Probab. Th. Rel. Fields 86, 41–52 (1990)

Probability Theory and Related Fields

Heat flow and Brownian motion for a region in $\mathbb{R}^2$ with a polygonal boundary

M. van den Berg [1] **and S. Srisatkunarajah** [2]

[1] Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UK
[2] Department of Mathematics and Statistics, University of Jaffna, Thirunelvely, Jaffna, Sri Lanka

Received April 10, 1989; in revised form February 12, 1990

Summary. Consider the following heat conduction problem. Let D be an open, bounded and connected set in euclidean space $\mathbb{R}^2$ with a polygonal boundary. Suppose that D has temperature 1 at time $t=0$, while the boundary is kept at temperature 0 for all time $t>0$. We obtain the asymptotic behaviour for the amount of heat in D at time t up to $O(e^{-q/t})$ as $t \to 0$.

1. Introduction

Consider the following problem in heat conduction. Let K be a compact set in $\mathbb{R}^m$. Suppose that K is held at temperature 1 while $\mathbb{R}^m \backslash K$ is at temperature 0 at time 0. Let $E_K(t)$ represent the amount of heat flown from K into $\mathbb{R}^m \backslash K$ up to time t. Compute the asymptotic behaviour of $E_K(t)$ as $t \to \infty$. This problem has been studied in [5, 8, 9, 12].

The following problem has received little attention. Let D be an open, bounded and connected set in $\mathbb{R}^m$. Suppose that D has temperature 1 at time 0 while $\mathbb{R}^m \backslash D$ is held at temperature 0. Let $Q_D(t)$ represent the amount of heat in D at time t. Compute the asymptotic behaviour of $Q_D(t)$ as $t \to 0$.

Both E_K and Q_D have a probabilistic representation. Let $(B(t), t \geqq 0; \mathbb{P}_A, A \in \mathbb{R}^m)$ be a brownian motion associated to $-\Delta + \frac{\partial}{\partial t}$. Define for any open set $D \ni A$

$$T_D = \inf\{t \geqq 0 \colon B(t) \in \mathbb{R}^m \backslash D\}. \tag{1.1}$$

Then

$$E_K(t) = \int_{\mathbb{R}^m \backslash K} \mathbb{P}_A [T_{\mathbb{R}^m \backslash K} < t] \, \mathrm{d}A, \tag{1.2}$$

and

$$Q_D(t) = \int_D \mathbb{P}_A [T_D > t] \, \mathrm{d}A. \tag{1.3}$$

In this paper we will study the case where D is an open bounded and connected set in $\mathbb{R}^2$ with a polygonal boundary ∂D (resp. K is a connected compact set in $\mathbb{R}^2$ with a polygonal boundary ∂K). We will show that (Theorem 1) for some positive constant q (depending on D)

$$Q_D(t) = |D| - \frac{2t^{1/2}}{\pi^{1/2}} |\partial D| + t \sum_{i=1}^{n} c(\gamma_i) + O(e^{-q/t}), \qquad t \downarrow 0, \tag{1.4}$$

where $|D|$ is the area of D, $|\partial D|$ is the length of the boundary ∂D, $\gamma_1, \ldots, \gamma_n$ are the angles at the vertices $P_1, \ldots, P_n$, and $c: (0, 2\pi] \to \mathbb{R}$ is defined by

$$c(\beta) = \int_0^\infty \frac{4 \sinh((\pi - \beta) x)}{(\sinh(\pi x))(\cosh(\beta x))} \, dx. \tag{1.5}$$

For a compact set K in $\mathbb{R}^2$ with a polygonal boundary ∂K and exterior angles $\phi_1, \ldots, \phi_n$ we obtain

$$E_K(t) = \frac{2t^{1/2}}{\pi^{1/2}} |\partial K| - t \sum_{i=1}^{n} c(\phi_i) + O(e^{-r/t}), \qquad t \downarrow 0, \tag{1.6}$$

where r is some positive constant (depending on K).

Suppose D is an open, bounded and connected set in $\mathbb{R}^2$ with a smooth boundary ∂D. Then by approximating D with a polygon (in which most angles are $\pi \pm 2\pi/n$) we are led to conjecture that

$$Q_D(t) = |D| - \frac{2t^{1/2}}{\pi^{1/2}} |\partial D| + \pi t \, \chi(D) + o(t), \qquad t \downarrow 0, \tag{1.7}$$

where $\chi(D)$ is the Euler-Poincaré characteristic of D. Similarly (1.6) suggests that for a connected and compact set K in $\mathbb{R}^2$ with a smooth boundary ∂K

$$E_K(t) = \frac{2t^{1/2}}{\pi^{1/2}} |\partial K| + \pi t \, \chi(K) + o(t), \qquad t \downarrow 0. \tag{1.8}$$

However, the constants q and r are $O(n^{-2})$ as $n \uparrow \infty$ so that (1.7) and (1.8) do not follow from Theorem 1. Before we state Theorem 1 we introduce some further notation. Let W_i $(i = 1, \ldots, n)$ be the infinite open wedge of angle γ_i with vertex P_i such that the boundary of the wedge contains the two edges adjacent to P_i. Denote the distance of non-empty sets G and H in $\mathbb{R}^2$ by $d(G, H)$. Define for $y > 0$,

$$B_i(y) = \{A \in W_i : d(A, P_i) < y\}. \tag{1.9}$$

Furthermore let

$$\gamma = \min \gamma_i, \tag{1.10}$$

$$R = \tfrac{1}{2} \sup \left\{ y\colon B_i(y) \cap B_j(y) = \phi \text{ for all } i \neq j, \bigcup_{k=1}^{n} B_k(y) \subset D \right\}. \tag{1.11}$$

Theorem 1. *Let D be an open, bounded and connected set in* $\mathbb{R}^2$ *with a polygonal boundary* ∂D. *Then for all* $t > 0$

$$\left| Q_D(t) - |D| + \frac{2}{\pi^{1/2}} |\partial D| \, t^{1/2} - t \sum_{i=1}^{n} c(\gamma_i) \right|$$
$$\leqq (4|D| + 2|\partial D| \, t^{1/2}/\pi^{1/2} + 4\pi^3 n t/\gamma^2) \, e^{-(R \sin(\gamma/2))^2/(32t)}, \tag{1.12}$$

where γ, R *and* c *are as in* (1.10), (1.11) *and* (1.5).

Note that (1.12) is a bound which implies the asymptotic result (1.4). Furthermore note that (1.4) and (4a) in [10] are quantitatively of a similar type (also (1.7) and (4b) in [10]). For general regions we have no proof of such a relationship.

This paper is organized as follows. In Sect. 2 we compute

$$Q_\gamma(t; R) = \int_{\{A \in W : d(A,P) < R\}} \mathrm{d}A \, \mathbb{P}_A[T_W > t], \tag{1.13}$$

using the Kontorovich-Lebedev representation of the Green's function for an infinite open wedge W with angle γ and vertex P. Despite the fact that various expressions for $\mathbb{P}_A[T_W > t]$ are available [2, 7, 11] we are unable to extract the vertex contribution $t\,c(\gamma)$ from these. The Kontorovich-Lebedev representation of the Green function for a wedge has been used in [1] and [10] (p. 44). In Sect. 3 we obtain upper and lower bounds for $\mathbb{P}_A[T_D > t]$. In Sect. 4 we prove Theorem 1. The proof of (1.6) is along similar lines.

2. Computation of $Q_\gamma(t; R)$

In this section we prove the following.

Theorem 2. *For* $Q_\gamma(t; R)$ *as in* (1.13) *and* $M = 1, 2, \ldots$

$$Q_\gamma(t; R) = \frac{\gamma R^2}{2} - \frac{4 R t^{1/2}}{\pi^{1/2}} + c(\gamma)\, t + \frac{4 R t^{1/2}}{\pi^{1/2}} \int_1^\infty \frac{\mathrm{d}v}{v^2} \int_0^1 \frac{y \, \mathrm{d}y}{(1-y^2)^{1/2}} \, e^{-R^2 v^2 y^2/(4t)} - S(t), \tag{2.1}$$

where

$$S(t)=\frac{2}{\pi}\int_R^\infty a\,\mathrm{d}a\int_0^\infty \mathrm{d}u\,\mathrm{Erfc}\left(\frac{a\cosh u}{2t^{1/2}}\right)$$
$$\cdot\ln\left(\frac{\cosh(\pi u/\gamma)-\cos(\pi^2/(2\gamma))}{\cosh(\pi u/\gamma)+\cos(\pi^2/(2\gamma))}\right),\qquad \frac{\pi}{2}<\gamma\leqq 2\pi, \tag{2.2}$$

$$S(t)=4\sum_{n=1}^{M}(-1)^{n+1}\int_R^\infty a\,\mathrm{d}a\int_0^{(\pi-2n\gamma)/2}\mathrm{d}\alpha\,\mathrm{Erfc}\left(\frac{a\cos\alpha}{2t^{1/2}}\right)$$
$$+\frac{2}{\pi}\int_R^\infty a\,\mathrm{d}a\int_0^\infty \mathrm{d}u\,\mathrm{Erfc}\left(\frac{a\cosh u}{2t^{1/2}}\right)$$
$$\cdot\ln\left(\frac{\cosh(\pi u/\gamma)-\cos(\pi^2/(2\gamma))}{\cosh(\pi u/\gamma)+\cos(\pi^2/(2\gamma))}\right),\qquad \frac{\pi}{2M+2}<\gamma\;\frac{\pi}{2M}, \tag{2.3}$$

$$S(t)=4\sum_{n=1}^{M-1}(-1)^{n+1}\int_R^\infty a\,\mathrm{d}a\int_0^{(\pi-2n\gamma)/2}\mathrm{d}\alpha\,\mathrm{Erfc}\left(\frac{a\cos\alpha}{2t^{1/2}}\right)$$
$$+(-1)^{M+1}\frac{4}{\pi}\int_R^\infty a\,\mathrm{d}a\int_0^\infty \mathrm{d}u\,\mathrm{Erfc}\left(\frac{a\cosh u}{2t^{1/2}}\right)\ln\left(\coth\frac{\pi u}{2\gamma}\right),$$
$$\gamma=\frac{\pi}{2M},\ \sum_{n=1}^{M-1}\ldots=0\quad \text{for } M=1, \tag{2.4}$$

and

$$\mathrm{Erfc}(x)=2\pi^{-1/2}\int_x^\infty e^{-t^2}\,\mathrm{d}t. \tag{2.5}$$

Proof. Let $-\Delta_W$ be the Dirichlet laplacian for W. Let $p_W(A_1, A_2; t)$ denote the heat kernel associated to the parabolic operator $-\Delta_W+\frac{\partial}{\partial t}$ and define for $s>0$

$$G_W(A_1, A_2; s)=\int_0^\infty \mathrm{d}t\,e^{-st}\,p_W(A_1, A_2; t). \tag{2.6}$$

Then (see [10] p. 44)

$$G_W(A_1, A_2; s)=\frac{1}{\pi^2}\int_0^\infty \mathrm{d}x\,K_{ix}(\sqrt{s}\,a_1)\,K_{ix}(\sqrt{s}\,a_2)$$
$$\cdot\left\{\cosh((\pi-|\alpha_1-\alpha_2|)x)-\frac{\sinh(\pi x)}{\sinh(\gamma x)}\cosh((\gamma-\alpha_1-\alpha_2)x)\right.$$
$$\left.+\frac{\sinh((\pi-\gamma)x)}{\sinh(\gamma x)}\cosh((\alpha_1-\alpha_2)x)\right\}, \tag{2.7}$$

where $a_1 = d(P, A_1)$, $a_2 = d(P, A_2)$ and α_1 (resp. α_2) are the angles between $A_1 P$ (resp. $A_2 P$) and one of the edges of the wedge, and K is the modified Bessel function. By Fubini's theorem

$$\int_0^\gamma \mathrm{d}\alpha_1 \int_0^\gamma \mathrm{d}\alpha_2\, G_W(A_1, A_2; s) = \frac{2}{\pi^2} \int_0^\infty \frac{\mathrm{d}x}{x} K_{ix}(\sqrt{s}\, a_1)\, K_{ix}(\sqrt{s}\, a_2)$$
$$\cdot \left\{ \gamma \sinh(\pi x) - \frac{2}{x}\left(\tanh \frac{\gamma x}{2}\right)(\sinh(\pi x))\right\}, \tag{2.8}$$

and

$$\int_0^\gamma \mathrm{d}\alpha_1 \int_{A_2 \in W} \mathrm{d}A_2\, G_W(A_1, A_2; s)$$
$$= \frac{2}{\pi^2} \int_0^\infty \frac{\mathrm{d}x}{x} K_{ix}(\sqrt{s}\, a_1) \int_0^\infty a_2\, \mathrm{d}a_2\, K_{ix}(\sqrt{s}\, a_2)$$
$$\cdot \left\{ \gamma \sinh(\pi x) - \frac{2}{x}\left(\tanh \frac{\gamma x}{2}\right)(\sinh(\pi x))\right\}$$
$$= \frac{2}{\pi s} \int_0^\infty \mathrm{d}x\, K_{ix}(\sqrt{s}\, a_1) \left\{ \gamma \cosh \frac{\pi x}{2} - \frac{2}{x} \sinh \frac{\pi x}{2} + \frac{2}{x} \frac{\sinh((\pi-\gamma)\,x/2)}{\cosh(\gamma\, x/2)} \right\}$$

[6, formula 6.561.16]

$$= \frac{\gamma}{s} - \frac{4}{\pi s} \int_0^\infty \frac{\mathrm{d}x}{x} K_{ix}(\sqrt{s}\, a_1) \sinh \frac{\pi x}{2} + \frac{4}{\pi s} \int_0^\infty \frac{\mathrm{d}x}{x} K_{ix}(\sqrt{s}\, a_1) \frac{\sinh((\pi-\gamma)\,x/2)}{\cosh(\gamma\, x/2)}$$

[6, 6.794.2]. (2.9)

Moreover by Fubini's theorem

$$-\frac{4}{\pi s} \int_0^\infty \frac{\mathrm{d}x}{x} K_{ix}(\sqrt{s}\, a_1) \sinh \frac{\pi x}{2} = -\frac{4}{\pi s} \int_0^{\pi/2} \mathrm{d}\alpha \int_0^\infty \mathrm{d}x (\cosh(\alpha x))\, K_{ix}(\sqrt{s}\, a_1)$$
$$= -\frac{2}{s} \int_0^{\pi/2} \mathrm{d}\alpha\, \mathrm{e}^{-a_1 \sqrt{s} \cos\alpha} \qquad [6, 6.795.1]. \tag{2.10}$$

Hence

$$\int_0^\gamma \mathrm{d}\alpha_1 \int_{A_2 \in W} \mathrm{d}A_2\, G_W(A_1, A_2; s)$$
$$= \frac{\gamma}{s} - \frac{2}{s} \int_0^{\pi/2} \mathrm{d}\alpha\, \mathrm{e}^{-a_1 \sqrt{s} \cos\alpha} + \frac{4}{\pi s} \int_0^\infty \frac{\mathrm{d}x}{x} K_{ix}(\sqrt{s}\, a_1) \frac{\sinh((\pi-\gamma)\,x/2)}{\cosh(\gamma\, x/2)}. \tag{2.11}$$

Define the inverse Laplace transform by $L^{-1}\{g(s)\}=f(t)$ if

$$g(s)=\int_0^\infty f(t)\,\mathrm{e}^{-st}\,\mathrm{d}t. \tag{2.12}$$

Then

$$L^{-1}\left\{\int_0^R a\,\mathrm{d}a\,\frac{\gamma}{s}\right\}=\frac{\gamma R^2}{2}, \tag{2.13}$$

$$\begin{aligned}
L^{-1}&\left\{-\frac{2}{s}\int_0^R a\,\mathrm{d}a\int_0^{\pi/2}\mathrm{d}\alpha\,\mathrm{e}^{-a\sqrt{s}\cos\alpha}\right\}=-2\int_0^R a\,\mathrm{d}a\,L^{-1}\left\{\frac{1}{s}\int_0^{\pi/2}\mathrm{d}\alpha\,\mathrm{e}^{-a\sqrt{s}\cos\alpha}\right\}\\
&=-2\int_0^R a\,\mathrm{d}a\int_0^{\pi/2}\mathrm{d}\alpha\,\mathrm{Erfc}\left(\frac{a\cos\alpha}{2t^{1/2}}\right)\quad [4, 5.6.3]\\
&=-\frac{2}{\pi^{1/2}t^{1/2}}\int_0^R a^2\,\mathrm{d}a\int_1^\infty \mathrm{d}v\int_0^1\frac{y\,\mathrm{d}y}{(1-y^2)^{1/2}}\,\mathrm{e}^{-\frac{v^2y^2a^2}{4t}}\\
&=\frac{4t^{1/2}}{\pi^{1/2}}\int_0^R \mathrm{d}a\int_1^\infty \mathrm{d}v\int_0^1 \mathrm{d}y\,\frac{1}{v^2}(1-y^2)^{1/2}\frac{\partial}{\partial y}(\mathrm{e}^{-\frac{v^2y^2a^2}{4t}})\\
&\quad-\frac{2}{\pi^{1/2}t^{1/2}}\int_0^R a^2\,\mathrm{d}a\int_1^\infty \mathrm{d}v\int_0^1 \mathrm{d}y\,y^3(1-y^2)^{-1/2}\,\mathrm{e}^{-\frac{v^2y^2a^2}{4t}}\\
&=-\frac{4Rt^{1/2}}{\pi^{1/2}}+\frac{4t^{1/2}}{\pi^{1/2}}\int_0^R \mathrm{d}a\int_1^\infty\frac{\mathrm{d}v}{v^2}\int_0^1 \mathrm{d}y\,y(1-y^2)^{-1/2}\,\mathrm{e}^{-\frac{v^2y^2a^2}{4t}}\\
&\quad-\frac{2}{\pi^{1/2}t^{1/2}}\int_0^R a^2\,\mathrm{d}a\int_1^\infty \mathrm{d}v\int_0^1 \mathrm{d}y\,y^3(1-y^2)^{-1/2}\,\mathrm{e}^{-\frac{v^2y^2a^2}{4t}}\\
&=-\frac{4Rt^{1/2}}{\pi^{1/2}}+\frac{4Rt^{1/2}}{\pi^{1/2}}\int_1^\infty\frac{\mathrm{d}v}{v^2}\int_0^1\frac{y\,\mathrm{d}y}{(1-y^2)^{1/2}}\,\mathrm{e}^{-\frac{R^2y^2v^2}{4t}},
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
L^{-1}&\left\{\frac{4}{\pi s}\int_0^\infty a\,\mathrm{d}a\int_0^\infty\frac{\mathrm{d}x}{x}K_{ix}(\sqrt{s}\,a)\,\frac{\sinh((\pi-\gamma)x/2)}{\cosh(\gamma x/2)}\right\}\\
&=L^{-1}\left\{\frac{2}{s^2}\int_0^\infty \mathrm{d}x\,\frac{\sinh((\pi-\gamma)x/2)}{(\cosh(\gamma x/2))(\sinh(\pi x/2))}\right\}=t\,c(\gamma)\quad [6, 6.561.16].
\end{aligned} \tag{2.15}$$

Since

$$Q_\gamma(t;R)=L^{-1}\{\int_{\{A_1\in W:d(P,A_1)<R\}}\mathrm{d}A_1\int_W \mathrm{d}A_2\,G_W(A_1,A_2;s)\} \tag{2.16}$$

we have by (2.11), (2.13), (2.14) and (2.15)

$$Q_\gamma(t;R)=\frac{\gamma R^2}{2}-\frac{4Rt^{1/2}}{\pi^{1/2}}+\frac{4Rt^{1/2}}{\pi^{1/2}}\int_1^\infty\frac{\mathrm{d}v}{v^2}\int_0^1\frac{y\,\mathrm{d}y}{(1-y^2)^{1/2}}\,\mathrm{e}^{-\frac{R^2y^2v^2}{4t}}+t\,c(\gamma)$$

$$-L^{-1}\left\{\frac{4}{\pi s}\int_R^\infty a\,\mathrm{d}a\int_0^\infty\frac{\mathrm{d}x}{x}K_{ix}(\sqrt{s}\,a)\,\frac{\sinh((\pi-\gamma)\,x/2)}{\cosh(\gamma x/2)}\right\}. \qquad (2.17)$$

In order to compute the fifth term in (2.17) we have the following:

(i) For $-\frac{\pi}{2}<\beta<\frac{\pi}{2}$

$$L^{-1}\left\{\int_R^\infty a\,\mathrm{d}a\int_0^\infty\frac{\mathrm{d}x}{x}K_{ix}(\sqrt{s}\,a)\,\frac{\sinh(\beta x)}{s}\right\}$$

$$=L^{-1}\left\{\int_R^\infty a\,\mathrm{d}a\,\frac{1}{s}\int_0^\beta\mathrm{d}\alpha\int_0^\infty\mathrm{d}x(\cosh(\alpha x))\,K_{ix}(\sqrt{s}\,a)\right\}$$

$$=L^{-1}\left\{\frac{\pi}{2s}\int_R^\infty a\,\mathrm{d}a\int_0^\beta\mathrm{d}\alpha\,\mathrm{e}^{-a\sqrt{s}\cos\alpha}\right\} \quad [6,\ 6.795.1]$$

$$=\frac{\pi}{2}\int_R^\infty a\,\mathrm{d}a\int_0^\beta\mathrm{d}\alpha\,\mathrm{Erfc}\left(\frac{a\cos\alpha}{2t^{1/2}}\right) \quad [4,\ 5.6.3]. \qquad (2.18)$$

(ii) For $-\frac{\gamma}{2}<\beta<\frac{\gamma}{2}$

$$L^{-1}\left\{\frac{1}{s}\int_R^\infty a\,\mathrm{d}a\int_0^\infty\frac{\mathrm{d}x}{x}\,\frac{\sinh(\beta x)}{\cosh(\gamma x/2)}\,K_{ix}(\sqrt{s}\,a)\right\}$$

$$=L^{-1}\left\{\frac{1}{s}\int_R^\infty a\,\mathrm{d}a\int_0^\infty\frac{\mathrm{d}x}{x}\,\frac{\sinh(\beta x)}{\cosh(\gamma x/2)}\int_0^\infty\mathrm{d}u(\cos(xu))\,\mathrm{e}^{-a\sqrt{s}\cosh u}\right\} \quad [6,\ 3.547.4]$$

$$=L^{-1}\left\{\frac{1}{s}\int_R^\infty a\,\mathrm{d}a\int_0^\infty\mathrm{d}u\,\mathrm{e}^{-a\sqrt{s}\cosh u}\int_0^\infty\mathrm{d}x\,\frac{\sinh(\beta x)}{x\cosh(\gamma x/2)}\cos(xu)\right\}$$

$$=L^{-1}\left\{\frac{1}{2s}\int_R^\infty a\,\mathrm{d}a\int_0^\infty\mathrm{d}u\,\mathrm{e}^{-a\sqrt{s}\cosh u}\ln\left(\frac{\cosh(\pi u/\gamma)+\sin(\pi\beta/\gamma)}{\cosh(\pi u/\gamma)-\sin(\pi\beta/\gamma)}\right)\right\}$$

[6, 4.114.2]

$$=\frac{1}{2}\int_R^\infty a\,\mathrm{d}a\int_0^\infty\mathrm{d}u\,\mathrm{Erfc}\left(\frac{a\cosh u}{2t^{1/2}}\right)\ln\left(\frac{\cosh(\pi u/\gamma)+\sin(\pi\beta/\gamma)}{\cosh(\pi u/\gamma)-\sin(\pi\beta/\gamma)}\right) \quad [4,\ 5.6.3].$$

(2.19)

(iii) For $\beta>0$

$$L^{-1}\left\{\frac{1}{s}\int_R^\infty a\,\mathrm{d}a\int_0^\infty \frac{\mathrm{d}x}{x}(\tanh(\beta x))\,K_{ix}(\sqrt{s}\,a)\right\}$$

$$=L^{-1}\left\{\frac{1}{s}\int_R^\infty a\,\mathrm{d}a\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-a\sqrt{s}\cosh u}\int_0^\infty \frac{\mathrm{d}x}{x}(\tanh(\beta x))\cos(xu)\right\}$$

$$=L^{-1}\left\{\frac{1}{s}\int_R^\infty a\,\mathrm{d}a\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-a\sqrt{s}\cosh u}\ln(\coth(\pi u/(4\beta)))\right\} \qquad [6, 4.116.2]$$

$$=\int_R^\infty a\,\mathrm{d}a\int_0^\infty \mathrm{d}u\,\mathrm{Erfc}\left(\frac{a\cosh u}{2t^{1/2}}\right)\ln(\coth(\pi u/(4\beta))) \qquad [4, 5.6.3]. \tag{2.20}$$

iv) For $M=1,2,\ldots$

$$\frac{\sinh((\pi-\gamma)x/2)}{\cosh(\gamma x/2)}=\begin{cases}2\sum_{n=1}^{M}(-1)^{n+1}\sinh((\pi-2n\gamma)x/2)+(-1)^M\dfrac{\sinh((\pi-2M\gamma-\gamma)x/2)}{\cosh(\gamma x/2)}, \\ \qquad \dfrac{\pi}{2M+2}<\gamma<\dfrac{\pi}{2M}, \\ 2\sum_{n=1}^{M-1}(-1)^{n+1}\sinh((\pi-2n\gamma)x/2)+(-1)^{M+1}\tanh(\gamma x/2), \qquad \gamma=\dfrac{\pi}{2M}.\end{cases} \tag{2.21}$$

The proof of (2.1), (2.2) follows from (2.17) and (2.19). The proof of (2.1), (2.3) and (2.1), (2.4) follows from (2.17)–(2.21).

Corollary 3. *For* $t>0$

$$|S(t)|\leqq\begin{cases}2t\gamma\,\mathrm{e}^{-R^2/(4t)}, & \dfrac{\pi}{2}<\gamma\leqq 2\pi, \\ \dfrac{5\pi^3 t}{4\gamma^2}\,\mathrm{e}^{-(R\sin\gamma)^2/(4t)}, & 0<\gamma\leqq\dfrac{\pi}{2}.\end{cases} \tag{2.22}$$

Proof.

$$\frac{2}{\pi}\int_R^\infty a\,\mathrm{d}a\,\mathrm{Erfc}\left(\frac{a\cosh u}{2t^{1/2}}\right)\leqq\frac{2}{\pi}\int_R^\infty a\,\mathrm{d}a\,\mathrm{e}^{-\frac{a^2}{4t}}=\frac{4t}{\pi}\,\mathrm{e}^{-\frac{R^2}{4t}}, \tag{2.23}$$

and

$$\int_0^\infty \mathrm{d}u\left|\ln\left(\frac{\cosh(\pi u/\gamma)-\cos(\pi^2/(2\gamma))}{\cosh(\pi u/\gamma)+\cos(\pi^2/(2\gamma))}\right)\right|\leqq\int_0^\infty \mathrm{d}u\left|\ln\left(\frac{\cosh(\pi u/\gamma)-1}{\cosh(\pi u/\gamma)+1}\right)\right|$$

$$=2\int_0^\infty \mathrm{d}u\ln(\coth(\pi u/(2\gamma)))=\frac{2\gamma}{\pi}\int_0^\infty \frac{u}{\sinh u}\,\mathrm{d}u=\pi\gamma/2 \qquad [6, 3.521.1], \tag{2.24}$$

so that (2.22) follows for $\frac{\pi}{2}<\gamma\leqq 2\pi$ from (2.2), (2.23) and (2.24). Since the series in (2.3) and (2.4) are alternating in n and the absolute values of its terms are decreasing we find that

$$\left|4\sum_{n=1}^{M}(-1)^{n+1}\int_{R}^{\infty} a\,\mathrm{d}a \int_{0}^{(\pi-2n\gamma)/2} \mathrm{d}\alpha\,\mathrm{Erfc}\left(\frac{a\cos\alpha}{2t^{1/2}}\right)\right|$$

$$\leqq 4\int_{R}^{\infty} a\,\mathrm{d}a \int_{0}^{(\pi-2\gamma)/2} \mathrm{d}\alpha\,\mathrm{Erfc}\left(\frac{a\cos\alpha}{2t^{1/2}}\right)$$

$$\leqq 2\pi\int_{R}^{\infty} a\,\mathrm{d}a\,\mathrm{Erfc}\left(\frac{a\sin\gamma}{2t^{1/2}}\right)\leqq\frac{4\pi t}{(\sin\gamma)^2}\,\mathrm{e}^{-(R\sin\gamma)^2/(4t)}\leqq\frac{\pi^3 t}{\gamma^2}\,\mathrm{e}^{-(R\sin\gamma)^2/(4t)} \tag{2.25}$$

and (2.22) follows for $0<\gamma\leqq\frac{\pi}{2}$ from (2.3), (2.4), (2.23)–(2.25).

3. Bounds for $\mathbb{P}_A[T_D>t]$

In this section we will obtain upper and lower bounds for $\mathbb{P}_A[T_D>t]$. First we show the "principle of not feeling the boundary" for $t\downarrow 0$.

Lemma 4. *Let D be an open set in $\mathbb{R}^m$. Then*

$$1\geqq\mathbb{P}_A[T_D>t]\geqq 1-2^{1+\frac{m}{2}}\,\mathrm{e}^{-d^2(A)/(8t)}, \tag{3.1}$$

where $d(A)=d(A,\partial D)$.

Proof. By Levy's maximal inequality [13, Theorem 3.6.5]

$$\mathbb{P}_A[T_D<t]\leqq 2\mathbb{P}_0[|B(t)|>d(A)]=2\int_{|x|>d(A)}\mathrm{e}^{-|x|^2/(4t)}\frac{\mathrm{d}x}{(4\pi t)^{m/2}}$$

$$\leqq 2^{1+\frac{m}{2}}\,\mathrm{e}^{-d^2(A)/(8t)}. \tag{3.2}$$

For A near a vertex we have the following bound.

Lemma 5. *Let R be as in* (1.11). *Then for all $A\in D$ with $d(A,P_i)<R$ $(i\in\{1,\ldots,n\})$*

$$|\mathbb{P}_A[T_D>t]-\mathbb{P}_A[T_{W_i}>t]|\leqq 4\,\mathrm{e}^{-R^2/(8t)}. \tag{3.3}$$

Proof. Suppose $A\in W_i$, $d(A,P_i)<R$. Then

$$\mathbb{P}_A[T_D>t]\leqq\mathbb{P}_A[T_{W_i}>t]+\mathbb{P}_A[T_{\mathbb{R}^2\setminus C_i(2R)}<t], \tag{3.4}$$

where $C_i(R)$ is the arc in W_i with centre P_i and radius R. Since R is as in (1.11) $C_i(2R)\subset D$. Since $d(A, P_i)<R$ we have $d(A, C_i(2R))\geqq R$. Hence by Lemma 4 $\mathbb{P}_A[T_{\mathbb{R}^2\setminus C_i(2R)}<t]\leqq 4\exp(-R^2/(8t))$. Similarly

$$\mathbb{P}_A[T_D>t]\geqq \mathbb{P}_A[T_{W_i}>t]-\mathbb{P}_A[T_{\mathbb{R}^2\setminus C_i(2R)}<t]. \tag{3.5}$$

For A near an edge of ∂D but not near a vertex we have the following.

Lemma 6. *Let γ and R be as in* (1.10) *and* (1.11) *and let*

$$\delta=\frac{R}{2}\sin\frac{\gamma}{2}. \tag{3.6}$$

Then for all $A\in D$ with $d(A)<\delta$ and $d(A, P_i)>R$ and for all $i\in\{1, \dots, n\}$

$$\left|\mathbb{P}_A[T_D>t]-\frac{1}{(\pi t)^{1/2}}\int_0^{d(A)} e^{-y^2/(4t)}\,dy\right|\leqq 4\,e^{-(R\sin(\gamma/2))^2/(32t)}. \tag{3.7}$$

Proof. Since $A\in D$, $d(A)<\delta$ and $\min_i d(A, P_i)>R$ there is a unique edge E_A of ∂D nearest to A. Let H_A be the halfspace such that $H_A\ni A$, $\partial H_A\supset E_A$. Then by Lemma 4

$$\begin{aligned}\mathbb{P}_A[T_D>t]&\leqq \mathbb{P}_A[T_{\mathbb{R}^2\setminus E_A}>t]\\ &\leqq \mathbb{P}_A[T_{H_A}>t]+\mathbb{P}_A[T_{\mathbb{R}^2\setminus(\overline{\partial H_A\setminus E_A})}<t]\\ &\leqq \frac{1}{(\pi t)^{1/2}}\int_0^{d(A)} e^{-y^2/(4t)}\,dy+4\,e^{-R^2/(8t)}.\end{aligned} \tag{3.8}$$

Let G_A be the convex hull of E_A and the component of

$$\{B\in D\colon d(B)<2\delta, \min_i d(B, P_i)>R\}$$

which contains A. Then $G_A\subset D$ and

$$d(A, \partial G_A\setminus E_A)=d(A, \mathbb{R}^2\setminus \operatorname{int}\{(\mathbb{R}^2\setminus H_A)\cup G_A\})>(R/2)\sin(\gamma/2).$$

Hence

$$\begin{aligned}\mathbb{P}_A[T_D>t]&\geqq \mathbb{P}_A[T_{G_A}>t]\geqq \mathbb{P}_A[T_{H_A}>t]-\mathbb{P}_A[T_{\operatorname{int}\{(\mathbb{R}^2\setminus H_A)\cup G_A\}}<t]\\ &\geqq \frac{1}{(\pi t)^{1/2}}\int_0^{d(A)} e^{-y^2/(4t)}\,dy-4\,e^{-(R\sin(\gamma/2))^2/(32t)}.\end{aligned} \tag{3.9}$$

4. Proof of Theorem 1

From Lemma 4 and (3.6) we obtain

$$\left|\int_{d(A)>\delta} dA\,\mathbb{P}_A[T_D>t]-\int_{d(A)>\delta} dA\right|\leqq 4\int_{d(A)>\delta} dA\,e^{-(R\sin(\gamma/2))^2/(32t)}, \tag{4.1}$$

$$\left|\sum_{i=1}^{n}\int_{\substack{d(A,P_i)<R\\ A\in W_i}} dA\,\mathbb{P}_A[T_D>t]-\sum_{i=1}^{n} Q_{\gamma_i}(t;R)\right|\leqq 4\sum_{i=1}^{n}\int_{\substack{d(A,P_i)<R\\ A\in W_i}} dA\,e^{-R^2/(8t)}. \tag{4.2}$$

Furthermore Lemma 6 gives

$$\left|\int_{E(\delta,R)} \mathrm{d}A\,\mathbb{P}_A[T_D>t]-\int_{E(\delta,R)}\mathrm{d}A\int_0^{d(A)}\frac{1}{(\pi t)^{1/2}}\,\mathrm{e}^{-y^2/(4t)}\,\mathrm{d}y\right| \leqq \int_{E(\delta,R)}\mathrm{d}A\,\mathrm{e}^{-(R\sin(\gamma/2))^2/(32t)}, \tag{4.3}$$

where

$$E(\delta,R)=\{A\in D\colon d(A)<\delta,\ \min_i d(A,P_i)>R\}. \tag{4.4}$$

Combining (4.1), (4.2) and (4.3):

$$\left|\int_{A\in D}\mathrm{d}A\,\mathbb{P}_A[T_D>t]-\int_{\substack{d(A)>\delta\\ A\notin\bigcup_{i=1}^n B_i(R)}}\mathrm{d}A-\sum_{i=1}^n Q_{\gamma_i}(t;R)-\int_{E(\delta,R)}\mathrm{d}A\,\frac{1}{(\pi t)^{1/2}}\int_0^{d(A)}\mathrm{e}^{-y^2/(4t)}\,\mathrm{d}y\right| \leqq 4\,|D|\,\mathrm{e}^{-(R\sin(\gamma/2))^2/(32t)}. \tag{4.5}$$

From Theorem 2 and Corollary 3 we obtain

$$\begin{aligned}&\left|\sum_{i=1}^n Q_{\gamma_i}(t;R)-\sum_{i=1}^n\frac{\gamma_i R^2}{2}+\frac{4nRt^{1/2}}{\pi^{1/2}}-\sum_{i=1}^n c(\gamma_i)\,t\right.\\ &\qquad\left.-\frac{4nRt^{1/2}}{\pi^{1/2}}\int_1^\infty\frac{\mathrm{d}v}{v^2}\int_0^1\frac{y\,\mathrm{d}y}{(1-y^2)^{1/2}}\,\mathrm{e}^{-R^2v^2y^2/(4t)}\right|\\ &\leqq\sum_{\left\{i:\,\gamma_i>\frac{\pi}{2}\right\}}2t\gamma_i\,\mathrm{e}^{-R^2/(4t)}+\sum_{\left\{i:\,\gamma_i\leqq\frac{\pi}{2}\right\}}\frac{5\pi^3 t}{4\gamma_i^2}\,\mathrm{e}^{-(R\sin(\gamma_i/2))^2/(4t)}\\ &\leqq\frac{4\pi^3 nt}{\gamma^2}\,\mathrm{e}^{-(R\sin(\gamma/2))^2/(32t)}.\end{aligned} \tag{4.6}$$

Finally we have

$$\begin{aligned}&\int_{E(\delta,R)}\mathrm{d}A\int_0^{d(A)}\frac{\mathrm{e}^{-y^2/(4t)}}{(\pi t)^{1/2}}\,\mathrm{d}y=\int_0^\delta\mathrm{d}z\,(|\partial D|-2n(R^2-z^2)^{1/2})\int_0^z\frac{\mathrm{e}^{-y^2/(4t)}}{(\pi t)^{1/2}}\,\mathrm{d}y\\ &=|E(\delta,R)|-\int_0^\delta\mathrm{d}z\,(|\partial D|-2nR)\int_z^\infty\frac{\mathrm{d}y}{(\pi t)^{1/2}}\,\mathrm{e}^{-y^2/(4t)}\\ &\quad-2\int_0^\delta\mathrm{d}z\,(nR-n(R^2-z^2)^{1/2})\int_z^\infty\frac{\mathrm{d}y}{(\pi t)^{1/2}}\,\mathrm{e}^{-y^2/(4t)}\\ &=|E(\delta,R)|-\frac{2t^{1/2}}{\pi^{1/2}}(|\partial D|-2nR)+\int_\delta^\infty\mathrm{d}z\,(|\partial D|-2nR)\int_z^\infty\frac{\mathrm{d}y}{(\pi t)^{1/2}}\,\mathrm{e}^{-y^2/(4t)}\end{aligned}$$

$$-\frac{4nRt^{1/2}}{\pi^{1/2}}\int_1^\infty \frac{dv}{v^2}\int_0^{(\sin(\gamma/2))/2} dy\, y(1-y^2)^{-1/2}\, e^{-R^2v^2y^2/(4t)}$$

$$+\frac{4nt^{1/2}}{\pi^{1/2}}(R-(R^2-\delta^2)^{1/2})\int_1^\infty \frac{dv}{v^2}\, e^{-\delta^2v^2/(4t)}, \tag{4.7}$$

where $|E(\delta, R)|$ is the volume of $E(\delta, R)$.

Theorem 1 follows from (4.5), (4.6) and (4.7):

$$\left|Q_D(t)-|D|+\frac{2}{\pi^{1/2}}|\partial D|\,t^{1/2}-\sum_{i=1}^n c(\gamma_i)t\right| \leq \frac{4\pi^3nt}{\gamma^2}\, e^{-(R\sin(\gamma/2))^2/(32t)}$$

$$+4|D|\,e^{-(R\sin(\gamma/2))^2/(32t)}+\int_\delta^\infty dz(|\partial D|-2nR)\int_z^\infty \frac{dy}{(\pi t)^{1/2}}\, e^{-y^2/(4t)}$$

$$+\frac{4nt^{1/2}}{\pi^{1/2}}(R-(R^2-\delta^2)^{1/2})\int_1^\infty \frac{dv}{v^2}\, e^{-\delta^2v^2/(4t)}$$

$$+\frac{4nRt^{1/2}}{\pi^{1/2}}\int_1^\infty \frac{dv}{v^2}\int_{(\sin(\gamma/2))/2}^1 \frac{y\,dy}{(1-y^2)^{1/2}}\, e^{-R^2v^2y^2/(4t)}$$

$$\leq \left(\frac{4\pi^3nt}{\gamma^2}+4|D|+\frac{2|\partial D|\,t^{1/2}}{\pi^{1/2}}\right) e^{-(R\sin(\gamma/2))^2/(32t)}. \tag{4.8}$$

References

1. Berg, M. van den, Srisatkunarajah, S.: Heat equation for a region in $\mathbb{R}^2$ with a polygonal boundary. J. London Math. Soc. (2) **37**, 119–127 (1988)
2. Cooke, J.C.: Conduction of heat in the wedge $r>0$, $0<\theta<\theta_0$, with zero initial temperature and unit surface temperature. Am. Math. Monthly **62**, 331–334 (1955)
3. Erdélyi, A.: Higher transcendental functions, vol. 2. New York: McGraw-Hill 1954
4. Erdélyi, A.: Tables of integral transforms, vol. 1. New York: McGraw-Hill 1954
5. Getoor, R.K.: Some asymptotic formulas involving capacity. Z. Wahrscheinlichkeitstheor. Verw. Geb. **4**, 248–252 (1965)
6. Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series, and products. New York: Academic Press 1980
7. Kac, M.: Can one hear the shape of a drum? Am. Math. Monthly **73**, 1–23 (1966)
8. Le Gall, J.F.: Sur une conjecture de M. Kac. Probab. Th. Rel. Fields **78**, 389–402 (1988)
9. Louchard, G.: Hitting probabilities for brownian motion in three dimensions. J. Math. Phys. **44**, 177–181 (1965)
10. McKean, H.P., Singer, I.M.: Curvature and eigenvalues of the laplacian. J. Diff. Geom. **1**, 43–69 (1967)
11. Spitzer, F.: Some theorems concerning 2-dimensional brownian motion. Trans. Am. Math. Soc. **87**, 187–197 (1958)
12. Spitzer, F.: Electrostatic capacity, heat flow, and brownian motion. Z. Wahrscheinlichkeitstheor. Verw. Geb. **3**, 110–121 (1964)
13. Simon, B.: Functional integration and quantum physics. New York: Academic Press 1980