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Solution of forward–backward stochastic differential equations

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Summary. In this paper, we study the existence and uniqueness of the solution to forward–backward stochastic differential equations without the non-degeneracy condition for the forward equation. Under a certain "monotonicity" condition, we prove the existence and uniqueness of the solution to forward–backward stochastic differential equations.

Mathematics Subject Classification: 60H10, 60H20

1 Introduction

Let $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\}_{t \geqq 0})$ be a filtered probability space satisfying the usual hypotheses, and let a standard d-dimensional Brownian motion $\{W_t\}_{t \geqq 0}$ be defined on it. Consider the following forward–backward stochastic differential equations:

$$X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s, Z_s)\,\mathrm{d}W_s\,, \tag{1}$$

$$Y_t = g(X_T) - \int_t^T h(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \quad t \in [0, T]\,, \tag{2}$$

where X, Y, Z take values in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n \times d}$, respectively, and b, σ, g, h are functions with appropriate dimensions; $T > 0$ is an arbitrarily fixed number. Our aim is to find a triple (X, Y, Z) which is $\{\mathscr{F}_t\}$-adapted, and satisfies the above forward–backward stochastic differential equations, on [0, T], P-almost surely. Note that it is the extra process Z that makes it possible for (1), (2) to have an adapted triple.

Non-linear backward stochastic differential equations were introduced by Pardoux and Peng [8]. It was then shown in ([10, 11, 9, 5], etc.) that this kind of backward stochastic differential equations gives a probabilistic representation for a large class of systems of quasilinear parabolic and elliptic PDEs, which generalized the classical Feynman–Kac formula for linear parabolic and elliptic PDEs. As for the forward–backward equations, Antonelli [1] first studied these equations, and he gave the existence and uniqueness when the time duration T is sufficiently small. Using a PDE approach, Ma et al. [7] gave the existence and uniqueness to a class of forward–backward SDEs in which the forward SDE is non-degenerate (i.e., the coefficient σ is non-degenerate).

Forward–backward equations are encountered when one applies the stochastic maximum principle to optimal stochastic control problems (see [4] for a linear version of such equations in an optimal stochastic control problem). Such equations are also encountered in the probabilistic interpretation of a general type of systems of quasilinear PDEs (see [12]), as well as in finance (see [2, 3]).

Here we shall establish an existence and uniqueness result for the forward–backward SDEs over an arbitrarily prescribed time duration, without the non-degeneracy condition of σ, instead we assume a kind of "monotonicity" condition. Note that without the non-degeneracy condition of σ, our conditions are essential for the existence of an adapted solution over an arbitrarily time interval $[0, T]$; in fact, Antonelli's counterexample in [1] shows that otherwise the adapted solution may not even exist when the time duration T is large. Also, we will give the "deterministic" version of our results which gives the existence and uniqueness to a two-point boundary value problem. We think that these results are of independent interest in the deterministic two-point boundary value problem.

The paper is organized as follows: in Sect. 2, we give the formulation of the problem and our standing assumptions; in Sect. 3, we give our main results about the existence and uniqueness to Eqs. (1) and (2), and we prove it; in Sect. 4, we prove some technical lemmas needed in Sect. 3; and in the last section, we give the "deterministic" version of our results.

2 Formulation of the problem

Let $(\Omega, \mathscr{F}, P)$ be a probability space carrying a standard d-dimensional Brownian motion $W = \{W_t : t \geqq 0\}$, and let $\{\mathscr{F}_t\}$ be the σ-field generated by W (that is, $\mathscr{F}_t = \sigma\{W_s : 0 \leqq s \leqq t\}$). We make the standard P-augmentation to each $\mathscr{F}_t$ such that $\mathscr{F}_t$ contains all the P-null sets of $\mathscr{F}$. Then $\{\mathscr{F}_t\}$ is right continuous and $\{\mathscr{F}_t\}$ satisfies the usual hypothesis. We consider the following forward–backward SDEs:

$$X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s, Z_s)\,\mathrm{d}W_s\,, \tag{3}$$

$$Y_t = g(X_T) - \int_t^T h(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s, \quad t \in [0, T]\,. \tag{4}$$

Here the processes X, Y, Z take values in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n\times d}$, respectively; and b, h, σ and g take values in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n\times d}$ and $\mathbb{R}^n$, respectively.

We will use the following notations:

(,) denotes the usual inner product in $\mathbb{R}^n$; we use the usual Euclidean norm in $\mathbb{R}^n$; and for $z \in \mathbb{R}^{n\times d}$, we define $|z| = \{\mathrm{tr}(zz^{\mathrm{T}})\}^{1/2}$, where "T" means transpose.

For $z^1 \in \mathbb{R}^{n\times d}$, $z^2 \in \mathbb{R}^{n\times d}$,

$$((z^1,z^2)) = \mathrm{tr}(z^1(z^2)^{\mathrm{T}}),$$

and for $u^1 = (x^1, y^1, z^1) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$, $u^2 = (x^2, y^2, z^2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$,

$$[u^1,u^2] = (x^1,x^2) + (y^1,y^2) + ((z^1,z^2))$$

for $u = (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$,

$$f(t,u) = (h(t,u), b(t,u), \sigma(t,u)).$$

Definition 2.1 We denote by $M^2(0,T;\mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued $\mathscr{F}_t$-adapted processes $v(\cdot)$ such that

$$E\int_0^T |v(s)|^2\,\mathrm{d}s < +\infty.$$

Definition 2.2 A triple of processes $(X,Y,Z) : [0,T] \times \Omega \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$ is called an adapted solution of the Eqs. (3) and (4), if $(X,Y,Z) \in M^2(0,T;\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$, and it satisfies (3) and (4) P-almost surely.

The adaptedness of the solution enables us to rewrite (3) and (4) in a differential form:

$$\begin{aligned}
\mathrm{d}X_t &= b(t,X_t,Y_t,Z_t)\,\mathrm{d}t + \sigma(t,X_t,Y_t,Z_t)\,\mathrm{d}W_t,\\
\mathrm{d}Y_t &= h(t,X_t,Y_t,Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t,\\
X_0 &= x, \qquad Y_T = g(X_T).
\end{aligned}$$

It is clear that the above equations is a stochastic two point boundary value problem. Especially, it contains deterministic two point boundary value problem as a special case when $\sigma = 0$.

Now we give the standing assumptions of our paper:

Assumption 2.1 For each $u \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$, $f(\cdot\,,u) \in M^2(0,T;\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$, and for each $x \in \mathbb{R}^n$, $g(x) \in L^2(\Omega,\mathscr{F}_T;\mathbb{R}^n)$; there exists a constant $c_1 > 0$, such that

$$|f(t,u^1) - f(t,u^2)| \leqq c_1|u^1 - u^2|, \quad P\text{-a.s., a.e. } t \in \mathbb{R}^+,$$

$$\forall u^1 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}, \quad u^2 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d};$$

and

$$|g(x_1) - g(x_2)| \leqq c_1|x_1 - x_2|, \quad P\text{-a.s., } \forall (x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Assumption 2.2 There exists a constant $c_2 > 0$, such that

$$[f(t,u^1) - f(t,u^2), u^1 - u^2] \leqq -c_2|u^1 - u^2|^2, \quad P\text{-a.s., a.e. } t \in \mathbb{R}^+,$$

$$\forall u^1 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}, \quad u^2 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d};$$

and

$$(g(x^1) - g(x^2), x^1 - x^2) \geqq c_2|x^1 - x^2|^2, \quad P\text{-a.s., } \forall x^1 \in \mathbb{R}^n, \quad x^2 \in \mathbb{R}^n.$$

3 Existence and uniqueness for forward–backward equations

In this section, we shall give the main result of our paper.

Theorem 3.1 *Let Assumptions* (2.1) *and* (2.2) *hold, then there exists a unique adapted solution* (X, Y, Z) *for Eqs.* (3) *and* (4).

Proof of Uniqueness. If $U^1 = (X^1, Y^1, Z^1)$, $U^2 = (X^2, Y^2, Z^2)$ are two adapted solutions of (3), (4), we set

$$\begin{aligned}(\hat{X}, \hat{Y}, \hat{Z}) &= (X^1 - X^2, Y^1 - Y^2, Z^1 - Z^2),\\ \hat{b}(t) &= b(t, U_t^1) - b(t, U_t^2),\\ \hat{\sigma}(t) &= \sigma(t, U_t^1) - \sigma(t, U_t^2),\\ \hat{h}(t) &= h(t, U_t^1) - h(t, U_t^2).\end{aligned}$$

From Assumption 2.1, it follows that $\{\hat{X}_t\}$ and $\{\hat{Y}_t\}$ are continuous, and

$$E(\sup_{t\in[0,T]} |\hat{X}_t|^2) + E(\sup_{t\in[0,T]} |\hat{Y}_t|^2) < +\infty.$$

Applying the Itô formula to $(\hat{Y}_t, \hat{X}_t)$ (the inner product of $\hat{Y}_t$ and $\hat{X}_t$),

$$d(\hat{Y}_t, \hat{X}_t) = [f(t, U_t^1) - f(t, U_t^2), U_t^1 - U_t^2]\, dt + (\hat{X}_t^T \hat{Z}_t + \hat{Y}_t^T \hat{\sigma}(t))\, dW_t$$

Let now v be a stopping time such that:

(α) $0 \leqq v \leqq T$,

(β) $E\int_0^v |\hat{X}_t^T \hat{Z}_t + \hat{Y}_t^T \hat{\sigma}(t)|^2\, dt < +\infty$,

we have then,

$$E(\hat{Y}_v, \hat{X}_v) = E\int_0^v [f(t, U_t^1) - f(t, U_t^2), U_t^1 - U_t^2]\, dt.$$

Let now $\{v_n\}$ be an increasing sequence of stopping times satisfying (α) and (β) which converges a.s. to T. It follows from the Lebesgue dominated convergence theorem that

$$E(g(X_T^1) - g(X_T^2), X_T^1 - X_T^2) = E\int_0^T [f(t, U_t^1) - f(t, U_t^2), U_t^1 - U_t^2]\, dt$$

By Assumptions (2.1) and (2.2), we get then

$$\begin{aligned}c_2|X_T^1 - X_T^2|^2 &\leqq E(g(X_T^1) - g(X_T^2), X_T^1 - X_T^2)\\ &\leqq -c_2 E\int_0^T |U_t^1 - U_t^2|^2 dt.\end{aligned}$$

So

$$U^1 = U^2 . \qquad \square$$

Remark 3.1 We can see that in the proof of uniqueness, we have applied Itô's formula to $(\hat{Y}, \hat{X})$, rather than $|\hat{X}|^2$ or $|\hat{Y}|^2$. That is the main difference between the method of [1, 7] and ours. One will see that the same is true for the proof of existence.

We shall use a kind of apriori method to give a proof to the existence part of Theorem 3.1. First we give two technical lemmas whose proof will be given in the next section.

Lemma 3.1 *Suppose that* $(b_0(\cdot), \sigma_0(\cdot), h_0(\cdot)) \in M^2(0,T;\mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n), g_0 \in L^2(\Omega, \mathscr{F}_T; \mathbb{R}^n)$, *then the following linear forward–backward stochastic differential equations*

$$X_t = x + \int_0^t (-Y_s + b_0(s))\,\mathrm{d}s + \int_0^t (-Z_s + \sigma_0(s))\,\mathrm{d}W_s \,, \tag{5}$$

$$Y_t = (X_T + g_0) - \int_t^T (-X_s + h_0(s))\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s \tag{6}$$

have a unique adapted solution (X, Y, Z).

Now we define, for any given $\alpha \in \mathbb{R}$,

$$\begin{aligned} b^\alpha(t,x,y,z) &= \alpha b(t,x,y,z) + (1-\alpha)(-y)\,, \\ \sigma^\alpha(t,x,y,z) &= \alpha\sigma(t,x,y,z) + (1-\alpha)(-z)\,, \\ h^\alpha(t,x,y,z) &= \alpha h(t,x,y,z) + (1-\alpha)(-x)\,, \\ g^\alpha(x) &= \alpha g(x) + (1-\alpha)(x)\,, \end{aligned}$$

and consider the following equations:

$$X_t = x + \int_0^t [b^\alpha(s, U_s) + b_0(s)]\,\mathrm{d}s + \int_0^t [\sigma^\alpha(s, U_s) + \sigma_0(s)]\,\mathrm{d}W_s \,, \tag{7}$$

$$Y_t = (g^\alpha(X_T) + g_0) - \int_t^T [h^\alpha(s, U_s) + h_0(s)]\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s \,, \tag{8}$$

where $U = (X, Y, Z)$.

Lemma 3.2 *We assume that, for a given* $\alpha_0 \in [0,\ 1)$ *and for any*

$$(b_0(\cdot), \sigma_0(\cdot), h_0(\cdot)) \in M^2(0,T;\mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n), \quad g_0 \in L^2(\Omega, \mathscr{F}_T; \mathbb{R}^n)\,,$$

Eqs. (7) *and* (8) *have an adapted solution. Then there exists a* $\delta_0 \in (0,1)$ *which depends only on* c_1, c_2 *and* T, *such that for all* $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$, *and for any* $(b_0(\cdot), \sigma_0(\cdot), h_0(\cdot)) \in M^2(0,T;\mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n)$, $g_0 \in L^2(\Omega, \mathscr{F}_T; \mathbb{R}^n)$, *Eqs.* (7) and (8) *have an adapted solution.*

Now we can give

Proof of Existence. From Lemma 3.1, we see immediately that, when $\alpha = 0$, for any $(b_0(\cdot), \sigma_0(\cdot), h_0(\cdot)) \in M^2(0,T;\mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n)$, $g_0 \in L^2(\Omega, \mathscr{F}_T;$

$\mathbb{R}^n$) Eqs. (7) and (8) have an adapted solution. According to Lemma 3.2, for any $(b_0(\cdot), \sigma_0(\cdot), h_0(\cdot)) \in M^2(0,T;\mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n)$, $g_0 \in L^2(\Omega, \mathscr{F}_T;\mathbb{R}^n)$, we can solve Eqs. (7) and (8) successively for the case $\alpha \in [0,\delta_0], [\delta_0, 2\delta_0], \ldots$ It turns out that, when $\alpha = 1$, for any $(b_0(\cdot), \sigma_0(\cdot), h_0(\cdot)) \in M^2(0,T;\mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^n)$, $g_0 \in L^2(\Omega, \mathscr{F}_T;\mathbb{R}^n)$, the adapted solution of Eqs. (7) and (8) exists, then we deduce immediately that the adapted solution of Eqs. (3) and (4) exists. □

In the applications of the stochastic maximum principle, the existence of the optimal stochastic control gives the existence of solutions to the forward–backward equations. So the uniqueness of the solutions to the forward–backward equations is more important there. If we examine further the proof of uniqueness in Theorem 3.1, we can see easily that Assumption (2.2) can be weakened to

Assumption 3.1 There exists a constant $c_2 > 0$, such that

$$[f(t,u^1) - f(t,u^2), u^1 - u^2] \leqq -c_2|x^1 - x^2|^2, \quad P\text{-a.s., a.e. } t \in \mathbb{R}^+,$$

$$\forall u^1 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}, \quad u^2 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d};$$

and

$$(g(x^1) - g(x^2), x^1 - x^2) \geqq 0, \quad P\text{-a.s., } \forall x^1 \in \mathbb{R}^n, \quad x^2 \in \mathbb{R}^n.$$

Or

Assumption 3.2 There exists a constant $c_2 > 0$, such that

$$[f(t,u^1) - f(t,u^2), u^1 - u^2] \leqq -c_2|y^1 - y^2|^2, \quad P\text{-a.s., a.e.} t \in \mathbb{R}^+,$$

$$\forall u^1 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d}, \quad u^2 \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d};$$

and

$$(g(x^1) - g(x^2), x^1 - x^2) \geqq 0, \quad P\text{-a.s., } \forall x^1 \in \mathbb{R}^n, \quad x^2 \in \mathbb{R}^n.$$

Theorem 3.2 *Let Assumptions* (2.1) *and* (3.1) *hold, then there exists at most one adapted solution* (X,Y,Z) *for Eqs.* (3) *and* (4).

Proof. We use the same notations as in Theorem 3.1. Applying the Itô formula to $(\hat{Y}_t, \hat{X}_t)$, then the same procedure as that in the proof of uniqueness in Theorem 3.1 yields,

$$E(g(X_T^1) - g(X_T^2), X_T^1 - X_T^2) = E\int_0^T [f(t,U_t^1) - f(t,U_t^2), U_t^1 - U_t^2]\,dt.$$

By Assumption (3.1), we get then

$$\begin{aligned} 0 &\leqq E(g(X_T^1) - g(X_T^2), X_T^1 - X_T^2) \\ &\leqq -c_2 E\int_0^T |X_t^1 - X_t^2|^2\,dt. \end{aligned}$$

So

$$X^1 = X^2 .$$

From the uniqueness of the backward equation (4), we finally get

$$U^1 = U^2 . \quad \square$$

Theorem 3.3 *Let Assumptions* (2.1) *and* (3.2) *hold, then there exists at most one adapted solution* (X, Y, Z) *for Eqs.* (3) *and* (4).

Proof. First, the same proof as that of Theorem 3.2 yields,

$$\hat{Y} = 0 .$$

Applying the Itô formula to $|\hat{Y}_t|^2$ and integrating from 0 to T, it follows then,

$$\int_0^T |\hat{Z}_t|^2 \, \mathrm{d}t = 0 ,$$

that is

$$\hat{Z} = 0.$$

From the uniqueness of the forward equation (3), we finally get

$$U^1 = U^2 . \quad \square$$

4 Proof of Lemmas

In this section, we prove the technical lemmas needed in Sect. 3.

Proof of Lemma 3.1. We consider the following backward stochastic differential equation:

$$\bar{Y}_t = g_0 - \int_t^T (\bar{Y}_s + h_0(s) - b_0(s)) \, \mathrm{d}s - \int_t^T (2\bar{Z}_s - \sigma_0(s)) \, \mathrm{d}W_s .$$

By the result of [8], the above equation has a unique adapted solution $(\bar{Y}, \bar{Z})$.

Then we solve the following forward equation:

$$X_t = x + \int_0^t (-X_s - \bar{Y}_s + b_0(s)) \, \mathrm{d}s + \int_0^t (-\bar{Z}_s + \sigma_0(s)) \, \mathrm{d}W_s$$

and set

$$Y = X + \bar{Y}, \qquad Z = \bar{Z} ,$$

we easily see that (X, Y, Z) is a solution of Eqs. (5) and (6). Thus the existence is proved.

As for uniqueness, it suffices to use the method of the proof of uniqueness in Theorem 3.1 and we omit it. $\square$

Proof of Lemma 3.2. Observe that

$$b^{\alpha_0+\delta}(t, x, y, z) = b^{\alpha_0}(t, x, y, z) + \delta(y + b(t, x, y, z)),$$
$$\sigma^{\alpha_0+\delta}(t, x, y, z) = \sigma^{\alpha_0}(t, x, y, z) + \delta(z + \sigma(t, x, y, z)),$$
$$h^{\alpha_0+\delta}(t, x, y, z) = h^{\alpha_0}(t, x, y, z) + \delta(x + h(t, x, y, z)),$$
$$g^{\alpha_0+\delta}(x) = g^{\alpha_0}(x) + \delta(-x + g(x)).$$

We set $U^0 = (X^0, Y^0, Z^0) = 0$, and solve iteratively the following equations:

$$X_t^{i+1} = x + \int_0^t [b^{\alpha_0}(s, U_s^{i+1}) + \delta(Y_s^i + b(s, U_s^i)) + b_0(s)]\,\mathrm{d}s$$
$$+ \int_0^t [\sigma^{\alpha_0}(s, U_s^{i+1}) + \delta(Z_s^i + \sigma(s, U_s^i)) + \sigma_0(s)]\,\mathrm{d}W_s, \tag{9}$$

$$Y_t^{i+1} = [g^{\alpha_0}(X_T^{i+1}) + \delta(-X_T^i + g(X_T^i)) + g_0]$$
$$- \int_t^T (h^{\alpha_0}(s, U_s^{i+1}) + \delta(X_s^i + h(s, U_s^i)) + h_0(s))\,\mathrm{d}s - \int_t^T Z_s^{i+1}\,\mathrm{d}W_s, \tag{10}$$

where $U^i = (X^i, Y^i, Z^i)$.

We set
$$\hat{U}^{i+1} = (\hat{X}^{i+1}, \hat{Y}^{i+1}, \hat{Z}^{i+1}) = U^{i+1} - U^i.$$

Apply the Itô formula to $(\hat{Y}_t^{i+1}, \hat{X}_t^{i+1})$, then the same procedure as that in the proof of uniqueness in Theorem 3.1 yields,

$$E(g^{\alpha_0}(X_T^{i+1}) - g^{\alpha_0}(X_T^i), \hat{X}_T^{i+1})$$
$$= \delta E(\hat{X}_T^i - (g(X_T^i) - g(X_T^{i-1})), \hat{X}_T^{i+1})$$
$$+ E\int_0^T [f^{\alpha_0}(t, U_t^{i+1}) - f^{\alpha_0}(t, U_t^i), \hat{U}_t^{i+1}]\,\mathrm{d}t$$
$$+ \delta\left(E\int_0^T [\hat{U}_t^i + f(t, U_t^i) - f(t, U_t^{i-1}), \hat{U}_t^{i+1}]\,\mathrm{d}t\right).$$

From Assumptions (2.1) and (2.2), we get easily that

$$E|\hat{X}_T^{i+1}|^2 + E\int_0^T |\hat{U}_t^{i+1}|^2\,\mathrm{d}t \leqq \frac{\delta(1+c_1)}{c_2'}\left(E\left(|\hat{X}_T^i||\hat{X}_T^{i+1}|\right) + E\int_0^T |\hat{U}_t^i||\hat{U}_t^{i+1}|\,\mathrm{d}t\right),$$

where $c_2' = \min(1, c_2)$.

We know that for $\varepsilon > 0$
$$ab \leqq \frac{a^2}{4\varepsilon} + \varepsilon b^2$$
and we take $\varepsilon = (\delta(1+c_1)/c_2')^{-1} \cdot \frac{1}{2}$, then we derive

$$E|\hat{X}_T^{i+1}|^2 + E\int_0^T |\hat{U}_t^{i+1}|^2\,\mathrm{d}t \leqq \left(\frac{\delta(1+c_1)}{c_2'}\right)^2 \left(E|\hat{X}_T^i|^2 + E\int_0^T |\hat{U}_t^i|^2\,\mathrm{d}t\right).$$

Remember that $\forall i \geqq 1$,

$$\hat{X}_T^i = \int_0^T [b^{\alpha_0}(t, U_t^i) - b^{\alpha_0}(t, U_t^{i-1}) + \delta(\hat{Y}_t^{i-1} + b(t, U_t^{i-1}) - b(t, U_t^{i-2}))]\,\mathrm{d}t$$
$$+ \int_0^T [\sigma^{\alpha_0}(t, U_t^i) - \sigma^{\alpha_0}(t, U_t^{i-1}) + \delta(\hat{Z}_t^{i-1} + \sigma(t, U_t^{i-1})$$
$$- \sigma(t, U_t^{i-2}))]\,\mathrm{d}W_t .$$

By a standard method of estimation, we can derive easily that there exists a constant $c_3 > 0$ which depends only on c_1 and T, such that

$$E|\hat{X}_T^i|^2 \leqq c_3 \left(E\int_0^T |\hat{U}_t^i|^2\,\mathrm{d}t + E\int_0^T |\hat{U}_t^{i-1}|^2\,\mathrm{d}t \right), \quad \forall i \geqq 1 .$$

So there exists a constant $c_4 > 0$ which depends only on c_1, c_2 and T, such that

$$E\int_0^T |\hat{U}_t^{i+1}|^2\,\mathrm{d}t \leqq c_4\delta^2 \left(E\int_0^T |\hat{U}_t^i|^2\,\mathrm{d}t + E\int_0^T |\hat{U}_t^{i-1}|^2\,\mathrm{d}t \right)$$

So there exists a $\delta_0 \in (0, 1)$, which depends only on c_1, c_2 and T, such that when $0 < \delta \leqq \delta_0$,

$$E\int_0^T |\hat{U}_t^{i+1}|^2\,\mathrm{d}t \leqq \frac{1}{4} E\int_0^T |\hat{U}_t^i|^2\,\mathrm{d}t + \frac{1}{8} E\int_0^T |\hat{U}_t^{i-1}|^2\,\mathrm{d}t \quad \forall i \geqq 1 .$$

From the next lemma, it turns out that U^i is a Cauchy sequence in $M^2(0, T; \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$. We denote its limit by $U = (X, Y, Z)$. Passing to the limit in Eqs. (9) and (10), we see that, when $0 < \delta \leqq \delta_0$, $U = (X, Y, Z)$ solves Eqs. (7) and (8) for $\alpha = \alpha_0 + \delta$. The proof is completed. □

Lemma 4.1 *Suppose that a real sequence* $\{a_i\}_{i=0}^{\infty}$ *satisfies the following conditions*: $a_i \geqq 0$, $i = 0, 1, 2, \ldots$, *and*

$$a_{i+1} \leqq \frac{1}{4} a_i + \frac{1}{8} a_{i-1}, \quad \forall i \geqq 1 ,$$

then there exists a constant $c > 0$, *such that*

$$a_i \leqq c \left(\frac{1}{2}\right)^i, \quad \forall i \geqq 0 .$$

The proof of this lemma is elementary, so we omit it. □

5 Deterministic version

In this section, we will give the "deterministic" version of our results which give the existence and uniqueness of the solutions to the deterministic two-point boundary value problem under the "monotonicity" condition.

Consider the following equations:

$$X_t = x + \int_0^t b(s, X_s, Y_s)\,ds\,, \tag{11}$$

$$Y_t = g(X_T) - \int_t^T h(s, X_s, Y_s)\,ds, \quad t \in [0, T]\,, \tag{12}$$

where $f = (h, b) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $u = (x, y)$.

Definition 5.1 A couple of functions $(X, Y) : [0,T] \to \mathbb{R}^n \times \mathbb{R}^n$ is called a solution of Eqs. (11) and (12), if $(X, Y) \in C(0,T; \mathbb{R}^n \times \mathbb{R}^n)$, and it satisfies Eqs. (11) and (12).

We assume

Assumption 5.1 There exists a constant $c_1 > 0$, such that

$$|f(t, u^1) - f(t, u^2)| \leqq c_1|u^1 - u^2|, \quad \text{a.e.t} \in \mathbb{R}^+\,,$$

$$\forall u^1 \in \mathbb{R}^n \times \mathbb{R}^n, \quad u^2 \in \mathbb{R}^n \times \mathbb{R}^n\,;$$

and

$$|g(x_1) - g(x_2)| \leqq c_1|x_1 - x_2|, \quad \forall (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n\,.$$

Assumption 5.2 There exists a constant $c_2 > 0$, such that

$$(f(t, u^1) - f(t, u^2), u^1 - u^2) \leqq -c_2|u^1 - u^2|^2, \quad \text{a.e. } t \in \mathbb{R}^+\,,$$

$$\forall u^1 \in \mathbb{R}^n \times \mathbb{R}^n, \quad u^2 \in \mathbb{R}^n \times \mathbb{R}^n;$$

and

$$(g(x^1) - g(x^2), x^1 - x^2) \geqq c_2|x^1 - x^2|^2, \quad \forall x^1 \in \mathbb{R}^n, \quad x^2 \in \mathbb{R}^n\,.$$

We have

Theorem 5.1 *Let Assumptions* (5.1) *and* (5.2) *hold, then there exists a unique solution* (X, Y) *for Eqs.* (11) *and* (12).

As before, for the uniqueness of solutions to Eqs. (11) and (12), Assumption (5.2) can be weakened to

Assumption 5.3 There exists a constant $c_2 > 0$, such that

$$(f(t, u^1) - f(t, u^2), u^1 - u^2) \leqq -c_2|x^1 - x^2|^2, \quad \text{a.e. } t \in \mathbb{R}^+\,,$$

$$\forall u^1 \in \mathbb{R}^n \times \mathbb{R}^n, \quad u^2 \in \mathbb{R}^n \times \mathbb{R}^n;$$

and

$$(g(x^1) - g(x^2), x^1 - x^2) \geqq 0, \quad \forall x^1 \in \mathbb{R}^n, \quad x^2 \in \mathbb{R}^n\,.$$

Or

Assumption 5.4 There exists a constant $c_2 > 0$, such that

$$(f(t,u^1)-f(t,u^2),u^1-u^2) \leqq -c_2|y^1-y^2|^2, \quad \text{a.e. } t\in \mathbb{R}^+,$$

$$\forall u^1 \in \mathbb{R}^n \times \mathbb{R}^n, \quad u^2 \in \mathbb{R}^n \times \mathbb{R}^n;$$

and

$$(g(x^1)-g(x^2),x^1-x^2) \geqq 0, \quad \forall x^1 \in \mathbb{R}^n, \quad x^2 \in \mathbb{R}^n .$$

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