t)\}(\cdot, v) - \Phi(t, \cdot)||_{L^2} \leq C\sqrt{\text{dist}(v, M)}e^{-ct}, \quad t \geq 0,$$

holds for every $v \in \mathcal{V}_{\beta_2,0}$, where

$$\Phi(t, y) := D\{\Lambda(t)\}(y, m) = \frac{1}{||\nabla m||_{L^2}^2} e^{-t\mathcal{A}}(-\mathcal{A})\nabla^2 m(y). \tag{9.48}$$

Proof. From Proposition 9.1, we see for $\tilde{\Lambda}(t)$ defined in Lemma 9.9

$$|\tilde{\Lambda}(t)| \leq C\sqrt{\text{dist}(v, M)}e^{-ct}, \quad v \in \mathcal{V}_{\beta_2,0},$$

while $||D\eta_t(\cdot, v)||_{L^2}$ is bounded in $t \in [0, \infty)$ and v; see the proof of Theorem 7.2. Again from Proposition 9.1, we have

$$||\bar{\Xi}(t, \cdot)||_{L^2} \leq C\sqrt{\text{dist}(v, M)}e^{-ct}, \quad v \in \mathcal{V}_{\beta_2,0},$$

where $\bar{\Xi}(t, y) = \tilde{\Xi}(t, y) + \tilde{\varphi}_1(t)^{-1}\psi_2(t, y)$, i.e., $\tilde{\Xi}(t, y)$ without its last term. Therefore, since Corollary 9.1-(iii) and (9.15) can be used to see

$$\begin{aligned} |\tilde{\varphi}_1(t) - \langle m, \nabla^2 m\rangle| &\leq |\langle v_t - m, \nabla^2 m_{\eta_t}\rangle| + |\langle\nabla^2 m, m_{\eta_t} - m\rangle| \\ &\leq C\sqrt{\text{dist}(v, M)}e^{-ct}, \quad v \in \mathcal{V}_{\beta_2,0}, \end{aligned} \tag{9.49}$$

recalling Lemma 9.9, it is sufficient for completing the proof to show

$$||\psi_2(t, \cdot) - e^{-t\mathcal{A}}(-\mathcal{A})\nabla^2 m||_{L^2} \leq C\sqrt{\text{dist}(v, M)}e^{-ct}, \quad v \in \mathcal{V}_{\beta_2,0}. \tag{9.50}$$

However, the left hand side of (9.50) can be rewritten in $||U + V||_{L^2}$, where

$$\begin{aligned} U &= T_{0,t;v}\left[\{\Delta - \tilde{F}''(v_t)\}\nabla^2 m_{\eta_t} - \{\Delta - \tilde{F}''(m)\}\nabla^2 m\right], \\ V &= \{T_{0,t;v} - e^{-t\mathcal{A}}\}(-\mathcal{A})\nabla^2 m. \end{aligned}$$

Using Lemma 9.6-(ii) with $\lambda = 0$,

$$\begin{aligned} ||U||_{L^2} &\leq C\left[||\nabla^4\{m_{\eta_t} - m\}||_{L^2} + ||\tilde{F}''(v_t)\nabla^2\{m_{\eta_t} - m\}||_{L^2}\right. \\ &\qquad \left. + ||\{\tilde{F}''(v_t) - \tilde{F}''(m)\}\nabla^2 m||_{L^2}\right] \\ &\leq C\sqrt{\text{dist}(v, M)}e^{-ct}. \end{aligned}$$

Here, the second inequality is shown similarly to (9.49). On the other hand, from Lemma 9.6-(iv) by noting that $\mathcal{A}\nabla^2 m \in \{\nabla m\}^\perp$, we have

$$||V||_{L^2} \leq C\sqrt{\text{dist}(v, M)}e^{-ct}.$$

These two estimates prove (9.50). □

9.7.2. Estimates on $I_1 - I_3$

Here, we prove (9.47;1)-(9.47;3). We always assume $v \in \mathscr{V}_{\beta_2,0}$ and $\lambda > 0$ is taken sufficiently small.

Proof of (9.47;1). From (9.43) and (9.48), we have

$$I_1(y,y,m) = \int_0^\infty \Phi(t,y)\,dt \cdot D\eta(y,m), \tag{9.51}$$

and therefore

$$I_1(y,y,v) - I_1(y,y,m) = A + B + C,$$

where

$$A \equiv A(y,v) = \int_0^\infty \{DA(t)(y,v) - \Phi(t,y)\}\,dt \cdot e^{\int_0^\infty A(s)ds} D\eta(y,v),$$

$$B \equiv B(y,v) = \int_0^\infty \Phi(t,y)\,dt \times \left\{e^{\int_0^\infty A(s)ds} - 1\right\} D\eta(y,v),$$

$$C \equiv C(y,v) = \int_0^\infty \Phi(t,y)\,dt \times \{D\eta(y,v) - D\eta(y,m)\}.$$

Since $e^{\int_0^\infty A(s)ds}$ and $||D\eta(\cdot,v)||_{L^2_\lambda}$ are bounded, using Lemma 9.12, we have

$$||A||_{L^1_\lambda} \le C_1 \int_0^\infty ||DA(t)(\cdot,v) - \Phi(t,\cdot)||_{L^2}\,dt \le C_2\sqrt{\text{dist}(v,M)}. \tag{9.52}$$

For the second term B, since $\langle \mathscr{A}\nabla^2 m, \nabla m\rangle = 0$, we get

$$||\Phi(t,\cdot)||_{L^2} \le C_3 e^{-\mu_* t}, \tag{9.53}$$

see Lemma 3.1 for μ_* (spectral gap of $\mathscr{A}$). Therefore, noting (9.45), we obtain

$$||B||_{L^1_\lambda} \le C_4\sqrt{\text{dist}(v,M)}.$$

Finally, similar bound on the third term $||C||_{L^1_\lambda}$ is shown by proving

$$||D\eta(\cdot,v) - D\eta(\cdot,m)||_{L^2_\lambda} \le C_5\,\text{dist}(v,M), \tag{9.54}$$

cf. Paragraph 9.6. □

Proof of (9.47;2). First note that

$$I_2(y,y,m) = D^2\eta(y,y,m). \tag{9.55}$$

Then, the estimate (9.47;2) easily follows by showing $||D^2\eta(y,y,v)||_{L^1_\lambda(\mathbf{R}_y)} < \infty$ and

$$||D^2\eta(y,y,v) - D^2\eta(y,y,m)||_{L^1_\lambda(\mathbf{R}_y)} \le C_6\sqrt{\text{dist}(v,M)}$$

from the concrete form of $D^2\eta(y,y,v)$ given in Lemma 9.5. □

Proof of (9.47;3). Since $\Xi(t,y;m) = 0$ from (9.43), we see

(9.56) $$I_3(y, y, m) = 0.$$

Then, since (9.52) and (9.53) show that $\int_0^\infty ||DA(s)(\cdot, v)||_{L^2}\, ds$ is bounded and also since $e^{\int_0^\infty |A(t)|dt}$ is bounded, (9.47;3) is shown from Corollary 9.2 (estimate on $||\Xi(t,\cdot)||_{L^2_\lambda}$). □

9.7.3. Estimate on I_4

Finally we shall prove (9.47;4). To this end, it is sufficient to show the following two bounds:

(9.57) $$||D\Xi(t,y,y,v) - D\Xi(t,y,y,m)||_{L^1_\lambda} \le C\sqrt{\mathrm{dist}(v,M)}e^{-ct}\{1 + \frac{1}{\sqrt{t}}\},$$

(9.58) $$||D\Xi(t,y,y,m)||_{L^1_\lambda} \le Ce^{-ct}\{1 + \frac{1}{\sqrt{t}}\},$$

for some $C, c, \lambda > 0$. We always assume $v \in \mathscr{V}_{\beta_2,0}$ again. To show the estimate (9.57), from Lemma 9.10, it is enough to prove the following five estimates:

(9.59; i) $$||J_i(t,y,y,v)||_{L^1_\lambda} \le C\sqrt{\mathrm{dist}(v,M)}e^{-ct},$$

for $i = 1, 2, 3$ and

(9.59; 4) $$||J_4(t,y,y,v) - J_4(t,y,y,m)||_{L^1_\lambda} \le C\sqrt{\mathrm{dist}(v,M)}e^{-ct},$$

(9.59; 5) $$||J_5(t,y,y,v) - J_5(t,y,y,m)||_{L^1_\lambda} \le C\sqrt{\mathrm{dist}(v,M)}e^{-ct}\{1 + \frac{1}{\sqrt{t}}\}.$$

We prepare three lemmas under the assumption that $v \in \mathscr{V}_{\beta_2,0}$:

Lemma 9.13. *For sufficiently small $\lambda > 0$, we have*

(i) $$||D\eta_t(\cdot, v) - D\eta(\cdot, v)||_{L^2_\lambda} \le C\sqrt{\mathrm{dist}(v,M)}, \quad t \ge 0,$$

(ii) $$||D\eta_t(\cdot, v)||_{L^2_\lambda} \le C, \quad t \ge 0.$$

Proof. The assertion (i) follows from the expression of $D\eta_t(y, v)$, see (9.35), and the bounds on $A(t)$ and $\Xi(t,y)$ given in Corollary 9.2. The assertion (ii) follows from (i). □

Lemma 9.14. *There exist $C, \bar{c}, \bar{\lambda} > 0$ such that*

(9.60) $$||p_{0,t}(\cdot, z; v)||^2_{L^2_\lambda} \le C\left\{e^{\bar{c}\lambda t} + \frac{1}{\sqrt{t}}\right\}e^{2\lambda\theta(z)}, \quad t \ge 0,\ 0 \le \lambda \le \bar{\lambda}.$$

Proof. For $0 < t \le 1$, using the Gaussian bound in Lemma 9.3, the left hand side of (9.60) is bounded as follows:

$$\int_{\mathbf{R}} e^{2\lambda\theta(y)} p_{0,t}(y,z;v)^2\,dy \le \frac{C_1^2}{4\pi t}\int_{\mathbf{R}} e^{2\lambda\theta(y)-\frac{(y-z)^2}{2t}}\,dy$$
$$= \frac{C_1^2}{4\pi\sqrt{t}}\int_{\mathbf{R}} e^{2\lambda\theta(z+\sqrt{t}y)-\frac{y^2}{2}}\,dy \le \frac{C_2}{\sqrt{t}}e^{2\lambda\theta(z)}, 0 < t \le 1,\ 0 \le \lambda \le \bar\lambda.$$

For $t \ge 1$, using the semigroup property of $\{p_{s,t}(y,z)\}$ and then the Gaussian bound again, we have

$$p_{0,t}(y,z) = T_{0,t-1;v}\{p_{t-1,t}(\cdot,z)\}(y) \le \frac{C_1}{\sqrt{4\pi}}T_{0,t-1;v}\{e^{-\frac{1}{4}(\cdot-z)^2}\}(y).$$

Therefore, the left hand side of (9.60) is bounded by

$$\frac{C_1^2}{4\pi}||T_{0,t-1;v}\{e^{-\frac{1}{4}(\cdot-z)^2}\}||_{L^2_\lambda}^2$$
$$\le \frac{C^2C_1^2}{4\pi}e^{2\bar c\lambda(t-1)}||e^{-\frac{1}{4}(\cdot-z)^2}||_{L^2_\lambda}^2 \le C_3 e^{2\bar c\lambda t}e^{2\lambda\theta(z)},$$
$$t \ge 1,\ 0 \le \lambda \le \bar\lambda.$$

Here, we have used Lemma 9.6-(ii). □

Lemma 9.15. *There exist $C, \bar c, \bar\lambda > 0$ such that*

$$(9.61)\qquad ||\langle D^2\{v_t(\cdot)\}(y,y),\varphi\rangle||_{L^1_\lambda(\mathbf{R}_y)} \le Ce^{\bar c\lambda' t}||\varphi||_{L^2_{\lambda'}},$$

holds for all $\varphi = \varphi(y,v)$ and for every $0 < \lambda < \lambda' \le \bar\lambda$.

Proof. From the formula for $D^2\{v_t(\cdot)\}(y,y)$ given in Lemma 9.7-(ii) and using Lemma 9.14, the left hand side of (9.61) is bounded by

$$\int_0^t ds \int_{\mathbf{R}^2} p_{s,t}(z',z)|\tilde F'''(v_s(z'))|\,|\varphi(z)|\,dzdz' \int_{\mathbf{R}} e^{\lambda\theta(y)}p_{0,s}(y,z')^2\,dy$$
$$\le C_1||\tilde F'''||_{L^\infty}\int_0^t \{e^{c_1\lambda s/2} + \frac{1}{\sqrt{s}}\}||T_{s,t;v}|\varphi|\,||_{L^1_\lambda}\,ds.$$

However, noting $\lambda < \lambda'$ and using Lemma 9.6-(ii),

$$||T_{s,t;v}|\varphi|\,||_{L^1_\lambda} \le C_{\lambda,\lambda'}||T_{s,t;v}|\varphi|\,||_{L^2_{\lambda'}} \le C_2 e^{c_2\lambda'(t-s)}||\varphi||_{L^2_{\lambda'}}.$$

Therefore, the conclusion follows. □

Now, we can prove (9.59;1)–(9.59;3). Recall that we are assuming $v \in \mathscr{V}_{\beta_2,0}$.

Proof of (9.59;1). Using Proposition 9.1, we see

$$||J_{1,1}(t,y,y,v)||_{L^1_\lambda} \le 2||\tilde\psi_1(t,\cdot)||_{L^2}\{|\varphi_1(t)|\,||\tilde\psi_1(t,\cdot)||_{L^2_\lambda} + |\tilde\varphi_1(t)|\,||\psi_1(t,\cdot)||_{L^2_\lambda}\}$$
$$\le C_1\sqrt{\mathrm{dist}(v,M)}e^{-c_1 t}$$

and

$$\begin{aligned}||J_{1,2}(t,y,v)D\eta_t(y,v)||_{L^1_\lambda} \leq &||J_{1,2}(t,y,v)||_{L^2}||D\eta_t(y,v)||_{L^2_\lambda}\\ \leq &C_2\sqrt{\mathrm{dist}(v,M)}e^{-c_1 t}.\end{aligned}$$

Here, we have used Lemma 9.13-(ii). These two estimates complete the proof of (9.59;1). □

Proof of (9.59;2). If $\lambda > 0$ is sufficiently small, one can find $\lambda' > \lambda$ such that $||\nabla^2 m_{\eta_t}||_{L^2_{\lambda'}}$ is bounded in t and therefore, from Lemma 9.15,

$$||\langle D^2\{v_t(\cdot)\}(y,y), \nabla^2 m_{\eta_t}\rangle||_{L^1_\lambda} \leq C_3 e^{\bar{c}\lambda' t}.$$

Now the estimate (9.59;2) is shown from Proposition 9.1-(i) and (ii). □

Proof of (9.59;3). Taking $\lambda' > \lambda$ similarly as above, we have

$$\begin{aligned}&||\langle D^2\{v_t(\cdot)\}(y,y), \{\Delta - \tilde{F}''(v_t)\}\nabla m_{\eta_t}\rangle||_{L^1_\lambda}\\ &\leq C_4 e^{\bar{c}\lambda' t}||\{\Delta - \tilde{F}''(v_t)\}\nabla m_{\eta_t}||_{L^2_{\lambda'}}\\ &\leq C_5 e^{\bar{c}\lambda' t}\sqrt{\mathrm{dist}(v,M)}e^{-ct}.\end{aligned}$$

Here, for the second inequality, see the proof of Proposition 9.1-(iii), $k = 1$. Therefore, (9.59;3) is proved by taking sufficiently small λ'. □

Now, let us give the proof of (9.59;4): First we notice that

$$J_4(t,y,y,m) = -\Phi(t,y)D\eta(y,m), \tag{9.62}$$

where $\Phi(t,y)$ is the function defined in Lemma 9.12. In fact, (9.62) is shown by noting $\tilde{\varphi}_1(t,m) = \langle m, \nabla^2 m\rangle, \psi_2(t,y,m) = e^{-t\mathscr{A}}(-\mathscr{A})\nabla^2 m(y)$ and $D\eta_t(y,m) = D\eta(y,m)$. In order to prove (9.59;4), we decompose

$$J_4(t,y,y,v) - J_4(t,y,y,m) = A + B + C, \tag{9.63}$$

where

$$\begin{aligned}A &\equiv A(t,y,v) = \{\tilde{\varphi}_1(t)\}^{-1}\psi_2(t,y)\{D\eta_t(y,v) - D\eta(y,m)\},\\ B &\equiv B(t,y,v) = \{\tilde{\varphi}_1(t)\}^{-1}\{\psi_2(t,y) - e^{-t\mathscr{A}}(-\mathscr{A})\nabla^2 m(y)\}D\eta(y,m),\\ C &\equiv C(t,y,v) = \left\{\frac{1}{\tilde{\varphi}_1(t)} - \frac{1}{\langle m, \nabla^2 m\rangle}\right\}e^{-t\mathscr{A}}(-\mathscr{A})\nabla^2 m(y)D\eta(y,m).\end{aligned}$$

We give estimates on A, B and C separately, recall that $v \in \mathscr{V}_{\beta_2,0}$:

$$\begin{aligned}||A||_{L^1_\lambda} &\leq C_1||\psi_2(t,\cdot)||_{L^2}||D\eta_t(\cdot,v) - D\eta(\cdot,m)||_{L^2_\lambda}\\ &\leq C_2\sqrt{\mathrm{dist}(v,M)}e^{-ct},\end{aligned} \tag{9.64}$$

from Lemma 9.13-(i), (9.54) and $||\psi_2(t,\cdot)||_{L^2} \leq C_3 e^{-ct}$ for $v \in \mathscr{V}_{\beta_2,0}$, which follows from (9.50) and (9.53). Secondly,

$$(9.65)\qquad \begin{aligned}||B||_{L^1_\lambda} \le & C_1||\psi_2(t,\cdot) - e^{-t\mathscr{A}}(-\mathscr{A})\nabla^2 m||_{L^2}||D\eta(\cdot,m)||_{L^2_\lambda} \\ \le & C_4\sqrt{\mathrm{dist}(v,M)}e^{-ct},\end{aligned}$$

from (9.50). Finally, from (9.49), we have

$$(9.66)\qquad ||C||_{L^1_\lambda} \le C_5\sqrt{\mathrm{dist}(v,M)}e^{-ct}.$$

Combining (9.64)–(9.66), we have proved (9.59;4). □

Finally, let us prove (9.59;5). We are assuming $v \in \mathscr{V}_{\beta_2,0}$. Note that

$$(9.67)\qquad J_5(t,y,y,m) = \frac{1}{\langle m, \nabla^2 m\rangle}\int_{\mathbf{R}} p_{0,t}(y,z;m)^2\tilde{F}'''(m(z))\nabla m(z)\,dz,$$

and decompose

$$(9.68)\qquad J_5(t,y,y,v) - J_5(t,y,y,m) = \{\tilde{\varphi}_1(t)\}^{-1}A + \{\tilde{\varphi}_1(t)\}^{-1}B + C,$$

where

$$\begin{aligned}
A \equiv A(t,y,v) &= \int_{\mathbf{R}}\{p_{0,t}(y,z;v)^2 - p_{0,t}(y,z;m)^2\}\tilde{F}'''(m(z))\nabla m(z)\,dz,\\
B \equiv B(t,y,v) &= \int_{\mathbf{R}} p_{0,t}(y,z;v)^2\bar{B}(t,z,v)\,dz,\\
\bar{B}(t,z,v) &= \tilde{F}'''(v_t(z))\nabla m_{\eta_t}(z) - \tilde{F}'''(m(z))\nabla m(z)\\
C \equiv C(t,y,v) &= \left\{\frac{1}{\tilde{\varphi}_1(t)} - \frac{1}{\langle m, \nabla^2 m\rangle}\right\}\bar{C}(t,y),\\
\bar{C}(t,y) &= \int_{\mathbf{R}} p_{0,t}(y,z;m)^2\tilde{F}'''(m(z))\nabla m(z)\,dz.
\end{aligned}$$

Estimate on B: Using Lemma 9.14, we have

$$\begin{aligned}
||B(t,\cdot,v)||_{L^1_\lambda} &\le \int_{\mathbf{R}} ||p_{0,t}(\cdot,z;v)||^2_{L^2_{\lambda/2}}|\bar{B}(t,z,v)|\,dz\\
&\le C\left\{e^{\bar{c}\lambda t/2} + \frac{1}{\sqrt{t}}\right\}||\bar{B}(t,\cdot,v)||_{L^1_\lambda}, \quad 0<\lambda<\bar{\lambda}.
\end{aligned}$$

However, for sufficiently small $\lambda' > \lambda$,

$$||\bar{B}(t,\cdot,v)||_{L^1_\lambda} \le C_1||\bar{B}(t,\cdot,v)||_{L^2_{\lambda'}} \le C_2\sqrt{\mathrm{dist}(v,M)}e^{-ct},$$

use Corollary 9.1-(iii) and (9.15) to show the second inequality. Therefore, we obtain for sufficiently small $\lambda > 0$

$$(9.69)\qquad ||B(t,\cdot,v)||_{L^1_\lambda} \le C_3\sqrt{\mathrm{dist}(v,M)}e^{-c't}\left\{1 + \frac{1}{\sqrt{t}}\right\}.$$

Estimate on C: We prepare the next lemma. A much weaker estimate is sufficient here; however, this lemma will become necessary later again for the proof of (9.58).

Lemma 9.16.

$$||\bar{C}(t,\cdot)||_{L^1_\lambda} \le Ce^{-ct}\left\{1+\frac{1}{\sqrt{t}}\right\},$$

for sufficiently small $\lambda > 0$.

Proof. First we assume $t \ge 1$. Then,

$$\begin{aligned} p_{0,t}(y,z;m) &= p_{0,t}(z,y;m) = e^{-(t-1)\mathcal{A}}\{g(\cdot,y)\}(z) \\ &= \langle g(\cdot,y),\phi\rangle_{L^2}\phi(z) + e^{-(t-1)\mathcal{A}}\{\tilde{g}(\cdot,y)\}(z), \end{aligned}$$

where $\phi = \nabla m/||\nabla m||_{L^2}$, $g(z,y) = p_{0,1}(z,y;m)$ and $\tilde{g}(z,y) = g(z,y) - \langle g(\cdot,y),\phi\rangle_{L^2}\phi(z)$. Therefore, noting that $\int_{\mathbf{R}}\phi^2(z)\tilde{F}'''(m(z))\nabla m(z)\,dz = 0$ (m is odd from the assumption (1.2)-(b)), we have

$$\begin{aligned} |\bar{C}(t,y)| &= \Big| 2\langle g(\cdot,y),\phi\rangle_{L^2}\langle e^{-(t-1)\mathcal{A}}\{\tilde{g}(\cdot,y)\},\phi\tilde{F}'''(m)\nabla m\rangle_{L^2} \\ &\quad + \int_{\mathbf{R}}\left[e^{-(t-1)\mathcal{A}}\{\tilde{g}(\cdot,y)\}(z)\right]^2\tilde{F}'''(m(z))\nabla m(z)\,dz\Big| \\ &\le C_1\left[||e^{-(t-1)\mathcal{A}}\{\tilde{g}(\cdot,y)\}||_{L^2_{-\lambda'}} + ||e^{-(t-1)\mathcal{A}}\{\tilde{g}(\cdot,y)\}||^2_{L^2_{-\lambda'}}\right] \end{aligned}$$

for every sufficiently small $\lambda' > 0$. However, from Lemma 9.6-(iii), we see

$$\begin{aligned} ||e^{-(t-1)\mathcal{A}}\{\tilde{g}(\cdot,y)\}||_{L^2_{-\lambda'}} &\le e^{-c(t-1)}||\tilde{g}(\cdot,y)||_{L^2_{-\lambda'}} \\ &\le C_2e^{-ct}||g(\cdot,y)||_{L^2_{-\lambda'}} \le C_3e^{-ct}e^{-\lambda'\theta(y)}. \end{aligned}$$

For the last inequality, we have used the Gaussian bound; see Lemma 9.3. Now the desired bound is shown when $t \ge 1$. When $0 < t \le 1$, similarly as deriving the estimate on $||B(t,\cdot,v)||_{L^1_\lambda}$, we have for sufficiently small $\lambda > 0$

$$\begin{aligned} ||\bar{C}(t,\cdot)||_{L^1_\lambda} &\le C\left\{e^{\bar{c}\lambda t/2} + \frac{1}{\sqrt{t}}\right\}||\tilde{F}'''(m(\cdot))\nabla m(\cdot)||_{L^1_\lambda} \\ &\le C_4\left\{1+\frac{1}{\sqrt{t}}\right\}. \quad \square \end{aligned}$$

From Lemma 9.16 and (9.49), we have for sufficiently small $\lambda > 0$

$$||C(t,\cdot,v)||_{L^1_\lambda} \le C\sqrt{\mathrm{dist}(v,M)}e^{-ct}\left\{1+\frac{1}{\sqrt{t}}\right\}. \tag{9.70}$$

Estimate on A: We shall prove

$$||A(t,\cdot,v)||_{L^1_\lambda} \le C\sqrt{\mathrm{dist}(v,M)}e^{-ct}, \quad t \ge 0, \tag{9.71}$$

for sufficiently small $\lambda > 0$. To this end, we prepare

Lemma 9.17.

$$p_{s,t}(y,z) = \bar{p}_{s,t}(y,z) + \int_s^t dr \int_{\mathbf{R}} \bar{p}_{r,t}(z',z)\tilde{c}_r(z')p_{s,r}(y,z')\,dz',$$

where $p_{s,t}(y,z) = p_{s,t}(y,z;v)$ *and* $\bar{p}_{s,t}(y,z) = p_{s,t}(y,z;m)$.

Proof. This is just the kernel representation of the equation (9.21). □

From this lemma, we have a decomposition:

$$A(t,y,v) = A_1 + A_2, \tag{9.72}$$

where

$$A_1 \equiv A_1(t,y,v) = 2\int_0^t dr \int_{\mathbf{R}} \tilde{c}_r(z')p_{0,r}(y,z')\mathcal{A}_1(r,t,y,z')\,dz',$$

$$\mathcal{A}_1(r,t,y,z') = \int_{\mathbf{R}} \bar{p}_{0,t}(y,z)\bar{p}_{r,t}(z',z)\tilde{F}'''(m(z))\nabla m(z)\,dz$$

$$A_2 \equiv A_2(t,y,v) = \int_0^t dr_1 \int_0^t dr_2 \int_{\mathbf{R}^2} \tilde{c}_{r_1}(z_1')p_{0,r_1}(y,z_1')$$
$$\times \tilde{c}_{r_2}(z_2')p_{0,r_2}(y,z_2')\mathcal{A}_2(r_1,r_2,t,z_1',z_2')\,dz_1'dz_2',$$

$$\mathcal{A}_2(r_1,r_2,t,z_1',z_2') = \int_{\mathbf{R}} \bar{p}_{r_1,t}(z_1',z)\bar{p}_{r_2,t}(z_2',z)\tilde{F}'''(m(z))\nabla m(z)\,dz$$

Estimate on A_1: Notice that

$$\mathcal{A}_1(r,t,y,z') = e^{-t\mathcal{A}}\{\bar{\mathcal{A}}_1(r,t,\cdot,z')\}(y)$$

where

$$\bar{\mathcal{A}}_1(r,t,z,z') = \bar{p}_{r,t}(z',z)\tilde{F}'''(m(z))\nabla m(z).$$

Therefore, from Lemma 9.6-(iii),

$$\|\mathcal{A}_1(r,t,\cdot,z')\|_{L^2_\lambda(\mathbf{R}_y)} \le \|\langle\bar{\mathcal{A}}_1(r,t,\cdot,z'),\phi\rangle_{L^2}\phi\|_{L^2_\lambda}$$
$$+e^{-ct}\|\bar{\mathcal{A}}_1(r,t,\cdot,z')\|_{L^2_\lambda(\mathbf{R}_z)},$$

and noting that $\langle\bar{\mathcal{A}}_1(r,t,\cdot,z'),\phi\rangle_{L^2} = e^{-(t-r)\mathcal{A}}\{\tilde{F}'''(m)\nabla m\cdot\phi\}(z')$, we have

$$\|\mathcal{A}_1(r,t,\cdot,\cdot)\|^2_{L^2_\lambda\otimes L^2_\lambda} \le 2\|\phi\|^2_{L^2_\lambda}\|e^{-(t-r)\mathcal{A}}\{\tilde{F}'''(m)\nabla m\cdot\phi\}\|^2_{L^2_\lambda}$$
$$+2e^{-2ct}\|\bar{\mathcal{A}}_1(r,t,\cdot,\cdot)\|^2_{L^2_\lambda\otimes L^2_\lambda}, \tag{9.73}$$

where

$$\|\mathcal{A}_1\|^2_{L^2_\lambda\otimes L^2_\lambda} = \int_{\mathbf{R}^2} \mathcal{A}_1(y,z)^2 e^{2\lambda\theta(y)+2\lambda\theta(z)}\,dydz.$$

Here, since $\langle\tilde{F}'''(m)\nabla m\cdot\phi,\phi\rangle = 0$, again from Lemma 9.6-(iii), we have

$$\|e^{-(t-r)\mathcal{A}}\{\tilde{F}'''(m)\nabla m\cdot\phi\}\|_{L^2_\lambda}$$
$$\le e^{-c(t-r)}\|\tilde{F}'''(m)\nabla m\cdot\phi\|_{L^2_\lambda} \le C_1 e^{-c(t-r)}, \tag{9.74}$$

while, using Lemma 9.14 (one can replace $p_{0,t}$ by $p_{r,t}$ there),

$$(9.75)\quad \|\bar{A}_1(r,t,\cdot,\cdot)\|^2_{L^2_\lambda\otimes L^2_\lambda} = \left\| \tilde{F}'''(m(z))\nabla m(z)\|p_{r,t}(\cdot,z;m)\|_{L^2_\lambda}\right\|^2_{L^2_\lambda(\mathbf{R}_z)}$$
$$\le C\left\{e^{\bar{c}\lambda(t-r)} + \frac{1}{\sqrt{t-r}}\right\}\|\tilde{F}'''(m)\nabla m\|^2_{L^2_{2\lambda}}$$
$$\le C_2\left\{e^{\bar{c}\lambda(t-r)} + \frac{1}{\sqrt{t-r}}\right\}.$$

From (9.73)–(9.75), we obtain

$$(9.76)\quad \|A_1(r,t,\cdot,\cdot)\|_{L^2_\lambda\otimes L^2_\lambda} \le C_3\left[e^{-c(t-r)} + e^{-ct}\left\{e^{\bar{c}\lambda(t-r)/2} + (t-r)^{-\frac{1}{4}}\right\}\right].$$

To complete the estimate on $A_1(t,y,v)$, we also need the following: Using Lemma 9.14 with $\lambda = 0$ and (9.29),

$$(9.77)\quad \|\tilde{c}_r(z')p_{0,r}(y,z')\|_{L^2(\mathbf{R}_y)\otimes L^2(\mathbf{R}_{z'})} \le C^{1/2}\left\{1 + \frac{1}{\sqrt{r}}\right\}^{1/2}\|\tilde{c}_r\|_{L^2}$$
$$\le C_4\sqrt{\operatorname{dist}(v,M)}e^{-cr}\left\{1 + r^{-\frac{1}{4}}\right\}.$$

Then, (9.76) and (9.77) can be combined to get

$$(9.78)\quad \|A_1(t,\cdot,v)\|_{L^1_\lambda}$$
$$\le 2\int_0^t \|\tilde{c}_r(z')p_{0,r}(y,z')\|_{L^2(\mathbf{R}_y)\otimes L^2(\mathbf{R}_{z'})}\|A_1(r,t,\cdot,\cdot)\|_{L^2_\lambda(\mathbf{R}_y)\otimes L^2(\mathbf{R}_{z'})}\,dr$$
$$\le C_5\sqrt{\operatorname{dist}(v,M)}e^{-c't}.$$

Estimate on A_2: First, in a quite similar manner to deriving (9.76), we can prove

$$(9.79)\quad \|A_2(r_1,r_2,t,\cdot,\cdot)\|_{L^2_\lambda\otimes L^2_\lambda}$$
$$\le C_6\left[e^{-c(t-r_2)} + e^{-c(t-r_1)}\left\{e^{\bar{c}\lambda(t-r_2)/2} + (t-r_2)^{-\frac{1}{4}}\right\}\right].$$

On the other hand, using Lemma 9.14 and (9.29),

$$(9.80)\quad \|\tilde{c}_{r_1}(z_1')p_{0,r_1}(y,z_1')\tilde{c}_{r_2}(z_2')p_{0,r_2}(y,z_2')\|_{L^1_\lambda(\mathbf{R}_y)\otimes L^2_{-\lambda}(\mathbf{R}_{z_1'})\otimes L^2_{-\lambda}(\mathbf{R}_{z_2'})}$$
$$\le \prod_{i=1}^2 \|\tilde{c}_{r_i}(z)p_{0,r_i}(y,z)\|_{L^2_{\lambda/2}(\mathbf{R}_y)\otimes L^2_{-\lambda}(\mathbf{R}_z)}$$
$$\le \prod_{i=1}^2\left[\|\tilde{c}_{r_i}\|_{L^2}\sup_z\left\{\|p_{0,r_i}(\cdot,z)\|_{L^2_{\lambda/2}}e^{-\lambda\theta(z)}\right\}\right]$$
$$\le C_7\operatorname{dist}(v,M)\prod_{i=1}^2 e^{-cr_i}\{1 + r_i^{-\frac{1}{4}}\},$$

for sufficiently small $\lambda > 0$. Therefore, from (9.79) and (9.80), we have

$$\begin{aligned}
&||A_2(t,\cdot,v)||_{L^1_\lambda} \\
&\le \int_0^t \int_0^t ||\tilde{c}_{r_1}(z_1')p_{0,r_1}(y,z_1')\tilde{c}_{r_2}(z_2')p_{0,r_2}(y,z_2')||_{L^1_\lambda(\mathbf{R}_y)\otimes L^2_{-\lambda}(\mathbf{R}_{z_1'})\otimes L^2_{-\lambda}(\mathbf{R}_{z_2'})} \\
&\qquad \times ||A_2(r_1,r_2,t,\cdot,\cdot)||_{L^2_\lambda\otimes L^2_\lambda}\, dr_1 dr_2 \\
(9.81)\quad &\le C_8\sqrt{\mathrm{dist}(v,M)}e^{-c't}.
\end{aligned}$$

Two estimates (9.78) and (9.81) complete the proof of the desired estimate (9.71) on $A(t,y,v)$ and consequently, from (9.69)-(9.71), we obtain (9.59;5) and therefore (9.57). □

Proof of (9.58). Noting that $J_i(t,y,y,m) = 0$ for $i = 1,2,3$, from Lemma 9.10, (9.62) and (9.67), we see that

$$(9.82)\quad D\Xi(t,y,y,m) = \frac{1}{||\nabla m||_{L^2}^2}\int_{\mathbf{R}} p_{0,t}(y,z;m)^2\tilde{F}'''(m(z))\nabla m(z)\,dz + \Phi(t,y)D\eta(y,m).$$

Using Lemma 9.16 for the first term and (9.53) and $||D\eta(\cdot,m)||_{L^2_\lambda} < \infty$ with small $\lambda > 0$ for the second, (9.58) is proved. □

9.7.4. Proof of Theorem 7.3; completion

Only task what is left before completing the proof of Theorem 7.3 is to show the equality (7.9). By the shift-invariance (cf. Lemma 7.1-(ii)), we may suppose $\eta = 0$. Noting that

$$I_4(y,y,m) = \int_0^\infty D\Xi(t,y,y,m)\,dt,$$

from (9.51), (9.55), (9.56), (9.82) and (9.42), we obtain

$$(9.83)\quad D^2\zeta(y,y,m) = \frac{1}{||\nabla m||_{L^2}^2}\int_0^\infty dt\int_{\mathbf{R}} p_{0,t}(y,z;m)^2\tilde{F}'''(m(z))\nabla m(z)\,dz + 2\int_0^\infty \Phi(t,y)\,dt\cdot D\eta(y,m) + D^2\eta(y,y,m).$$

However, since $\langle\nabla^2 m, \nabla m\rangle = 0$ (recall that m is odd), we have

$$\int_0^\infty \Phi(t,y)\,dt = \frac{1}{||\nabla m||_{L^2}^2}\int_0^\infty \frac{d}{dt}e^{-t\mathcal{A}}\nabla^2 m(y)\,dt = -\frac{\nabla^2 m(y)}{||\nabla m||_{L^2}^2}.$$

Therefore, from Lemma 9.5, the sum of the second and the third terms in the right hand side of (9.83) cancels and we obtain (7.9); note that $\tilde{F}(v) = F(v)$ for $|v| \le 1$. □

9.8 Proof of Theorem 7.4

Let $H^\alpha \equiv H^\alpha(\mathbf{R}), \alpha \geq 0$, be the Sobolev spaces equipped with the norms $||\cdot||_{H^\alpha}$ defined by (3.1) for $\alpha = n \in \mathbb{Z}_+$ and by interpolation for general α. The proof of Theorem 7.4 can be completed with the help of Proposition 9.3 stated below. In this paragraph, we always mean by $v' \to v$ that $v', v \in \mathscr{V}_{\beta_2}$ and $||v' - v||_{L^2} \to 0$.

Proposition 9.3. *(i) For every $v \in \mathscr{V}_{\beta_2}$, we have $D\zeta(\cdot, v) \in \cap_{\delta>0} H^{2-\delta}$ and*

$$\lim_{v' \to v} ||D\zeta(\cdot, v') - D\zeta(\cdot, v)||_{H^{2-\delta}} = 0, \quad \delta > 0.$$

(ii) Let $v_t = v_t(\cdot; v_0)$ be the solution of the PDE (7.2) and suppose $v_0 \in H^\delta + m, 0 < \delta < 2$, for its initial data. Then,

$$\lim_{t \downarrow 0} ||v_t - v_0||_{H^\delta} = 0.$$

Since $\zeta = \zeta(v)$ is constant along the classical flow v_t, we have

$$0 = \frac{d}{dt}\zeta(v_t) = \langle D\zeta(\cdot, v_t), \Delta v_t - \tilde{F}'(v_t)\rangle, \quad t > 0. \tag{9.84}$$

However, assuming that Proposition 9.3 is already proved, we see

$$\lim_{t \downarrow 0}\langle D\zeta(\cdot, v_t), \Delta v_t\rangle = \langle D\zeta(\cdot, v_0), \Delta v_0\rangle,$$

if $v_0 \in \mathscr{V}_{\beta_2} \cap (H^\delta + m)$. Furthermore, since $\lim_{t\downarrow 0} ||\tilde{F}'(v_t) - \tilde{F}'(v_0)||_{L^2} = 0$, we have

$$\lim_{t \downarrow 0}\langle D\zeta(\cdot, v_t), \tilde{F}'(v_t)\rangle = \langle D\zeta(\cdot, v_0), \tilde{F}'(v_0)\rangle.$$

Hence, Theorem 7.4 is shown by letting $t \downarrow 0$ in (9.84).

Let us give the proof of Proposition 9.3-(i). We prepare some lemmas.

Lemma 9.18.
(i) $\lim_{v' \to v} \sup_{0 \leq t \leq T} ||v_t(\cdot; v') - v_t(\cdot; v)||_{L^2} = 0, \quad T \geq 0,$
(ii) $\lim_{v' \to v} \eta_t(v') = \eta_t(v), \quad t \geq 0,$
(iii) $\lim_{v' \to v} ||D\eta(\cdot, v') - D\eta(\cdot, v)||_{H^2} = 0.$

Proof. The assertion (i) is a consequence of Lemma 9.1-(i), while the assertion (ii) is shown from

$$|\eta_t(v') - \eta_t(v)| = \left| \int_0^1 \langle D\eta_t(\cdot, av' + (1-a)v), v' - v\rangle \, da \right| \leq C_1 ||v' - v||_{L^2}.$$

We have used $\sup_{v \in \mathscr{V}_{\beta_2}} ||D\eta_t(\cdot, v)||_{L^2} < \infty$ which follows from (9.35). Finally, (iii) is a consequence of (9.11); notice $\lim_{v' \to v} \eta(v') = \eta(v)$ as a special case of (ii) with $t = 0$. □

Corollary 9.3.
(i) $\lim_{v'\to v} \varphi_k(t,v') = \varphi_k(t,v), \quad t \ge 0,$
(ii) $\lim_{v'\to v} \tilde\varphi_k(t,v') = \tilde\varphi_k(t,v), \quad t \ge 0,$
(iii) $\lim_{v'\to v} \Lambda(t,v') = \Lambda(t,v), \quad t \ge 0.$

Proof. The conclusion follows from Lemma 9.18 by recalling the definition (9.13) of $\varphi_k, \tilde\varphi_k$ and (9.14) of $\Lambda(t)$. □

In particular, from Corollary 9.3-(iii), we obtain

$$\lim_{v'\to v} e^{\int_s^\infty \Lambda(r,v')dr} = e^{\int_s^\infty \Lambda(r,v)dr}, \quad s \ge 0. \tag{9.85}$$

We apply Lebesgue's dominated convergence theorem by noting the estimate on $|\Lambda(t,v)|$ given in Corollary 9.2. Therefore, from Lemma 9.18-(iii), we have

$$\lim_{v'\to v} \left\| e^{\int_0^\infty \Lambda(s,v')ds} D\eta(\cdot,v') - e^{\int_0^\infty \Lambda(s,v)ds} D\eta(\cdot,v) \right\|_{H^2} = 0. \tag{9.86}$$

Namely, the first term of $D\zeta(\cdot,v)$ given in (9.41) converges in a desired manner. Hence, for completing the proof of Proposition 9.3-(i), noting (9.85), it is sufficient to show the following two assertions:

$$\lim_{v'\to v} \|\Xi(t,\cdot,v') - \Xi(t,\cdot,v)\|_{H^{2-\delta}} = 0, \quad t > 0, \tag{9.87}$$

$$\|\Xi(t,\cdot,v)\|_{H^{2-\delta}} \le Ce^{-ct}\{1 + t^{-\frac{2-\delta}{2}}\}, \quad t > 0,\ v \in \mathscr{V}_{\beta_2}. \tag{9.88}$$

However, from (9.14) and the bounds in Proposition 9.1, the proof of (9.87) and (9.88) can be reduced to showing

$$\lim_{v'\to v} \|\psi_1(t,\cdot,v') - \psi_1(t,\cdot,v)\|_{H^{2-\delta}} = 0, \quad t > 0, \tag{9.89;1}$$

$$\lim_{v'\to v} \|\tilde\psi_1(t,\cdot,v') - \tilde\psi_1(t,\cdot,v)\|_{H^{2-\delta}} = 0, \quad t > 0, \tag{9.89;2}$$

respectively

$$\|\psi_1(t,\cdot,v)\|_{H^{2-\delta}} \le \bar Ce^{-ct}\{1 + t^{-\frac{2-\delta}{2}}\}, \tag{9.90;1}$$

$$\|\tilde\psi_1(t,\cdot,v)\|_{H^{2-\delta}} \le \bar C_\lambda e^{\lambda t}\{1 + t^{-\frac{2-\delta}{2}}\}, \quad t > 0, v \in \mathscr{V}_{\beta_2}, \tag{9.90;2}$$

for every $\lambda > 0$.

Lemma 9.19. *There exists $c > 0$ such that*

$$c^{-1}\|v\|_{H^\alpha} \le \|(1+\mathscr{A})^{\alpha/2}v\|_{L^2} \le c\|v\|_{H^\alpha}, \quad v \in H^\alpha,\ 0 \le \alpha \le 2.$$

Proof. The result is trivial for $\alpha = 0$ and can be shown for $\alpha = 2$ by direct calculation noting that $\tilde F''(m)$ is bounded. Therefore, by interpolation technique [9, p.115], we get the conclusion. □

We introduce weighted Sobolev spaces:

$$H_\lambda^\alpha = \{v; ve^{\lambda\theta} \in H^\alpha\}, \quad \alpha \geq 0, \ \lambda \in \mathbf{R},$$

equipped with norms $||v||_{H_\lambda^\alpha} = ||ve^{\lambda\theta}||_{H^\alpha}$. Notice that this norm is equivalent to $|v|_{H_\lambda^\alpha}$ which is defined by $|v|_{H_\lambda^\alpha}^2 = \sum_{k=0}^\alpha ||\nabla^k v||_{L_\lambda^2}^2$ for $\alpha \in \mathbb{Z}_+$ and by interpolation for general $\alpha \geq 0$; see Remark 2.1 of [33, p.502] though the sign of λ is converse there.

Lemma 9.20. *Assume* $0 < \alpha < 2$.
(i) For every $\mu > 0$, *there exists* $C = C_\mu > 0$ *such that*

$$||T_{0,t;v}||_{L^2 \to H^\alpha} \leq C\{1 + t^{-\alpha/2}\}e^{\mu t}, \quad t > 0, \ v \in \mathscr{V}_{\beta_2}^{\cdot}.$$

(ii) For every $\lambda, T > 0$, *there exists* $C = C_{\lambda,T} > 0$ *such that*

$$||T_{0,t;v}||_{L_\lambda^2 \to H_\lambda^\alpha} \leq Ct^{-\alpha/2}, \quad 0 < t \leq T, \ v \in \mathscr{V}_{\beta_2}^{\cdot}.$$

Proof. Let us first show two estimates

$$(9.91) \qquad ||e^{-t\mathscr{A}}||_{L^2 \to H^\alpha} \leq C_\alpha\{1 + t^{-\alpha/2}\}, \quad t > 0,$$

$$(9.92) \qquad ||e^{-t\mathscr{A}}||_{L_\lambda^2 \to H_\lambda^\alpha} \leq C_{\alpha,\lambda,T}t^{-\alpha/2}, \quad 0 < t \leq T.$$

In fact, noting Lemma 9.19, the left hand side of (9.91) is bounded by $\sup_{a \geq 0}(1 + a)^{\alpha/2}e^{-ta}$ and then, an elementary calculation shows that this quantity is bounded by the right hand side of (9.91). To prove (9.92), noting that the coefficient $\tilde{F}''(m)$ of the differential operator $\mathscr{A}$ is smooth, we see that the kernel function $p_{0,t}(y, z; m)$ of $e^{-t\mathscr{A}}$ has Gaussian bound with $C = C_T, c = c_T > 0$:

$$|\nabla_y^k p_{0,t}(y, z; m)| \leq Ct^{-\frac{k+1}{2}}e^{-\frac{(y-z)^2}{ct}}, \qquad 0 < t \leq T, y, z \in \mathbf{R}, k \in \mathbb{Z}_+,$$

see, e.g., [17]. Based on this estimate and using the bound (2.3) of [33, p.498], one can verify

$$(9.93) \qquad ||\nabla^k e^{-t\mathscr{A}}v||_{L_\lambda^2}^2 \leq C_1 t^{-k}||v||_{L_\lambda^2}^2.$$

This proves (9.92) for $\alpha \in \mathbb{Z}_+$ and therefore for general $\alpha \geq 0$ by interpolation.

Now, let us give the proof of (i). From (9.21) with $s = 0$, we obtain

$$(9.94) \qquad ||T_{0,t;v}||_{L^2 \to H^\alpha} \leq C_\alpha\{1 + t^{-\alpha/2}\} + C_2 \int_0^t e^{-cr}||T_{0,r;v}||_{L^2 \to H^\alpha}\, dr, \quad t > 0.$$

We have used (9.91), (9.24) with $\lambda = 0$ and $||e^{-(t-r)\mathscr{A}}||_{L^2 \to L^2} \leq 1$. This bound implies the assertion (i). Indeed, first prove it for $0 < t \leq t_0$ by noting that $t^{-\alpha/2}$ is integrable near $t = 0$ because of $\alpha < 2$; t_0 is defined by $C_2 e^{-ct_0} = \mu$, and then, prove for $t \geq t_0$. For the assertion (ii), we derive a similar inequality for $||T_{0,t;v}||_{L_\lambda^2 \to H_\lambda^\alpha}$ to (9.94) but only for $0 < t \leq T$ by using (9.92) and Lemma 9.6-(i). □

We need the following extension of Rellich's theorem.

Lemma 9.21. *The imbedding of the space H_λ^α into $H_{\lambda'}^{\alpha'}$ is compact if $\lambda > \lambda'$ and $\alpha > \alpha'$.*

Proof. The result might be already known so that we give only the sketch of the proof. Let $\tilde{\chi}_M \in C_0^\infty(\mathbf{R}), M > 0$, be a cut-off function satisfying $0 \le \tilde{\chi}_M \le 1, \tilde{\chi}_M(y) = 1$ for $|y| \le M$, $\tilde{\chi}_M(y) = 0$ for $|y| \ge M+1$ and $\sup_{M,y} |\nabla^k \tilde{\chi}_M(y)| < \infty$ for each $k \in \mathbb{Z}_+$. Then, we can prove

$$(9.95) \qquad ||\tilde{\chi}_M v||_{H_\lambda^\alpha} \le C||v||_{H_\lambda^\alpha},$$

with $C > 0$ independent of M and

$$(9.96) \qquad \lim_{M\to\infty} \sup\{||(1-\tilde{\chi}_M)v||_{H_{\lambda'}^\alpha}; \ ||v||_{H_\lambda^\alpha} \le 1\} = 0,$$

if $\lambda > \lambda'$. In fact, (9.95) is shown first for $\alpha = n \in \mathbb{Z}_+$ by direct calculation and then for all $\alpha \ge 0$ by interpolation. On the other hand, for showing (9.96), we notice that the norm $||g||_{H^\alpha}$ is equivalent to an another norm $|g|_{H^\alpha}$ defined by

$$(9.97) \qquad |g|_{H^\alpha}^2 = ||g||_{H^n}^2 + \int_{\mathbf{R}}\int_{\mathbf{R}} \frac{|\nabla^n g(y) - \nabla^n g(z)|^2}{|y-z|^{1+2\sigma}}\, dydz,$$

where $\alpha = n+\sigma, n \in \mathbb{Z}_+, 0 < \sigma < 1$; see Theorem 7.48 of [1, p.214]. Then, based on this norm and by direct computation, we can prove (9.96). Now, assume that a bounded sequence $\{v_n\}_{n=1}^\infty$ in the space H_λ^α is given. Then, one can find its subsequence converging in the space $H_{\lambda'}^{\alpha'}$, by using (9.95), (9.96) and applying Rellich's theorem which is valid for bounded domains; see, e.g., [1]. □

Let us return to the proof of (9.89) and (9.90). The two bounds (9.90;1) and (9.90;2) are shown in a similar manner to the proof of Proposition 9.1-(iii), the case of $k = 1$, and Proposition 9.1-(iv), respectively, based on Lemma 9.20-(i) with $\alpha = 2 - \delta$. Furthermore, based on Lemma 9.20-(ii), we obtain

$$(9.98) \qquad \sup_{v\in\mathscr{V}_{\beta_2}} ||\psi_1(t,\cdot,v)||_{H_\lambda^\alpha} < \infty \quad \text{and} \quad \sup_{v\in\mathscr{V}_{\beta_2}} ||\tilde{\psi}_1(t,\cdot,v)||_{H_\lambda^\alpha} < \infty$$

for every $t > 0, 0 < \alpha < 2$ with sufficiently small $\lambda > 0$; again we use the same argument as the proof of Proposition 9.1. Therefore, from Lemma 9.21, it is sufficient for completing the proof of (9.89) to show the weak convergence of $\psi_1(t,\cdot,v')$ and $\tilde{\psi}_1(t,\cdot,v')$ to $\psi_1(t,\cdot,v)$ and $\tilde{\psi}_1(t,\cdot,v)$, respectively, as $v' \to v$. To this end, let us take a test function $h \in C_0^\infty(\mathbf{R})$. Then, since

$$\langle h, \psi_1(t,\cdot,v)\rangle = \langle T_{0,t;v}^* h, \{\Delta - \tilde{F}''(v_t)\}\nabla m_{\eta_t}\rangle,$$
$$\langle h, \tilde{\psi}_1(t,\cdot,v)\rangle = \langle T_{0,t;v}^* h, \nabla^2 m_{\eta_t}\rangle,$$

noting Lemma 9.18-(i) and (ii), the conclusion is completed if we can prove

$$(9.99) \qquad u_t' := T_{0,t;v'}^* h \longrightarrow u_t := T_{0,t;v}^* h \quad \text{in } L^2 \quad \text{as } v' \to v.$$

However, since $\tilde{u}_t := u_t' - u_t$ is a solution of the PDE (9.8) with inhomogeneous term $r_t(\cdot) = \{\tilde{F}''(v_t(\cdot; v)) - \tilde{F}''(v_t(\cdot; v'))\}u_t'$ and initial data $\tilde{u}_0 = 0$. Therefore, we obtain (9.99) from Lemmas 9.2, 9.18-(i) and $\sup_{0 \le s \le t} \|u_s'\|_{L^\infty} < \infty$; use Gaussian bound (Lemma 9.3) by noting that h is bounded. This completes the proof of (9.89).

Finally, we prove Proposition 9.3-(ii). To this end, we rewrite the PDE (7.2) (recall $\tilde{f} = -\tilde{F}'$) in an equivalent integral equation:

$$v_t = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} \tilde{F}'(v_s)\, ds$$

from which we obtain

$$\|v_t - v_0\|_{H^\delta} \le \|e^{t\Delta} v_0 - v_0\|_{H^\delta} + \int_0^t \|e^{(t-s)\Delta}\|_{L^2 \to H^\delta} \|\tilde{F}'(v_s)\|_{L^2}\, ds. \tag{9.100}$$

However, we can prove similarly to (and even much simpler than) (9.92):

$$\|e^{t\Delta}\|_{L^2 \to H^\delta} \le C_T t^{-\delta/2}, \quad 0 < t \le T, \tag{9.101}$$

$$\|e^{t\Delta}\|_{H^\delta \to H^\delta} \le C_T, \quad 0 < t \le T. \tag{9.102}$$

Moreover, since $\|\tilde{F}''\|_{L^\infty} < \infty$ and $v_0 \in L^2 + m$,

$$\|\tilde{F}'(v_s)\|_{L^2} \le \|\tilde{F}'(v_s) - \tilde{F}'(v_0)\|_{L^2} + \|\tilde{F}'(v_0)\|_{L^2} \le C_2\{\|v_s - v_0\|_{L^2} + 1\}. \tag{9.103}$$

Inserting (9.101) and (9.103) into (9.100), since $\delta < 2$, we see

$$\|v_t - v_0\|_{H^\delta} \le \|e^{t\Delta} v_0 - v_0\|_{H^\delta} + C_3 + C_4 \int_0^t (t-s)^{-\delta/2} \|v_s - v_0\|_{H^\delta}\, ds.$$

Therefore, the conclusion of Proposition 9.3-(ii) follows if one can prove the strong continuity of $e^{t\Delta}$ on the space H^δ:

$$\lim_{t \downarrow 0} \|e^{t\Delta} v_0 - v_0\|_{H^\delta} = 0, \quad v_0 \in H^\delta. \tag{9.104}$$

However, we may assume $v_0 \in C_0^\infty(\mathbf{R})$ to show (9.104), since $C_0^\infty(\mathbf{R})$ is dense in H^δ and by noting (9.102). Then, (9.104) is easily shown. This completes the proof of Proposition 9.3-(ii).

Remark 9.7. Falconer [19] gave a general theory: If $v_t(\cdot; v)$ is C^2 in v and its second derivative is Lipschitz continuous, then its limit $m_{\zeta(v)}$ as $t \to \infty$ is C^2 in v. However, this general theory does not work effectively for our purpose.

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