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Probability Theory and Related Fields

Markov fields on branching planes *

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Summary. We investigate Ising models indexed by the sites of a branching plane $\mathbb{T} \times \mathbb{Z}$, which is the product of a regular tree $\mathbb{T}$ and the line $\mathbb{Z}$. There are three parameter regimes corresponding to:

(1) a unique Gibbs distribution;

(2) nonunique Gibbs distributions with treelike structure – the free boundary condition field is not a mixture of the plus and minus b.c. fields;

(3) nonunique Gibbs distributions with planelike structure – the free b.c. field is a mixture of the plus and minus b.c. fields.

Our analysis is based on earlier work by Grimmett and Newman concerning independent percolation on $\mathbb{T} \times \mathbb{Z}$, the Fortuin-Kasteleyn representation of Ising (and Potts) systems as dependent percolation models, and a "finite island" property of percolation models on $\mathbb{T} \times \mathbb{Z}$.

1. Introduction

The simplest examples of Markov random fields with finite state spaces are ± 1 valued nearest neighbor Ising models $\{S_x\}$ indexed by the sites x of some lattice $\mathbb{L}$ with a multidimensional graph structure. These have been extensively studied both for ordinary d-dimensional lattices such as $\mathbb{Z}^d$ [D2, LR] and for regular trees $\mathbb{T}_k$, where every site has exactly $k+1$ neighbors [Pr]. (We will generally write $\mathbb{T}$ rather than $\mathbb{T}_k$ below.) Trees are sometimes regarded as infinite dimensional because the volume of the "ball" of radius R grows exponentially in R. For both types of lattices, phase transitions occur providing $d \geqq 2$ [P, G, D1] or $k \geqq 2$ [KT, Pr]; i.e., there is nonuniqueness of the Gibbs distributions corresponding to a fixed family of conditional probabilities. However, trees and ordinary d-dimensional lattices exhibit different types of nonuniqueness, as we now explain.

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For ferromagnetic (i.e., positively dependent) Ising models, it is convenient to focus on the translation invariant Gibbs distributions ν^+, ν^- and ν^f obtained as the limit of finite volume systems $\{S_x: x \in \Lambda \subset \mathbb{L}\}$ as $\Lambda \to \mathbb{L}$ with respectively plus, minus and free boundary conditions. The Gibbs distribution is unique if and only if the distributions ν^+ and ν^- (which are necessarily extremal) are identical [LM, R]. In the models we consider there is no term in the Hamiltonian (which defines the conditional probabilities) to break the $\{S_x\} \to \{-S_x\}$ symmetry. Thus, ν^+ is transformed to ν^- and ν^f is unchanged by this symmetry operation; furthermore, $\nu^+ \neq \nu^-$ if and only if $M \equiv \langle S_x \rangle_+ > 0$ where $\langle \cdot \rangle_*$ denotes expectation with respect to the measure ν^* ($*$ is $+$ or $-$ or f). If $\nu^+ \neq \nu^-$ and ν^f is a mixture of ν^+ and ν^-, it must be the symmetric mixture $(\nu^+ + \nu^-)/2$. The decomposition of ν^f as a mixture of ν^+ and ν^- seems to be the case for $\mathbb{Z}^d$ – at least, it has been proved for $d=2$ [MM] and for $d>2$ at all low temperatures [GM], and at all but countably many temperatures [L]. However, this decomposition of ν^f is definitely not valid on the tree. To see this, consider the decay as x and y separate of $\langle S_x S_y \rangle_f$, the ν^f-covariance of S_x and S_y. In the language of [Pr, S], ν^f defines a Markov chain on $\mathbb{T}$ which in particular is an ordinary Markov chain along any linear path within $\mathbb{T}$. Thus $\langle S_x S_y \rangle_f = r^{\delta_{\mathbb{T}}(x,y)}$ where $0<r<1$ and $\delta_{\mathbb{T}}$ is the natural distance in $\mathbb{T}$ (the number of bonds on the unique linear path connecting x and y). Since $\langle S_x S_y \rangle_f \to 0$ as $\delta_{\mathbb{T}}(x, y) \to \infty$, it follows that ν^f is not a mixture of ν^+ and ν^-, when nonuniqueness occurs, since $\langle S_x S_y \rangle_\pm \geqq M^2 > 0$ for all x, y.

In this paper, motivated by the work of [GN], we consider Ising models (and the related Potts models with q states) on a lattice $\mathbb{L}$ which is a branching plane intermediate between $\mathbb{Z}^2$ and $\mathbb{T}$. $\mathbb{L}$ is the product $\mathbb{T} \times \mathbb{Z}$, with two types of bonds, $\mathbb{T}$-bonds between sites $x=(t, z)$ and $x'=(t', z)$ with $\delta_{\mathbb{T}}(t, t')=1$ and $\mathbb{Z}$-bonds between sites $x=(t, z)$ and $x'=(t, z')$ with $|z-z'|=1$. The distance $\delta(x, x')$ between $x=(t, z)$ and $x'=(t', z')$ is defined as $\delta_{\mathbb{T}}(t, t')+|z-z'|$. It is natural to have two coupling parameters (into which the temperature has already been absorbed) in the Hamiltonian, J_t for $\mathbb{T}$-bonds and J_z for $\mathbb{Z}$-bonds, both assumed to be positive so that the model is ferromagnetic. $\mathbb{T} \times \mathbb{Z}$ may be regarded both as a branching plane and as a stack of trees. Although it is infinite dimensional in the same sense as the tree, the existence of loops in its graphical structure means that Ising models on $\mathbb{T} \times \mathbb{Z}$ do not have simple embedded Markov chains.

A natural question to ask concerning Ising models on such a branching plane is whether they are tree-like or plane-like in their phase transition behavior. Such a question for independent percolation on $\mathbb{T} \times \mathbb{Z}$ was studied by Grimmett and Newman [GN] who discovered that as the parameters are varied both types of behavior occur. In the percolation context, plane-like behavior means a unique infinite cluster and tree-like behavior means infinitely many distinct infinite clusters. (Closely related results concerning the contact process on a tree have since been obtained in [Pe].)

It is known that there are strong analogies between nonuniqueness of infinite clusters in percolation and the occurrence of translation invariant Gibbs distributions for Ising ferromagnets which are not mixtures of ν^+ and ν^- [AKN]. Thus the results of [GN] suggest that also for Ising models on $\mathbb{T} \times \mathbb{Z}$ there should occur both tree-like and plane-like behavior. The main results of this paper substantiate this suggestion. We prove for any $k \geqq 2$ that:

(1) for small J_t and J_z, $\nu^+ = \nu^-$ (and so there is a unique Gibbs distribution);

(2) for intermediate values of J_t and J_z, $M>0$ (so $\nu^+ \neq \nu^-$) and $\langle S_x S_y \rangle_f \to 0$ as $\delta(x, y) \to \infty$ (so $\nu^f \neq (\nu^+ + \nu^-)/2$); the extent of this intermediate region depends on k but always has a nonempty interior;

(3) for $J_z>0$ and large J_t, $M>0$ and $\nu^f = (\nu^+ + \nu^-)/2$.

In Sect. 2 of the paper we present more detailed versions of our results, valid for Potts as well as Ising models. (However the intermediate region for q-state Potts models with $q \geqq 3$ has been proved nonempty only for sufficiently large k, depending on q.) The methods, which can be applied also to lattices such as $\mathbb{T} \times \mathbb{Z}^d$ (see [GN]), are based primarily on the independent percolation model results of [GN]. These results are carried over to Ising and Potts models by use of the Fortuin-Kasteleyn representation of Ising and Potts models as dependent percolation models [FK] and Fortuin's comparison inequalities relating these dependent percolation models to independent ones [F, ACCN]. However, in order to show that $\nu^f = (\nu^+ + \nu^-)/2$ in some parameter region, the result of [GN] that the infinite cluster for independent percolation is unique in a corresponding parameter region was not sufficient. This led us to investigate the independent percolation model again and verify a "finite island" property stronger than uniqueness of the infinite cluster (see Lemma 3.3 and preceding discussion below).

2. Statement of main results

2.1. Ising and Potts models

The ferromagnetic Ising model on the lattice $\mathbb{L} = \mathbb{T} \times \mathbb{Z}$ is described by the "spin" random variables $\{S_x\}_{x \in \mathbb{L}}$. Each S_x takes on the values ± 1. The interaction between the spins is described by the Hamiltonian

$$H = -\tfrac{1}{2} \sum_{\langle x, y \rangle} J_{x,y}(S_x S_y - 1) \tag{2.1}$$

in which the sum is over nearest neighbor bonds $\langle x, y \rangle$. The couplings $\{J_{x,y}\}$ are nonnegative (ferromagnetic) and may take different value on the $\mathbb{T}$-bonds and the $\mathbb{Z}$-bonds. I.e.,

$$J_{x,y} = \begin{cases} J_t, & \text{if } \langle x, y \rangle \text{ is a } \mathbb{T}\text{-bond} \\ J_z, & \text{if } \langle x, y \rangle \text{ is a } \mathbb{Z}\text{-bond} \end{cases} \tag{2.2}$$

where $J_t \geqq 0$ and $J_z \geqq 0$. We will assume that $J_t > 0$ and $J_z > 0$ unless stated otherwise. Such a Hamiltonian is translation invariant (i.e., with respect to all lattice translations).

Potts models are a generalization of Ising models where each spin variable can take on one of q distinct values. The basic feature of the interaction is that the energy between any fixed nearest neighbor pair of spins depends only on whether or not the spin values agree. When the interaction always favors agreement, the model is said to be ferromagnetic.

Two convenient representations for the spin variables are as taking values in the set $\{1, 2, \ldots, q\}$, or as unit vectors $\{e_1, e_2, \ldots, e_q\}$ pointing to the vertices of a fixed $(q-1)$-dimensional "tetrahedron". We will use the latter representation and use $\sigma_x \in \mathbb{R}^{q-1}$ to describe the state of a spin. For $q=2$, the values 1 and 2 or e_1 and e_2 correspond respectively to the values $+1$ and -1 for the usual Ising variables, S_x. The inner product of any two such vectors assumes only two values and satisfies

$$\sigma_x \cdot \sigma_y = \frac{q}{q-1}\left(\delta_{\sigma_x,\sigma_y} - \frac{1}{q}\right)$$

where

$$\delta_{\sigma_x,\sigma_y} = \begin{cases} 1, & \text{if } \sigma_x = \sigma_y \\ 0, & \text{otherwise.} \end{cases}$$

A Potts model is described therefore by a Hamiltonian

$$H = -\sum_{\langle x,y \rangle} J_{x,y}(\delta_{\sigma_x,\sigma_y} - 1) = -\sum_{\langle x,y \rangle} \tilde{J}_{x,y}(\sigma_x \cdot \sigma_y - 1) \tag{2.3}$$

with $\tilde{J}_{x,y} = \left(\frac{q-1}{q}\right) J_{x,y}$. The case $q=2$ coincides with the previously defined Ising model.

We denote by v_Λ^f the finite volume free b.c. Gibbs distribution whose configuration probabilities are proportional to $\exp(-H)$ with the sum in (2.3) restricted to x, $y \in \Lambda$, where Λ is a finite subset of $\mathbb{L}$. v_Λ^i denotes the finite volume "i" $(i=1, 2, \ldots, q)$ b.c. Gibbs distribution in which only x is restricted to Λ while σ_y is set to e_i for each y in Λ^c, the complement of Λ. The free and "i" distributions all have limits v^* as $\Lambda \to \mathbb{L}$, where $* = f, 1, 2, \ldots, q$ [ACCN]. The expectation w.r.t. v^* is denoted by $\langle \cdot \rangle_*$. The infinite volume quantities of primary interest to us are the magnetization

$$M = \langle e_i \cdot \sigma_x \rangle_i$$

(this quantity, which does not depend on x, is just $\langle S_x \rangle_+$ for $q=2$) and the two-point function with $*$ boundary conditions,

$$\langle \sigma_x \cdot \sigma_y \rangle_* = \left\langle \frac{q}{q-1}\left(\delta_{\sigma_x,\sigma_y} - \frac{1}{q}\right)\right\rangle_*$$

(which is just $\langle S_x S_y \rangle_*$ for $q=2$).

Let $\tau = 1 - \mathrm{e}^{-J_t}$ and $\lambda = 1 - \mathrm{e}^{-J_z}$. We shall use τ and λ instead of J_t and J_z as the parameters which describe the interaction between two spins. The reader should keep in mind that the measure v^* and quantities like M and $\langle \sigma_x \cdot \sigma_y \rangle_*$ depend on the parameters q, τ and λ. However, we will write, for example, M as $M_{q,\tau,\lambda}$ only when we think it is necessary. Some of the main results in this paper are about the Gibbs distributions v^*. We state them as several propositions.

In these propositions, there is a quantity $\phi(\tau', \lambda')$ defined as follows (see Sect. 2 of [GN] for more details). Consider independent nearest neighbor bond percolation on $\mathbb{Z}^2$ in which horizontal bonds are occupied with probability τ and vertical ones with probability λ. Then $\phi(\tau, \lambda)$ is the inverse correlation length in the horizontal direction. Two inequalities for ϕ [GN] are the trivial one

$$\exp(-\phi(\tau, \lambda)) \geqq \tau \tag{2.4}$$

and the less trivial, but still elementary one

$$\exp(-\phi(\tau, \lambda)) \geqq 1 - \frac{(1-\tau)(1-\lambda)^2}{(1-\lambda(1-\tau))^2}. \tag{2.5}$$

Proposition 2.1.1. *If*

$$\tau k(1+\lambda+\sqrt{2\lambda(1+\lambda)}) < 1-\lambda, \tag{2.6}$$

then $M=0$ *and consequently the Gibbs distribution is unique.*

Proposition 2.1.2. *If*

$$k\,\mathrm{e}^{-\phi(\tau',\lambda')} > 1, \tag{2.7}$$

where

$$\tau' = \frac{\tau}{\tau+(1-\tau)q} \quad \textit{and} \quad \lambda' = \frac{\lambda}{\lambda+(1-\lambda)q}, \tag{2.8}$$

then $M>0$ *(and hence there is more than one Gibbs distribution).*

Proposition 2.1.3. *If*

$$\tau\sqrt{k}(1+\lambda+\sqrt{2\lambda(1+\lambda)}) < 1-\lambda, \tag{2.9}$$

then $\langle \sigma_x \cdot \sigma_y \rangle_f \to 0$ *as* $\delta(x, y) \to \infty$. *Consequently, if also* $M>0$, *then*

$$v^f \neq \frac{1}{q}(v^1 + \ldots + v^q). \tag{2.10}$$

Proposition 2.1.4. *If*

$$\sqrt{k}\,\mathrm{e}^{-\phi(\tau',\lambda')} > 1 \tag{2.11}$$

where τ' *and* λ' *are given by* (2.8), *then*

$$\langle \sigma_x \cdot \sigma_y \rangle_f \geqq \varepsilon > 0, \quad \textit{for any } x \textit{ and } y \textit{ in } \mathbb{L}.$$

Proposition 2.1.5. *If* $\lambda>0$ *and* τ *is close to* 1, *then* ($M>0$ *and*)

$$v^f = \frac{1}{q}(v^1 + \ldots + v^q). \tag{2.12}$$

Before we introduce the Fortuin-Kasteleyn random cluster models, we make several remarks about the above propositions.

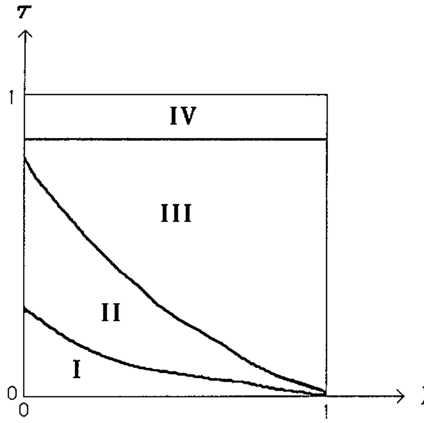

Fig. 1. A Schematic phase diagram. In region *I*, the Gibbs distribution is unique. In region *II*, there is tree-like nonuniqueness and in regions *III* and *IV* plane-like nonuniqueness. In region *IV*, the free b.c. Gibbs dist. has been proved to be a mixture of the plus and minus b.c. ones

1. Proposition 2.1.1 provides a lower bound $\underline{\tau}_L(\lambda)$ for the curve between regions I and II of Fig. 1 with the endpoint properties

$$\underline{\tau}_L(0)=\frac{1}{k}, \quad \dot{\underline{\tau}}_L(0)=-\infty, \quad \underline{\tau}_L(1)=0 \quad \text{and} \quad \dot{\underline{\tau}}_L(1)=-\frac{1}{4k}, \tag{2.13}$$

where $\dot{\tau}(\lambda)$ denotes the derivative of $\tau(\lambda)$.

2. Proposition 2.1.2 provides an upper bound $\bar{\tau}_L(\lambda)$ for the curve between regions I and II. The quality of this upper estimate depends on how accurate a bound is used for $\phi(\tau', \lambda')$. The trivial bound (2.4) yields the result (also easily obtained by other methods) that $\tau' > \frac{1}{k}$ $\left(\text{or equivalently } \tau > \frac{q}{k+q-1}\right)$ implies $M>0$. The improved bound (2.5) for $\phi(\tau', \lambda')$ can be shown to give a $\bar{\tau}_L(\lambda)$ with

$$\begin{aligned} &\bar{\tau}_L(0)=\frac{q}{k+q-1}, \quad \dot{\bar{\tau}}_L(0)=\frac{-2(k-1)}{(k+q-1)^2}, \\ &\bar{\tau}_L(1)=0 \quad \text{and} \quad \dot{\bar{\tau}}_L(1)=-q^2\left(\sqrt{\frac{k}{k-1}}-1\right). \end{aligned} \tag{2.14}$$

3. Note that (2.9) differs from (2.6) only in the replacement of k by $\sqrt{k}$. Proposition 2.1.3 provides a lower bound $\underline{\tau}_U(\lambda)$ for the curve between regions II and III of Fig. 1 with

$$\underline{\tau}_U(0)=\frac{1}{\sqrt{k}}, \quad \dot{\underline{\tau}}_U(0)=-\infty, \quad \underline{\tau}_U(1)=0 \quad \text{and} \quad \dot{\underline{\tau}}_U(1)=-\frac{1}{4\sqrt{k}}. \tag{2.15}$$

For any given $q \geqq 1$, when k is sufficiently large (depending on q)

$$\bar{\tau}_L(0)<\underline{\tau}_U(0) \quad \text{and} \quad \dot{\underline{\tau}}_U(1)<\dot{\bar{\tau}}_L(1)<0 \tag{2.16}$$

and hence the lower curve is really below the upper curve for sufficiently large k. Note that for $k=2$ (the smallest relevant value of k), the first inequality of (2.16) is valid only for $q<(\sqrt{2}-1)^{-1}\approx 2.41$; this includes the Ising model case where $q=2$.

4. Equation (2.11) differs from (2.7) only in the replacement of k by $\sqrt{k}$. Proposition 2.1.4 provides an upper bound $\bar{\tau}_U(\lambda)$ for the curve between regions II and III of Fig. 1 satisfying

$$\begin{aligned} &\bar{\tau}_U(0)=\frac{q}{\sqrt{k}+q-1}, \qquad \dot{\bar{\tau}}_U(0)=\frac{-2(\sqrt{k}-1)}{(\sqrt{k}+q-1)^2}, \\ &\bar{\tau}_U(1)=0 \quad \text{and} \quad \dot{\bar{\tau}}_U(1)=-q^2\left(\sqrt{\frac{\sqrt{k}}{\sqrt{k}-1}}-1\right). \end{aligned} \tag{2.17}$$

5. Proposition 2.1.5 states that

$$\nu^f=\frac{1}{q}(\nu^1+\ldots+\nu^q) \tag{2.18}$$

when τ and λ are in the region IV of Fig. 1. We remark that it is natural to conjecture that the region IV should extend all the way down to region II; i.e., (2.18) should be also true in region III; however this has not been proved.

2.2. *Fortuin-Kasteleyn random cluster models*

Propositions 2.1.1–2.1.5 will be proved together with analogous results about the Fortuin-Kasteleyn (FK) random cluster models [KF, FK, F]. The FK models are dependent percolation models which are closely related to the Potts/Ising models introduced in the previous section but are defined with a real positive parameter q, not necessarily an integer. The FK random cluster models are described by probability measures on the configurations of bond occupation variables, $n=\{n_b\}$, which take the value 1 – meaning the bond $b=\{x,y\}$ is occupied, or 0 – meaning b is vacant. For a finite Λ, the free b.c. measure $\mu_\Lambda^f=\mu_{q,\tau,\lambda,\Lambda}^f$ (restricted to bonds with x, y both in Λ) has bond-configuration probabilities proportional to

$$q^{\mathscr{C}(n)}\,\tau^{\mathcal{O}_{\mathbb{T}}(n)}(1-\tau)^{|\Lambda|_{\mathbb{T}}-\mathcal{O}_{\mathbb{T}}(n)}\,\lambda^{\mathcal{O}_{\mathbb{Z}}(n)}(1-\lambda)^{|\Lambda|_{\mathbb{Z}}-\mathcal{O}_{\mathbb{Z}}(n)} \tag{2.19}$$

where $\mathscr{C}(n)$ denotes the number of distinct clusters of sites (defined by the bond configuration), $|\Lambda|_{\mathbb{Z}}$ the number of $\mathbb{Z}$-bonds of Λ, $|\Lambda|_{\mathbb{T}}$ the number of $\mathbb{T}$-bonds of Λ, $\mathcal{O}_{\mathbb{T}}(n)$ the number of occupied $\mathbb{T}$-bonds of Λ and $\mathcal{O}_{\mathbb{Z}}(n)$ the number of occupied $\mathbb{Z}$-bonds of Λ.

For $q=1$, this is just an independent bond percolation model; for $q=2, 3, \ldots$, one has, for g any function of the spin variables in Λ, the identity

$$\langle g(\sigma)\rangle_{f,\Lambda}=\sum_n \mu_\Lambda^f(n)\,E_n^f(g(\sigma)) \tag{2.20}$$

where for each configuration n of bond variables, $E_n^f(-)$ is a very simple average over the spins σ – the spins constrained to be constant on each cluster with the values for different clusters being independent and symmetric (i.e., with all q values equally likely). A special case of (2.20) is

$$\langle \sigma_x \cdot \sigma_y \rangle_{f,\Lambda} = \mu_\Lambda^f(x \leftrightarrow y) \tag{2.21}$$

where $x \leftrightarrow y$ denotes the event consisting of those bond configurations in which x and y belong to the same cluster. The analogue of the Gibbs distribution ν_Λ^*, $*=1, 2, \ldots, q$, of the Potts/Ising model is the "wired" b.c. measure μ_Λ^w (for nearest neighbor bonds with x in Λ and any y) in which $\mathscr{C}(n)$ is determined by regarding all the sites in Λ^c, as well as those sites in Λ which are connected to Λ^c by an occupied path, as connected. The wired version of (2.20) is

$$\langle g(\sigma) \rangle_{*,\Lambda} = \sum_n \mu_\Lambda^w(n)\, E_n^*(g(\sigma)), \qquad *=1, 2, \ldots, q \tag{2.22}$$

where $E_n^*(-)$ is defined similarly to $E_n^f(-)$ but with σ_x set to e_* for every x in Λ^c or connected by an occupied path to Λ^c.

For $q \geqq 1$, infinite volume measures μ^f and μ^w exist [F, ACCN]. Again, μ^* ($*=f$ or w) depends on q, τ and λ. We will write μ^* as $\mu_{q,\tau,\lambda}^*$ only when necessary. μ^f and μ^w are equal for $q=1$; in this case we will write them as P, the Bernoulli product measure. For $q=2, 3, \ldots$, the following identities are valid:

$$M = \mu^w(x \leftrightarrow \infty), \tag{2.23}$$

$$\langle \sigma_x \cdot \sigma_y \rangle_f = \mu^f(x \leftrightarrow y), \tag{2.24}$$

$$\langle \sigma_x \cdot \sigma_y \rangle_i = \mu^w(x \leftrightarrow y, \text{ but } x \not\leftrightarrow \infty \text{ and } y \not\leftrightarrow \infty) + \mu^w(x \leftrightarrow \infty \text{ and } y \leftrightarrow \infty), \tag{2.25}$$

where $x \leftrightarrow \infty$ ($x \not\leftrightarrow \infty$) means that the cluster of x is infinite (finite). (For further discussion of these identities, see [ACCN] and [IN].)

We state our results about Fortuin-Kasteleyn random cluster models as several propositions. First define

$$\theta_{\mathbb{L}}^*(q, \tau, \lambda) = \mu_{q,\tau,\lambda}^*(0 \leftrightarrow \infty). \tag{2.26}$$

In all the following propositions, q is assumed to be real and $\geqq 1$ and $*$ can be either f or w.

Proposition 2.2.1. *If* (2.6) *is satisfied, then* $\theta_{\mathbb{L}}^*(q, \tau, \lambda) = 0$, *so there is* a.s. *no infinite occupied cluster in* $\mathbb{L}$ *with respect to* (w.r.t.) $\mu_{q,\tau,\lambda}^*$.

Proposition 2.2.2. *If* (2.7) *is satisfied, then* $\theta_{\mathbb{L}}^*(q, \tau, \lambda) > 0$, *and there is* a.s. *an infinite occupied cluster in* $\mathbb{L}$ w.r.t. $\mu_{q,\tau,\lambda}^*$.

Proposition 2.2.3. *If* (2.9) *is satisfied and* $\theta_{\mathbb{L}}^*(q, \tau, \lambda) > 0$, *then there are* a.s. *infinitely many infinite occupied clusters in* $\mathbb{L}$ w.r.t. $\mu_{q,\tau,\lambda}^*$.

Remark. The proofs of these propositions given below also show that when (2.9) is satisfied $\mu^*(x \leftrightarrow y) \to 0$ as $\delta(x, y) \to \infty$ but when (2.11) is satisfied $\mu^*(x \leftrightarrow y) \geqq \varepsilon > 0$ for all x, y.

Proposition 2.2.4. *If $\lambda > 0$ and τ is close to* 1, *then there exists* a.s. *a unique infinite occupied cluster in* $\mathbb{L}$ w.r.t. $\mu^*_{q,\tau,\lambda}$.

3. Proofs of the results

The identities (2.23)–(2.25) allow one to rewrite Propositions 2.1.1–2.1.4 as statements about FK random cluster models. Our strategy will then be to prove these statements as well as Propositions 2.2.1–2.2.3 by comparison to independent percolation where the analogous results were proved in [GN]. Propositions 2.1.5 and 2.2.4 will require us to obtain some new results for the independent percolation case before appealing to a comparison argument for FK models (see Lemma 3.3). We first present the following lemma which is a consequence of Harris' original version [H] of the FKG inequalities. It is also a special case of Fortuin's comparison inequalities between μ^*_q and $\mu^*_{q'}$ [F, ACCN]. Following standard practice, we say an event A is increasing if its indicator function is nondecreasing in each occupation variable n_b and then we indicate stochastic ordering between probability measures μ and μ' by writing $\mu \ll \mu'$ (resp. $\mu \gg \mu'$) if $\mu(A) \leqq \mu'(A)$ (resp., $\mu(A) \geqq \mu'(A)$) for every increasing event A.

Lemma 3.1. *For $q \geqq 1$, let $\mu^*_{q,\tau,\lambda}$ be a free or wired* b.c. *measure of the* FK *random cluster model in* $\mathbb{L}$ *and let $P_{\tau,\lambda}$ be the corresponding independent percolation measure* (*for $q = 1$*). *Then*

(a) $$\mu^*_{q,\tau,\lambda} \ll P_{\tau,\lambda}$$

and

(b) $$\mu^*_{q,\tau,\lambda} \gg P_{\tau',\lambda'}$$

where τ' and λ' are given by (2.8).

Proof of Propositions 2.1.1 and 2.2.1. We first prove Proposition 2.2.1. By Lemma 3.1,

$$\theta^*_{\mathbb{L}}(q, \tau, \lambda) = \mu^*_{q,\tau,\lambda}(0 \leftrightarrow \infty) \leqq P_{\tau,\lambda}(0 \leftrightarrow \infty).$$

From Proposition 1 of [GN], the RHS vanishes whenever (2.6) holds. So there is a.s. no infinite occupied cluster in $\mathbb{L}$. This proves Proposition 2.2.1. Proposition 2.1.1 then follows because $M_{q,\tau,\lambda} = \theta^w_{\mathbb{L}}(q, \tau, \lambda)$ (see Theorem 2.4(b) of [ACCN]). □

Before proceeding with the proof of Propositions 2.1.2 and 2.2.2, we introduce the following lemma.

Lemma 3.2. *Suppose $q \geqq 1$ and $* = f$ or w, then*

$$\mu^*(\textit{there exists an infinite cluster}) = 0 \ \textit{or}\ 1.$$

Proof. We will only prove the case of $*=w$. The proof is similar for $*=f$. Let A denote the event that there exists an infinite cluster, and let $\alpha=\mu^w(A)$. Suppose that $0<\alpha<1$. Then we could decompose μ^w as

$$\mu^w=\alpha\,\mu_1^w+(1-\alpha)\,\mu_2^w$$

where $\mu_1^w(B)=\dfrac{\mu^w(B\cap A)}{\mu^w(A)}$ and $\mu_2^w(B)=\dfrac{\mu^w(B\cap A^c)}{\mu^w(A^c)}$. By the FKG inequality property of μ^w, $\mu_1^w\gg\mu^w$; on the other hand, μ^w, as a Gibbs distribution, is the maximal one (see, for example, Theorem A.2 of [ACCN]), so $\mu_1^w\ll\mu^w$ and hence $\mu_1^w=\mu^w$, which implies that $\alpha=1$, a contradiction. This finishes the proof of the lemma. □

Proof of Propositions 2.1.2 and 2.2.2. We first prove Proposition 2.2.2. Again by Lemma 3.1,

$$\theta_{\mathbb{L}}^*(q,\tau,\lambda)=\mu_{q,\tau,\lambda}^*(0\leftrightarrow\infty)\geqq P_{\tau',\lambda'}(0\leftrightarrow\infty). \tag{3.1}$$

From Proposition 2 of [GN], the RHS of (3.1) is positive when (2.7) holds. Hence by Lemma 3.2 there is a.s. an infinite occupied cluster in $\mathbb{L}$. Proposition 2.1.2 follows from the above and (2.23). □

Proof of Proposition 2.2.3. By Lemma 3.2 and the assumption of the proposition, we have that

$$\mu_{q,\tau,\lambda}^*\text{ (there exists an infinite cluster)}=1. \tag{3.2}$$

So to prove the proposition, it is sufficient to show that when (2.9) holds,

$$\mu_{q,\tau,\lambda}^*\text{ (there are exactly } i \text{ infinite clusters)}=0,\text{ for any positive integer } i. \tag{3.3}$$

Now it follows from (3.2) and the $\mathbb{Z}$-translation invariance of μ^* that

$$\mu_{q,\tau,\lambda}^*\text{ (there is a } \mathbb{Z}\text{-line in } \mathbb{L} \text{ which has infinitely many points touching an infinite cluster)}=1. \tag{3.4}$$

Denote the event in (3.4) by A and the one in (3.3) by I; then

$$\begin{aligned}\mu_{q,\tau,\lambda}^*(I)&=\mu_{q,\tau,\lambda}^*(I\cap A)\\ &\leqq\mu_{q,\tau,\lambda}^*\text{ (there is a } \mathbb{Z}\text{-line in } \mathbb{L} \text{ which touches a single infinite cluster infinitely often)}\\ &\leqq P_{\tau,\lambda}\text{ (there is a } \mathbb{Z}\text{-line in } \mathbb{L} \text{ which touches a single infinite cluster infinitely often).}\end{aligned} \tag{3.5}$$

By Lemma 3 and Proposition 4 of [GN], the last probability in (3.5) vanishes when (2.9) holds. This completes the proof. □

Proof of Proposition 2.1.3. By (2.24) and Lemma 3.1,

$$\nu^f(\sigma_x\cdot\sigma_y)\equiv\langle\sigma_x\cdot\sigma_y\rangle_f=\mu_{q,\tau,\lambda}^f(x\leftrightarrow y)\leqq P_{\tau,\lambda}(x\leftrightarrow y). \tag{3.6}$$

By Lemma 3 of [GN], the last probability in (3.6) tends to zero as $\delta(x, y) \to \infty$ whenever (2.9) holds. Now we turn to the proof of (2.10). By (2.25) and the FKG inequalities, we have for $i = 1, 2, \ldots, q$,

$$\begin{aligned} v^i(\sigma_x \cdot \sigma_y) \equiv \langle \sigma_x \cdot \sigma_y \rangle_i & \\ &= \mu^w(x \leftrightarrow y, \text{ but } x \not\leftrightarrow \infty \text{ and } y \not\leftrightarrow \infty) \\ &\quad + \mu^w(x \leftrightarrow \infty \text{ and } y \leftrightarrow \infty) \\ &\geqq M^2 > 0. \end{aligned}$$

This together with the just proved fact that $v^f(\sigma_x \cdot \sigma_y) \to 0$ as $\delta(x, y) \to \infty$ yields (2.10). □

Proof of Proposition 2.1.4. By (2.24) and Lemma 3.1

$$\langle \sigma_x \cdot \sigma_y \rangle_f = \mu^f_{q,\tau,\lambda}(x \leftrightarrow y) \geqq P_{\tau',\lambda'}(x \leftrightarrow y). \tag{3.7}$$

By Proposition 5 of [GN], there exists a.s. a unique infinite cluster for the independent percolation model with parameters τ' and λ' whenever (2.11) holds. So by the Harris (FKG) inequalities [H],

$$P_{\tau',\lambda'}(x \leftrightarrow y) \geqq P_{\tau',\lambda'}(x \leftrightarrow \infty \text{ and } y \leftrightarrow \infty) \geqq [P_{\tau',\lambda'}(x \leftrightarrow \infty)]^2 > 0. \tag{3.8}$$

The proposition follows by combining (3.7) and (3.8). □

Before proceeding to the proofs of Propositions 2.1.5 and 2.2.4, we introduce the notion of a "finite island" property. We say a bond percolation model has the finite island property if the removal of all sites in all infinite occupied bond clusters leaves only finite site components. More precisely, if we color each site which belongs to any infinite occupied bond cluster red and color all the others white, we can define $I(x)$, the (white) site component containing x by

$$I(x) = \{y \in \mathbb{L}: y \text{ is connected to } x \text{ by a path of white sites}\}. \tag{3.9}$$

$I(x)$ is empty if x is red; otherwise $I(x)$ includes x. The model is said to have the finite island property if for each site x in $\mathbb{L}$, $I(x)$ is finite.

Lemma 3.3. *For independent percolation on $\mathbb{L}$ or for the* FK *random cluster model on $\mathbb{L}$ with $q \geqq 1$, if τ is close to* 1, *then the model satisfies the finite island property almost surely.*

The proof of this lemma will use the following fact. For any site $x = (t, z)$ in $\mathbb{L}$, there are $k+1$ bonds on the tree $\mathbb{T} \times \{z\} \equiv \{(t', z): t' \in \mathbb{T}\}$ passing through x. Each bond leads to a "one-sided" tree we call a *branch of* x. The original $\mathbb{T}$ may be thought of as composed of $k+1$ such branches joined together by the site t and $k+1$ bonds. Let S be any finite subset of sites in $\mathbb{L}$. For any $x = (t, z) \in S$, we call x a boundary point of S if x has a branch in $\mathbb{T} \times \{z\}$ which contains no point in S, and otherwise call x an interior point of S. Denote by $\partial(S)$ the set of boundary points of S and by $\mathrm{i}(S)$ the set of interior points of S. (Note: this is different than the usual definition of boundary.)

Lemma 3.4. *For any finite subset of sites $S \subset \mathbb{L}$, $|\partial(S)| > |\mathrm{i}(S)|$.*

Proof. If $|S|=1$, the only point of S is a boundary point and there is no interior point, so the lemma is true. Assume that the lemma is true for $|S| \leqq m$. When $|S|=m+1$, it is clear that S has at least one boundary point, say x. Consider the two sets $\partial(S-\{x\})$ and $\partial(S)$. It is not hard to see that

$$\partial(S)-\partial(S-\{x\})=\{x\}$$

and

$$|\partial(S-\{x\})-\partial(S)|=0 \text{ or } 1.$$

If $|\partial(S-\{x\})-\partial(S)|=0$, then $\partial(S)=\partial(S-\{x\}) \cup \{x\}$ and $\mathrm{i}(S)=\mathrm{i}(S-\{x\})$, so

$$|\partial(S)|=|\partial(S-\{x\})|+1>|\mathrm{i}(S-\{x\})|+1=|\mathrm{i}(S)|+1. \tag{3.10}$$

If $|\partial(S-\{x\})-\partial(S)|=1$, define y by $\{y\}=\partial(S-\{x\})-\partial(S)$; then $\partial(S)=\partial(S-\{x, y\}) \cup \{x\}$ and $\mathrm{i}(S)=\mathrm{i}(S-\{x, y\}) \cup \{y\}$, so

$$|\partial(S)|=|\partial(S-\{x, y\})|+1>|\mathrm{i}(S-\{x, y\})|+1=|\mathrm{i}(S)|. \tag{3.11}$$

The lemma is proved by combining the two cases (3.10) and (3.11). □

Proof of Lemma 3.3. We first prove the independent case. The FK model result then easily follows by the second part of Lemma 3.1. In order that

$$P(\omega\colon |I(x)|<\infty \text{ for each } x \in \mathbb{L})=1,$$

it is sufficient that $E(|I((\phi, 0))|)<\infty$, where $E(\cdot)$ is the expectation w.r.t. P and $I((\phi, 0))$ is the white component of the origin.

$$E(|I((\phi, 0))|)=\sum_{y \in \mathbb{L}} P(\text{there is a self-avoiding path } \{x_i\colon i=0, 1, \ldots, n,$$
$$\text{with } x_0=(\phi, 0),\ x_n=y\} \text{ such that } x_i \not\leftrightarrow \infty \text{ in } \mathbb{L} \text{ for each } i)$$
$$\leqq \sum_{n=0}^{\infty} \sum P\left(\bigcap_{i=0}^{n} \{x_i \not\leftrightarrow \infty \text{ in } \mathbb{L}\}\right), \tag{3.12}$$

where the inner sum of the rightmost expression in (3.12) is over self-avoiding paths $S=\{x_i\colon i=0, 1, \ldots, n\}$ with $x_0=(\phi, 0)$. Now we will estimate the probability in (3.12). By Lemma 3.4, one has that $|\partial(S)|>\frac{n+1}{2}$. For each $x_i \in \partial(S)$, there is a branch of x_i which contains no point of S, which we denote B_{x_i}. (If $x_0 \notin \partial(S)$, we define B_{x_0} to be some fixed branch of x_0.) Then the events $\{x_i \not\leftrightarrow \infty \text{ in } B_{x_i}\}$, for $x_i \in \partial(S)$ are independent. So

$$\begin{aligned} P\left(\bigcap_{i=0}^{n} \{x_i \not\leftrightarrow \infty \text{ in } \mathbb{L}\}\right) &\leqq P\Big(\bigcap_{x_i \in \partial(S)} \{x_i \not\leftrightarrow \infty \text{ in } \mathbb{L}\}\Big) \\ &\leqq P\Big(\bigcap_{x_i \in \partial(S)} \{x_i \not\leftrightarrow \infty \text{ in } B_{x_i}\}\Big) \\ &\leqq [P(x_0 \not\leftrightarrow \infty \text{ in } B_{x_0})]^{(n+1)/2} \\ &= [\sqrt{\eta(\tau)}]^{n+1} \end{aligned} \tag{3.13}$$

where $\eta(\tau)=P(x_0 \not\leftrightarrow \infty$ in $B_{x_0})$. Denote by $N(n)$ the number of self-avoiding paths on $\mathbb{L}$ of length n starting from the origin; then we have

$$N(n) \leqq (k+3)(k+2)^{n-1}. \tag{3.14}$$

From (3.12)–(3.14) we have

$$E(|I((\phi, 0))|) \leqq \sum_{n=0}^{\infty} (k+3)(k+2)^{n-1} [\sqrt{\eta(\tau)}]^{n+1}.$$

Since $\eta(\tau) \to 0$ as $\tau \to 1$, we can take τ sufficiently close to 1 so that

$$\eta(\tau) < \frac{1}{(k+2)^2},$$

and hence $E(|I((\phi, 0))|) < \infty$. This completes the proof. □

Proof of Proposition 2.2.4. Let $A = \{$there exists an unique infinite occupied cluster in $\mathbb{L}\}$. Note that A itself is not an increasing event but $\bar{A} = A \cap \{\omega\colon |I(x)| < \infty$ for each $x \in \mathbb{L}\}$ is increasing. Thus by Lemma 3.1, it suffices to prove $P_{\tau', \lambda'}(\bar{A}) = 1$ for $\lambda' > 0$ and τ' close to 1. This follows from Proposition 5 of [GN] and Lemma 3.3. □

Proof of Proposition 2.1.5. The proof is based on an analysis of clusters in the FK model. Given an FK (bond) configuration, corresponding site configurations are generated by assigning spin values independently and symmetrically to each (bond) cluster; i.e., by assigning with probability $\frac{1}{q}$ all sites in the same (bond) cluster one of the q spin values. By Proposition 2.2.4 and Lemma 3.3, when $\lambda > 0$ and τ is close to 1, the FK model satisfies the finite island property and the infinite occupied (bond) cluster is unique. So there is only one infinite site cluster with like spin values which "traps" $\mathbb{L}$; i.e., the complement of this infinite site cluster contains only finite connected components.

Let A_i be the event that the infinite trapping site cluster has spin value i, $i = 1, 2, \ldots, q$. Clearly the A_i's are disjoint and $v^f(A_i) = \frac{1}{q}$ for each i. So we can decompose v^f as

$$v^f(B) = v^f(A_1 \cap B) + \ldots + v^f(A_q \cap B) = \frac{1}{q}\left[\frac{v^f(A_1 \cap B)}{v^f(A_1)} + \ldots + \frac{v^f(A_q \cap B)}{v^f(A_q)}\right].$$

$\dfrac{v^f(A_i \cap B)}{v^f(A_i)}$ is a Gibbs distribution, and is equal to $v^i(B)$ by a variant of Lemma 1 of [Ru] (see also Lemma 1 and Proposition 4 of [A]). This completes the proof. □

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