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Large deviations w.r.t. quasi-every starting point for symmetric right processes on general state spaces

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Summary. In the work of Donsker and Varadhan, Fukushima and Takeda and that of Deuschel and Stroock it has been shown, that the lower bound for the large deviations of the empirical distribution of an ergodic symmetric Markov process is given in terms of its Dirichlet form. We give a short proof generalizing this principle to general state spaces that include, in particular, infinite dimensional and non-metrizable examples. Our result holds w.r.t. quasi-every starting point of the Markov process. Moreover we show the corresponding weak upper bound w.r.t. quasi-every starting point.

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1 Introduction

Already in the work [Do-V2] of Donsker and Varadhan the Dirichlet form appears naturally as the function governing the large deviations of the empirical distribution $L_t(\omega, dx)$ defined by $L_t(\omega, A) := \frac{1}{t} \int_0^t I_A(X_s(\omega))\,ds$. Fukushima and Takeda were the first to derive the lower bound for quasi-every starting point $x \in E$, in case of an m-symmetric ergodic Hunt process associated with a regular Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E; m)$ on a locally compact separable metric space E in their well known paper [F-T] [1]. Their result is a corollary to a highly non-trivial representation formula for the Dirichlet form associated to a Girsanov-transformation. The first result

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[1] More precisely, the irreducibility of the Dirichlet space is assumed instead of the ergodicity of the process. However, by [F2, p. 201, 202], these assumptions are equivalent

of this paper extends the large deviation result in [F-T], but we avoid the mentioned representation formula so that our proof is shorter and uses more L^2-semigroup theory. Our objective was to generalize their result to infinite dimensional situations. Deuschel and Stroock already considered large deviations on a polish state space E of a Markov family $(P_x)_{x\in E}$ on the Skorohod space $D([0,\infty);E)$ that admits an m-symmetric process with P_x-a.s. left continuous paths in their book [D-St]. They showed that for any probability measure ν such that there is a $t>0$ such that νP_t is not singular w.r.t. m, the lower bound of the large deviations of $P_\nu\circ L_t^{-1}, t>0$, is governed by the Dirichlet form, or, more precisely, the functional $J_{\mathscr{E}}$ defined in Theorem 1 below (cf. [D-St, 5.3.10]). Thus their result holds for $P_x\circ L_t$, $t>0$, if for some $t>0$ the measure $p_t(x,\cdot)$ is not singular (e.g. absolutely continuous) w.r.t. the measure m. This is typically not fulfilled for any x in the infinite dimensional examples from quantum field theory we consider in Sect. 6 (compare Remark 16(i) below).

By the development in the theory of Dirichlet forms as presented in the book of Ma and Röckner [M-R] it is now possible to complement their result. However, we need the quasi-regularity of the underlying Dirichlet form, which is related to more regularity of the associated Markov process, namely the strong Markov property. Moreover, it is the decisive assumption that allows to consider all semigroups and corresponding generators on $L^2(E;m)$ instead of the subspace B^0 of the space of bounded measurable functions as in [D-St, p. 122] and [Do-V2], [Do-V3], and still to get quasi-everywhere results. The main idea of proof is here as well as in the cited literature a drift transformation and goes back to Donsker and Varadhans paper "Asymptotic evaluation of certain Wiener integrals for large time", [Do-V1].

For the weak upper bound the following results are known. In [D-St, 5.3.2] it was shown that the large deviations of $P_m\circ L_t^{-1}, t>0$, are governed by $J_{\mathscr{E}}$ in any topology w.r.t. which $J_{\mathscr{E}}$ is lower semicontinuous. By [D-St, Ex. 4.2.63] $J_{\mathscr{E}}$ is indeed lower semicontinuous (and henceforth a rate function) w.r.t. the τ-topology on $\mathscr{M}_1(E)$ (:= the space of all probability measures on E), i.e., the topology generated by open sets of the form

$$U(\mu;\delta,f):=\{\nu\in\mathscr{M}_1(E):|\int f\,d\mu-\int f\,d\nu|<\delta\},$$

$\mu\in\mathscr{M}_1(E)$, f bounded and measurable, $\delta>0$. Secondly, in case the transition kernel is Feller continuous, the functional $J_{\mathscr{E}}$ can be identified with various other rate functions (see [Do-V2, Thm. 5], [D-St, 4.2.58]), that govern the large deviations of $P_x\circ L_t^{-1}, t>0$, $\sup_{x\in E} P_x\circ L_t^{-1}, t>0$, respectively (see [Do-V2], [D-St, 4.2.16, 4.2.17]) if the Markov process is uniformly ergodic (which implies the full upper bound, too) and satisfies some further regularity assumptions. Thus our weak upper bound (Theorem 2 below) is stronger, on the expense of a "quasi-everywhere statement".

Let us now describe our main results more precisely, that apply even for non metrizable state spaces. We consider an m-symmetric right process $\mathbb{M}$. Let $(\mathscr{E}, D(\mathscr{E}))$ be the corresponding symmetric quasi-regular Dirichlet form on $L^2(E; m)$, where m is a probability measure and the state space E is only assumed to be a topological Hausdorff space on which the Borel- and the Baire-σ-algebra coincide. Let $(P_x)_{x \in E}$ be the Markov family of $\mathbb{M}$ and set $P_m := \int_E P_x m(dx)$.

Theorem 1 *Assume that P_m is ergodic and let $U \subseteq \mathscr{M}_1(E)$ be τ-open. Then for $\mathscr{E}$-quasi-every $x \in E$*

$$\liminf_{t \to \infty} \frac{1}{t} \log P_x[L_t \in U] \geqq - \inf_{\mu \in U} J_{\mathscr{E}}(\mu),$$

where $J_{\mathscr{E}}(\mu) := \mathscr{E}(\varphi, \varphi)$, if $\mu = \varphi^2 m$, $\varphi \in D(\mathscr{E})$, $J_{\mathscr{E}}(\mu) := \infty$ else.

Then it is standard to derive the lower bound of Varadhan's integral lemma, see Corollary 12 below. Once $J_{\mathscr{E}}$ is identified as an appropriate functional (see Proposition 13 below), we show the corresponding weak upper bound by the standard Cramer method. For the form applied here, we refer to Liming Wu [W1].

Theorem 2 *Let $K \subset \mathscr{M}_1(E)$ be τ-compact. Then*

$$\inf_{N: Cap(N) = 0} \ \sup_{x \in E \setminus N} \ \limsup_{t \to \infty} \frac{1}{t} \log P_x[L_t \in K] \leqq - \inf_{\mu \in K} J_{\mathscr{E}}(\mu). \tag{1}$$

For some concluding remarks on the upper bound we refer to the Appendix. The organization of the paper is as follows: in Sect. 2, we give the basic definitions. In Sect. 3 we give a construction of a Markov process $\mathbb{M}^\varphi$ that exhibits as typical behaviour what is considered as a large deviation for $\mathbb{M}$. In spite of the generality of the state space the transformed process can be constructed on a subset of the original path space (Theorem 21 below). The method of proof is standard apart from technicalities, so we only give it in the appendix. The proof of conservativeness uses ideas from the theory of L^2-semigroups presented in A. Pazy's book [P]. In addition, it gives immediately the symmetry of the transformed semigroup, thus shortening the argument, and moreover it gives the existence of an associated Dirichlet form. The condition in [D-St, 5.3.10] on the Dirichlet form ensures (and is in fact equivalent to) the ergodicity of the process $\mathbb{M}$ (see Lemma 9 below). In Sect. 4 we prove Theorem 1. The reader's attention is also directed to Lemma 11 which describes a method how to obtain $\mathscr{E}$-q.e. statements from m-a.e. statements. This we apply to the ergodic theorem (see the proof of Theorem 1 and compare [F-T, after (4.3)]). In Sect. 5 we prove Theorem 2. In Sect. 6 we consider some examples from quantum field theory demanding the generality of our results: the time zero free field, the space time free field and perturbations of both. For these examples a (full) upper bound holds, see Theorem 19 below.

2 Definitions and notation

Let E be a Hausdorff topological space, $C(E)$ the space of continuous real valued functions on $E, \mathscr{B}(E)$ the Borel-σ-Algebra on E and assume that $\mathscr{B}(E)=\sigma(C(E))$. By $\mathscr{M}_1(E)$ we denote the space of probability measures on E and by $\mathscr{B}_b(E)$ the bounded $\mathscr{B}(E)/\mathscr{B}(\mathbb{R})$-measurable functions. Let $m\in\mathscr{M}_1(E)$. By $(\cdot,\cdot)$ we denote the inner product in $L^2(E;m)$.

Definition 3 A symmetric bilinear form $\mathscr{E}(\cdot,\cdot)$ on $L^2(E;m)$ with $D(\mathscr{E})$ dense in $L^2(E;m)$ is called a *Dirichlet form* if it is closed (i.e., $D(\mathscr{E})$ is complete with respect to $\mathscr{E}_1:=\mathscr{E}+(\cdot,\cdot)$) and if every normal contraction operates on $(\mathscr{E}, D(\mathscr{E}))$, i.e., $\mathscr{E}(T(u), T(u))\leqq\mathscr{E}(u,u)$ for every $u\in D(\mathscr{E})$ and every T: $\mathbb{R}\to\mathbb{R}$ satisfying $T(0)=0$, $|T(x)-T(y)|\leqq|x-y|$, x, $y\in\mathbb{R}$.

From now on we fix a Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E;m)$. Let $(L, D(L))$ denote the unique selfadjoint negative definite operator such that $D(L)\subset D(E)$ and $\mathscr{E}(u,v)=(-Lu,v)$, $u\in D(L)$, $v\in D(\mathscr{E})$. The generator $(L, D(L))$ is in one-to-one correspondence with a strongly continuous semigroup of symmetric contractions $T_t=e^{tL}, t\geqq 0$, and a strongly continuous symmetric contraction resolvent $G_\alpha=(\alpha-L)^{-1}$, $\alpha>0$, on $L^2(E;m)$ (see, e.g., [M-R, Diagram 3, remarks after I.2.22]).

We assume from now on that $T_t 1=1$ for all $t>0$. So we have in particular $1\in D(L)$. Let us recall the definition of the *(1-) Capacity* associated with $(\mathscr{E}, D(\mathscr{E}))$:

For $U\subseteq E$, U open, set

$$Cap(U):=\inf\{\mathscr{E}_1(u,u): u\geqq 1\,m-\text{a.e. on } U\},$$

and for $A\subseteq E$ arbitrary

$$Cap(A):=\inf\{Cap(U): A\subseteq U, U \text{ open}\}.$$

A set $N\subseteq E$ is called *$\mathscr{E}$-exceptional*, iff $Cap(N)=0$. An increasing sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets is called an *$\mathscr{E}$-nest*, iff $\lim_{n\to\infty} Cap(E\setminus F_n)=0$. We say that a property holds *$\mathscr{E}$-quasi-everywhere* ($\mathscr{E}$-q.e.) iff there is an $\mathscr{E}$-exceptional set N, such that it holds everywhere on $E\setminus N$. A function f: $A\to\mathbb{R}$, $A\subseteq E$, is called *$\mathscr{E}$-quasi-continuous*, iff there exists an $\mathscr{E}$-nest $(F_n)_{n\in\mathbb{N}}$, such that f is in

$$C(\{F_n\}):=\{f: B\to\mathbb{R}: \bigcup F_n\subseteq B\subseteq E, f|_{F_n},\ n\in\mathbb{N}, \text{ is continuous}\}.$$

Remark. Since we assume $T_t 1=1$, this capacity is the capacity $Cap_{h,g}$ defined in [M-R, III.2.4] for $h=g=G_1 1=1$. By virtue of [M-R, III.2.11] the definitions of $\mathscr{E}$-nests and $\mathscr{E}$-exceptional sets given here are equivalent to those in [M-R].

Definition 4 *The Dirichlet form* $(\mathscr{E}, D(\mathscr{E}))$ *is called* quasi-regular, *iff*
(i) *There is an* $\mathscr{E}$*-nest* $(E_n)_{n\in\mathbb{N}}$ *consisting of compact sets.*
(ii) *There is an* $\mathscr{E}_1^{1/2}$*-dense subset of* $D(\mathscr{E})$*, whose elements have* $\mathscr{E}$*-quasi-continuous m-versions.*
(iii) *There exist* $u_n\in D(\mathscr{E})$, $n\in\mathbb{N}$, *having* $\mathscr{E}$*-quasi-continuous m-versions* $\tilde{u}_n, n\in\mathbb{N}$, *and an* $\mathscr{E}$*-exceptionel set* $N\subseteq E$, *such that* $\{\tilde{u}_n | n\in\mathbb{N}\}$ *separates the points of* $E\backslash N$.

We adjoin a point Δ (the cemetery) as an isolated point to E, if E is not locally compact, else we take $E_\Delta := E\cup\{\Delta\}$ to be the one point compactification of E. From now on we fix a quasi-regular Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$. Let $\mathbb{M}=(\Omega, (\mathscr{F}_t)_{t\geqq 0}, (X_t)_{t\geqq 0}, (P_x)_{x\in E_\Delta}, \zeta)$ be a right process with state space E and life time ζ, which is properly associated to $(\mathscr{E}, D(\mathscr{E}))$, i.e., $p_t f(x) := E_x[f(X_t)]$ is an $\mathscr{E}$-quasi-continuous m-version of $T_t f$ for all $f\in\mathscr{B}_b(E)$, $t>0$. Here, $E_x[\cdot]$ denotes expectation w.r.t. P_x. By [M-R, IV.3.5] there always exists such a right process, which is, in particular, m-tight, m-special standard.

Remark. We emphasize, that in most cases we do not have to worry wether a quasi-regular Dirichlet form is given a priori. We can always start right away with a right process $\mathbb{M}$ which is symmetric w.r.t. the measure m, if the state space E is a Borel-subset of a polish space. In that case the symmetric Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E;m)$ corresponding to it is automatically quasi-regular and the process $\mathbb{M}$ is properly associated to it (cf. [M-R, IV.6.7]).

We assume w.l.o.g. that we are in the following canonical situation. Let Ω' be the set of all maps $\omega\colon \mathbb{R}_0^+ \to E_\Delta$ and define $\zeta(\omega) := \inf\{t>0\colon \omega(t) = \Delta\}$, $\omega\in\Omega'$. Then let Ω be the set of all $\omega\in\Omega'$ with $\omega(t)=\Delta$ for all $t\geqq\zeta$, that are right continuous on $[0,\infty)$ and have left limits in E on $(0,\zeta)$. Further let $X_t(\omega) := \omega(t)$, $\mathscr{F}_t^0 := \sigma(X_s; s\leqq t)$ and let $\mathscr{F}_t$ denote the universal completion of $\mathscr{F}_t^0$, i.e., the intersection of the P_ν-completions of $\mathscr{F}_t^0$, as ν runs over all probability measures on $(E_\Delta, \mathscr{B}(E_\Delta))$. Finally let $\mathscr{F}$ be the universal completion of $\sigma(X_s; s\geqq 0)$ and let $\{\theta_t\colon \Omega\to\Omega, t\geqq 0\}$, denote the time-shift semigroup defined by $X_s(\omega\circ\theta_t) = X_{s+t}(\omega)$.

Let $\overline{X_0^t}(\omega)$ denote the closure of $\{X_s(\omega)\colon 0\leqq s\leqq t\}$ in E. A Borel set $S\in\mathscr{B}(E)$ is called $\mathbb{M}$-*invariant*, iff there exists $\Omega_{E\backslash S}\in\mathscr{F}$ such that

$$\Omega_{E\backslash S} \supseteq \{\overline{X_0^t}\cap(E\backslash S)\neq\emptyset \quad \text{for some } 0\leqq t<\zeta\}$$

and $P_x[\Omega_{E\backslash S}]=0$ for all $x\in S$. We will call $\Omega\backslash\Omega_{E\backslash S}$ the *defining set for S*.

For convenience we state the following

Lemma 5 *Let $f\colon E\to\mathbb{R}$ be an $\mathscr{E}$-quasi-continuous function that is constant m-a.e. Then there is an $\mathscr{E}$-exceptional Borel set N, such that $E\backslash N$ is $\mathbb{M}$-invariant and such that f is constant on $E\backslash N$.*

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be an $\mathscr{E}$-nest such that $f\in C(\{F_n\})$ and let $(E_n)_{n\in\mathbb{N}}$ be the $\mathscr{E}$-nest consisting of compact sets of Definition 4. Set $K_n:=\operatorname{supp}[I_{E_n\cap F_n}m]$. Then, by [M-R, III.3.8], $(K_n)_{n\in\mathbb{N}}$ is a regular $\mathscr{E}$-nest and by [MR 92, III.3.9] f is constant on $\bigcup_{n\in\mathbb{N}} K_n$. By [M-R, IV.6.5] there is an $\mathscr{E}$-exceptional set $N, E\backslash N\subset\bigcup_{n\in\mathbb{N}} K_n$, such that $E\backslash N$ is $\mathbb{M}$-invariant. □

We call a Markov process corresponding to a Markov family $(Q_x)_{x\in E}$ *conservative*, iff $Q_x[\zeta<\infty]=0$ for $\mathscr{E}$-q.e. $x\in E$.

Since we assumed $T_t1=1$, Lemma 5 gives $p_t1(x)=1$ and thus $P_x[\zeta<\infty]=0$ on an $\mathbb{M}$-invariant set $S\subseteq E$ with defining set Ω_S, such that $E\backslash S$ is $\mathscr{E}$-exceptional. Now, w.l.o.g. we may identify $\mathbb{M}$ with the trivial extension to E of its restriction to S (cf. the appendix for details). Then we even have $P_x[\zeta<\infty]=0$ for all $x\in E$.

3 A transformation

The resolvent $(R_\alpha)_{\alpha>0}$ of $(p_t)_{t\geqq 0}$ is defined by $R_\alpha g(x):=\int e^{-\alpha t}p_tf(x)dt$, $\alpha>0$, $g\in\mathscr{B}_b(E)$. Let $\alpha, c>0$, $g\in\mathscr{B}_b(E)$, $g\geqq c$, and set $\varphi:=R_\alpha g$. Then $\|\varphi\|_\infty\leqq 1/\alpha\,\|g\|_\infty$, so $\varphi\in\mathscr{B}_b(E)\cap D(L)$. By [M-R, IV.2.8] $R_\alpha\varphi$ and $R_\alpha 1$ are $\mathscr{E}$-quasi-continuous. Since $p_t1=1$ m-a.e., by Lemma 5 there is an $\mathscr{E}$-exceptional set N_0, such that $E\backslash N_0$ is $\mathbb{M}$-invariant, and $p_t1=1$, $t>0$, hence $R_\alpha 1=1/\alpha$ on $E\backslash N_0$. Hence $\varphi\geqq R_\alpha c\geqq c/\alpha$ on $E\backslash N_0$. Finally $L\varphi=\alpha\varphi-g$ is bounded by $2\|g\|_\infty$. In the following we will study the semigroup $(p_t^{(\varphi)})_{t>0}$ defined by

$$p_t^{(\varphi)}f(x):=\begin{cases} E_x\left[f(X_t)\dfrac{\varphi(X_t)}{\varphi(X_0)}\exp\left[-\int\limits_0^t\dfrac{L\varphi}{\varphi}(X_s)ds\right]\right] & :x\in E\backslash N_0,\\ f(x) & :x\in N_0.\end{cases}$$

Lemma 6 *Let $f\in\mathscr{B}_b(E)$. Then $p_t^{(\varphi)}f$ is $\mathscr{E}$-quasi-continuous for all $t>0$.*

Proof. It suffices to consider positive $f\in\mathscr{B}_b(E)$. Let $V:=-\dfrac{L\varphi}{\varphi}$, $s>0$ and set

$$g_{\pm s}(x):=\exp[\pm s\|V\|_\infty]E_x\left[f(X_t)\frac{\varphi(X_t)}{\varphi(X_0)}\exp\left[\int\limits_s^t V(X_r)dr\right]\right].$$

Then $g_{-s}(x)\leqq p_t^{(\varphi)}f(x)\leqq g_{+s}(x)$. By the Markov property

$$g_{\pm s}(x)=\exp[\pm s\|V\|_\infty]\frac{1}{\varphi(x)}\left(p_sE\left[f(X_{t-s})\varphi(X_{t-s})\exp\left[\int\limits_0^{t-s}V(X_r)dr\right]\right]\right)(x).$$

Thus, since $\mathbb{M}$ is properly associated to $(\mathscr{E}, D(\mathscr{E}))$ and φ is $\mathscr{E}$-quasi-continuous, $g_{\pm s}$ is $\mathscr{E}$-quasi-continuous too. So, by [M-R, III.3.3], there exists an $\mathscr{E}$-nest $(F_n)_{n\in\mathbb{N}}$ such that $g_{\pm 1/n}|_{F_k}$ is continuous for all $n\in\mathbb{N}$ and such that (w.l.o.g.) $\bigcup F_n \subseteq E\setminus N_0$, N_0 as at the beginning of the section. Since $(p_t^{(\varphi)} f)|_{F_k} = \inf_{n\in\mathbb{N}} g_{+1/n}|_{F_k}$, it is upper semi-continuous. Correspondingly we have $(p_t^{(\varphi)} f)|_{F_k} = \sup_{n\in\mathbb{N}} g_{-1/n}|_{F_k}$, so it is lower semi-continuous, hence continuous. □

Lemma 7 *The semigroup $(p_t^{(\varphi)})_{t>0}$ induces a strongly continuous semigroup $(T_t^{\varphi})_{t>0}$ on $L^2(E;\varphi^2 m)$ satisfying*

$$\|T_t^{\varphi}\| \leqq \frac{\|\varphi\|_\infty}{\inf \varphi} \exp\left[t\left\|\frac{L\varphi}{\varphi}\right\|_\infty\right].$$

We shall see below, that $(T_t^{\varphi})_{t>0}$ is in fact a contraction semigroup.

Proof. Let $f, g\in\mathscr{B}_b(E)$, $f=g\,m$-a.e., and set $C:=\left\|\frac{L\varphi}{\varphi}\right\|_\infty$. Then

$$\int p_t^{(\varphi)}(f-g)\varphi^2\,dm \leqq \|\varphi^2\|_\infty \exp[tC]\int p_t|f-g|\,dm = 0,$$

so $p_t^{(\varphi)}$ respects $\varphi^2 m$-classes. Let $[\cdot]$ denote the $\varphi^2 m$-class of a function and define $T_t^{(\varphi)}[f] := [p_t^{(\varphi)} f]$. Since for positive $f\in\mathscr{B}_b(E)$

$$\frac{1}{\varphi} e^{-tC} p_t(f\varphi) \leqq p_t^{(\varphi)} f \leqq \frac{1}{\varphi} e^{tC} p_t(f\varphi),$$

the strong continuity of $(T_t)_{t>0}$ implies that of $(T_t^{\varphi})_{t>0}$. The asserted inequality is obvious. □

Proposition 8 (i) *The generator L^{φ} of $(T_t^{\varphi})_{t>0}$ on $L^2(E;\varphi^2 m)$ is given by*

$$D(L^{\varphi}) = \{u\in L^2(E;\varphi^2 m) : u\varphi\in D(L)\}$$

$$L^{\varphi} u = \frac{1}{\varphi}\{L(u\varphi) - uL\varphi\}, \qquad u\in D(L^{\varphi}).$$

In particular, $(L^{\varphi}, D(L^{\varphi}))$ is self-adjoint, hence $(T_t^{\varphi})_{t>0}$ symmetric.
(ii) *There is an $\mathscr{E}$-exceptional set N_1 such that $E\setminus N_1$ is $\mathbb{M}$-invariant and $p_t^{\varphi} I_E(x) = 1$ for all $x\in E\setminus N_1$ and all $t>0$ simultaneously.*
(iii) *The semigroup $(T_t^{\varphi})_{t>0}$ is a contraction semigroup.*

Proof. (i) Set $V := \frac{L\varphi}{\varphi}$ and define

$$p_t^V f(x) := E_x\left[f(X_t)\exp\left[\int_0^t V(X_s)\,ds\right]\right], \qquad f\in\mathscr{B}_b(E).$$

Then $(p_t^V)_{t>0}$ is the Feynman-Kac semigroup associated with V, which is selfadjoint on $L^2(E;m)$. Since V is bounded, it is well known that its generator

L^V on $L^2(E;m)$ is given by $L^V = L + V$. Since $\frac{1}{\varphi} p_t^V(\varphi u) = p_t^{(\varphi)} u$ for all $u \in L^2(E;\varphi^2 m)$ and $D(L^V) = D(L)$ the first part of assertion (i) follows. Note, that $(L^\varphi, D(L^\varphi))$ is the sum of an operator that is unitary equivalent to the self-adjoint operator $(L, D(L))$ and a bounded multiplication operator. Hence it is self-adjoint.

(ii) By Lemma 7 and [P, I.5.3] the resolvent set $\rho(L^\varphi)$ of L^φ contains the ray $\left]\left\|\frac{L\varphi}{\varphi}\right\|_\infty, \infty\right[$. Let $G_\alpha^\varphi = (\alpha - L^\varphi)^{-1}$, $\alpha \in \rho(L^\varphi)$, denote the resolvent of L^φ. Obviously $L^\varphi 1 = 0$. So $\alpha G_\alpha^\varphi 1 = G_\alpha^\varphi(\alpha - L^\varphi) 1 = 1$ and $\alpha L^\varphi G_\alpha^\varphi 1 = 0$. Henceforth, by the representation of T_t^φ that holds due to (i) and [P, I.5.5], $T_t^\varphi 1 = \lim_{\alpha \to \infty} e^{t\alpha L^\varphi G_\alpha^\varphi} 1 = 1$. So $p_t^{(\varphi)} 1 = 1\, m$-a.e. Now let $(t_n)_{n \in \mathbb{N}}$ be an enumeration of the positive rational numbers. By [M-R, III.3.3] there is an $\mathscr{E}$-nest $(F_n)_{n \in \mathbb{N}}$ w.r.t. which $p_{t_n}^{(\varphi)} 1$ is $\mathscr{E}$-quasi-continuous for all $n \in \mathbb{N}$. So Lemma 5 gives an $\mathscr{E}$-exceptional set N_1, such that $E \setminus N_1$ is $\mathbb{M}$-invariant and $p_{t_n}^{(\varphi)} 1 = 1$ on $E \setminus N_1$. By the right-continuity of $\mathbb{M}$ and Lebesgue's theorem of dominated convergence then follows $p_t^{(\varphi)} 1 = 1$ on $E \setminus N_1$ for all $t > 0$.

(iii) By (ii) and symmetry of $(T_t^\varphi)_{t>0}$ we see that $\int (T_t^\varphi g)^2 \varphi^2 \, dm = \int (p_t^{(\varphi)} g)^2 \varphi^2 \, dm \leq \int (p_t^{(\varphi)} g^2) \varphi^2 \, dm = \int g^2 (p_t^{(\varphi)} 1) \varphi^2 \, dm = \int g^2 \varphi^2 \, dm$, $\quad g \in \mathscr{B}_b(E)$, which gives the assertion. □

Remark. By Proposition 8 it is clear that there is a Dirichlet form $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ correspondiong to $(T_t^\varphi)_{t>0}$.

Lemma 9 *The following conditions are equivalent:*

(i) P_m *is ergodic, i.e., any random variable* Z *satisfying* $P_m[Z = Z \circ \theta_t] = 1$ *for all* $t > 0$ *is* P_m*-a.s. constant.*

(ii) *If* $g \in \mathscr{B}_b(E)$ *satisfies* $p_t g = g$, *then* g *is constant* m*-a.e.*

(iii) *If* $u \in D(\mathscr{E})$ *satisfies* $\mathscr{E}(u, u) = 0$, *then* u *is constant* m*-a.e.*

Proof. (i)$\Leftrightarrow$(ii) follows by the well known fact (see, e.g., [F2]), that a bounded measurable random variable Z is shift-invariant, i.e., $P_m[Z = Z \circ \theta_t] = 1$ for all $t \geq 0$ if and only if $Z = g(X_0) P_m$-a.s. for a function $g \in \mathscr{B}_b(E)$ satisfying $p_t g = g$.

(ii)$\Rightarrow$(iii). Let $u \in D(\mathscr{E})$, $\mathscr{E}(u, u) = 0$. Then $|\mathscr{E}(u, v)| \leq \mathscr{E}(u, u)^{1/2} \mathscr{E}(v, v)^{1/2} = 0$ for all $v \in D(\mathscr{E})$, hence $u \in D(L)$ and $Lu = 0$. Consequently $\alpha G_\alpha u = G_\alpha(\alpha - L) u = u$ and $T_t u = \lim_{\alpha \to \infty} e^{t\alpha(\alpha G_\alpha - 1)} u = u$.

(iii)$\Rightarrow$(ii). Let $p_t g = g$. Then $g \in D(L)$ and $\mathscr{E}(g, g) = \lim_{t \to 0} \frac{1}{t} (g - T_t g, g) = 0$. Hence g is constant m-a.e. by (iii). □

Now there is an $\mathbb{M}$-invariant set $S^\varphi \subset E$ such that $E \setminus S^\varphi$ is $\mathscr{E}$-exceptional and such that one can show by the standard multiplicative functional technique that there is a unique family of (conservative) Markov kernels $(P_x^\varphi)_{x \in S^\varphi}$ rendering $(X_t)_{t \geq 0}$ Markov with semigroup $(p_t^{(\varphi)})_{t \geq 0}$. For details we refer to the appendix.

The idea of the proof of the following Proposition is as in [D-St, 5.3.9].

Proposition 10 *The measure* $P_{\varphi^2 m}^\varphi := \int P_x^\varphi \varphi^2(x) \, m(dx)$ *is ergodic, if* P_m *is ergodic.*

Proof. Let $f\in\mathscr{B}_b(E)$, $f=p_t^{(\varphi)}f$ m-a.e. Then $f\in D(L^\varphi)$ and, since $p_t^{(\varphi)}1=1$,

$$\int (f(x)-f(y))^2\, p_t^{(\varphi)}(x,dy)\,\varphi^2(x)\,m(dx)=0.$$

Set $C:=\left\|\frac{L\varphi}{\varphi}\right\|_\infty$, $\gamma:=\frac{\inf\varphi}{\|\varphi\|_\infty}$. Now $\gamma e^{-tC}p_t^{(\varphi)}\leqq p_t\leqq\gamma^{-1}e^{tC}p_t^{(\varphi)}$ and $p_t1=1$ imply $E_m[(f(X_0)-f(X_t))^2]=\int (f(x)-f(y))^2 p_t(x,dy)m(dx)=0$. Hence by symmetry and conservativeness

$$\begin{aligned}0&=\lim_{t\to 0}\frac{1}{t}E_m[(f(X_0)-f(X_t))^2]\\&=\lim_{t\to 0}\frac{1}{t}E_m[2f^2(X_0)-2f(X_0)f(X_t)]\\&=\lim_{t\to 0}\frac{1}{t}2(f-p_tf,f)\\&=2\mathscr{E}(f,f).\end{aligned}$$

The assumption and Lemma 9 lead to the conclusion. □

4 Proof of the lower bound

Let $(P_x^\varphi)_{x\in S^\varphi}$ with semigroup $(p_t^{(\varphi)})_{t>0}$ be the Markov family constructed in the previous chapter.

Lemma 11 *Let F be $(\bigvee_{t\geqq 0}\mathscr{F}_t)/\mathscr{B}(\mathbb{R})$-measurable, bounded and such that $P_x^\varphi[F=F\circ\theta_t]=1$ for $\mathscr{E}$-q.e. $x\in E$ and all $t>0$. Then $[x\mapsto E_x^\varphi[F]]$, where $E_x^\varphi[\cdot]$ denotes expectation w.r.t. P_x^φ, is $\mathscr{E}$-quasi-continuous. If, in particular, $F=I_A$ and $P_x^\varphi[A]=c$, $c\in[0,1]$, for m-a.e. $x\in E$, then $P_x^\varphi[A]=c$ for $\mathscr{E}$-q.e. $x\in E$.*

Proof. Set $f(x):=E_x^\varphi[F]$. By shift-invariance of F it follows with the Markov property that $p_t^{(\varphi)}f(x)=f(x)$ P_x^φ-a.s. for $\mathscr{E}$-q.e. $x\in E$. Now, by Lemma 6, $p_t^{(\varphi)}f$ and hence f is $\mathscr{E}$-quasi-continuous. □

Proof of Theorem 1 Let $U\subseteq\mathscr{M}_1(E)$ be τ-open. W.l.o.g. we may assume $\inf_{\mu\in U}J_{\mathscr{E}}(\mu)<\infty$, i.e., there is $\varphi\in D(\mathscr{E})$ such that $\varphi^2m\in U$. Now, let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $D(\mathscr{E})$ with $\varphi=\lim_{n\to\infty}\varphi_n$ w.r.t. $\mathscr{E}_1^{1/2}$, then

$$\left|\int f(\varphi_n^2-\varphi^2)\,dm\right|\leqq\|f\|_\infty(\|\varphi_n\|_2+\|\varphi\|_2)(\|\varphi_n-\varphi\|_2)$$

for all $f\in\mathscr{B}_b(E)$, hence φ_n^2m tends to φ^2m in the τ-topology. So for n_0 large enough, $\varphi_n^2m\in U$ for all $n>n_0$. Denote by $D_b(\mathscr{E})$, $D_b(L)$ the bounded functions in $D(\mathscr{E})$, $D(L)$ respectively. Then $\varphi_n:=\varphi\wedge n$ is in $D_b(\mathscr{E})$ and converges in $D(\mathscr{E})$ to φ. Any $\varphi \in D_b(\mathscr{E})$ can be approximated w.r.t. $\mathscr{E}_1^{1/2}$ by $nR_n\varphi \in D_n(L)$ and any $\varphi \in D_b(L)$ by $(\varphi + 1/n)$. Hence any $\varphi \in D(\mathscr{E})$ can be approximated w.r.t. $\mathscr{E}_1^{1/2}$ by $\varphi_n \in D(\mathscr{E})$ which are of type $\varphi = R_\alpha g$, $g \in \mathscr{B}_b(E)$, $g \geqq c > 0$. Thus, as will be clear by (2) below, we only have to consider the case $\varphi = R_\alpha g$, $g \in \mathscr{B}_b(E)$, $g \geqq c > 0$, such that $\int \varphi^2 dm = 1$ and we choose $U_\delta := \bigcap_{i=1}^{n} U(\varphi^2 m; \delta, f_i)$ such that $U_\delta \subseteq U$ for some $\delta > 0$, $f_i \in \mathscr{B}_b(E)$, $i = 1, \ldots, n$. Set $A(t,\delta) := \{L_t \in U_\delta\}$, $S(t,\delta) := \left\{L_t \in U\left(\varphi^2 m; \delta, \frac{L\varphi}{\varphi}\right)\right\}$ and $S'(t,\delta) := S(t,\delta) \cap A(t,\delta)$. Let (P_x^φ) be the associated conservative Markov family constructed in Sect. 3. By Proposition 10 and the ergodic theorem we have $\lim_{t\to\infty} \int f(x) L_t(\cdot, dx) = \int f \varphi^2 dm$ $P^\varphi_{\varphi^2 m}$-a.s. for all $f \in \mathscr{B}_b(E)$, hence, in particular, $\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{L\varphi}{\varphi}(X_s)\, ds = \int_E \varphi L\varphi\, dm$ $P^\varphi_{\varphi^2 m}$-a.s. Consequently $P_x^\varphi[\liminf_{t\to\infty} S'(t,\delta)] = 1$ for $\varphi^2 m$-a.e. $x \in E$. By Lemma 11 this holds for $\mathscr{E}$-q.e. $x \in E$. Then

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{t} \log P_x[L_t \in U_\delta] &= \liminf_{t\to\infty} \frac{1}{t} \log E_x^\varphi\left[\frac{\varphi(X_0)}{\varphi(X_t)} \exp\left[\int_0^t \frac{L\varphi}{\varphi}(X_s)\, ds\right]; A(t,\delta)\right] \\
&\geqq \liminf_{t\to\infty} \frac{1}{t} \log \left\{\exp\left[t\left(\int_E \varphi L\varphi\, dm - \delta\right)\right]\right. \\
&\quad \left. \cdot E_x^\varphi\left[\frac{\varphi(X_0)}{\varphi(X_t)}; S'(t,\delta)\right]\right\} \\
&\geqq \int_E \varphi L\varphi\, dm - \delta + \liminf_{t\to\infty} \frac{1}{t} \log \left\{\frac{\varphi(x)}{\|\varphi\|_\infty} \cdot P_x^\varphi[S'(t,\delta)]\right\} \\
&= \int_E \varphi L\varphi\, dm - \delta
\end{aligned}$$

for $\mathscr{E}$-q.e. $x \in E$. So, since $U_\delta \subseteq U$ and δ was arbitrary,

$$(2) \qquad \liminf_{t\to\infty} \frac{1}{t} \log P_x[L_t \in U] \geqq -\mathscr{E}(\varphi, \varphi) \qquad \text{for } \mathscr{E}\text{-q.e. } x \in E. \quad \square$$

Corollary 12 *Let $F: E \to \mathbb{R}$ be measurable and such that $\nu \mapsto \langle F, \nu\rangle := \int F\, d\nu$, $\nu \in \mathscr{M}_1(E)$, is lower semicontinuous w.r.t. the τ-topology. Then*

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t} \log E_x[\exp[t\langle F, L_t\rangle]] \\
&\quad \geqq \sup\{\langle F, \nu\rangle - J_{\mathscr{E}}(\nu) : \nu \in \mathscr{M}_1(E), \langle F, \nu\rangle \wedge J_{\mathscr{E}}(\nu) < \infty\}
\end{aligned}$$

for $\mathscr{E}$-q.e. $x \in E$.

Proof. Let $\delta>0$ and $\varphi\in D(\mathscr{E})$ such that $\mu:=\varphi^2 m\in\mathscr{M}_1(E)$. By lower semicontinuity there is a τ-open neighborhood G of μ such that $\inf_{\nu\in G}\langle F,\nu\rangle\geqq\langle F,\mu\rangle$ δ. Hence

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log E_x[\exp[t\langle F,L_t\rangle]]&\geqq\liminf_{t\to\infty}\frac{1}{t}\log E_x[I_G(L_t)\exp[t\langle F,L_t\rangle]]\\&\geqq\inf_{\nu\in G}\langle F,\nu\rangle+\liminf_{t\to\infty}\frac{1}{t}\log P_x[L_t\in G]\\&\geqq\langle F,\mu\rangle-J_{\mathscr{E}}(\mu)-\delta\end{aligned}$$

for $\mathscr{E}$-q.e. $x\in E$. □

5 The upper bound w.r.t. the τ-topology

A prominent role for the large deviations of $P_x\circ L_t^{-1}$, $t>0$, is played by the logarithmic spectral radius

$$\rho(p_1^V):=\lim_{t\to\infty}\frac{1}{t}\log\|p_t^V\|_{L^2(E;m)\to L^2(E;m)}$$

of the Feynman-Kac transform $(p_t^V)_{t>0}$ (introduced in the proof of Proposition 8(i)) by a function $V\in\mathscr{B}_b(E)$. The key to Theorem 2 is that it dominates the Cramer-functional Λ_q, that we introduce here: Let

$$\Lambda_q(V):=\inf_{N:Cap(N)=0}\ \sup_{x\in E\setminus N}\ \limsup_{t\to\infty}\frac{1}{t}\log E_x\left[\exp\left[\int_0^t V(X_s)ds\right]\right],\qquad V\in\mathscr{B}_b(E).$$

Note, that this functional takes values in the compact interval $[\inf_E V,\sup_E V]$. The corresponding Legendre-transformation Λ_q^* w.r.t. the τ-topology is defined by

$$\Lambda_q^*(\mu):=\sup\left\{\int_E V\,d\mu-\Lambda_q(V):V\in\mathscr{B}_b(E)\right\},\qquad \mu\in\mathscr{M}_1(E).$$

Let finally

$$\Lambda_{\mathscr{E}}(V):=\sup\left\{\int_E V\,d\mu-J_{\mathscr{E}}(\mu):\mu\in\mathscr{M}_1(E)\right\},\qquad V\in\mathscr{B}_b(E),$$

where $J_{\mathscr{E}}$ is defined as in Theorem 1.

Proposition 13 *Let $V \in \mathscr{B}_b(E)$. Then $\Lambda_q(V) \leqq \Lambda_{\mathscr{E}}(V)$ and $J_{\mathscr{E}} \leqq \Lambda_q^*$.*

Proof. Let $V \in \mathscr{B}_b(E)$. First we will show, that $\rho(p_1^V) \geqq \Lambda_q(V)$. Therefore observe that

$$\rho(p_1^V) \geqq \inf\{\lambda > 0: \lim_{t\to\infty} e^{-\lambda t} \| p_t^V 1 \|_{L^2(E;m)} = 0\} \tag{3}$$

$$\geqq \inf\left\{\lambda > 0: \int_0^\infty e^{-\lambda t} p_t^V 1\, dt \in L^2(E;m)\right\},$$

where the first inequality is obvious and the second is seen as follows: Let $\lambda > \inf\{\lambda > 0: \lim_{t\to\infty} e^{-\lambda t} \| p_t^V 1 \|_{L^2(E;m)} = 0\}$ and $\delta > 0$. Then

$$\left\| \int_0^\infty e^{-(\lambda+2\delta)t} p_t^V 1\, dt \right\|_{L^2(E;m)} = \int_0^\infty \int_s^\infty e^{-(\lambda+2\delta)t} \| p_t^V 1 \|_{L^1(E;m)} dt\, ds$$

$$\leqq \int_0^\infty e^{-\delta s} \int_s^\infty e^{-\delta t} (e^{-\lambda t} \| p_t^V 1 \|_{L^2(E;m)}) dt\, ds$$

$$< \infty.$$

Let $\Lambda_{L^2}(V)$ denote the right hand side of (3) and let $\lambda > \Lambda_{L^2}(V)$. Then $R_\lambda^V 1 := \int_0^\infty e^{-\lambda t} p_t^V 1\, dt$ is well-defined and in $L^2(E;m)$. We trivially have that

$$R_\lambda^V 1 = R_\lambda(V R_\lambda^V 1 + 1). \tag{4}$$

Hence, in particular, $R_\lambda^V 1 \in D(L) \subset D(\mathscr{E})$. Therefore, $\lim_{n\to\infty} p_{1/n} R_\lambda^V 1 = R_\lambda^V 1$ w.r.t. $\mathscr{E}_1^{1/2}$. By [M-R, III.3.5] there is a subsequence $(p_{1/n_k} R_\lambda^V 1)_{k\in\mathbb{N}}$ converging to an $\mathscr{E}$-quasi-continuous m-version of the limit $R_\lambda^V 1$. But, by (4), $p_{1/n} R_\lambda^V 1 = \int_0^\infty e^{-\lambda t} p_{t+1/n}(V R_\lambda^V 1 + 1) dt = e^{\lambda/n} \int_{1/n}^\infty e^{-\lambda t} p_t (V R_\lambda^V 1 + 1) dt$ converges pointwise to $R_\lambda^V 1$, hence $R_\lambda^V 1$ is $\mathscr{E}$-quasi-continuous. So, in particular, $R_\lambda^V 1 < \infty$ $\mathscr{E}$-q.e. and henceforth

$$\Lambda_{L^2}(V) \geqq \inf\left\{\lambda > 0: \int_0^\infty e^{-\lambda t} p_t^V 1(\cdot)\, dt < \infty\ \mathscr{E}\text{-q.e.}\right\}$$

$$= \inf\{\lambda > 0: \lim_{t\to\infty} e^{-\lambda t} p_t^V 1(\cdot) = 0\ \mathscr{E}\text{-q.e.}\}$$

$$= \Lambda_q(\mathrm{V}).$$

To see, that the first infimum dominates the second, first observe that for $t\in[k,k+1)$, $k\in\mathbb{N}$,

$$(5)\qquad e^{-\|V\|_\infty} p_k^V 1 \leqq p_t^V 1 \leqq e^{\|V\|_\infty} p_k^V 1 .$$

Let $\lambda > \inf\left\{\lambda>0\colon \int_0^\infty e^{-\lambda t} p_t^V 1(\cdot)\,dt<\infty\ \mathscr{E}\text{-q.e.}\right\}$. Then there is an $\mathscr{E}$-exceptional set N such that for all $x\in E\setminus N$

$$\sum_{k=1}^\infty e^{-\|V\|_\infty} e^{-\lambda(k+1)} p_k^V 1(x) \leqq \int_0^\infty e^{-\lambda t} p_t^V 1(x)\,dt<\infty$$

by the first part of (5). Hence necessarily $\lim_{k\to\infty} e^{-\lambda k} p_k^V 1(x)=0$ and by the second part of (5) $\lim_{t\to\infty} e^{-\lambda t} p_t^V 1(x)=0$ for all $x\in E\setminus N$. The dual inequality and the last equality are obvious.

Since the logarithmic spectral radius $\rho(p_1^V)$ is equal to $-\lambda_V$, where λ_V is the lowest eigenvalue of the generator L^V of $(p_t^V)_{t\geqq 0}$ and $\Lambda_{\mathscr{E}}(V)$ is also equal to $-\lambda_V$ (c.f., e.g., [D-St, p. 131]), $\Lambda_q(V)\leqq\Lambda_{\mathscr{E}}(V)$. Moreover we know by [D-St, Ex. 4.2.63] $J_{\mathscr{E}}(\mu)=\Lambda_{\mathscr{E}}^*(\mu):=\sup\{\int_E V\,d\mu-\Lambda_{\mathscr{E}}(V)\colon V\in\mathscr{B}_b(E)\}$. Thus $J_{\mathscr{E}}\leqq\Lambda_q^*$. □

As announced, the following is now standard. Nevertheless we give it in full detail.

Proof of Theorem 2 Let $\mu\in K$, $0<\delta<1$ and choose $V\in\mathscr{B}_b(E)$ such that

$$\int_E V\,d\mu-\Lambda_q(V)\geqq\begin{cases}\Lambda_q^*(\mu)-\delta : \Lambda_q^*(\mu)<\infty,\\ \delta^{-1} \qquad :else.\end{cases}$$

Let $U=U(\mu;\delta,V)$ and let $\rho_t^x:=P_x\circ L_t^{-1}$ denote the distribution of L_t under P_x. Then

$$\limsup_{t\to\infty}\frac{1}{t}\log\rho_t^x(U)\leqq\limsup_{t\to\infty}\frac{1}{t}\log\int_{\mathscr{M}_1(E)}\exp\left[t\left(\int V\,d\nu-\int V\,d\mu+\delta\right)\right]\rho_t^x(d\nu)$$
$$\leqq\delta-\int V\,d\mu+\limsup_{t\to\infty}\frac{1}{t}\log E_x\left[\exp\left[\int_0^t V(X_s)\,ds\right]\right].$$

By τ-compactness, $K\subset\mathscr{M}_1(E)$ is contained in some finite union $\bigcup_{i=1}^n U_{\mu_i}$ of τ-open sets $U_{\mu_i}=U(\mu_i;\delta,V_i)$, $\mu_i\in K$, where $V_i\in\mathscr{B}_b(E)$ corresponds to δ and

μ_i as above. Then the previous inequality combined with $\rho_t^x(K) \leqq \sum_{i=1}^n \rho_t^x(U_{\mu_i}) \leqq n \max_{1 \leqq i \leqq n} \rho_t^x(U_{\mu_i})$ leads to

$$\begin{aligned}
&\inf_{N:Cap(N)=0} \sup_{x \in E \setminus N} \limsup_{t \to \infty} \frac{1}{t} \log P_x[L_t \in K] \\
&\leqq \inf_{N:Cap(N)=0} \sup_{x \in E \setminus N} \max_{1 \leqq i \leqq n} \Big\{ \delta - \int V_i \, d\mu_i + \limsup_{t \to \infty} \\
&\quad \cdot \frac{1}{t} \log E_x \Big[\exp \Big[\int_0^t V_i(X_s) \, ds \Big] \Big] \Big\} \\
&= \max_{1 \leqq i \leqq n} \Big\{ \delta - \int V_i \, d\mu_i + \inf_{N:Cap(N)=0} \sup_{x \in E \setminus N} \limsup_{t \to \infty} \\
&\quad \cdot \frac{1}{t} \log E_x \Big[\exp \Big[\int_0^t V_i(X_s) \, ds \Big] \Big] \Big\} \\
&\leqq \max_{1 \leqq i \leqq n} \{ \delta + \max(-\Lambda_q^*(\mu_i) + \delta, \, -\delta^{-1}) \}.
\end{aligned}$$

Now let δ tend to zero and next replace the maximum over the μ_i by the supremum over all $\mu \in K$. Then the assertion follows by Proposition 13. □

Remark 14 The standard argument to derive the (full) upper bound, i.e., the statement corresponding to Theorem 2 with closed sets $A \subseteq \mathcal{M}_1(E)$ in place of compacts K, is to show the (w.r.t. $x \in E$ uniform) exponential tightness of the family of measures $\rho_t^x := P_x \circ L_t^{-1}, t > 0$, which in general is hard. On polish spaces the following is known. Uniform exponential tightness holds, if the process is uniformly ergodic (see [D-St]). However, one has exponential tightness of the measures $P_m \circ L_t^{-1}$, $t > 0$, under weaker conditions. For example, it is sufficient that the semi-group $(p_t)_{t \geqq 0}$ is m-hypercontractive (cf. [D-St, Chap. VI]). The rate function is then again given by $J_{\mathscr{E}}$, which in this case equals the Donsker-Varadhan entropy J_m (c.f., e.g., [D-St, Ex. 5.4.36]). For some weaker conditions we refer to [W2, Thm. 3.4, Thm. 3.7(ii)]. Confer also Theorem 19 below.

6 Examples

Let $S(\mathbb{R}^d)$ denote the space of Schwartz test functions on $\mathbb{R}^d$ and $S'(\mathbb{R}^d)$ its dual. The examples in our first and second subsection were originally only studied on $S'(\mathbb{R}^d)$. But recently Röckner has given a treatment on a separable Banach subspace of $S'(\mathbb{R}^d)$, wherefore considerable effort was necessary. But all the examples could also (with less effort) be given on the non-metrizable space $S'(\mathbb{R}^d)$ itself (see [A-R1, Ex. 1.1.3(ii), (iii)]) and our results would apply even then.

6.1 *Time zero free field Dirichlet form*

So let $E=B_\alpha$, $\alpha>(d-1)/2$, be the separable Banach subspace of $S'(\mathbb{R}^d)$, with $S(\mathbb{R}^d)\subset B'_\alpha$ defined in [R1], [R2, (1.3)] and take $H=L^2(\mathbb{R}^d; dx)$, where dx denotes Lebesgue measure. It has been shown in [R1], [R2, (1.3)] that there exists a Gaussian measure μ on $\mathscr{B}(B_\alpha)$ such that

$$\int \exp[i_{S'}\langle x, l\rangle_S]\,\mu(dx)=\exp[-1/2\,\|(-\Delta+1)^{-1/4}l\|_{L^2(\mathbb{R}^d;dx)}]$$

for all $l\in S(\mathbb{R}^d)$, where Δ is the Laplacian on $\mathbb{R}^d$. The measure μ is just the time zero free field of quantum field theory. Define the linear space

$$\mathscr{F}C_b^\infty:=\{f(l_1,\ldots,l_m): m\in\mathbb{N}, f\in C_b^\infty(\mathbb{R}^m), l_1,\ldots,l_m\in E'\},$$

where $C_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable functions on $\mathbb{R}^m$ with all partial derivatives bounded. Now let for $k\in E$ and $u\in\mathscr{F}C_b^\infty$

$$\frac{\partial u}{\partial k}(x):=\frac{d}{ds}u(x+sk)|_{s=0}, \qquad x\in E.$$

Observing that $h\to\frac{\partial u}{\partial h}(x)$, $h\in H$, is a continuous linear functional on H, let $\nabla u(x)\in H$ be defined by $\langle\nabla u(x), h\rangle_H=\frac{\partial u}{\partial h}(x)$, $h\in H$.

Then $(\mathscr{E}, \mathscr{F}C_b^\infty)$ defined by

$$\mathscr{E}(u,v):=\int_{B_\alpha}\langle\nabla u, \nabla v\rangle_{L^2(\mathbb{R}^d;dx)}\,d\mu, \qquad u,v\in\mathscr{F}C_b^\infty,$$

is closable ([A-R2, Ex. 5.6(ii)]) and its closure $(\mathscr{E}, D(\mathscr{E}))$ is a quasi-regular symmetric Dirichlet form on $L^2(B_\alpha;\mu)$ (cf. [M-R, IV. Sect. 4b)]). Let $(L, D(L))$ be the corresponding generator. It is known that 0 is the infimum of the spectrum of $(L, D(L))$ and that it is a simple isolated eigenvalue with normalized eigenvector the constant function 1. (Cf. [A-HK, Sect. 5.3] and also [Re-S2]). Hence, if $u\in D(\mathscr{E})$ with $\mathscr{E}(u,u)=0$, then $u\in D(L)$, $Lu=0$ and henceforth $u=$const. This implies by Lemma 9 that the right process $\mathbb{M}$ properly associated to $(\mathscr{E}, D(\mathscr{E}))$ (which is in fact a diffusion and which exists according to [M-R, IV.3.5]) is ergodic. Clearly, it is symmetric, hence Theorems 1 and 2 apply.

Remark 15 By, e.g., [A-R4, Sect. 7] we can apply [Re-S1, Thm. X.61(b)] to conclude that the semigroup corresponding to the free field Dirichlet form is hypercontractive. Hence the full upper bound holds by Remark 14 for $P_m \circ L_t^{-1}$, $t>0$, and is given by $J_{\mathscr{E}}$. But cf. also Theorem 19 below.

Remark 16 (i) Let $A := (-\Delta+1)^{1/2}$. Then the transition semigroup corresponding to the free field Dirichlet form defined above is given by

$$p_t f(x) = \int f(e^{-tA}x + \sqrt{1-e^{1-2tA}}\,y)\mu(dy), \qquad x\in E, t>0,$$

$f\geqq 0$ $\mathscr{B}(E)$-measurable (cf. [B-R, Ex. 5.6(ii)]). Hence, by the Hajek-Feldman theorem, the measures $p_t(x,\cdot)$, $t>0$, are singular w.r.t. μ for all $x\in E$, which means that Theorem 1 sharpens indeed [D-St, 5.3.2]. The same is true for Sect. 6.2 below.
(ii) For another Banach space E in place of B_α see [B-R, Thm. 3.1].
(iii) The results in this section including the full upper bound, as specified in Remark 15, hold more generally for all Dirichlet forms associated with a semigroup $e^{-t\Gamma(A)}$, where $\Gamma(A)$ is the second quantization of a strictly positive definite self-adjoint operator A on a Hilbert space H (cf. [A-R4, Sect. 7.I]). Another concrete example is handled in Sect. 6.3 below.

6.2 *Perturbed time zero free field Dirichlet form*

Let E, H, μ be as in Sect. 6.1. Let $\varphi\in D(\mathscr{E})$, $\varphi\neq 0$ m-a.e. Then, by [A-R3, Thm. 4.7(i)], $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ defined by

$$(6) \qquad \mathscr{E}^\varphi(u,v) := \int_{B_\alpha} \langle \nabla u, \nabla v\rangle_{L^2(\mathbb{R}^d;\,dx)}\varphi^2\,d\mu, \qquad u,v\in\mathscr{F}C_b^\infty,$$

is closable on $L^2(E;\varphi^2 m)$ and its closure $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ is a quasi-regular Dirichlet form (cf. [M-R, IV. Sect. 4b)]). Note, that if $\mathscr{E}^\varphi(u,u)=0$ implies that $u=$const μ-a.e., the right process properly associated to it is ergodic and Theorems 1 and 2 apply.

Now let us consider particular examples of φ, namely ground states of Schrödinger operators $-L+V$.

Let $P^{(\leqq n)}$ be the closed linear span of $\{{}_{S'}\langle\cdot,k_1\rangle_S \cdots {}_{S'}\langle\cdot,k_m\rangle_S : m \leqq n, k_i\in S(\mathbb{R})\}$ in $L^2(E;\mu)$ and let: $:_n$ denote the orthogonal projection onto the nth homogeneous chaos $H_n := P^{(\leqq n)} \ominus P^{(\leqq n-1)}$.

Then let V be a renormalized polynomial (with cutoff h), i.e.,

$$V(x) = \sum_{n=0}^{2N} a_n :x^n:(h), \qquad x\in E.$$

Here $N\in\mathbb{N}$, $a_n\in\mathbb{R}$, $0\leqq n\leqq 2N$, with $a_{2N}>0$, $h\in L^{1+\varepsilon}(\mathbb{R};dx)$ for some $\varepsilon>0$, and $:x^n:(h)$ is defined as the unique element in H_n such that

$$\int :x^n:(h) : \prod_{j=1}^n X_{k_j} :_n d\mu = n! \int_{\mathbb{R}} \prod_{j=1}^n \Big(\int_{\mathbb{R}} (-\Delta+1)^{-1/2}(x-y_j)k_j(y_j)\,dy_j\Big)h(x)\,dx$$

for all $k_1, \dots, k_n \in S(\mathbb{R})$. (Cf., e.g., [S, Sect. V.1] for details). Then $V \in L^p(E; \mu)$ for all $p < \infty$ and $e^{-tV} \in L^1(E; \mu)$ for all $t > 0$ (cf. [S, Sect. V.2]). Therefore $H_V := -L + V$ is well-defined in the sense of forms and has a simple isolated lowest eigenvalue λ. Now let φ be the so called ground state, i.e., the eigenvector corresponding to this eigenvalue. Then it is known that $\varphi > 0$ μ-a.e. (cf. [A-HK, Sect. 5.3]) and clearly $\varphi \in D(\mathscr{E})$ (see e.g. [R-Z, Lemma 5.2]), hence all applies if $\mathscr{E}^\varphi(u, u) = 0$ implies that $u = \text{const}$ μ-a.e., which is, of course, the case if the generator of $(\mathscr{E}^\varphi, D(\mathscr{E}^\varphi))$ is unitary equivalent to $H_V - \lambda$. So far, this has only been shown in the case where $V(x) = :x^4:(h)$ with $h \in S(\mathbb{R})$ (cf. [A-HK, Remark after Lemma 5.7]).

6.3 Space time free field Dirichlet form and its perturbation

Another example shall now be obtained by the space time free field Dirichlet form constructed on $S'(\mathbb{R}^d)$ (cf. [A-R1, Ex. 1.13(ii)], [A-R3]). Here we only look at the case of "finite volume": Let μ be the space time free field on a finite volume Λ, $\Lambda \subset \mathbb{R}^2$ an open rectangle, i.e., μ is the unique mean zero gaussian measure on $E := H_{-\delta}$, $\delta > 0$, with covariance

$$\int ({}_{E'}\langle l, z\rangle_E)^2 \, \mu(dz) = \|l\|^2_{H_{-1}}, \qquad l \in E' = H_\delta.$$

For notation and more details we refer to [R-Z, Sect. 7]. Let

$$\mathscr{E}^*(u, v) := \int \langle \nabla u, \nabla v \rangle_{L^2(\Lambda; dx)} \, d\mu, \qquad u, v \in \mathscr{F}C_b^\infty.$$

Then $(\mathscr{E}^*, \mathscr{F}C_b^\infty)$ is closable (by [R-Z, Sect. 7]) and its closure $(\mathscr{E}^*, D(\mathscr{E}^*))$ is a symmetric quasi-regular Dirichlet form by [M-R, IV. Sect. 4b)]. As in the time zero case we have that 0 is the infimum of the spectrum of the generator of $(\mathscr{E}^*, D(\mathscr{E}^*))$, and that it is an isolated simple eigenvalue with (normalized) eigenvector the constant function 1. Thus the associated diffusion process is ergodic and Theorems 1 and 2 apply. Moreover we have the analogue of Remark 15 (cf. Remark 16(iii)), hence Theorem 19 below applies.

As in the time zero case one can now introduce perturbations with concrete functions φ as follows: Let $\varphi := \exp(-1/2\,V)$ (>0 μ-a.e.), where V is a Wick polynomial in two dimensions (for details see [A-R-Z, Sect. 7]). As above, it follows that $\varphi \in D(\mathscr{E}^*)$ and that $(\mathscr{E}^{*,\varphi}, \mathscr{F}C_b^\infty)$ defined by

$$\mathscr{E}^{*,\varphi}(u, v) := \int \langle \nabla u, \nabla v \rangle_{L^2(\Lambda; dx)} \, \varphi^2 \, d\mu, \qquad u, v \in \mathscr{F}C_b^\infty,$$

is closable and its closure $(\mathscr{E}^{*,\varphi}, D(\mathscr{E}^{*,\varphi}))$ is a symmetric quasi-regular Dirichlet form. Let $(P_x^*)_{x \in E}$, $(P_x^{*,\varphi})_{x \in E}$ denote the Markov family of the diffusion properly associated to $(\mathscr{E}^*, D(\mathscr{E}^*))$, $(\mathscr{E}^{*,\varphi}, D(\mathscr{E}^{*,\varphi}))$ respectively. As is shown in [R-Z, Sect. 7], the function φ satisfies the assumptions for [A-R-Z, Thm. 1.3], by which $P_\mu^* := \int P_x^* \mu(dx)$ is absolutely continuous w.r.t. $P_{\varphi^2 m}^{*;\varphi} := \int P_x^{*,\varphi} \varphi^2 m(dx)$ and vice versa on each $\mathscr{F}_t^0$, $t>0$. Thus also $P_{\varphi^2 m}^{*;\varphi}$ is ergodic and Theorems 1 and 2 apply.

A Concluding remarks on upper bounds

Since the submission of this paper it has turned out that a stronger result than Theorem 2 holds under even weaker conditions. For the proof neither the symmetry of the process, nor the strong Markov property or the quasi-regularity and properly associatedness are required. Instead, consider a Markov process $\mathbb{M}$, to which there is associated a (not necessarily symmetric) Semi-Dirichletform $(\mathscr{E}, D(\mathscr{E}))$, i.e., $p_t f$ is an m-version of $T_t f$ for all $f \in \mathscr{B}_b(E)$ (cf. [M-O-R]). Assume that $\mathbb{M}$ is (strictly) conservative, i.e. $P_x[\zeta < \infty] = 0$ for all $x \in E$.

Theorem 17 *Let $K \subset \mathscr{M}_1(E)$ be τ-compact. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in E} P_x[L_t \in K] \leqq -\inf_{\mu\in K} J_{\mathscr{E}}(\mu).$$

First, we introduce some notation. Let Φ denote the set of all functions φ of type $\varphi = R_\alpha g$, $\alpha > 0$, for some $g \in \mathscr{B}_b(E)$, $g \geqq c > 0$. For $V \in \mathscr{B}_b(E)$ let

$$\Lambda(V) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in E} E_x\left[\exp\left[\int_0^t V(X_s)\,ds\right]\right].$$

The following Lemma is a substitute of Proposition 13. The latter was based on the symmetry of the Dirichlet form in an essential way.

Lemma 18 *Let $\varphi \in \Phi$. Then $\Lambda^*(\varphi^2 m) \geqq \mathscr{E}(\varphi, \varphi)$.*

Proof. Let $\varphi \in \Phi$, φ corresponding to $g \in \mathscr{B}_b(E)$, $c > 0$. Set $V_\varphi = -\frac{L\varphi}{\varphi}$. Then φ satisfies the integral equation

$$\psi = p_t \varphi + \int_0^t p_{t-s}(V_\varphi \psi)\,ds,$$

which has as unique solution the Feynman-Kac transform $p_t^{V_\varphi}$ applied to φ (cf. [D-St, 4.2.23]). Thus $p_t^{V_\varphi}\varphi = \varphi$. By (strict) conservativeness $\varphi = R_\alpha g \geqq R_\alpha c = c/\alpha$ and henceforth

$$E_x\left[\exp\left[\int_0^t V_\varphi(X_s)\,ds\right]\right] \leqq \frac{\alpha}{c} p_t^{V_\varphi} \varphi(x) = \frac{\alpha}{c} \varphi(x).$$

Thus $\Lambda(V_\varphi) \leqq 0$ and thereby $\Lambda^*(\varphi^2 m) \geqq \{\int V_\varphi \varphi^2\,dm - \Lambda(V_\varphi)\} \geqq \mathscr{E}(\varphi, \varphi)$. □

Theorem 17 now follows analogously to the way Theorem 2 follows from Proposition 13, since Φ is dense in $\{\varphi \in D(\mathscr{E}): \varphi \geqq 0\}$. However, Proposition 13 is still useful to show the (full) upper bound, now even for *$\mathscr{E}$-quasi-every* starting point. Consider again the situation of Sect. 5. (In particular, $(\mathscr{E}, D(\mathscr{E}))$ is now again symmetric.)

Theorem 19 *Assume, in addition, E is polish and $(p_t)_{t \geqq 0}$ is m-hypercontractive, i.e., $\|p_T\|_{L^2(m) \to L^4(m)} = 1$ for some $T > 0$. Then there is an $\mathscr{E}$-exceptional set N such that for every $A \subseteq \mathscr{M}_1(E)$, A closed w.r.t. the weak topology,*

$$\limsup_{t \to \infty} \frac{1}{t} \log P_x[L_t \in A] \leqq - \inf_{v \in A} J_{\mathscr{E}}(v)$$

for all $x \in E \setminus N$.

Proof. Hypercontractivity implies $H(v|m) \leqq \alpha J_{\mathscr{E}}(v)$. Henceforth the sets $\{J_{\mathscr{E}} \leqq L\}$ are weakly compact. In particular, $J_{\mathscr{E}}$ is lower semicontinuous w.r.t. the weak topology. Thus $J_{\mathscr{E}}(v) = \sup\{\int V dv - \Lambda_{\mathscr{E}}(V): V \in C_b(E)\}$. Proposition 13 implies the weak upper bound for $J_{\mathscr{E}}$ w.r.t. the weak topology. Again by hypercontractivity and by Proposition 13 $\Lambda_q(V) \leqq 1/\alpha \log \int \exp[\alpha V] dm$, $V \in \mathscr{B}_b(E)$. As in the proof of [D-St, Lemma 3.2.7] it follows that there is an $\mathscr{E}$-exceptional set N such that $P_x \circ L_t^{-1}, t > 0$, is exponentially tight w.r.t. the weak topology for every $x \in E \setminus N$. □

Remark 20 Confer Remark 15, Remark 16 (iii) and Sect. 6.3 for examples. More detailed proofs are given in the authors thesis ([Mü]).

B Appendix

There are two procedures of modifying a right processes that we use several times. Given an $\mathbb{M}$-invariant subspace $S \subset E$ with defining set Ω_S we can always restrict $\mathbb{M}$ to S, Ω_S. The *restriction* to S, Ω_S is again a right process (below we will call it simply the restriction to S). Secondly, given a right process $\mathbb{M}_S = (\Omega_S, (\mathscr{F}_t \cap \Omega_S)_{t \geqq 0} (X_t)_{t \geqq 0}, (P_x)_{x \in S_\Delta}, \zeta)$ on a state space S and given a space $S' \supset S$ one can always define a *trivial extension* $\mathbb{M}_{S'}$ *(of* $\mathbb{M}_S$*) to* S' by defining each $x \in S' \setminus S$ as a trap, i.e., we adjoin $S' \setminus S$ to Ω_S such that $P_x[X_t = x \, \forall t \geqq 0] = 1$, $x \in S' \setminus S$. (Cf. [M-R, Thm. 4.1.3] and [M-R, IV.(3.48)] for details). Of particular importance is the case of an $\mathbb{M}$-invariant subspace $S \subset E$, where $E \setminus S$ is $\mathscr{E}$-exceptional: Since our main objective is an $\mathscr{E}$-quasi-everywhere statement for the lower bound of large deviations of $\mathbb{M}$, we may identify, w.l.o.g., $\mathbb{M}$ with the trivial extension to E of its restriction to S.

For a space S and a point Δ, S_Δ will denote $S \cup \{\Delta\}$. For a topological subspace $S \subset E$ let $\mathscr{B}^u(S)$ denote the universal completion of $\mathscr{B}(S)$, i.e., the intersection of the v-completions of $\mathscr{B}(S)$, as v runs over all probability measures on $(S, \mathscr{B}(S))$ and set $\mathscr{F}^u := \sigma(f(X_s); s \geqq 0, f \in \mathscr{B}^u(E))$. By an extension of the Ionescu-Tulcea theorem in [Sh] we get the following

Theorem 21 *There is an* $\mathbb{M}$*-invariant set* $S^\varphi \subset E$ *with defining set* Ω^φ*, such that* $E \setminus S^\varphi$ *is* $\mathscr{E}$*-exceptional and such that there is a family of unique Markov kernels* $(P_x^\varphi)_{x \in S^\varphi}$ *from* $(S^\varphi, \mathscr{B}^u(S^\varphi))$ *to* $(\Omega^\varphi, \mathscr{F}^u \cap \Omega^\varphi)$ *rendering* $(X_t)_{t \geqq 0}$ *Markov with semigroup* $(p_t^{(\varphi)})_{t \geqq 0}$ *and* $P_x^\varphi[X_0 = x] = 1$, $x \in S^\varphi$. *In addition: 1) For all bounded* $\mathscr{F}_t^0$*-measurable* F *the expectation under* P_x^φ*, denoted by* $E_x^\varphi[\cdot]$*, is given by*

$$E_x^\varphi[F I_{\{t<\zeta\}}] = E_x\left[F \frac{\varphi(X_t)}{\varphi(X_0)} \exp\left[-\int_0^t \frac{L\varphi}{\varphi}(X_s)\,ds\right]\right], \qquad x \in S^\varphi.$$

In particular, P_x^φ *is absolutely continuous w.r.t.* P_x, $x \in S^\varphi$, *on the trace of* $\mathscr{F}_t \cap \Omega^\varphi$ *on* $\{t < \zeta\}$ *and vice versa.*
2) Let $\mathscr{F}_t^\varphi$ *denote the universal completion of* $\mathscr{F}_t^0$ *w.r.t. the measures* $\int P_x^\varphi \nu(dx)$*, where* ν *runs over all probability measures on* $(S_\Delta^\varphi, \mathscr{B}(S_\Delta^\varphi))$*. The process* $\mathbb{M}^\varphi = (\Omega^\varphi, (\mathscr{F}_t^\varphi)_{t \geqq 0}, (X_t)_{t \geqq 0}, (P_x^\varphi)_{x \in S_\Delta^\varphi}, \zeta)$ *thus defined is a conservative right process.*

Proof. We want to apply [Sh, Thm. (62.19)]. Therefore, we need some preparations. By [M-R, VI.1.2, VI.1.6] there exists an $\mathbb{M}$-invariant subset $S \subseteq E$, which is a Borel subset of a locally compact separable metric space $E^\#$, such that $E \setminus S$ is $\mathscr{E}$-exceptional and such that $\mathbb{M}^\#$ is a Hunt process, where $\mathbb{M}^\#$ denotes the trivial extension to $E^\#$, $\Omega^\#$ of the restriction of $\mathbb{M}$ to S. Then its restriction $\mathbb{M}_S = (\Omega_S, (\mathscr{F}_t \cap \Omega_S)_{t \geqq 0} (X_t)_{t \geqq 0}, (P_x)_{x \in S_\Delta}, \zeta)$ to S, Ω_S (which clearly equals the restriction of $\mathbb{M}$ to S, Ω_S) is again a Hunt process (cf. [F1, Thm. 4.1.2]). As a Borel-subspace of $E^\#$, S is a Radon space. We now transform this Hunt process $\mathbb{M}_S$.

Let N_0, N_1 be the $\mathscr{E}$-exceptional sets defined at the beginning of Sect. 3 and in Proposition 8(ii). By [M-R, IV.6.5] there exists an $\mathscr{E}$-exceptional set $N^* \supseteq (N_0 \cup N_1 \cup (E \setminus S))$ such that $S^\varphi := E \setminus N^* \in \mathscr{B}(E)$ is $\mathbb{M}$-invariant. Let Ω^φ denote the corresponding defining set. Define

$$m_t(\omega) := \begin{cases} \dfrac{\varphi(X_t(\omega))}{\varphi(X_0(\omega))} \exp\left[-\displaystyle\int_0^t \frac{L\varphi}{\varphi}(X_s(\omega))\,ds\right] & : \omega \in \Omega^\varphi, \\ 0 & : \omega \in (\Omega_S \setminus \Omega^\varphi). \end{cases}$$

Since $\bigcup_{s,t} \{\omega : m_{t+s}(\omega) \neq m_t(\omega) m_s(\omega \circ \theta_t)\} = \emptyset$ and since by conservativeness $t \mapsto m_t(\cdot)$ is P_x-a.s. right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$ (cf., e.g., [M-R, IV.5.13 and Claim 1 in the proof of IV.5.14]), it is a (strong) perfect multiplicative functional [Sh, (54.2)–(54.4) and following remarks]. Obviously $(m_t)_{t \geqq 0}$ is adapted to the filtration $\mathscr{F}_t^0 \cap \Omega^\varphi$ and by Proposition 8 (ii) it is a P_x-martingale for all $x \in S$. Note, that by the $\mathbb{M}$-invariance of S^φ, and since $P_x[X_t = x \,\forall t \geqq 0] = 1$ for $x \in S \setminus S^\varphi$, $I_{S^\varphi}(X_t)$ is P_x-a.s. right continuous on $[0, \infty)$ and has left limits in S on $(0, \infty)$ for all $x \in S$, hence S^{φ} is nearly optional. Thus Hypothesis (62.9) in [Sh] is satisfied with S, S^{φ} in place of E and E_m. As the space Ω^{φ} is clearly projective ([Sh. (62.4) and the subsequent remarks]), [Sh, Thm. (62.19)] now gives the assertion. Note, that $m_t > 0$ P_x-a.s. for $x \in S^{\varphi}$, so $P_x \ll P_x^{\varphi}$ on the trace of $\mathscr{F}_t \cap \Omega^{\varphi}$ on $\{t < \zeta\}$. Since for every $x \in S^{\varphi}$ we have $P_x^{\varphi}[\zeta > t] = P_x^{\varphi}[X_t \in S^{\varphi}] = P_x^{\varphi}[X_t \in E] = p_t^{(\varphi)} I_E(x) = 1$, the transformed process $\mathbb{M}^{\varphi}$ is conservative as a process on S^{φ}. □

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