Probab. Th. Rel. Fields 86, 1–19 (1990)

Probability Theory and Related Fields

Association and random measures

Steven N. Evans*

University of California at Berkeley, Department of Statistics, 367 Evans Hall, Berkeley, CA 94720, USA

Received May 17, 1989; in revised form January 5, 1990

Summary. Our point of departure is the result, due to Burton and Waymire, that every infinitely divisible random measure has the property variously known as association, positive correlations, or the FKG property. This leads into a study of stationary, associated random measures on $\mathbb{R}^d$. We establish simple necessary and sufficient conditions for ergodicity and mixing when second moments are present. We also study the second moment condition that is usually referred to as finite susceptibility. As one consequence of these results, we can easily rederive some central limit theorems of Burton and Waymire. Using association techniques, we obtain a law of the iterated logarithm for infinitely divisible random measures under simple moment hypotheses. Finally, we apply these results to a class of stationary random measures related to measure-valued Markov branching processes.

1. Introduction

We begin by recalling the general notion of association given, for instance, in (Lindqvist 1988). Suppose that $\mathscr{X}$ and $\mathscr{Y}$ are complete, separable, metric spaces, each furnished with a closed order that we will write in both cases as $\leqq$. We say that a map $f\colon \mathscr{X} \to \mathscr{Y}$ is *non-decreasing* if $x_1 \leqq x_2$ implies $f(x_1) \leqq f(x_2)$. We say that an $\mathscr{X}$-valued random variable X is *associated* if for each pair of bounded, Borel measurable, non-decreasing functions $f\colon \mathscr{X} \to \mathbb{R}$ and $g\colon \mathscr{X} \to \mathbb{R}$, we have

$$(*) \qquad \operatorname{Cov}(f(X), g(X)) \geqq 0.$$

This property is also known as the FKG (following (Fortuin et al. 1971)) or the positive correlations property.

In this paper we are interested in studying this property for random measures. Let S be a locally compact, separable, metric space. Denote by $M(S)$ the space

* Research supported in part by NSF Grant DMS-8701212 at the University of Virginia

of Radon measures on S topologised by vague convergence, and let $\mathscr{M}(S)$ denote the corresonding Borel σ-field. We can give $M(S)$ a partial order which is easily shown to be closed by declaring that $\mu \leqq \nu$ if $\mu(B) \leqq \nu(B)$ for all Borel sets B, or, equivalently, $\mu(f) \leqq \nu(f)$ for every non-negative, continuous function with compact support. A *random measure* is an $M(S)$-valued random variable defined over some underlying probability space $(\Omega, \mathscr{F}, P)$. We refer the reader to (Kallenberg 1983) for an extensive treatment of the theory of random measures.

The following result, which appears in (Burton and Waymire 1986) along with a sketch of a proof, shows that a large class of random measures are indeed associated. See also Theorem 5.2 of (Burton and Waymire 1985). (We say that a random measure, X, is infinitely divisible if for each $n \in \mathbb{N}$ we can construct independent, identically distributed random measures $X_1, \ldots, X_n$ such that X has the same distribution as $X_1 + \ldots + X_n$.)

Theorem 1.1. *Each infinitely divisible random measure on S is associated.*

There are, of course, associated random measures which are not infinitely divisible. For instance, if μ is a fixed Radon measure and Y is any non-negative, real random variable, then, by Theorems 3.2 and 3.4 of (Lindqvist 1988), the random measure $Y\mu$ is associated. Clearly, $Y\mu$ will be infinitely divisible if and only if Y is. For a more interesting class of examples, note that if X_1 and X_2 are independent associated random measures on $\mathbb{R}^d$ such that $X_1 * X_2$ defines a random measure (for example, if X_1 and X_2 both have finite total mass almost surely), then, by Theorems 3.2 and 3.3 of (Lindqvist 1988), $X_1 * X_2$ is associated. Typically, $X_1 * X_2$ will not be infinitely divisible, even if X_1 and X_2 are.

We give a complete proof of Theorem 1.1 in Sect. 2 and indicate how the result can be derived using the general techniques in (Harris 1977) or (Herbst and Pitt 1988). With the class of examples provided by the infinitely divisible random measures in mind, we then proceed to investigate the properties of associated random measures using techniques developed for associated sequences of random variables.

In Sect. 3, we give simple necessary and sufficient conditions when second moments exist for stationary, associated random measures to be ergodic and mixing. These conditions are expressed in terms of the second moment structure of the random measure. There is some overlap between the class of random measures covered by these results and the family of general, stationary, infinitely divisible random point measures for which necessary and sufficient conditions for ergodicity and mixing are given in (Matthes et al. 1978). On the overlap, our results seem to be less complex and easier to apply.

In Sect. 4 we study the so-called *finite susceptibility* condition for use in later sections. We also use our results to give quick rederivations of the central limit theorems in (Burton and Waymire 1985, 1986).

We give a corresponding law of the iterated logarithm for the special case of stationary, infinitely divisible random measures under an extra fourth moment hypothesis in Sect. 5. This type of result does not appear to have been discussed in the literature.

Finally, in Sect. 6, we apply the results of earlier sections to study measure-valued Markov branching processes.

In a forthcoming paper (Evans 1990), we use some of our results here to study random sets such as the Boolean coverage process.

2. A proof of Theorem 1.1

We intend to prove Theorem 1.1 using an approximation procedure. For this reason, it will be convenient to know the following three results, which are also used in the sequel.

Lemma 2.1. *Suppose that* $(\mathscr{X}, d)$ *is a complete, separable, metric space furnished with a closed order. Suppose that the metric* d *is such that for every closed set* $C \subset \mathscr{X}$ *which has non-decreasing indicator function, we have* $d(x, C) \geqq d(y, C)$ *whenever* $x \leqq y$. *Then, for any such set, we can find a sequence* $\{h_n\}_{n=1}^{\infty}$ *of non-decreasing, continuous functions such that* $0 \leqq h_n \leqq 1$ *and* $h_n(x) \downarrow 1_C(x)$ *for each* x *as* $n \to \infty$.

Proof. We follow (Esary et al. 1967) and set $h_n(x) = (1 - n d(x, C))^+$. It is clear that $\{h_n\}$ has the required properties. □

Lemma 2.2. *Suppose that* $\mathscr{X}$ *satisfies the conditions of Lemma* 2.1. *Then the following hold.*

(i) *A random variable* X *is associated if (and only if)* $(*)$ *holds for all* f *and* g *that are bounded, continuous, and non-decreasing.*

(ii) *If* $\{X_n\}$ *is a sequence of associated random variables such that* $X_n \Rightarrow X$ *in distribution, then* X *is also associated.*

Proof. (i) Suppose that $(*)$ holds for all f and g that are bounded, continuous, and non-decreasing. From Theorem 3.1 of (Lindqvist 1988), it suffices to show that $(*)$ holds when f and g are non-decreasing indicator functions of closed sets, but this is clear from Lemma 2.1 and monotone convergence.

(ii) This is clear from (i) (cf. the proof of Theorem 3.5 in (Lindqvist 1988)). □

Lemma 2.3. *Suppose that* $(\mathscr{X}, d)$ *is a complete, separable, metric space furnished with a closed order. Suppose further that* $\mathscr{X}$ *is a semigroup and the semigroup structure is compatible with the metric and order structures in the sense that*

$$d(x+z, y+z) = d(x, y),$$

and

$$x \leqq y \Leftrightarrow \exists z \in \mathscr{X}, \quad y = x + z.$$

Then the conditions of Lemma 2.1 *hold.*

Proof. Let x, y and C be as in the statement of Lemma 2.1. By assumption $y = x + z$ for some $z \in \mathscr{X}$. Consider any $w \in C$, then $w \leqq w + z$, and so $w + z \in C$ also. Since $d(y, w) = d(x, w + z)$, it is clear that $d(y, C) \geqq d(x, C)$, as required. □

Remarks. Suppose that $\{h_n\}_{n=1}^{\infty}$ is a countable, vague convergence determining class of continuous functions with compact support. We can choose as our complete metric for $M(S)$ the metric given by

$$d(\mu, \nu) = \sum_{n=1}^{\infty} (|\mu(h_n) - \nu(h_n)| \wedge 1)\, 2^{-n}.$$

It is easy to see that with this metric $M(S)$ satisfies the conditions of Lemma 2.3. Similarly, if we give $M(S)^{\mathbb{N}}$ the complete metric

$$d'((\mu_n),(\nu_n))=\sum_{n=1}^{\infty}(d(\mu_n,\nu_n)\wedge 1)\,2^{-n}$$

(which induces the product topology) and the order

$$(\mu_n)\leqq(\nu_n)\Leftrightarrow\mu_n\leqq\nu_n,\qquad\forall n,$$

then $M(S)^{\mathbb{N}}$ satisfies the conditions of Lemma 2.3. Also, if we give the Skorohod space $D(M(S),[0,T])$ the complete metric, d''_T, based on d in the manner described in the proof of Lemma 6.2 in Lindqvist (1988) and the order

$$(\mu_t)\leqq(\nu_t)\Leftrightarrow\mu_t\leqq\nu_t,\qquad\forall t,$$

then the conditions of Lemma 2.3 are satisfied. A similar comment is true for $D(M(S),[0,\infty[)$ with the complete metric

$$d''((\mu_t),(\nu_t))=\sum_{n=1}^{\infty}(d''_n((\mu_t)|_{[0,n]},(\nu_t)|_{[0,n]})\wedge 1)\,2^{-n}.$$

Thus the conclusion of Lemma 2.2 holds for all of the above spaces.

Proof of Theorem 1.1. Choose $S_1\subset S_2\subset\ldots\subset S$ such that S_n is compact for all n and $\bigcup_{n=1}^{\infty}S_n=S$. Define random measures X_n by $X_n(A)=X(A\cap S_n)$. As $X_n\to X$ almost surely, it suffices by the Remarks above to show that each X_n is associated; but we can regard X_n as an infinitely divisible random measure on S_n, so this amounts to showing that the theorem holds when S is compact.

From Lemma 6.5 of Kallenberg (1983), we can assume without loss of generality that over our underlying probability space there is a Poisson point measure, η, on $M(S)\backslash\{0\}$ and a fixed measure $\alpha\in M(S)$ such that $X=\alpha+\int\mu\eta(\mathrm{d}\mu)$. Recall that when S is compact, $M(S)\backslash\{0\}$ is locally compact. Choose $K_1\subset K_2\subset\ldots\subset M(S)\backslash\{0\}$ such that K_n is compact for all n and $\bigcup_{n=1}^{\infty}K_n=M(S)\backslash\{0\}$. Then $\alpha+\int_{K_n}\mu\eta(\mathrm{d}\mu)\to X$ almost surely, and so it further suffices to show that the random measure $\int_K\mu\eta(\mathrm{d}\mu)$ is associated whenever $K\subset M(S)\backslash\{0\}$ is compact.

Note that for such a set K we must have $\inf\{\mu(S):\mu\in K\}=\varepsilon>0$, and since

$$\alpha(S)+\varepsilon\eta(K)\leqq X(S)<\infty\qquad\text{a.s.}$$

we have $\eta(K)<\infty$ a.s. Suppose that we have metrised $M(S)$ using a convergence determining class $\{h_k\}_{k=1}^{\infty}$ as in the Remarks above. For each $n\in\mathbb{N}$, let

$A_{n,1}, \ldots, A_{n,m(n)} \in \mathscr{M}(S) \backslash \{\emptyset\}$ be a partition of K into sets of diameter at most 2^{-n}. Pick $\mu_{n,i} \in A_{n,i}$, $i=1, \ldots, m(n)$. Then, once $n \geqq k$,

$$\begin{aligned} &|\sum_i \mu_{n,i}(f_k)\,\eta\,(A_{n,i}) - \int \mu(f_k)\,\eta(\mathrm{d}\mu)| \\ &\quad \leqq \sum_i \int_{A_{n,i}} |\mu_{n,i}(f_k) - \mu(f_k)|\,\eta(\mathrm{d}\mu) \\ &\quad \leqq \sum_i \sup\{|\mu_{n,i}(f_k) - \mu(f_k)| : \mu \in A_{n,i}\}\,\eta(A_{n,i}) \\ &\quad \leqq \sum_i 2^{k-n}\eta(A_{n,i}) \\ &\quad = 2^{k-n}\eta(K) \\ &\quad \to 0 \quad \text{a.s.,} \qquad \text{as } n \to \infty. \end{aligned}$$

Thus

$$\sum_{i=1}^{m(n)} \mu_{n,i}\,\eta(A_{n,i}) \to \int_K \mu\,\eta(\mathrm{d}\mu) \quad \text{a.s.}$$

as $n \to \infty$, and it further suffices to show that if $\mu_1, \ldots, \mu_m \in M(S) \backslash \{0\}$, and $A_1, \ldots, A_m \in \mathscr{M}(S) \backslash \{\emptyset\}$ are disjoint, then the random measure $\sum_i \mu_i\,\eta(A_i)$ is associated.

Now each real-valued random variable $\eta(A_i)$ is associated (cf. Theorem 3.4 of Lindqvist (1988)) and, since $\eta\,(A_1), \ldots, \eta(A_m)$ are independent, the random vector $(\eta(A_1), \ldots, \eta(A_m))$ is also associated (cf. Theorem 3.3 of Lindqvist (1988)). Finally, since the function $F\colon \mathbb{R}^m \to M(S)$ defined by $F(x_1, \ldots, x_m) = \sum_i x_i\,\mu_i$ is non-decreasing, the random measure $F(\eta(A_1), \ldots, \eta(A_m))$ is associated, as required (cf. Theorem 3.2 of Lindqvist (1988)). $\square$

Remarks. Our proof seems to be in the spirit of the sketch outlined in Burton and Waymire (1986). We could have proved Theorem 1.1 in a number of different ways. For instance, by an appropriate succession of approximations, we could reduce the problem to one of showing that each $[0, \infty[^d$-valued infinitely divisible random variable with no drift component and discrete Lévy measure supported on a finite set is associated.

For this latter problem, it certainly suffices to show that the corresponding convolution semigroup $\{P_t\}$ preserves association; that is, if μ is an associated probability measure on $\mathbb{R}^d$, then so is μP_t for each $t \geqq 0$. Note that it is clear from the infinite divisibility that $\{P_t\}$ preserves increasing functions; that is, if $f\colon \mathbb{R}^d \to \mathbb{R}$ is a bounded, non-decreasing, Borel function, then so is $P_t f$ for each $t \geqq 0$.

One approach to showing that $\{P_t\}$ preserves association would now be to use methods similar to those employed in the proof of Proposition 4.1 of Herbst and Pitt (1988). To do this, one essentially has to show that if A denotes the infinitesimal generator of $\{P_t\}$, then for a sufficiently large class of bounded increasing functions in the domain of A,

$$\Gamma_1(f, g)(x) = A(fg)(x) - f(x)\,Ag(x) - g(x)\,Af(x) \geqq 0,$$

for all pairs (f, g) drawn from the class. Here, however, if ν denote the Lévy measure of $\{P_t\}$, then for any pair (f, g) of bounded, continuous functions on $\mathbb{R}^d$ we have

$$\Gamma_1(f, g) = \int [f(x+y) - f(x)][g(x+y) - g(x)]\, \nu(dy) \geqq 0,$$

and this is certainly enough to use the techniques in Herbst and Pitt (1988).

Alternatively, one observes that $\{P_t\}$ can be restricted to be the semigroup of a Markov chain with state space consisting of points of the form $\sum_{k=1}^{m} n_k e_k$, where $n_k \in \{0, 1, \ldots\}$ and $\{e_1, \ldots, e_m\} = \operatorname{supp} \nu$. The chain only takes jumps in the non-decreasing direction, and so the desired result for $\{P_t\}$ follows from Harris (1977).

3. Independence, ergodicity, and mixing

For the sake of reference, we record the following trivial observation.

Lemma 3.1. *If (U_1, U_2, V_1, V_2) is an associated random vector with all elements possessing finite second moments, then*

$$\operatorname{Cov}(U_1, V_1) \leqq \operatorname{Cov}(U_1 + U_2, V_1 + V_2).$$

Theorem 3.2. *Let X be an associated random measure. Suppose that A and B are two Borel sets such that $EX(A)^2 < \infty$ and $EX(B)^2 < \infty$. Then the random measures $X(\cdot \cap A)$ and $X(\cdot \cap B)$ are independent if and only if* $\operatorname{Cov}(X(A), X(B)) = 0$.

Proof. If $X(\cdot \cap A)$ and $X(\cdot \cap B)$ are independent, then, of course, $\operatorname{Cov}(X(A), X(B)) = 0$.

In the other direction, it will suffice to show that if $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are arbitrary, non-negative, continuous functions with compact support, then random vectors $(X(f_i 1_A))_{i=1}^m$ and $(X(g_j 1_B))_{j=1}^n$ are independent. The map H: $M(S) \to \mathbb{R}^{m+n}$ defined by

$$H(\mu) = ((\mu(f_i 1_A))_{i=1}^m, (\mu(g_j 1_B))_{j=1}^n)$$

is non-decreasing, and so the random vector $H(X)$ is associated (cf. Theorem 3.2 of Lindqvist (1988)). By Corollary 3 of Newman (1984), it therefore suffices to show that $\operatorname{Cov}(X(f_i 1_A), X(g_j 1_B)) = 0$ for all (i, j). However, by the same reasoning as above, the random vector

$$(X(f_i 1_A), X([\|f_i\|_\infty - f_i] 1_A), X(g_j 1_B), X([\|g_j\|_\infty - g_j] 1_B))$$

is associated and the result follows from Lemma 3.1. □

Definition. Given $T \in \mathbb{R}^d$, we define the *T-shift* to be the map τ_T: $M(\mathbb{R}^d) \to M(\mathbb{R}^d)$ given by $(\tau_T \mu)(A) = \mu(T + A)$. A random measure X on $\mathbb{R}^d$ is *stationary* if $\tau_T X$ has the same law as X for all $T \in \mathbb{R}^d$. If X is stationary, then the T-shift is *ergodic for X* if for all $B \in \mathcal{M}(\mathbb{R}^d)$ such that $\tau_T B = B$, we have $(P \circ X^{-1})(B) = 0$ or 1; and X is *ergodic* if for all $T \neq 0$, the T-shift is ergodic for X. Similarly, if X is stationary, then the T-shift is *mixing for* X if for all A, $B \in \mathscr{M}(\mathbb{R}^d)$, we have $\lim_{k\to\infty} (P\circ X^{-1})(A\cap\tau_T^{-k}B)=(P\circ X^{-1})(A)(P\circ X^{-1})(B)$; and X is *mixing* if for all $T\neq 0$ the T-shift is mixing for X.

Notation. For $n\in\mathbb{N}$, let I and $S(n)$ be the subsets of $\mathbb{R}^d$ given by

$$I=[-\tfrac{1}{2},\tfrac{1}{2}[^d$$

and

$$S(n)=[-\tfrac{1}{2},n-\tfrac{1}{2}[\times[-\tfrac{1}{2},\tfrac{1}{2}]^{d-1}.$$

Theorem 3.3. *Let X be a stationary, associated random measure on $\mathbb{R}^d$ such that $EX(I)^2<\infty$. The* (1, 0, ..., 0)*-shift is ergodic for X if and only if for all $T>0$,*

$$\lim_{n\to\infty} n^{-1}\operatorname{Cov}(X(TI),X(TS(n)))=0.$$

Proof. We follow the approach outlined in Newman (1980) and Newman (1984) for proving ergodicity of stationary sequences and arrays.

Let τ denote the (1, 0, ..., 0)-shift. Suppose τ is ergodic. Given $T>0$, choose $N\in\mathbb{N}$ such that $N>T$. Then, by the mean ergodic theorem applied to the stationary sequence $\{\tau^{kN}X(NI)\}_{k=0}^{\infty}$ (see, for example, Theorem 2.1.1 of Petersen (1983)), we have that

$$\begin{aligned}\lim_{n\to\infty} n^{-1}X(NS(n)) &= \lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}\tau^{kN}X(NI)\\ &= EX(NI)\end{aligned}$$

in $L^2(P)$. Thus

$$\lim_{n\to\infty} n^{-1}\operatorname{Cov}(X(NI),X(NS(n)))=0,$$

and the necessity of the condition follows from Lemma 3.1.

Conversely, suppose that the condition holds. By a standard test for ergodicity (see, for example, Proposition 2.4.5 of Petersen (1983)) and a dense subspace argument (cf. Proposition 2.4.2 of Petersen (1983)), it suffices to show that if $f_1,\dots f_p$ (respectively, $g_1,\dots,g_q$) are non-negative continuous functions with compact support and $F\colon\mathbb{R}^p\to\mathbb{R}$ (respectively, $G\colon\mathbb{R}^q\to\mathbb{R}$) has bounded partial derivatives $F^1,\dots,F^p$ (respectively, $G^1,\dots,G^q$), then

$$\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1}\operatorname{Cov}(F[X(f_1),\dots,X(f_p)],G[\tau^kX(g_1),\dots,\tau^kX(g_q)])=0. \quad (3.3.1)$$

Choose $T>0$ sufficiently large so that $\operatorname{supp} f_j\subset TI$, $i=1,\dots,p$, and $\operatorname{supp} g_j\subset TI$, $j=1,\dots,q$. Then, by Lemma 3.1(i) of Birkel (1988) and our Lemma 3.1, there exists a constant K not depending on k such that

$$\begin{aligned}&|\operatorname{Cov}(F[X(f_1),\dots,X(f_p)],G[\tau^kX(g_1),\dots,\tau^kX(g_q)])|\\ &\leqq\sum_{i=1}^{p}\sum_{j=1}^{q}\|F^i\|_\infty\|G^j\|_\infty\operatorname{Cov}(X(f_i),\tau^kX(g_j))\\ &\leqq K\operatorname{Cov}(X(TI),\tau^kX(TI)),\end{aligned}$$

and (3.3.1) follows easily from this. □

Remark. By a change of coordinates, Theorem 3.3 can be used to derive a necessary and sufficient condition for a shift in any direction to be ergodic for X. All of these conditions will hold if and only if X is ergodic. □

We state without proof the following analogue of Theorem 3.3 for mixing (see, for example, Proposition 2.5.2 of Petersen (1983) for the relevant L^2 formulation of mixing).

Theorem 3.4. *Let X be a stationary, associated random measure on $\mathbb{R}^d$ such that $EX(I)^2<\infty$. The* $(1, 0, \ldots, 0)$*-shift is mixing for X if and only if for all $T>0$,*

$$\lim_{n\to\infty} \operatorname{Cov}(X(TI), X(T[S(n)\setminus S(n-1)]))=0.$$

Remark. As in the Remark following Theorem 3.3, by changes of coordinates, we can use Theorem 3.4 to derive a set of necessary and sufficient conditions for X to be mixing.

4. Finite susceptibility

Definition. Suppose that X is a stationary, associated random measure on $\mathbb{R}^d$ such that $EX(I)^2<\infty$. Following Newman (1980), we will say that X has *finite susceptibility*, Γ, if

$$\sup_{T>0} \operatorname{Cov}(X(I), X(TI))=\Gamma<\infty.$$

Lemma 4.1. *Let X be a stationary, associated random measure on $\mathbb{R}^d$ such that $EX(I)^2<\infty$ and X has finite susceptibility, Γ. Let f be a non-negative, bounded, Borel function with compact support. Then*

$$\operatorname{Cov}(X(f), X(NI))\uparrow\Gamma\int f(x)\,\mathrm{d}x$$

as $N\to\infty$.

Proof. For $N\in\mathbb{N}$, it is clear, using Lemma 3.1, that there is a (unique) Radon measure μ_N such that

$$\mu_N(A)=\operatorname{Cov}(X(A), X(NI)),$$

when A is a set with compact closure. Suppose that g is a non-negative, continuous function with compact support. Then, by Lemma 3.1, it follows that $\{\mu_N(g)\}_{N=1}^{\infty}$ is a bounded, non-decreasing sequence, and hence $\mu_N(g)$ converges to a finite limit as $N\to\infty$. Thus there exists a Radon measure μ such that $\mu_N\to\mu$ as $N\to\infty$. From the stationarity of X it is clear that μ is translation invariant, and so μ is a multiple of Lebesgue measure. Since the boundary of I has zero Lebesgue measure, we must have $\mu_N(I)\uparrow\mu(I)$, and hence $\mu(\mathrm{d}x)=\Gamma\,\mathrm{d}x$.

The result now follows by a fairly standard monotone class argument. □

Lemma 4.2. *Let X be a stationary, associated random measure on $\mathbb{R}^d$ such that $EX(I)^2 < \infty$, and X has finite susceptibility, Γ. For $f \in L^1 \cap L^\infty$ (Lebesgue), put $Z(f) = X(f) - EX(f)$. Then*

$$EZ(f)^2 \leqq \Gamma \|f\|_1 \|f\|_\infty.$$

Proof. Note first of all that, by stationarity, the measure $EX(\cdot)$ is a multiple of Lebesgue measure, and so $Z(f)$ is well defined. By the same observation, $Z(f) = Z((f \wedge \|f\|_\infty) \vee (-\|f\|_\infty))$, and so we may suppose that f is, in fact, bounded.

For each $N \in \mathbb{N}$, we have from Lemma 4.1 and Lemma 3.1 that

$$\begin{aligned} EZ(f^+ 1_{NI})^2 &= \operatorname{Var} X(f^+ 1_{NI}) \\ &\leqq \operatorname{Cov}(X(f^+ 1_{NI}), X(\|f\|_\infty 1_{NI})) \\ &\leqq \Gamma \|f^+\|_1 \|f\|_\infty, \end{aligned}$$

and so, by Fatou's lemma,

$$EZ(f^+)^2 \leqq \Gamma \|f^+\|_1 \|f\|_\infty.$$

A similar inequality holds for f^-, and so

$$\begin{aligned} EZ(f)^2 &= EZ(f^+)^2 + EZ(f^-)^2 - 2E(Z(f^+) Z(f^-)) \\ &\leqq \Gamma \|f^+\|_1 \|f\|_\infty + \Gamma \|f^-\|_1 \|f\|_\infty \\ &= \Gamma \|f\|_1 \|f\|_\infty. \quad \square \end{aligned}$$

Lemma 4.3. *Let X be a stationary, associated random measure on $\mathbb{R}^d$ such that $EX(I)^2 < \infty$ and X has finite susceptibility, Γ. Supose that $f \in L^1 \cap L^\infty$, and $\{g_N\}_{N=1}^\infty \subset L^1 \cap L^\infty$ are non-negative functions such that $g_N \uparrow 1$ pointwise as $N \to \infty$. Then*

$$\operatorname{Cov}(X(f), X(g_N)) \uparrow \Gamma \int f(x)\, \mathrm{d}x$$

as $N \to \infty$.

Proof. From Lemmas 3.1, 4.1, and 4.2, we have

$$\begin{aligned} &\sup_N \operatorname{Cov}(X(f), X(g_N)) \\ &\quad = \sup_N \sup_L \sup_M \operatorname{Cov}(X(f 1_{LI}), X(g_N 1_{MI})) \\ &\quad = \sup_L \sup_M \sup_N \operatorname{Cov}(X(f 1_{LI}), X(g_N 1_{MI})) \\ &\quad = \sup_L \sup_M \operatorname{Cov}(X(f 1_{LI}), X(MI)) \\ &\quad = \sup_L \Gamma \int_{LI} f(x)\, \mathrm{d}x \\ &\quad = \Gamma \int f(x)\, \mathrm{d}x. \quad \square \end{aligned}$$

The following result was proved in Burton and Waymire (1985) for the case of indicator functions of non-overlapping rectangles; and a similar result was stated in Burton and Waymire (1986) for an unspecified class of functions.

The observation that such results can be proved using the central limit theorem for arrays of random variables in Newman (1980) is due to Burton and Waymire.

Theorem 4.4. *Let X be a stationary, associated random measure on $\mathbb{R}^d$ such that $EX(I)^2<\infty$, and X has finite susceptibility, Γ. For $T>0$ and $f\in L^1\cap L^\infty$, put $X_T(f)=X(f(\cdot/T))$, and $Z_T(f)=X_T(f)-EX_T(f)$. Then $T^{-d/2}Z_T(f)$ converges in distribution to a $N(0, \Gamma\|f\|_2^2)$ random variable as $T\to\infty$.*

Proof. Choose m, $n\in\mathbb{N}$ and $k_1, \dots, k_m\in\mathbb{Z}^d$ with $k_i\neq k_j$, $i\neq j$. Set $I(n;\ k_i)=n^{-1}(k_i+I)$, $i=1, \dots, m$. We begin by showing that the random vector $T^{-d/2}(Z_T(I(n;k_i)))_{i=1}^m$ converges in distribution to a vector of independent, identically distributed, Gaussian random variables with common mean zero and common variance Γn^{-d}. If we restrict T to the integers, then this follows directly from Theorem 2 in Newman (1980) and Lemma 4.1. For arbitrary T, if we let $[T]$ denote the integer part of T, then it is clear from Lemma 4.2 that for $i=1, \dots, m$,

$$E(T^{-d/2}Z_T(I(n;k_i))-[T]^{-d/2}Z_{[T]}(I(n;k_i)))^2\to 0, \qquad \text{as } T\to\infty,$$

and our claim holds.

From this we see that the conclusion of the theorem holds when f is of the form

$$\sum_{i=1}^m a_i 1_{I(n;k_i)},\ a_1, \dots, a_m\in\mathbb{R}.$$

Let $\mathscr{C}$ denote the class of functions obtained in this way as m, n and $k_1, \dots, k_m$ vary.

The theorem is obviously true if $\Gamma=0$ or $f=0$ a.e., so let us suppose that this is not the case. Then we can find a sequence $\{f_p\}_{p=1}^\infty\subset\mathscr{C}$ such that $f_p\neq 0$ for all p, $f_p\to f$ in L^1 and $\|f_p\|_\infty\leq\|f\|_\infty$ for all p. Note that $f_p\to f$ in L^2 also. The theorem now follows from Lemma 4.2 and Chebyshev's inequality. □

Theorem 4.4 has the following distributional corollary, which follows almost immediately from the Minlos-Sazonov theorem. We let $S(\mathbb{R}^d)$ denote the usual Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$ and let $S'(\mathbb{R}^d)$ denote the corresponding dual space of tempered distributions.

Corollary 4.5. *Under the conditions of Theorem 4.4, the map $f\mapsto Z_T(f)$, $f\in S(\mathbb{R}^d)$, defines an $S'(\mathbb{R}^d)$-valued random variable. As $T\to\infty$, $T^{-d/2}Z_T$ converges in distribution to a Gaussian $S'(\mathbb{R}^d)$-valued random variable W with mean 0 and the variance given by $EW(f)^2=\Gamma\int f^2(x)\,dx, f\in S(\mathbb{R}^d)$.*

5. A law of the iterated logarithm

From Theorem 4.4, we see that under suitable conditions $T^{-d/2}Z_T(I)$ converges in distribution to a $N(0, \Gamma)$ random variable as $T\to\infty$, and it is natural to inquire if there is a law of the iterated logarithm "corresponding" to this central limit theorem. We could obtain such a result for associated random measures in general by using the law of the iterated logarithm for sums of associated random variables given in Dabrowski (1985). Unfortunately, such an approach would require that we place conditions on the covariance structure of X which are not satisfied for the applications we have in mind in Sect. 6. We therefore restrict attention to the infinitely divisible case where we have the following estimate, which is probably well known in the literature, but for which we have been unable to find a reference.

Lemma 5.1. *Let Y be an infinitely divisible random variable such that $EY=0$, $EY^2=1$ and $EY^4<\infty$. If we set $\kappa=EY^4-3$, then $\kappa\geqq 0$, and there exists a universal constant c such that*

$$\sup_{y\in\mathbb{R}}|P(Y\leqq y)-\Phi(y)|\leqq(c\kappa^{1/2})\wedge 1,$$

where Φ is the standard normal c.d.f.

Proof. Fix $n\in\mathbb{N}$. Let $Y_1, \ldots, Y_n$ be independent, identically distributed random variables such that $Y_1+\ldots+Y_n$ has the same distribution as Y. Observe that

$$EY^2=nEY_1^2,$$

and

$$EY^4=nEY_1^4+3n(n-1)(EY_1^2)^2,$$

so that

$$(EY_1^4)/(EY_1^2)^2=n\kappa+3. \tag{5.1.1}$$

Dividing both sides of (5.1.1) by n and letting $n\to\infty$, we see that $\kappa\geqq 0$.

Applying the Berry-Esseen theorem (see Theorem XVI.5.1 of Feller (1971)), we have from (5.1.1) that

$$\begin{aligned}\sup_y|P(Y\leqq y)-\Phi(y)|&\leqq\frac{3E|Y_1|^3}{(EY_1^2)^{3/2}n^{1/2}}\\&\leqq\frac{3(EY_1^4)^{3/4}}{(EY_1^2)^{3/2}n^{1/2}}\\&=3(n\kappa+3)^{3/4}n^{-1/2}\\&\leqq 3(n^{1/4}\kappa^{3/4}+3^{3/4}n^{-1/2}).\end{aligned}$$

From this we see that Y will actually be standard normal when $\kappa=0$, so we need only consider $\kappa>0$. If we put $n=[\kappa^{-1}]$, then the bound above becomes

$$3([\kappa^{-1}]^{1/4}\kappa^{3/4}+3^{3/4}[\kappa^{-1}]^{-1/2})\leqq 3(2^{1/4}+3^{3/4})\kappa^{1/2},$$

when $\kappa\leqq 1$, and so we can take $c=3(2^{1/4}+3^{3/4})$. □

Theorem 5.2. *Let X be a stationary, infinitely divisible random measure on $\mathbb{R}^d$ such that $EX(I)^4<\infty$, and X has finite susceptibility, Γ. Suppose further that $\lim_{T\to\infty}T^\delta\kappa(T)=0$ for some $\delta>0$, where $\kappa(T)=(EZ_T(I)^4)(EZ_T(I)^2)^{-2}-3$. Then there exists a constant $\gamma\in[\Gamma^{1/2}, (2\Gamma)^{1/2}]$ such that*

$$\limsup_{T\to\infty}\frac{Z_T(I)}{(T^d\log\log T)^{1/2}}=\gamma\quad\text{a.s.}$$

Proof. It follows from Theorem 3.3 (and, of course, Theorem 1.1) that the lim sup is almost surely constant.

Set $V_t = Z_t(I)$, $t > 0$. Observe that $t \mapsto V_t$ is lower semicontinuous. Observe also that if we consider any partition $0 = t_0 \leqq t_1 \leqq \ldots \leqq t_m = T$, then the random variables $V_{t_i} - V_{t_{i-1}}$, $i = 1, \ldots, m$ are associated. Taking limits over such partitions with the mesh converging to 0, we can therefore conclude from the one-sided version of the maximal inequality on p. 674 of Newman and Wright (1981) that

$$\begin{aligned} P(\sup_{0 \leqq t \leqq T} V_t \geqq \lambda (\operatorname{Var} V_T)^{1/2}) \\ \leqq 2P(V_T \geqq (\lambda - \sqrt{2})(\operatorname{Var} V_T)^{1/2}) \end{aligned}$$

for any $\lambda \in \mathbb{R}$. Applying Lemma 5.1, we see that if T is sufficiently large, then

$$\begin{aligned} P(\sup_{0 \leqq t \leqq T} V_t \geqq (\theta T^d \log\log T)^{1/2}) \\ \leqq 2[1 - \Phi((\theta \Gamma^{-1} \log\log T)^{1/2} - \sqrt{2}) + T^{-\delta/2}] \end{aligned}$$

for any $\theta > 0$. Since $1 - \Phi(x) \sim (2\pi)^{-1/2} x^{-1} \exp(-x^2/2)$ as $x \to \infty$, we conclude from the Borel-Cantelli lemma that if $\theta > 2\Gamma$ and $\alpha > 1$, then

$$P(\sup_{0 \leqq t \leqq \alpha^n} V_t \geqq (\theta \alpha^{nd} \log\log \alpha^n)^{1/2} \text{ i.o.}) = 0,$$

and hence

$$\limsup_{T \to \infty} \frac{V_T}{(T^d \log\log T)^{1/2}} \leqq \theta^{1/2} \alpha^{d/2}.$$

Letting $\alpha \downarrow 1$ and $\theta \downarrow 2\Gamma$, we get the desired upper bound.

For the lower bound, fix $\xi < 1$ and define events $E_1, E_2, \ldots, E'_1, E'_2, \ldots$ by

$$E_n = \{V_{e^n} \geqq (\xi v_n \log n)^{1/2}\}$$

and

$$E'_n = \{V_{e^n} \geqq (\xi v_n)^{1/2} [(\log n)^{1/2} - (\log n)^{-1/2}]\},$$

where we put $v_n = \operatorname{Var}(V_{e^n})$. If $\xi < \xi' < 1$, then from Lemma 5.1 and the approximation to the normal tail mentioned above, we have

$$P(E_n) \geqq n^{-\xi'/2} \tag{5.2.1}$$

for n sufficiently large. Similarly

$$P(E_n)/P(E'_n) \sim e^{\xi} \tag{5.2.2}$$

as $n \to \infty$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing differentiable function with

$$\begin{aligned} &f(x) = 0, \quad x \leqq -1, \\ &f(x) \in]0, 1[, \quad x \in]-1, 0[, \\ &f(x) = 1, \quad x \geqq 0, \end{aligned}$$

and $f'(x) \leqq 2$, $x \in \mathbb{R}$. Set $f_n(x) = f((\xi^{-1} \log n)^{1/2} x)$, $x \in \mathbb{R}$.

From Lemma 3 of Newman (1980), we have for $m < n$ that

$$\begin{aligned} P(E_m \cap E_n) &\leqq E f_m(v_m^{-1/2} V_{e^m} - (\xi \log m)^{1/2}) f_n(v_n^{-1/2} V_{e^n} - (\xi \log n)^{1/2}) \\ &\leqq [E f_m(v_m^{-1/2} V_{e^m} - (\xi \log m)^{1/2})] [E f_n(v_n^{-1/2} V_{e^n} - (\xi \log n)^{1/2})]. \end{aligned}$$

Note that $E f_m(v_m^{-1/2} V_{e^m} - (\xi \log m)^{1/2}) \leqq P(E'_m)$ and similarly for E'_n. Using (5.2.2), we see that for m, n sufficiently large, we can find a constant c^* such that

$$P(E_m \cap E_n) \leqq c^* [P(E_m) P(E_n) + (\log m)^{1/2} (\log n)^{1/2} e^{(m-n)d/2}].$$

Combining this with (5.2.1) implies

$$\sum_{n=1}^{\infty} P(E_n) = \infty,$$

and

$$\limsup_{N \to \infty} \frac{\sum_{m=1}^{N} \sum_{n=1}^{N} P(E_m \cap E_n)}{\left(\sum_{n=1}^{N} P(E_n) \right)^2} < \infty.$$

Applying Proposition 3, Sect. 26 of Spitzer (1976) gives

$$P(E_n \text{ i.o.}) > 0,$$

and hence

$$\limsup_{T \to \infty} \frac{V_T}{(\operatorname{Var}(V_T) \log \log T)^{1/2}} \geqq \xi,$$

almost surely. Since $\operatorname{Var} V_T \sim \Gamma T^d$ as $T \to \infty$, the desired lower bound follows once we let $\xi \uparrow 1$. □

6. Applications to measure-valued Markov branching processes

Given a Borel set $E \subset S$, let $M_F(E)$ denote the subspace of $M(S)$ consisting of measures with finite total mass which is concentrated on E. Let $\mathscr{M}_F(E)$ denote the Borel σ-field of $M_F(E)$ induced by the relative topology inherited from $M(S)$. Note that $\mathscr{M}_F(E)$ is just the trace of $\mathscr{M}(S)$ on $M_F(E)$.

We say that a Markov semigroup $\{Q_t\}_{t \geqq 0}$ on $(M_F(E), \mathscr{M}_F(E))$ is a *measure-valued Markov branching semigroup* if for each pair $\mu, \nu \in M_F(E)$, we have

$$\delta_{\mu+\nu} Q_t = (\delta_\mu Q_t) * (\delta_\nu Q_t), \qquad t \geqq 0,$$

where we write δ_η for the unit point mass at $\eta \in M_F(E)$. It is easy to see that for each $t \geqq 0$, the kernel Q_t is *stochastically monotone*, in that if f: $M_F(E) \to \mathbb{R}$ is a bounded, non-decreasing Borel function, then so is $Q_t f$. Moreover, for each $\mu \in M_F(E)$ the probability measure $\delta_\mu Q_t$ is obviously the law of an infinitely divisible and hence, by Theorem 1.1, associated random measure.

Suppose that Y is a Markov process with state space $(M_F(E), \mathcal{M}_F(E))$ and semigroup $\{Q_t\}_{t\geqq 0}$. We say that Y is a *finite measure-valued Markov branching process*. If the paths of Y lie in $D([0, \infty[, M(S))$ almost surely, then one can argue, using the monotonicity and asserciation properties of $\{Q_t\}_{t\geqq 0}$, that Y is an associated $D([0, \infty[, M(S))$-valued random variable whenever Y_0 is an associated, finite, random measure concentrated on E (cf. Sects. 5, 6 of Lindqvist 1988). The arguments in Lindqvist (1988) require that the path space be "normally ordered", but a reading of the proofs shows that this property is only used to ensure that the path space satisfies the conclusions of Lemma 2.2, and, as we noted in the Remarks following Lemma 2.3, this is the case for $D([0, \infty[, M(S))$.

Following earlier, more specialised constructions, a broad class of finite measure-valued Markov branching processes, the so-called (ξ, φ)-superprocesses, were constructed in Fitzsimmons (1988). We refer the reader to Fitzsimmons (1988) for a full account, but recall the following details.

Suppose that $\xi=(\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, \xi_t, P^x)$ is a Borel right Markov process with state space $(E, \mathcal{E})$, where $\mathcal{E}$ is the Borel σ-field of E, and semigroup $\{P_t\}_{t\geqq 0}$. Assume that $P_t 1=1$. Let $\varphi\colon E\times[0, \infty[\to\mathbb{R}$ be given by

$$\varphi(x, \lambda)=-b(x)\lambda-c(x)\lambda^2+\int n(x, \mathrm{d}u)(1-\mathrm{e}^{-\lambda u}-\lambda u),$$

where $c\geqq 0$ and b are bounded and $\mathcal{E}$-measurable, and $n\colon E\times\mathcal{B}([0, \infty[)\to[0, \infty[$ is a kernel such that $\int n(\cdot, \mathrm{d}u)u\vee u^2)$ is bounded.

For each bounded non-negative, $\mathcal{E}$-measurable function $f\colon E\to\mathbb{R}$, the integral equation

$$v_t(x)=P_t f(x)+\int_0^t P_s(x, \varphi(\cdot, v_{t-s}))\,\mathrm{d}s, \qquad t\geqq 0, \quad x\in E,$$

has a unique solution which we denote by $(t, x)\mapsto V_t f(x)$; and there exists a unique Markov kernel $\{Q_t\}_{t\geqq 0}$ on $(M_F(E), \mathcal{M}_F(E))$ with Laplace functionals

$$\int Q_t(\mu, \mathrm{d}\nu)\exp(-\nu(f))=\exp(-\mu V_t f).$$

There is a Markov process $Y=(W, \mathcal{G}, \mathcal{G}_t, \Theta_t, Y_t, \mathbb{P}^m)$ with state space $(M_F(E), \mathcal{M}_F(E))$ and semigroup $\{Q_t\}_{t\geqq 0}$. One can embed E as a Borel set in a locally compact, separable space which may be different from S (but such that $\mathcal{M}_F(E)$ remains unchanged) in such a way that Y is a Hunt process in the new topology for $M_F(E)$. The conclusion of the discussion above then applies to Y. A special case of this result was proved using different methods in Evans (1989).

Most of the theory we have developed in earlier sections is for stationary associated random measures. We now proceed to construct a class of such random measures using the semigroup $\{Q_t\}$.

Suppose that ξ is a Lévy process on $\mathbb{R}^d$ and $\varphi(x, \lambda)$ does not depend on x. That is,

$$\varphi(x, \lambda)=-b\lambda-c\lambda^2+\int_0^\infty n(\mathrm{d}u)(1-\mathrm{e}^{-\lambda u}-\lambda u).$$

Let N be the law of a stationary, associated random measure on $\mathbb{R}^d$ such that $\int v(I)\,N(\mathrm{d}v)<\infty$, and hence, for some constant $\rho\geqq 0$, we have $\int v(f)\,N(\mathrm{d}v)=\rho\int f(x)\,\mathrm{d}x$ for all $f\in L^1$.

Fix $t\geqq 0$. Using the Kolmogorov extension theorem, we may suppose that we can construct on our probability space a family, $\{U_k\}_{k\in\mathbb{Z}^d}$, of finite random measures such that for $k_1,\ldots,k_m\in\mathbb{Z}^d$ and $A_1,\ldots,A_m\in\mathcal{M}(\mathbb{R}^d)$, we have

$$\begin{aligned}&P(U_{k_1}\in A_1,\ldots,U_{k_m}\in A_m)\\&\quad=\int N(\mathrm{d}v)\,Q_t(\pi_{k_1}v,A_1)\ldots Q_t(\pi_{k_m}v,A_m),\end{aligned}$$

where $(\pi_{k_i}v)(\cdot)=v(\cdot\cap(k_i+I))$, $i=1,\ldots,m$. Observe that, since $\{Q_t\}$ is a measure-valued Markov branching kernel, the law of $U_{k_1}+\ldots+U_{k_m}$ is just $(N\circ(\pi_{k_1}+\ldots+\pi_{k_m})^{-1})\,Q_t$. For $n=1,2,\ldots$, let $R(n)=\{k\in\mathbb{Z}^d\colon |k^j|\leqq n,\ j=1,\ldots,d\}$. We claim first of all that the sequence of random measures

$$X_n=\sum_{k\in R(n)}U_k,\qquad n=1,2,\ldots,$$

converges almost surely to a random measure X. Let $\{f_l\}_{l=1}^{\infty}$ be a countable, vague convergence determining class of non-negative continuous functions with compact support. For each l, the sequence $\{X_n(f_l)\}_{n=1}^{\infty}$ is non-decreasing, and so it possesses a (possibly infinite) limit. By the monotone convergence theorem and the first moment calculation given in Proposition 2.7 of Fitzsimmons (1988), we have

$$\begin{aligned}E(\lim_{n\to\infty}X_n(f_l))&=\lim_{n\to\infty}E(X_n(f_l))\\&=\lim_{n\to\infty}\rho\int_{[-n-\frac12,n+\frac12[^d}e^{-bt}P_tf(x)\,\mathrm{d}x\\&=\rho e^{-bt}\int_{\mathbb{R}^d}f_l(x)\,\mathrm{d}x<\infty,\end{aligned}$$

and so the limit is finite almost surely for each l and X_n does indeed converge almost surely as $n\to\infty$ to a random measure we will denote as X.

The Laplace functional of X is given by

$$E\exp(-X(f))=\int N(\mathrm{d}v)\exp(-vV_tf).$$

It is easy to see from the integral equation defining V_tf that if we set $f_y(\cdot)=f(\cdot+y)$ for $y\in\mathbb{R}^d$, then $V_tf_y(\cdot)=V_tf(\cdot+y)$, and so X is stationary.

As the map $(\sum_{k\in R(n)}\pi_k)\colon M(\mathbb{R}^d)\to M_F(\mathbb{R}^d)$ is non-decreasing, we see from Theorem 3.2 for Lindqvist (1988) that $N\circ(\sum_{k\in R(n)}\pi_k)^{-1}$ is the law of an associated finite random measure. We have already observed that Q_t is stochastically monotone and that $\delta_\mu Q_t$ is the law of an associated random measure for each $\mu\in M_F(\mathbb{R}^d)$; and so, by Theorem 4.1 of Lindqvist (1988), X_n is also an associated random measure. If we now apply Lemmas 2.2 and 2.3 and the Remarks following Lemma 2.3, we find that X is an associated random measure.

Lemma 6.1. *Suppose that N and X are as above. If $\int v(I)^2 N(\mathrm{d}v)<\infty$, and N is the law of a random measure with finite susceptibility, Γ_N, then $EX(I)^2<\infty$ and X has finite susceptibility,*

$$\Gamma_X=\begin{cases}\Gamma_N+\rho\hat{c}t, & b=0,\\ \mathrm{e}^{-2bt}\Gamma_N+\rho\hat{c}b^{-1}(\mathrm{e}^{-bt}-\mathrm{e}^{-2bt}), & b\neq 0,\end{cases}$$

where $\hat{c}=2c+\int u^2 n(\mathrm{d}u)$.

Proof. We have already seen that $EX(I)=\rho\mathrm{e}^{-bt}<\infty$, so $\mathrm{Cov}(X(I), X(TI))$ is well defined for $T>0$, and it suffices to show that $\sup_T \mathrm{Cov}(X(I), X(TI))<\infty$. From the second moment calculation in Proposition 2.7 of Fitzsimmons (1988), we have

$$\begin{aligned}&\mathrm{Cov}(X(I), X(TI))\\&\quad=\int(v\mathrm{e}^{-bt}P_t 1_I)(v\mathrm{e}^{-bt}P_t 1_{TI})\,N(\mathrm{d}v)\\&\qquad-[\int(\mathrm{e}^{-bt}vP_t 1_I)\,N(\mathrm{d}v)][\int(\mathrm{e}^{-bt}vP_t 1_{TI})\,N(\mathrm{d}v)]\\&\qquad+\rho\hat{c}\int\int_0^t \mathrm{e}^{-bs}P_s[(\mathrm{e}^{-b(t-s)}P_{t-s}1_I)(\mathrm{e}^{-b(t-s)}P_{t-s}1_{TI})]\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

By Lemma 4.3, the sum of the first two terms converges to $\mathrm{e}^{-2bt}\Gamma_N\int P_t 1_I(x)\,\mathrm{d}x=\mathrm{e}^{-2bt}\Gamma_N$ as $T\to\infty$. The last term converges to

$$\rho\hat{c}\int_0^t \mathrm{e}^{-bs}\mathrm{e}^{-2b(t-s)}\mathrm{d}s=\begin{cases}\rho\hat{c}t, & b=0,\\ \rho\hat{c}b^{-1}(\mathrm{e}^{-bt}-\mathrm{e}^{-2bt}), & b\neq 0,\end{cases}$$

as $T\to\infty$. □

We are now in a position to apply the results of Sects. 3 and 4. If N has finite susceptibility, then it is clear from Lemma 6.1 and Theorem 3.4 that X is mixing, and hence ergodic. Moreover, the central limit results Theorem 4.4 and Corollary 4.5 hold.

It is not difficult to see that if X is a stationary associated random measure with finite susceptibility, then the Cox process directed by X has the same property. In particular, when $n\equiv 0$, $\xi\equiv$ Brownian motion, and N is the unit point mass concentrated at a multiple of Lebesgue measure, we can use Theorem 9.2 in Dawson and Ivanoff (1978) along with Theorem 4.4 and Corollary 4.5 to recover the central limit theorem for a system of binary branching Brownian motions given as Theorem 6.2 in Dawson and Ivanoff (1978). We could apply the results of Sect. 4 directly to systems of branching Markov processes with more complex branching mechanisms provided we are able to make the necessary moment estimates.

Returning to the general case, we would also like to be able to apply the law of the iterated logarithm given in Theorem 5.2. In order to avoid messy conditions on n and N, we content ourselves with the following result.

Lemma 6.2. *Suppose that $n\equiv 0$ and N is the unit point mass at Lebesgue measure. Then X satisfies the conditions of Theorem 5.2.*

Proof. We have already seen that X is a stationary, infinitely divisible random measure with finite susceptibility.

We can calculate the higher moments of $Z_T(I)$ using the appraoch of Theorem 1.1′ in Dynkin (1988). Setting $P_u^b = \mathrm{e}^{-bu} P_u$, $u \geqq 0$, we have

$$EZ_T(I)^2 = 2c \int \mathrm{d}x \int_0^t \mathrm{d}s\, P_s^b(x, (P_{t-s}^b 1_{TI})^2)$$

$$= 2c \int \mathrm{d}x \int_0^t \mathrm{d}s\, \mathrm{e}^{-bs} (P_{t-s}^b(x, 1_{TI}))^2$$

and

$$EZ_T(I)^4 = (2c)^2 (3J_T + 4K_T) + 3(EZ_T(I)^2)^2,$$

where

$$J_T = \int \mathrm{d}x \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \int_{s_1}^t \mathrm{d}s_3\, \mathrm{e}^{-bs_1} P_{s_2-s_1}^b(x, (P_{t-s_2}^b 1_{TI})^2) \cdot P_{s_3-s_1}^b(x, (P_{t-s_3}^b 1_{TI})^2)$$

and

$$K_T = \int \mathrm{d}x \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \int_{s_2}^t \mathrm{d}s_3\, \mathrm{e}^{-bs_1} P_{t-s_1}^b(x, 1_{TI}) \cdot P_{s_2-s_1}^b(x, (P_{t-s_2}^b 1_{TI})(P_{s_3-s_2}^b((P_{t-s_3}^b 1_{TI})^2))),$$

and we have already used the fact that $\int P_u f(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x$, $u \geqq 0$, to simplify our expressions.

Now

$$J_T \leqq \mathrm{e}^{3|b|t} \int \mathrm{d}x \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \int_{s_1}^t \mathrm{d}s_3\, \mathrm{e}^{-bs_1} (P_{t-s_1}^b(x, 1_{TI}))^2$$
$$\leqq t^2 \mathrm{e}^{3|b|t} EZ_T(I)^2,$$

and

$$K^T \leqq e^{3|b|t} \int \mathrm{d}x \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \int_{s_2}^t \mathrm{d}s_3\, \mathrm{e}^{-bs_1} (P_{t-s_1}^b(x, 1_{TI}))^2$$
$$\leqq t^2 \mathrm{e}^{3|b|t} EZ_T(I)^2.$$

So, in the notation of Theorem 5.2,

$$\kappa(T) = O((EZ_T(I)^2)^{-1}) = O(T^{-d}) \qquad \text{as } T \to \infty,$$

since X has finite susceptibility. □

One can show that if X satisfies the conditions of Theorem 5.2, then so does the Cox process directed by X. We can therefore argue as we did above for the central limit theorem to obtain a law of the iterated logarithm for binary branching Brownian motions "corresponding" to the central limit theorem, Theorem 6.2 in Dawson and Ivanoff (1978). We could also apply the results of

Sect. 5 directly to more complex systems, provided we are able to make the necessary moment estimates.

When $b=0$, $n\equiv 0$, ξ is a symmetric stable process with index $\alpha<d$, and N is the unit point mass at Lebesgue measure, then Dawson (1977) shows that, as $t\to\infty$, the law of X converges to that of a stationary, infinitely divisible random measure X_∞ for which $\operatorname{Cov}(X_\infty(A), X_\infty(B))=\int_A\int_B |x-y|^{\alpha-d}\,dx\,dy$. It is clear from Theorem 3.4 that X_∞ is mixing and hence ergodic. As X_∞ obviously does not have finite susceptibility, our central limit theorem and law of the iterated logarithm do not apply. Indeed, Dawson (1977) shows that the appropriate norming sequence in a central limit theorem for X_∞ is $T^{-(d+\alpha)/2}$, rather than $T^{-d/2}$, and the resulting Gaussian limit is not a white noise, but rather has long-range correlations.

Finally, we remark that the procedure which we used to construct X from the superprocess $\{Y_t\}_{t\geqq 0}$ can also be used to construct stationary, associated random measures based on the "man hours" process $\left\{\int_0^t Y_s\,ds\right\}_{t\geqq 0}$ (see (Iscoe 1986) for a discussion of this process). One can then use our results to analyse this class of random measures in the same way that we have studied X.

Acknowledgements. The author would like to thank Nat Martin and Loren Pitt for some helpful discussions on the subject of the paper. He would also like to thank the Center for Stochastic Processes at the University of North Carolina, Chapel Hill, for supporting a short visit, during which some of the research was carried out.

References

Birkel, T.: On the convergence rate in the central limit theorem for associated processes. Ann. Probab. **16**, 1685–1698 (1988)

Burton, R., Waymire, E.: Scaling limits for associated random measures. Ann. Probab. **13**, 1267–1278 (1985)

Burton, R., Waymire, E.: The central limit problem for infinitely divisible random measures. In: Taqqu, M., Eberlein, E. (eds.) Dependence in probability and statistics. Boston: Birkhauser 1986

Dabrowski, A.R.: A functional law of the iterated logarithm for associated sequences. Statist. Probab. Lett. **3**, 209–212 (1985)

Dawson, D.A.: The critical measure diffusion process. Z. Wahrscheinlichkeitstheor. Verw. Geb. **40**, 125–145 (1977)

Dawson, D.A., Ivanoff, G.: Branching diffusions and random measures. In: Joffe, A., Ney, P. (eds.) Advances in probability and related topics, vol. 5, pp. 61–103. New York: Dekker 1978

Dynkin, E.B.: Representation of functionals of superprocesses by multiple stochastic integrals, with applications to self-intersection local times. In: Colloque Paul Lévy sur les processus stochastiques, Astérisque 157–158 (Société Mathématique de France) (1988)

Esary, J., Proschan, F., Walkup, D.: Association of random variables with applications. Ann. Math. Statist. **38**, 1466–1474 (1967)

Evans, S.N.: Superprocesses and association. Preprint (1989)

Evans, S.N.: Rescaling the vacancy of a Boolean coverage process. In: Seminar on stochastic processes, 1989. Boston: Birkhauser 1990

Feller, W.: An introduction to probability theory and its applications, vol. II. New York: Wiley 1971

Fitzsimmons, P.J.: Construction and regularity of measure-valued Markov branching processes. Isr. J. Math. **64**, 337–361 (1988)
Fortuin, C.M., Kasteleyn, P.W., Ginibre, J.: Correlation inequalities on some partially ordered spaces. Commun. Math. Phys. **22**, 89–103 (1971)
Harris, T.E.: A correlation inequality for Markov processes in partially ordered state spaces. Ann. Probab. **5**, 451–454 (1977)
Herbst, I.W., Pitt, L.D.: Diffusion equation techniques in stochastic monotonicity and positive correlations. Probab. Th. Rel. Fields (to appear)
Iscoe, I.: A weighted occupation time for a class of measure-valued critical branching Brownian motion. Probab. Th. Rel. Fields **71**, 85–116 (1986)
Kallenberg, O.: Random measures. Berlin: Akademie-Verlag and New York: Academic Press 1983
Lindqvist, B.H.: Association of probability measures on partially ordered spaces. J. Multivar. Anal. **26**, 111–132 (1988)
Matthes, K., Kerstan, J., Mecke, J.: Infinitely divisible point processes. New York: Wiley 1978
Newman, C.M.: Normal fluctuations and the FKG inequalities. Commun. Math. Phys. **74**, 119–128 (1980)
Newman, C.M.: Asymptotic independence and limit theorems for positively and negatively dependent random variables. In: Tong, Y.L. (ed.) Inequalities in statistics and probability, pp. 127–140. Hayward: IMS
Newman, C.M., Wright, A.L.: An invariance principle for certain dependent sequences. Ann. Probab. **9**, 671–675 (1981)
Petersen, K.: Ergodic theory. Cambridge: Cambridge University Press 1983
Spitzer, F.: Principles of random walk, 2nd edn. Berlin Heidelberg New York: Springer 1976