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Shift-coupling in continuous time

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Summary. The result linking shift-coupling to time-average total variation convergence and to the invariant σ-field is extended to continuous time and an analogous result established linking ε-couplings to smooth total variation convergence and to a smooth tail σ-field. Shift- and ε-coupling inequalities are presented.

Mathematics Subject Classification: 60G99

1 Introduction

Shift-coupling means coupling two stochastic processes in such a way that their paths eventually coincide up to a random time-shift. In discrete time shift-coupling has been linked to time-average total variation convergence and to the invariant σ-field $\mathscr{I}$, see Berbee [3], Greven [4] and Aldous and Thorisson [1]. This parallels the better known result linking zero-shift-coupling (coupling in such a way that the paths eventually coincide, without the random time-shift) to total variation convergence and to the tail σ-field $\mathscr{T}$, cf. Lindvall [6].

In the present paper we extend this to continuous time and also treat an issue that does not arise in discrete time: what happens when the random time-shift can be made arbitrarily small, i.e. when ε-couplings exist. This is known to imply weak convergence, see Asmussen [2]. Here we link ε-couplings to *smooth* total variation convergence and to a *smooth* tail σ-field $\mathscr{S}$ lying in between $\mathscr{I}$ and $\mathscr{T}$.

For both shift- and ε-coupling we introduce inequalities which play a similar key role for time-average and smooth total variation convergence as the standard coupling time inequality does for plain total variation convergence. Applications to stationary processes, Markov processes, regenerative processes and processes with stationary cycles (Palm theory) are indicated.

2 Notation

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a common probability space supporting all the random elements in this paper. Let $Z=(Z_s)_{s\in[0,\infty)}$ and $Z'=(Z'_s)_{s\in[0,\infty)}$ be stochastic processes on a Polish state space $(E, \mathscr{E})$ with right-continuous paths. We shall regard Z and Z' as random elements in $(H, \mathscr{H})$ where H is the set of all right-continuous functions $z=(z_s)_{s\in[0,\infty)}$ from $[0,\infty)$ to E and $\mathscr{H}$ is generated by the projection mappings. For $t\in[0,\infty)$ define the shifts θ_t by $\theta_t z=(z_{t+s})_{s\in[0,\infty)}$. The total variation norm of a signed measure v is $\|v\|=$mass of $|v|$. For a measure μ on $\mathscr{H}$ and a sub-σ-field $\mathscr{A}$ of $\mathscr{H}$ let $\mu_{\mathscr{A}}$ denote the restriction of μ to $\mathscr{A}$. Let U be a random variable which is uniformly distributed on $[0,1]$ and independent of Z, Z' and the shift-coupling $(\hat{Z}, \hat{Z}', T, T')$ introduced below.

3 Inequalities

A pair of processes $\hat{Z}$ and $\hat{Z}'$ is a coupling of Z and Z' if $\hat{Z}$ has the same distribution as Z and $\hat{Z}'$ has the same distribution as Z'. An event C is a coupling event if $\hat{Z}=\hat{Z}'$ on C. Then (obviously):

$$\|\mathbf{P}(Z\in\cdot)-\mathbf{P}(Z'\in\cdot)\|\leqq 2\mathbf{P}(C^c). \quad \text{(coupling event inequality)}$$

A random time T is a coupling time if $\theta_T\hat{Z}=\theta_T\hat{Z}'$ on $\{T<\infty\}$. Clearly $\theta_t\hat{Z}$ and $\theta_t\hat{Z}'$ is a coupling of $\theta_t Z$ and $\theta_t Z'$ with coupling event $\{T\leqq t\}$ and thus it holds that: *for* $t\in[0,\infty)$

$$\|\mathbf{P}(\theta_t Z\in\cdot)-\mathbf{P}(\theta_t Z'\in\cdot)\|\leqq 2\mathbf{P}(T>t). \quad \text{(coupling time inequality)}$$

A shift-coupling of Z and Z' is a coupling $\hat{Z}$ and $\hat{Z}'$ and two random times T and T' such that $\{T<\infty\}=\{T'<\infty\}$ and $\theta_T\hat{Z}=\theta_{T'}\hat{Z}'$ on $\{T<\infty\}$ (i.e. when $T<\infty$ the paths coincide eventually, up to the time-shift $T-T'$). Clearly $U'=(U+(T'-T)/t)_{\text{mod}\,1}$ is uniform on $[0,1]$ and independent of $(\hat{Z}, \hat{Z}', T, T')$. Thus $\theta_{Ut}\hat{Z}$ and $\theta_{U't}\hat{Z}'$ is a coupling of $\theta_{Ut}Z$ and $\theta_{Ut}Z'$. On $C=\{T\leqq Ut<t-(T'-T)\}$ we have $U'=U+(T'-T)/t$ which yields the last identity in: $\theta_{Ut}\hat{Z}=\theta_{Ut-T}\theta_T\hat{Z}=\theta_{Ut-T}\theta_{T'}\hat{Z}'=\theta_{Ut+T'-T}\hat{Z}=\theta_{U't}\hat{Z}'$ on C. Now $\mathbf{P}(C)=\mathbf{P}(Ut\geqq T\vee T')$ and the coupling event inequality yields: *if there is a shift-coupling of* Z *and* Z' *with times* T *and* T' *then for* $t\in[0,\infty)$

$$\|\mathbf{P}(\theta_{Ut}Z\in\cdot)-\mathbf{P}(\theta_{Ut}Z'\in\cdot)\|\leqq 2\mathbf{P}(T\vee T'>Ut). \quad \text{(shift-coupling inequality)}$$

An ε-coupling, $\varepsilon>0$, of Z and Z' is a shift-coupling $\hat{Z}$ and $\hat{Z}'$ with times T and T' such that $|T-T'|\leqq\varepsilon$ on $\{T<\infty\}$. Clearly $U'=(U+(T'-T)/h)_{\text{mod }1}$ is uniform on $[0,1]$ and independent of $(\hat{Z},\hat{Z}',T,T')$. Thus $\theta_{t+Uh}\hat{Z}$ and $\theta_{t+U'h}\hat{Z}'$ is a coupling of $\theta_{t+Uh}Z$ and $\theta_{t+Uh}Z'$. On $C=\{T\leqq t, (T-T')\leqq Uh<h-(T'-T)\}$ we have $U'=U+(T'-T)/h$ which yields the last identity in: $\theta_{t+Uh}\hat{Z}=\theta_{t-T+Uh}\theta_T\hat{Z}=\theta_{t-T+Uh}\theta_{T'}\hat{Z}'=\theta_{t+(T'-T)+Uh}\hat{Z}'=\theta_{t+U'h}\hat{Z}'$ on C. Now $\mathbf{P}((T-T')\leqq Uh<h-(T'-T))\geqq 1-\varepsilon/h$ which yields $\mathbf{P}(C^c)\leqq\mathbf{P}(T>t)+\varepsilon/h$. Apply the coupling event inequality to obtain that: *if there is an ε-coupling of Z and Z' with times T and T' then for $h, t\in[0,\infty)$*

$$\|\mathbf{P}(\theta_{t+Uh}Z\in\cdot)-\mathbf{P}(\theta_{t+Uh}Z'\in\cdot)\|\leqq 2\mathbf{P}(T>t)+2\varepsilon/h. \quad (\varepsilon\text{-coupling inequality})$$

The couplings are called successful if $\mathbf{P}(T<\infty)=1$ (for all $\varepsilon>0$ in the ε-coupling case) and then the above inequalities yield obvious limit results. In particular, if Z' is stationary then

$$\theta_t Z\to_{tv}Z', \quad \theta_{Ut}Z\to_{tv}Z' \quad \text{and} \quad \theta_{t+Uh}Z\to_{tv}Z' \text{ for } h>0,$$

respectively, as $t\to\infty$.

4 Maximality and equivalences

Let

$$\mathcal{T}=\bigcap_{t\in[0,\infty)}\theta_t^{-1}\mathcal{H}$$

be the tail σ-field. Applying the tail maximality result in discrete time (Proposition 11 in [1]) to the random sequences $(\theta_n Z)_{n\geqq 0}$ and $(\theta_n Z')_{n\geqq 0}$ yields that: *there exists a coupling of Z and Z' with coupling time T such that*

$$\|\mathbf{P}(Z\in\cdot)_{\mathcal{T}}-\mathbf{P}(Z'\in\cdot)_{\mathcal{T}}\|=2\mathbf{P}(T=\infty). \quad (\mathcal{T}\text{-maximal coupling})$$

Since

$$\|\mathbf{P}(Z\in\cdot)_{\mathcal{T}}-\mathbf{P}(Z'\in\cdot)_{\mathcal{T}}\|\leqq\|\mathbf{P}(\theta_t Z\in\cdot)-\mathbf{P}(\theta_t Z'\in\cdot)\|\leqq 2\mathbf{P}(T>t)$$

this yields:

$$(1)\quad \|\mathbf{P}(\theta_t Z\in\cdot)-\mathbf{P}(\theta_t Z'\in\cdot)\|\to\|\mathbf{P}(Z\in\cdot)_{\mathcal{T}}-\mathbf{P}(Z'\in\cdot)_{\mathcal{T}}\| \quad \text{as} \quad t\to\infty.$$

This and the coupling time inequality yields that: *the following statements are equivalent*

(*a*) *there exists a coupling of Z and Z' with a finite coupling time;*
(*b*) $\|\mathbf{P}(\theta_t Z\in\cdot)-\mathbf{P}(\theta_t Z'\in\cdot)\|\to 0$ *as* $t\to\infty$*;*
(*c*) $\mathbf{P}(Z\in\cdot)_{\mathcal{T}}=\mathbf{P}(Z'\in\cdot)_{\mathcal{T}}$.

Let

$$\mathscr{I}=\{A\in\mathscr{H}: \theta_t^{-1}A=A, t\in[0,\infty)\}$$

be the invariant σ-field. In Sect. 6 we prove that: *there exists a shift-coupling of Z and Z' with times T and T' such that*

$$\|\mathbf{P}(Z\in\cdot)_{\mathscr{I}}-\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}\|=2\mathbf{P}(T=\infty). \quad (\mathscr{I}\text{-maximal shift-coupling})$$

Since

$$\|\mathbf{P}(Z\in\cdot)_{\mathscr{I}}-\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}\|\leqq\|\mathbf{P}(\theta_{Ut}Z\in\cdot)-\mathbf{P}(\theta_{Ut}Z'\in\cdot)\|\leqq 2\mathbf{P}(T\vee T'>Ut)$$

this yields:

$$\|\mathbf{P}(\theta_{Ut}Z\in\cdot)-\mathbf{P}(\theta_{Ut}Z'\in\cdot)\|\to\|\mathbf{P}(Z\in\cdot)_{\mathscr{I}}-\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}\| \quad \text{as} \quad t\to\infty.$$

This and the shift-coupling inequality yields that: *the following statements are equivalent*

(a') *there exists a shift-coupling of Z and Z' with finite times;*
(b') $\|\mathbf{P}(\theta_{Ut}Z\in\cdot)-\mathbf{P}(\theta_{Ut}Z'\in\cdot)\|\to 0$ *as* $t\to\infty$*;*
(c') $\mathbf{P}(Z\in\cdot)_{\mathscr{I}}=\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}$.

Define the *smooth tail σ-field* by $\mathscr{S}=\sigma\{\mathscr{S}^\circ\}$ where $\mathscr{S}^\circ$ is the following class of tail functions

$$\mathscr{S}^\circ=\{f\in\mathscr{T}: f(\theta_s z)\to f(z) \text{ as } s\downarrow 0,\ z\in H\}.$$

In Sect. 7 we show that: *if for each $\varepsilon>0$ there is an ε-coupling of Z and Z' with times T_ε and T'_ε then*

(2) $$\|\mathbf{P}(Z\in\cdot)_{\mathscr{S}}-\mathbf{P}(Z'\in\cdot)_{\mathscr{S}}\|\leqq 2\liminf_{\varepsilon\downarrow 0}\mathbf{P}(T_\varepsilon=\infty);$$

that: *for each $\varepsilon>0$ there exists an ε-coupling of Z and Z' with times T_ε and T'_ε such that*

$$\|\mathbf{P}(Z\in\cdot)_{\mathscr{S}}-\mathbf{P}(Z'\in\cdot)_{\mathscr{S}}\|=2\sup_{\varepsilon>0}\mathbf{P}(T_\varepsilon=\infty); \quad (\mathscr{S}\text{-maximal }\varepsilon\text{-couplings})$$

and that:

(3) $$\|\mathbf{P}(\theta_{Uh}Z\in\cdot)_{\mathscr{T}}-\mathbf{P}(\theta_{Uh}Z'\in\cdot)_{\mathscr{T}}\|\to\|\mathbf{P}(Z\in\cdot)_{\mathscr{S}}-\mathbf{P}(Z'\in\cdot)_{\mathscr{S}}\|, \quad h\downarrow 0.$$

Applying the ε-coupling inequality for (a'')$\Rightarrow$(b''), (1) and (3) for (b'')$\Rightarrow$(c''), and $\mathscr{S}$-maximality for (c'')$\Rightarrow$(a''), yields that: *the following statements are equivalent*

(a'') *for each $\varepsilon>0$, there is an ε-coupling of Z and Z' with finite times;*
(b'') *for each $h>0$,* $\|\mathbf{P}(\theta_{t+Uh}Z\in\cdot)-\mathbf{P}(\theta_{t+Uh}Z'\in\cdot)\|\to 0$ *as* $t\to\infty$*;*
(c'') $\mathbf{P}(Z\in\cdot)_{\mathscr{S}}=\mathbf{P}(Z'\in\cdot)_{\mathscr{S}}$.

5 Comments

If Z and Z' are stationary then $\mathbf{P}(Z\in\cdot)=\mathbf{P}(Z'\in\cdot)$ if and only if $\mathbf{P}(Z\in\cdot)_{\mathscr{I}} = \mathbf{P}(Z'\in\cdot)_{\mathscr{I}}$, due to the equivalence of (b') and (c').

For Markov processes the c-parts of the above equivalences are easily seen to be equivalent to $\mathbf{P}(Z\in\cdot)_{\mathscr{T}}$, $\mathbf{P}(Z\in\cdot)_{\mathscr{I}}$, $\mathbf{P}(Z\in\cdot)_{\mathscr{S}}=0$ or 1, respectively, for all initial distributions.

For wide-sense regenerative processes with spread-out inter-regeneration times it is well-known that (a) holds and thus (b) and (c), while if the inter-regeneration times are only non-lattice then (a'') holds and we obtain (b'') and (c''). If the inter-regeneration times have finite mean then we can choose Z' stationary and (b) becomes $\theta_t Z \to_{tv} Z'$ as $t\to\infty$ (which is well-known), and (b'') becomes $\theta_{t+Uh} Z \to_{tv} Z'$ as $t\to\infty$, $h>0$, which Glynn and Iglehart [5] recently established by renewal theoretic methods.

If Z is split by a point process into a stationary sequence of cycles and the conditional mean of the cycle lengths given the invariant σ-field of the cycles is finite then there exists a stationary process Z' such that (c') holds, see [9]. Thus (a') holds and (b') in the form $\theta_{Ut} Z \to_{tv} Z'$, $t\to\infty$.

6 Proof of the existence of an $\mathscr{I}$-maximal shift-coupling

Put $\mathscr{I}_1=\{A\in\mathscr{H}: \theta_1^{-1}A=A\}$ and assume that $\mathbf{P}(Z\in\cdot)_{\mathscr{I}_1}=\mathbf{P}(Z'\in\cdot)_{\mathscr{I}_1}$. This implies $\mathbf{P}((\theta_n Z)_{n\geqq 0}\in\cdot)=\mathbf{P}((\theta_n Z')_{n\geqq 0}\in\cdot)$ on the invariant σ-field of $(H^\infty, \mathscr{H}^\infty)$. By the equivalence result in discrete time (Corollary 16 in [1]) this yields the existence of a shift-coupling $(\theta_n \hat{Z})_{n\geqq 0}$ and $(\theta_n \hat{Z}')_{n\geqq 0}$ of $(\theta_n Z)_{n\geqq 0}$ and $(\theta_n Z')_{n\geqq 0}$ with finite (integer valued) times. Then $\hat{Z}$ and $\hat{Z}'$ is a shift-coupling of Z and Z' with the same finite times.

Now assume only $\mathbf{P}(Z\in\cdot)_{\mathscr{I}}=\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}$. Let f be a bounded function in $\mathscr{I}_1$ and define

$$f^{(1)}(z)=\int_0^1 f(\theta_s z)\,ds,\ z\in H.$$

It is readily checked that $f^{(1)}$ is in $\mathscr{I}$ which yields the second identity in $\mathbf{E}[f(\theta_U Z)]=\mathbf{E}[f^{(1)}(Z)]=\mathbf{E}[f^{(1)}(Z')]=\mathbf{E}[f(\theta_U Z')]$. Thus $\mathbf{P}(\theta_U Z\in\cdot)_{\mathscr{I}_1} = \mathbf{P}(\theta_U Z'\in\cdot)_{\mathscr{I}_1}$ which due to the first part of this proof yields the existence of a shift-coupling $\hat{Y}$ and $\hat{Y}'$ of $\theta_U Z$ and $\theta_U Z'$ with finite times K andK', say. Since $(E, \mathscr{E})$ is Polish and the paths right-continuous there is a regular version of $\mathbf{P}((Z, U)\in\cdot\,|\,\theta_U Z=\cdot)$ and thus we can (see Construction 1.1 in [7]) extend the underlying probability space to support a copy $(\hat{Z}, V)$ of (Z, U) such that $\theta_V \hat{Z}=\hat{Y}$. Again extend the underlying probability space now to support a copy $(\hat{Z}', V')$ of (Z', U) such that $\theta_{V'} \hat{Z}'=\hat{Y}'$. Then $\hat{Z}$ and $\hat{Z}'$ is a shift-coupling of Z and Z' with finite times $V+K$ and $V'+K'$.

Finally, drop the assumption that $\mathbf{P}(Z\in\cdot)_{\mathscr{I}}=\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}$. By the Lemma in [8] there is a component μ of $\mathbf{P}(Z\in\cdot)$ and μ' of $\mathbf{P}(Z'\in\cdot)$ such that

(4) $\quad \mu_{\mathscr{I}}=\mu'_{\mathscr{I}}=$ greatest common component of $\mathbf{P}(Z\in\cdot)_{\mathscr{I}}$ and $\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}$.

Due to $\mu_{\mathscr{I}}=\mu'_{\mathscr{I}}$ and the middle part of this proof there are processes $\hat{Y}$ and $\hat{Y}'$ and finite random times S and S' such that $\mathbf{P}(\hat{Y}\in\cdot)=\mu/\|\mu\|$, $\mathbf{P}(\hat{Y}'\in\cdot)=\mu'/\|\mu\|$ and $\theta_S\hat{Y}=\theta_{S'}\hat{Y}'$. Let C be an event such that C is independent of $\hat{Y}$ and $\hat{Y}'$ and $\mathbf{P}(C)=\|\mu\|$. Put $(\hat{Z},\hat{Z}',T,T')=(\hat{Y},\hat{Y}',S,S')$ on C while on C^c put $T=T'=\infty$ and let

$$\mathbf{P}(\hat{Z}\in\cdot,\hat{Z}'\in\cdot;C^c)=(\mathbf{P}(Z\in\cdot)-\mu)(\mathbf{P}(Z'\in\cdot)-\mu')/(1-\|\mu\|).$$

Then $\hat{Z}$ and $\hat{Z}'$ is a shift-coupling of Z and Z' with times T and T' and (4) yields the first step in

$$\|\mathbf{P}(Z\in\cdot)_{\mathscr{I}}-\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}\|=\|\mathbf{P}(Z\in\cdot)_{\mathscr{I}}-\mu_{\mathscr{I}}\|+\|\mathbf{P}(Z'\in\cdot)_{\mathscr{I}}-\mu_{\mathscr{I}}\|=2\mathbf{P}(T=\infty).$$

7 Proof of (2), (3) and the existence of $\mathscr{S}$-maximal ε-couplings

Put

$$v=\mathbf{P}(Z\in\cdot)_{\mathscr{T}}-\mathbf{P}(Z'\in\cdot)_{\mathscr{T}},$$

$$v^{(h)}=\mathbf{P}(\theta_{Uh}Z\in\cdot)_{\mathscr{T}}-\mathbf{P}(\theta_{Uh}Z'\in\cdot)_{\mathscr{T}}$$

and

$$f^{(h)}(z)=h^{-1}\int_0^h f(\theta_s z)\,ds$$

for $h>0$, bounded f in $\mathscr{T}$ and $z\in H$. Note that $f^{(h)}$ is in $\mathscr{S}^\circ$ and that if f is in $\mathscr{S}^\circ$ then $f^{(h)}\to f$ pointwise as $h\downarrow 0$.

From the ε-coupling inequality and (1) we deduce

$$\|v^{(h)}\|\leqq 2\liminf_{\varepsilon\downarrow 0}\mathbf{P}(T_\varepsilon=\infty),\ h>0.$$

It is readily checked that $\|v^{(h)}\|$ is continuous in h and increases as h goes to 0 through $h=2^{-n}$ which yields

$$\lim_{h\downarrow 0}\|v^{(h)}\|=\sup_{h>0}\|v^{(h)}\|.$$

Thus (2) and (3) follow if we can establish that

(5) $$\|v_{\mathscr{S}}\|=\sup_{h>0}\|v^{(h)}\|.$$

For that purpose take an $A \in \mathcal{S}$ such that $\|\nu_{\mathcal{S}}\| = 2\nu(A)$ and fix an $\varepsilon > 0$. There is an $n \geqq 1$, a Borel subset B of $(-\infty, \infty)^n$ and functions $f_1, \ldots, f_n$ in $\mathcal{S}^\circ$ such that $\int |1_A - 1_B(f_1, \ldots, f_n)| \, d|\nu| \leqq \varepsilon$. Thus there is a continuous function g from $(-\infty, \infty)^n$ to $[0, 1]$ such that $\int |1_A - f| d|\nu| \leqq 2\varepsilon$ where $f = g(f_1, \ldots, f_n)$. Clearly this f is in $\mathcal{S}^\circ$ which implies $f^{(h)} \to f$ pointwise as $h \downarrow 0$. Hence there is an $h > 0$ such that $\int |1_A - f^{(h)}| \, d|\nu| \leqq 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary and

$$\|\nu^{(h)}\| = 2 \sup_{f \in \mathcal{T}, \, 0 \leqq f \leqq 1} \int f^{(h)} \, d\nu$$

this yields

$$\|\nu_{\mathcal{S}}\| \leqq \sup_{h > 0} \|\nu^{(h)}\|.$$

The converse holds since $f \in \mathcal{T}$ implies $f^{(h)} \in \mathcal{S}$, and (5) is established.

It only remains to prove the $\mathcal{S}$-maximality result. For each $h > 0$ there is a coupling $\hat{Y}$ and $\hat{Y}'$ of $\theta_{Uh} Z$ and $\theta_{Uh} Z'$ with a coupling time S_h such that $\mathbf{P}(S_h = \infty) = \|\nu^{(h)}\|/2$. Extend the underlying probability space (see Construction 1.1 in [7]) to support a copy $(\hat{Z}, V)$ of (Z, U) such that $\theta_{Vh} \hat{Z} = \hat{Y}$. Again extend the underlying probability space now to support a copy $(\hat{Z}', V')$ of (Z', U) such that $\theta_{V'h} \hat{Z}' = \hat{Y}'$. Then $\hat{Z}$ and $\hat{Z}'$ is an h-coupling of Z and Z' with times $T_h = Vh + S_h$ and $T_h' = V'h + S_h$. Now $\mathbf{P}(T_h = \infty) = \mathbf{P}(S_h = \infty) = \|\nu^{(h)}\|/2$ and a reference to (5) completes the proof.

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