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Probability Theory and Related Fields

A generalization of Lévy's concentration-variance inequality

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Summary. Sharp lower bounds are found for the concentration of a probability distribution as a function of the expectation of any given convex symmetric function ϕ. In the case $\phi(x)=(x-c)^2$, where c is the expected value of the distribution, these bounds yield the classical concentration-variance inequality of Lévy. An analogous sharp inequality is obtained in a similar linear search setting, where a sharp lower bound for the concentration is found as a function of the maximum probability swept out from a fixed starting point by a path of given length.

1. Introduction

For a (Borel) probability distribution P on the real line, and $d>0$, the *d-concentration of P* is

$$Q_d(P):=\sup\{P([x, x+d]): x\in\mathbb{R}\}.$$

This concept was introduced by P. Lévy [6], who proved that the concentration is decreased by convolution

$$Q_d(P_1 * P_2)\leq \min\{Q_d(P_1), Q_d(P_2)\}. \tag{1}$$

Let ϕ be a convex function which attains its minimum, and define the ϕ-moment of P

$$P(\phi):=\inf(\int \phi(x+y)\,\mathrm{d}P(y): x\in\mathbb{R}\}.$$

If ϕ is strictly convex, then a small d-concentration implies a large ϕ-moment, and a small ϕ-moment implies a large d-concentration. For example, if $\phi(x)=x^2$, then $P(\phi)$ is the variance of P, and clearly a small d-concentration implies a

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large variance, and a small variance implies a large d-concentration. In general, there exist largest functions $c(\cdot)$ and $m(\cdot)$ (depending on d and ϕ) such that

$$Q_d(P) \leqq c \Rightarrow P(\phi) \geqq m(c),$$

and

$$P(\phi) \leqq m \Rightarrow Q_d(P) \geqq c(m),$$

and it is clear that $c(\cdot)$ is the inverse of $m(\cdot)$.

It is easy to see what happens for $c=\frac{1}{n}$ $(n=2, 3, 4, \ldots)$. Let a_n be the ϕ-moment of the distribution

$$P_n := \frac{1}{n}\,\delta_{x^*} + \frac{1}{n}\,\delta_{x^*+d} + \ldots + \frac{1}{n}\,\delta_{x^*+(n-1)d}$$

(x^* is arbitrary; concentration and ϕ-moment are translation invariant).

Although P_n does not have d-concentration $\frac{1}{n}$, it is the weak limit of probability measures with d-concentration $<\frac{1}{n}$, and it is plausible that P_n represents the extremal case among all P with $Q_d(P) \leqq \frac{1}{n}$, that is, $m\left(\frac{1}{n}\right) \geqq a_n$. Similarly,

$$c \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \Leftrightarrow m \in [a_n, a_{n+1}),$$

and the question arises of how $c(\cdot)$ behaves in the interior of the intervals from $\frac{1}{n+1}$ to $\frac{1}{n}$, or, equivalently, how $m(\cdot)$ behaves in the interior of the intervals from a_n to a_{n+1}. In the present paper this question is answered for all convex functions ϕ which are symmetric about the origin, in which case $a_n(d, \phi)$ is easy to calculate:

$$(2n+1)\, a_{2n+1} = \phi(0) + 2[\phi(d) + \phi(2d) + \ldots + \phi(nd)];$$

and

$$2n \cdot a_{2n} = 2\left[\phi(\tfrac{1}{2}d) + \phi(\tfrac{3}{2}d) + \ldots + \phi\left(\frac{2n-1}{2}\, d\right)\right].$$

The following theorem gives a complete answer for this case (the case where ϕ is symmetric about a point other than the origin is easy to do by translation, but the case where ϕ is not symmetric does not seem to yield as simple and clean a solution).

Throughout this note $\mathscr{P}$ denotes the set of Borel probability measures on the real line.

Theorem 1. *Let $P \in \mathscr{P}$, and $\phi: \mathbb{R} \to \mathbb{R}$ be convex, and symmetric (about 0). Then*

$$Q_d(P) \geqq \frac{a_n - P(\phi)}{n(n+1)(a_{n+1} - a_n)} + \frac{1}{n} \quad \text{for } a_n \leqq P(\phi) < a_{n+1}, \tag{2}$$

and these bounds are sharp.

For $P \in \mathscr{P}$, let σ_P^2 denote the variance of P; the next corollary is Lévy's concentration-variance inequality.

Corollary 2 (Lévy (1937) – see [5], p. 27). *Suppose* $0 < Q_d(P) \leqq 1$. *Then*

$$\sigma_P^2 \geqq \frac{d^2}{12} n(n+1)(3 - Q_d(P)(2n+1)),$$

where $n = \max\{j \in \mathbb{N}: j < (Q_d(P))^{-1}\}$, *and this bound is sharp.*

The equivalent formulation of Corollary 2 in terms of a sharp lower bound for Q in terms of σ^2 is as follows.

Corollary 2'. *Suppose* $0 < \sigma_P^2 < \infty$. *Then*

$$Q_d(P) \geqq \frac{3}{2n+1}\left(1 - \frac{4}{n(n+1)} \frac{\sigma_P^2}{d^2}\right),$$

where n is the positive integer satisfying $\frac{d^2}{12}(n^2-1) < \sigma_P^2 \leqq \frac{d^2}{12}(n^2+2n)$, *and this inequality is sharp.*

Proof of Corollary 2. If P has infinite first absolute moment, or variance, the conclusion is trivial, so assume $\int x \, dP(x) = 0$, and apply Theorem 1 with $\phi = x^2$. Converting (2) to the corresponding lower bound for σ_P^2 in terms of $Q_d(P)$ completes the proof. □

It should be observed that there is no requirement for the convex function in Theorem 1 to be centered about the expectation of P, which was the case in the last corollary. The next corollary is the corresponding first-absolute-moment lower bound for the concentration of P; the equivalent lower bound for the first-absolute-moment in terms of the concentration is left to the interested reader.

Corollary 3. *Suppose* $\mu = \int |x| P(dx) < \infty$. *Then*

$$Q_d(P) \geqq \frac{1}{n} - \frac{\mu}{dn^2},$$

where n is the positive integer satisfying $\frac{d(n-1)n}{2n-1} \leqq \mu < \frac{d(n+1)n}{2n+1}$, *and this bound is sharp.*

Proof. Apply Theorem 1 with $\phi(x) = |x|$, calculate the a_n's, and simplify. □

2. Proof of Theorem 1

Fix $d > 0$, and let ϕ be convex and symmetric about zero. It follows easily from the convexity and symmetry of ϕ that the $\{a_n\}$ are non-decreasing with $a_1 = 0$. If $\phi \equiv 0$ then $a_n = 0$ for all n and the conclusion is trivial, so assume without loss of generality that ϕ is not identically zero, in which case $a_n \to \infty$ as $n \to \infty$ and there is an interval $[a_n, a_{n+1})$ containing every non-negative real number.

It is easy to see that a sharp lower bound for $Q_d(P)$ in terms of $P(\phi)$ is equivalent to a sharp lower bound for $P(\phi)$ in terms of $Q_d(P)$, and it is in this latter framework that the proof will be given.

For each $\lambda \in (0, 1]$, define the symmetric purely atomic probability measure $P_{\lambda,d} \in \mathscr{P}$ as follows. Let n be the largest nonnegative integer such that $n\lambda < 1$, define probability weights $\{\beta_k\}_{k=0}^{+2n}$ by

$$\beta_k = \begin{cases} 1 - n\lambda & k \text{ even} \\ (n+1)\lambda - 1 & k \text{ odd}, \end{cases}$$

and define $P_{\lambda,d} = \sum_{k=0}^{2n} \beta_k \, \delta\left((k-n)\frac{d}{2}\right)$ where $\delta(x)$ is the measure with unit mass at x. (These are the same extremal distributions that Lévy found for the variance (e.g. [5], p. 27).)

It is easy to check that $P_{\lambda,d}(\phi)$ is strictly decreasing in λ, is linear on $((n+1)^{-1}, n^{-1})$ for each n, that $P_{n^{-1},d}(\phi) = a_n$ for all positive integers n, and hence that the inequality (2) is equivalent to the inequality

$$P(\phi) \geqq P_{\lambda,d}(\phi), \qquad \text{where } \lambda = Q_d(P). \tag{3}$$

The proof of (3) will parallel Lévy's proof for the variance bound (e.g. [5], p. 27), except that the symmetry of ϕ and convexity of Q_d will be exploited to avoid his maximization subproblem.

Fix P with $P(\phi) < \infty$ (otherwise the conclusion is trivial), and fix $\varepsilon > 0$. To establish (3), it suffices to show that $P(\phi) \geqq P_{\lambda,d}(\phi) - \varepsilon$.

Let $\mathscr{M} \subset \mathscr{P}$ be the set of all real Borel probability distributions $\hat{P}$ with $Q_d(\hat{P}) \leqq Q_d(P) =: \lambda$. Then there exists $P_1 \in \mathscr{M}$ such that P_1 is continuous with density which is strictly positive and continuous everywhere, and which satisfies $P(\phi) \geqq P_1(\phi) - \varepsilon$. (One way to see this is convolve P with a distribution with everywhere continuous density and sufficiently small tails, such as $c e^{-nx^2}(1+\phi(x))^{-1}$, and apply (1).) Replacing the density function $f_1(x)$ of P_1 by $(f_1(x) + f_1(-x))/2$, if necessary, it may be further assumed (by convexity of $Q_d(\cdot)$) that P_1 is symmetric.

Next, define $-\infty = r_0 \leqq r_1 \leqq \dots \leqq r_{2n+1} = +\infty$ by $\mu_1(r_0, r_1) = \beta_n = \beta_{-n}$, $\mu_1(r_1, r_2) = \beta_{n-1} = \beta_{-n+1}, \dots, \mu_1(r_{2n-1}, r_{2n}) = \beta_n = \beta_{-n}$, which is possible since P_1 is continuous.

Since $P_1(\phi) < \infty$ and ϕ is not identically zero, it follows easily from the convexity of ϕ that $\int |x| \, dP_1(x)$ is finite. Let $z_0, z_1, \dots, z_{2n}$ be the P_1-barycenters of the intervals $(r_0, r_1], [r_1, r_2], \dots, [r_{2n}, r_{2n+1})$ respectively, and define the symmetric purely atomic measure $P_2 \in \mathscr{M}$ by $P_2 = \sum_{k=0}^{2n} \beta_k \, \delta(z_k)$; and observe that by Jensen's inequality (conditional version), $P_1(\phi) \geqq P_2(\phi)$. The next key step is to show that $P_2 \in \mathscr{M}$, which follows as in Lévy's argument (e.g. [5], p. 28) by several changes of variables using the invertibility of the c.d.f. P_1.

It remains only to show that $P_2(\phi) \geqq P_{\lambda,d}(\phi)$. To see this, note first that $P_2 \in \mathscr{M}$ implies $z_{k+1} - z_{k-1} > d$ for $k = 1, 2, \dots, 2n-1$, since the masses at adjacent points z_k and z_{k-1} sum to λ for all $k = 1, \dots, 2n$. In particular, $z_{n+1} - z_{n-1} > d$, and since P_2 is symmetric about $z_n=0$, this implies $z_{n+1}>\frac{d}{2}$, and $z_{n-1}<-\frac{d}{2}$. Define $P_3\in\mathscr{P}$ by $P_3=\sum_{k=0}^{2n}\beta_k\,\delta(\hat{z}_k)$, where $\hat{z}_k=z_k$ if $k\neq n+1, n-1$, and $\hat{z}_{n+1}=\frac{d}{2}$, $\hat{z}_{n-1}=-\frac{d}{2}$. Since ϕ is symmetric and convex, moving these same masses closer to the origin decreases the integral, so $P_2(\phi)\geqq P_3(\phi)$. (Note that at this step, $P_3\notin\mathscr{M}$ in general, since there are now three atoms, namely those at $\hat{z}_{n-1}$, $\hat{z}_n$, and $\hat{z}_{n+1}$ whose sum is more than λ.) Next, observe that $z_n-z_{n-2}=z_{n+2}-z_n>d$, and define $P_4\in\mathscr{P}$ by $P_4=\sum_{k=0}^{2n}\beta_k\,\delta(\tilde{z}_k)$, where $\tilde{z}_k=\hat{z}_k$ if $k\neq n+2, n-2$, and $\tilde{z}_{n+2}=d$, $\tilde{z}_{n-2}=-d$. As before $P_3(\phi)\geqq P_4(\phi)$. Continuing in this manner, define $P_5, P_6, \ldots$ with $P_2(\phi)\geqq P_3(\phi)\geqq P_4(\phi)\geqq\ldots$, and observe that after n steps, $P_{n+2}=P_{\lambda,d}$, which completes the proof of (2).

Although $P_{\lambda,d}\notin\mathscr{M}$, taking distributions $\sum_{k=0}^{2n}\beta_k\,\delta(y_k)$ with y_k arbitrarily close to the extremal case $(k-n)\frac{d}{2}$, and with $y_{k+1}-y_k>\frac{d}{2}$, shows the bound is sharp. □

Theorem 1 allows *any* moment information concerning a probability distribution to be translated into a lower bound for the concentration; application of Corollary 3 yields the following fact.

Example. Every probability distribution with first-absolute-moment π (or less) places mass at least $\frac{6-\pi}{36}$ on some closed interval of length 1, and this bound is sharp. Conversely, any distribution with 1-concentration $\frac{6-\pi}{36}$ (or less) has a first-absolute-moment at least π.

3. An analogous inequality for fixed starting point and total variation

Another way of viewing $Q_d(P)$, more in the spirit of a linear search problem (e.g., Beck [1]), is this. Suppose an object is placed on the real line according to the distribution P, and a searcher is allowed to choose any starting point on the line he wishes and then move not more than d units from his starting point; $Q_d(P)$ then represents the best the searcher can do, i.e., the maximum P-probability a search of length d can "sweep out". Thus the above inequalities translate directly into inequalities relating optimal-search probabilities and, say, variance or other moments of the distribution.

Suppose now that the searcher may still move at most d units, but that his starting point is fixed at some real number s. What is a lower bound for the optimal-search probability under these circumstances? The purpose of this section is to derive an analog of Theorem 1 which answers this question.

Definition. For $d>0$ and $s\in\mathbb{R}$, $Q_{d,s}:\mathscr{P}\to[0,1]$ is the function given by

$$Q_{d,s}(P)=\sup\{P([x,y]):x\leqq s\leqq y\quad\text{and}\quad\min\{(y-x)+s-x,(y-x)+y-s\}\leqq d\}. \tag{4}$$

Intuitively, $Q_{d,s}(P)$ is the maximum P-probability a search can sweep out, given that it starts at s and may move no more than d units total (i.e., in total variation). Note that for all $P \in \mathscr{P}$,

$$\sup\{Q_{d,s}(P): s \in \mathbb{R}\} = Q_d(P).$$

The next lemma, which asserts the upper semicontinuity of $Q_{d,s}$, will be used in the proof of the main inequality in the fixed-starting-point setting.

Lemma 4. *For fixed $d>0$ and $s \in \mathbb{R}$, $Q_{d,s}$ is upper semicontinuous in P; that is, if $P_n \to P$ weakly (see Billingsley [2]) then* $\limsup_{n\to\infty} Q_{d,s}(P_n) \leqq Q_{d,s}(P)$.

Proof. Let $[x_n, y_n]$ be the endpoints of an optimal search of length d for P_n starting at s, i.e., $[x_n, y_n]$ is the closed interval attaining the supremum in (4) (with P replaced by P_n). (That this supremum is attained is routine; a more general result is in [4].) By taking subsequences if necessary it can be assumed that $Q_{d,s}(P_n) \to Q_{d,s} = \limsup_{n\to\infty} Q_{d,s}(P_n)$, $x_n \to x$, and $y_n \to y$, where $x \leqq s \leqq y$ and $\min\{(y-x)+s-x, (y-x)+y-s\} \leqq d$. (This follows easily from the tightness of the sequence $\{P_n\}$; see [2].)

Let $\delta_k \downarrow 0$, where $x-\delta_k$ and $y+\delta_k$ are in the continuity set of P. For every k,

$$\limsup_{n\to\infty} Q_{d,s}(P_n) = \lim_{n\to\infty} P_n([x_n, y_n]) \leqq \limsup_{n\to\infty} P_n([x-\delta_k, y+\delta_k])$$
$$= P([x-\delta_k, y+\delta_k]).$$

Taking limits on k yields

$$\limsup_{n\to\infty} Q_{d,s}(P_n) \leqq P([x, y]) \leqq Q_{d,s}(P),$$

which completes the proof. □

The next result is the main inequality of this section, the analog of Theorem 1 for $Q_{d,s}$ (recall that $P(\phi)$ is the expected value of ϕ with respect to P).

Theorem 5. *Let $\phi: \mathbb{R} \to \mathbb{R}$ be convex, and symmetric about s. Then*

$$Q_{d,s}(P) \geqq \begin{cases} \dfrac{2\phi\left(s+\frac{d}{3}\right) - \phi(s) - P(\phi)}{2\left[\phi\left(s+\frac{d}{3}\right) - \phi(s)\right]} & \text{if } \phi(s) < P(\phi) \leqq \phi\left(s+\frac{d}{3}\right) \\ \dfrac{\phi(s+d) - P(\phi)}{2\left[\phi(s+d) - \phi\left(s+\frac{d}{3}\right)\right]} & \text{if } \phi\left(s+\frac{d}{3}\right) < P(\phi) \leqq \phi(s+d) \\ 0 & \text{if } P(\phi) \geqq \phi(s+d) \end{cases} \qquad (5)$$

and these bounds are sharp.

Corollary 6. *Let* $m_1=\int|x-s|\,\mathrm{d}P$. *Then*

$$Q_{d,s}(P)\geqq\begin{cases}1-\dfrac{3}{2}\left(\dfrac{m_1}{d}\right) & \text{if } 0\leqq m_1\leqq\dfrac{d}{3}\\ \dfrac{3}{4}\left(1-\dfrac{m_1}{d}\right) & \text{if } \dfrac{d}{3}\leqq m_1\leqq d\\ 0 & \text{if } m_1\geqq d\end{cases}$$

Corollary 7. *Let* $m_2=\int(x-s)^2\,\mathrm{d}P$. *Then*

$$Q_{d,s}(P)\geqq\begin{cases}1-\dfrac{9}{2}\left(\dfrac{m_2}{d^2}\right) & \text{if } 0\leqq m_2\leqq\dfrac{d^2}{9}\\ \dfrac{9}{16}\left(1-\dfrac{m_2}{d^2}\right) & \text{if } \dfrac{d^2}{9}\leqq m_2\leqq d^2\\ 0 & \text{if } m_2\geqq d^2.\end{cases}$$

Proof of Theorem 5. It is easy to check that (5) is equivalent to

$$\inf_P\{P(\phi)\colon Q_{d,s}(P)\leqq\lambda\}=\begin{cases}2\lambda\phi\left(s+\dfrac{d}{3}\right)+(1-2\lambda)\,\phi(s+d), & 0\leqq\lambda<\frac{1}{2}\\ (2\lambda-1)\,\phi(s)+2(1-\lambda)\,\phi\left(s+\dfrac{d}{3}\right), & \frac{1}{2}\leqq\lambda\leqq 1\end{cases}. \tag{6}$$

By centering, rescaling and adding a constant, assume without loss of generality that $s=0$, $d=1$, and $\phi(0)=0$. First, make a discrete approximation to (6) and restrict P to the set $\mathscr{P}_k\subset\mathscr{P}$ of discrete distribution functions which place mass only at points $0, \pm\dfrac{1}{6k}, \pm\dfrac{2}{6k}, \ldots$. Thus, for each positive integer k, we have the problem

$$\min_{p\in\mathscr{P}_k}\{P(\phi)\colon Q_{d,s}(P)\leqq\lambda\}. \tag{7}$$

We will show that a probability mass function corresponding to an optimal $P\in\mathscr{P}_k$ is, for $0\leqq\lambda<\frac{1}{2}$,

$$p_i^*=\begin{cases}\dfrac{\lambda}{2} & \text{for } i=\pm 2k,\ \pm(2k+1)\\ \frac{1}{2}-\lambda & \text{for } i=\pm(6k+1)\\ 0 & \text{otherwise;}\end{cases} \tag{8}$$

and for $\frac{1}{2}\leqq\lambda\leqq 1$

$$p_i^*=\begin{cases}2\lambda-1 & \text{for } i=0\\ \dfrac{1-\lambda}{2} & \text{for } i=\pm 2k,\ \pm(2k+1)\\ 0 & \text{otherwise.}\end{cases} \tag{9}$$

To show that this is the optimal solution, we formulate (7) as a linear program (LP). Let $\phi_i=\phi\left(\frac{i}{6k}\right)$. Since $d=1$, the support of P can clearly be narrowed to $\left[-\left(1+\frac{1}{6k}\right), \left(1+\frac{1}{6k}\right)\right]$. The objective function of the LP is Min $\sum_{i=-6k}^{6k} \phi_i p_i + \left(1-\sum_{i=-6k}^{6k} p_i\right)\phi_{6k+1}$, or, equivalently,

$$\text{Max} \sum_{i=-6k}^{6k} (\phi_{6k+1}-\phi_i)\, p_i.$$

There are $(4k+1)$ constraints corresponding to $Q_{1,0}(P)\leqq\lambda$. Constraint -1 is

$$p_0+p_1+\ldots+p_{6k}\leqq\lambda$$

and corresponds to the only feasible search starting at 0 and ending at 1. Constraint -2 is

$$p_{-1}+p_0+p_1+\ldots+p_{6k-2}\leqq\lambda$$

and corresponds to turning at $\frac{-1}{6k}$. Similarly, constraint $-i$, $i=2, \ldots, 2k$, corresponds to turning at $\frac{-(i-1)}{6k}$. Let constraint $+i$, $i=1, \ldots, 2k$ be the constraint corresponding to reflecting the path in constraint $-i$. Thus, constraint $+i$, $i=2, \ldots, 2k$, corresponds to turning at $\frac{(i-1)}{6k}$. The last constraint $Q_{1,0}(P)\leqq\lambda$ is the symmetric case where the path turns at either $+\frac{1}{3}$ or $-\frac{1}{3}$, i.e.,

$$p_{-2k}+p_{-(2k-1)}+\ldots+p_{-1}+p_0+p_1+\ldots+p_{2k-1}+p_{2k}\leqq\lambda.$$

Label this constraint b. The remaining constraints ensure that P is a proper probability distribution. Constraint c is $p_{-6k}+\ldots+p_0+\ldots+p_{6k}\leqq 1$ and the non-negativity constraints are $p_i\geqq 0$, $i=0, \pm 1, \ldots, \pm 6k$. Thus, the LP has been formulated in the variables p_i, $i=0, \pm 1, \ldots, \pm 6k$, and any remaining probability mass is distributed arbitrarily between the points $\pm\left(1+\frac{1}{6k}\right)$.

To show that the solution to (7) is (8) when $0\leqq\lambda\leqq\frac{1}{2}$ and is (9) when $\frac{1}{2}<\lambda\leqq 1$, it suffices to find a feasible solution to the dual problem:

$$\min \sum_{j=1}^{2k} \lambda(y_j+y_{-j})+\lambda y_b+y_c$$

subject to

$$\begin{aligned}
&y_1+\dots+y_{2k}+y_{j-1}+y_{j-2}+\dots+y_{-2k}+y_b+y_c\geqq\phi_{6k+1}-\phi_j,\ -2k+1\leqq j\leqq 0\\
&y_{-1}+\dots+y_{-2k}+y_{j+1}+y_{j+2}+\dots+y_{2k}+y_b+y_c\geqq\phi_{6k+1}-\phi_j,\ 0<j\leqq 2k-1\\
&y_1+\dots+y_{2k}+y_b+y_c\geqq\phi_{6k+1}-\phi_{-2k}\\
&y_{-1}+\dots+y_{-2k}+y_b+y_c\geqq\phi_{6k+1}-\phi_{2k}\\
&y_1+\dots+y_{\left[\frac{j+6k+2}{2}\right]}+y_c\geqq\phi_{6k+1}-\phi_j,\\
&\qquad -6k\leqq j\leqq -2k-1\\
&y_{-1}+\dots+y_{-\left[\frac{-j+6k+2}{2}\right]}+y_c\geqq\phi_{6k+1}-\phi_j,\ 2k+1\leqq j\leqq 6k\\
&y_j\geqq 0,\ j=0,\ \pm 1,\ \dots,\ \pm 6k
\end{aligned}$$

which has the same value for its objective function (cf., Chvatal [3]). A solution y which works when $\lambda\leqq\frac{1}{2}$ is $y_{\pm1}=\phi_{6k+1}-\phi_{6k-1}$, $y_{\pm2}=\phi_{6k-1}-\phi_{6k-3},\dots$, $y_{\pm2k}=\phi_{2k+3}-\phi_{2k+1}$, $y_b=\phi_{2k+1}-\phi_{2k}$, $y_c=0$; and for $\lambda\geqq\frac{1}{2}$, $y_{\pm1}=\phi_1-\phi_0$, $y_{\pm2}=\phi_2-\phi_1,\dots$, $y_{\pm2k}=\phi_{2k}-\phi_{2k-1}$, $y_b=\phi_{2k+1}-\phi_{2k}$, $y_c=\phi_{6k+1}-\phi_{2k+1}-\phi_{2k}$, where y_j corresponds to constraint j, $j\in\{\pm1,\dots,\pm2k,b,c\}$.

So far it has been shown that the inequality of Theorem 5, which we write here as $Q_{d,s}(P)\geqq H(P(\phi))$, holds for P in the set $\mathscr{P}_k$. Now this is extended to arbitrary P. If $P(\phi)=\infty$ then the inequality holds trivially since ϕ is real valued; so assume that $P(\phi)<\infty$ and let $\varepsilon>0$. Since the function H is a uniformly continuous (piecewise linear) function, there is a $\delta>0$ such that $|a-b|<\delta$ implies $|H(b)-H(a)|<\varepsilon$, and since $P(\phi)<\infty$ there is an $A>d=1$ for which

$$\sum_{-A}^{A}\phi(x)\,\mathrm{d}P(x)>P(\phi)-\delta.$$

If $\hat{P}$ agrees with P on $(-A, A)$ and places mass $P([A,\infty))$ at A and $P(-\infty,-A])$ at $-A$, then $Q_{d,s}(\hat{P})=Q_{d,s}(P)$ and $P(\phi)\geqq\hat{P}(\phi)\geqq\int_{-A}^{A}\phi(x)\,\mathrm{d}P(x)$. Now let $P_k\in\mathscr{P}_k$ be chosen so that P_k converges weakly to $\hat{P}$, in the sense of Billingsley [2], as $k\to\infty$. Then from Lemma 4 and the continuity and boundedness of ϕ on $[-A, A]$,

$$Q_{d,s}(P)=Q_{d,s}(\hat{P})\geqq\limsup Q_{d,s}(P_k)\geqq\lim H(P_k(\phi))=H(\hat{P}(\phi))>H(P(\phi))-\varepsilon.$$

Since ε and P were arbitrary, this concludes the proof of the theorem. □

Next is an example of a concrete application of Corollary 7.

Example. If an object is placed randomly on the real line according to any probability distribution with second moment $+3$ (or less), then there is always a search of length (total variation) 2 beginning at the origin which will find the object with probability at least $\frac{9}{64}$, and this bound is sharp.

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