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Probability Theory and Related Fields

Exponential integrability of sub-Gaussian vectors

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Summary. In this paper we define two classes of Banach space $(B, \|\cdot\|)$-valued random vectors called sub-Gaussian vectors and γ-sub-Gaussian vectors. The main purpose of this paper is to prove the exponential integrability of a sub-Gaussian vector X, that is, $\mathbb{E}[e^{\varepsilon\|X\|^2}]<\infty$ for some $\varepsilon>0$, in the case where $B=L_p$. On the other hand, using the arguments of X. Fernique and M. Talagrand, we also show that the exponential integrability of a γ-sub-Gaussian vector in an arbitrary separable Banach space.

These two definitions of sub-Gaussian vectors and γ-sub-Gaussian vectors are not comparable, and neither of these definitions is a necessary condition for the exponential integrability. We shall give illuminating examples.

§1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be the underlying probability space, $\mathbb{E}[\ ]$ denote the expectation, $B=(B, \|\cdot\|)$ be a real separable Banach space, $B^*=(B^*, \|\cdot\|_*)$ be the topological dual space of B and $\langle\ ,\ \rangle$ denote the canonical bilinear form of $B^*\times B$.

A real random variable X is sub-Gaussian if there exists $C>0$ such that,

$$\mathbb{E}[e^{\lambda X}]\leqq e^{\frac{C}{2}\lambda^2}$$

for any $\lambda\in\mathbb{R}$. Kahane [6] proved that a real random variable X is sub-Gaussian if and only if $\mathbb{E}[X]=0$ and for some $\varepsilon>0$

$$\mathbb{E}[e^{\varepsilon X^2}]<\infty.$$

The main purpose of this paper is to define a *B-valued sub-Gaussian random vector* (in short *sub-Gaussian vector*) as a B-valued random vector X for which there exists $C>0$ such that

$$\mathbb{E}[e^{\langle y, X\rangle}]\leqq e^{\mathbb{E}[\langle y, X\rangle^2]}\langle\infty \tag{1.1}$$

for any $y \in B^*$, and to prove

$$\mathbb{E}[e^{\varepsilon \|X\|^2}] < \infty$$

for some $\varepsilon > 0$, in the case where $B = L_p$ (Theorem 4.3). We call this type of the integrability "*exponential integrability*". A B-valued Gaussian vector and a Rademacher series in B are typical examples of exponentially integrable sub-Gaussian vectors (Fernique [2], Kwapien [9]).

A sub-Gaussian vector, as a random process defined on B^*, is a special case of a random process with sub-Gaussian increment defined by Jain and Marcus [10] who gave a sufficient condition for continuity, and also an estimation for the tail distribution of the supremum norm.

There is another definition of sub-Gaussian vectors defined by Talagrand [14]. A B-valued random vector X is a *γ-sub-Gaussian vector* if there exists a B-valued Gaussian vector G such that

$$\mathbb{E}[e^{\langle y, X\rangle}] \leqq \mathbb{E}[e^{\langle y, G\rangle}] (= e^{\frac{1}{2}\mathbb{E}[\langle y, G\rangle^2]}) \tag{1.2}$$

for any $y \in B^*$. He proved the necessity of the existence of a majorizing measure for the boundedness of a Gaussian process, and as an application proved that $\|X\|$ is integrable for any γ-sub-Gaussian vector X. Using the Fernique's estimation [2] we shall prove the exponential integrability of a γ-sub-Gaussian vector in an arbitrary Banach space (Theor. 3.4). A same type of estimation was given by Heinkel [3] who gave a sufficient condition for sample continuity of random processes using this estimation.

In the case where $B = L_p$ every sub-Gaussian vector is a γ-sub-Gaussian vector (Lemma 4.6). But, in general, these two definitions of sub-Gaussian vectors turn out to be incomparable. We shall give illuminating examples in §5 (Example 5.1 and Example 5.2). Moreover, the two examples are also exponentially integrable, and this implies that, neither of "sub-Gaussian" is a necessary condition for the exponential integrability.

§2. Regularity conditions

First of all, we define two norms on the class of all real sub-Gaussian random variables and the class of all exponentially integrable random vectors.

Definition 2.1. (1) Let $\mathscr{G}_0(\mathbb{R})$ be the class of all real sub-Gaussian random variables, and for any element Y of $\mathscr{G}_0(\mathbb{R})$, define

$$\tau(Y) \rightleftarrows \inf\{C < 0 \colon \mathbb{E}[e^{\lambda Y}] \leqq e^{\frac{c^2}{2}\lambda^2} \text{ for any } \lambda \in C\}.$$

(2) For a B-valued random vector X, define

$$\tilde{\tau}(X) \rightleftarrows \sup_{n \in \mathbb{N}} \alpha_n \mathbb{E}[\|X\|^{2n}]^{1/2n},$$

where $\alpha_n = \sqrt{2}\{n!/(2n)!\}^{1/2n}$ and $\mathbb{N}$ denotes the collection of all natural numbers.

Remarks. (1) τ is a norm on $\mathscr{G}_0(\mathbb{R})$ and $(\mathscr{G}_0(\mathbb{R}), \tau)$ is a Banach space (Buldygin and Kozachenko [1]).

(2) A B-valued random vector X is exponentially integrable if and only if $\tilde{\tau}(X) < \infty$. Let $\mathscr{G}(B)$ *be the class of all* B-valued exponentially integrable random vectors, then $(\mathscr{G}(B), \tilde{\tau})$ is also a Banach space. Moreover, on the space $\mathscr{G}_0(\mathbb{R})$ $(\subset \mathscr{G}(\mathbb{R}))$, τ and $\tilde{\tau}$ are equivalent and stronger than any L_p-norms $(p \geqq 1)$. (Buldygin and Kozachenko [1], Kahane [6]).

Every B-valued sub-Gaussian vector X defined by (1.1) statisfies

$$\tau(\langle y, X\rangle) \leqq C\mathbb{E}[|\langle y, X\rangle|^2]^{\frac{1}{2}} \tag{2.1}$$

for any $y \in B^*$, where the positive constant C is independent of y. The above condition is called the "regularity condition" which plays a fundamental role in the succeeding arguments.

In the following, we shall state several properties derived from the regularity condition. They are proved in Kahane [6], Kadec and Pełczyński [7] and Jain and Marcus [10], but for the completeness of the paper we shall give the proofs of them.

Lemma 2.1. *Let X be a non-negative random variable in $L_p(\Omega, P)$ with $p>0$. Then for any $q \in (0, p)$ and $\lambda \in (0, 1)$ we have*

$$\{(1-\lambda^q)^{1/q} C^{-1}\}^{pq/(p-q)} \leqq P(X > \lambda \mathbb{E}[X^q]^{1/q}),$$

where $C = \dfrac{\mathbb{E}[X^p]^{1/p}}{\mathbb{E}[X^q]^{1/q}}$.

Proof. Put $r \equiv p/q > 1$, and $s \equiv p/(p-q)$, $(1/r + 1/s = 1)$. Then for any $a > 0$ we have

$$\begin{aligned}\mathbb{E}[X^q] &\leqq \mathbb{E}[X^q 1_{\{X>a\}}] + a^q \\ &\leqq \mathbb{E}[X^p]^{1/r} P(X>a)^{1/s} + a^q,\end{aligned}$$

$$\mathbb{E}[X^q] - a^q \mathbb{E}[X^p]^{1/r} P(X>a)^{1/s},$$

$$\left(\frac{\mathbb{E}[X^q]-a^q}{\mathbb{E}[X^p]^{1/r}}\right)^s \leqq P(X>a).$$

Choose $a = \lambda \mathbb{E}[X^q]^{1/q}$, and we have the required. □

Proposition 2.2. *Let $\{X_n\}$ be a sequence of B-valued random variables which converges to X in probability.*

(1a) *Suppose that there exist $C>0$ and $p>q>0$ such that*

$$\mathbb{E}[\|X_n\|^p]^{1/p} \leqq C\mathbb{E}[\|X_n\|^q]^{1/q} < \infty$$

for any $n \in \mathbb{N}$. Then,

$$\mathbb{E}[\|X\|^p]^{1/p} \leqq C\mathbb{E}[\|X\|^q]^{1/q} < \infty.$$

(1b) *If there exist $C>0$ and $q>0$ such that,*

$$\tilde{\tau}(X_n) \leqq C\mathbb{E}[\|X_n\|^q]^{1/q} < \infty$$

for any $n \in \mathbb{N}$. Then,

$$\tilde{\tau}(X) \leqq C\mathbb{E}[\|X\|^q]^{1/q} < \infty.$$

(2a) *Besides the hypothesis of* (1a), *we suppose*

$$\mathbb{E}[\|X_n - X_m\|^p]^{1/p} \leqq C\mathbb{E}[\|X_n - X_m\|^q]^{1/q} < \infty.$$

for any $n, m \in \mathbb{N}$. *Then,* X_n *converges to* X *in* L_p.

(2b) *Besides the hypothesis of* (1b), *we suppose*

$$\tilde{\tau}(X_n - X_m) \leqq C\mathbb{E}[\|X_n - X_m\|^q]^{1/q} < \infty.$$

for any $n, m \in \mathbb{N}$. *Then,* X_n *converges to* X *with respect to* $\tilde{\tau}$.

Proof. We shall prove only (1a) and (2a). (1b) and (2b) can be proved by the similar way.

(1a) From Lemma 2.1 we obtain

$$P(\|X_n\| \geqq \lambda \mathbb{E}[\|X_n\|^q]^{1/q}) > ((1-\lambda^q)^{1/q} C^{-1})^{pq/(p-q)} \equiv C_0 \tag{2.1}$$

for any $n \in \mathbb{N}$, where λ is an arbitrary fixed number in (0, 1).

Since $\{X_n\}$ converges in probability, we have,

$$\sup P(\|X_n\| > M) \to 0 \quad \text{as } M \to \infty. \tag{2.2}$$

By the hypothesis, (2.1) and (2.2) $\{\mathbb{E}[\|X_n\|^q]^{1/q}\}$ and $\{\mathbb{E}[\|X_n\|^p]^{1/p}\}$ are bounded.

Subtracting a sub-sequence if necessary, we may assume that $\|X_n\|$ converges to $\|X\|$ almost surely. Then, since $\{\mathbb{E}[\|X_n\|^p]\}$ is bounded and $p > q$, $\{\|X_n\|^q\}$ is uniformly integrable. Therefore, $\|X_n\|$ converges in L_q and we have

$$\begin{aligned}\mathbb{E}[\|X\|^p]^{1/p} &\leqq \liminf_{n\to\infty} \mathbb{E}[\|X_n\|^p]^{1/p} \\ &\leqq C \liminf \mathbb{E}[\|X_n\|^q]^{1/q} \\ &= C\mathbb{E}[\|X\|^q]^{1/q} < \infty. \quad \square\end{aligned}$$

(2a) By the similar argument as above, we obtain

$$P(\|X_n - X_m\| > \lambda \mathbb{E}[\|X_n - X_m\|^q]^{1/q}) > C_0.$$

Then, since $\|X_n - X_m\|$ converges to 0 in probability as $n, m \to \infty$, $\mathbb{E}[\|X_n - X_m\|^q]$ converges to 0 as $n, m \to \infty$. Therefore, from the hypothesis, we obtain

$\lim_{n\to\infty} \mathbb{E}[\|X_n - X\|^p] = 0.$ $\square$

Proposition 2.3. *For a B-valued random variable X, we have:*

(a) *Suppose*

$$\mathbb{E}[\|X\|^p]^{1/p} \leqq C\mathbb{E}[\|X\|^q]^{1/q} < \infty \quad \textit{with } p > q > 0,$$

for some $C \geqq 1$, *then, for any* $0 < r, s \leqq p$ *there exists a constant K which depends only on r, s, p, q and C such that,*

$$\mathbb{E}[\|X\|^r]^{1/r} \leqq K\mathbb{E}[\|X\|^s]^{1/s}.$$

(b) *Suppose*

$$\tilde{\tau}(X) \leqq C\mathbb{E}[\|X\|^q]^{1/q} < \infty \qquad \text{with } q > 0,$$

for some $C \geqq 0$, *then, for any* $0 \leqq s < \infty$ *there exists a constant* K *which depends only on* s, q *and* C *such that,*

$$\tilde{\tau}(X) \leqq K\mathbb{E}[\|X\|^s]^{1/s}.$$

Proof. We shall give a proof only for (a) with $r = p$ and $s = 1$. From Lemma 2.1

$$\{(1-\lambda^q)^{1/q} C^{-1}\}^{pq/(p-q)} \leqq P(X > \lambda\mathbb{E}[X^q]^{1/q})$$

for any $\lambda \in (0, 1)$. Therefore,

$$\begin{aligned}
\mathbb{E}[\|X\|] &= \int_0^\infty P(\|X\| > x)\, dx \\
&\geqq \int_0^1 \mathbb{E}[\|X\|^q]^{1/q} P(\|X\| > \lambda\mathbb{E}[\|X\|^q]^{1/q})\, d\lambda \\
&\geqq \int_0^1 \{(1-\lambda^q)^{1/q} C^{-1}\}^{pq/(p-q)}\, d\lambda\, \mathbb{E}[\|X\|^q]^{1/q} \\
&\geqq C^{-1-pq/(p-q)} q^{-1} \int_0^1 (1-x)^{p/(p-q)} x^{(1-q)/q}\, dx\, \mathbb{E}[\|X\|^p]^{1/p} \\
&= C^{-1-pq/(p-q)} q^{-1} B(1/q, (p/(p-q)) + 1)\, \mathbb{E}[\|X\|^p]^{1/p},
\end{aligned}$$

where $B(\cdot, \cdot)$ is a beta function. □

Corollary 2.4. *Let* E *be a linear subspace of* $L_0(\Omega \to B)$, *and suppose that there exist* $p > q > 0$ *and* $C > 0$ *such that*

$$\mathbb{E}[\|X\|^p]^{1/p} \leqq C\mathbb{E}[\|X\|^q]^{1/q}$$

for any $X \in E$. *Then* L_0*-topology is equivalent to* L_p*-topology on* $\bar{E}$, *where* $\bar{E}$ *is an* L_0*-completion of* E. *Moreover, for any* $r \in (q, p]$ *there exists* $K = K(r, p, q, C) > 0$ *such that, for any* $X \in \bar{E}$,

$$\mathbb{E}[\|X\|^r]^{1/r} \leqq K\mathbb{E}[\|X\|^q]^{1/q}. \quad \square$$

§3. γ – sub-Gaussian vectors

In this section we prove the exponential integrability of a γ-sub-Gaussian vector valued in an arbitrary Banach space. In the proof we make use of the existence of a majorizing measure for a bounded Gaussian process (Talagrand [14]) and Fernique's arguments (Fernique [2]).

At first we state their results in our terminology.

Theorem 3.1 (Fernieque [2], Talagrand [14]). *Let G be a Gaussian process on a countable set T, and d be a pseudo-metric on T defined by*

$$d(t, s) = \mathbb{E}[|G(t) - G(s)|^2]^{\frac{1}{2}}.$$

Then, $\sup_{t \in T} G(t) < \infty$ *almost surely if and only if there exists a probability measure* μ *on* $(T, \mathscr{B}_d)$ *such that,*

$$\sup_{t \in T} \int_0^\infty \left(\log \frac{1}{\mu(B_d(t, \varepsilon))}\right)^{\frac{1}{2}} d\varepsilon < \infty, \tag{3.1}$$

where $\mathscr{B}_d$ *is a Borel field with respect to d-topology, and* $B_d(t, \varepsilon)$ *is the closed d-ball with center t and radius* ε. □

Definition 3.1. Let $(S, \mathscr{B}, \nu)$ be a probability space. For a B or $\mathbb{R}$-valued measurable function f, we define

$$\sigma_\nu(f) \rightleftarrows \inf\{\sigma > 0: \int_S e^{\sigma^{-2} \|f(s)\|^2} d\nu(s) < 2\}.$$

A random vector X is exponentially integrable if and only if $\sigma_P(X) < \infty$ and σ_P is a norm on $\mathscr{G}(B)$ (Fernique [2]), and it is not difficult to show that τ and σ_P are equivalent on $\mathscr{G}_0(\mathbb{R})$.

Theorem 3.2 (Fernique [2]). *Let* $X(t)$ *be a random process on a countable set* T *such that* $\tau(X(t)) < \infty$ *for any* $t \in T$ *and* $X(t_0) = 0$ *for some* $t_0 \in T$. *For a probability measure* μ *on* $(T, \mathscr{B}_\rho)$ *we define*

$$Y(\omega) \rightleftarrows \sigma_{\mu \times \mu}\left(1_{\{\rho(t, s) \neq 0\}} \frac{X(t) - X(s)}{\rho(t, s)}\right),$$

where $\rho(t, s) = \tau(X(t) - X(s))$ *and* $\mathscr{B}_\rho$ *is the Borel σ-algebra of* (T, ρ). *Then, we have*

$$\sup_{t \in T} |X(t)| \leq K Y(\omega) \sup_{t \in T} \int_0^{D/2} \left(\log\left(1 + \frac{1}{\mu(B_\rho(t, \varepsilon))}\right)\right)^{\frac{1}{2}} d\varepsilon, \tag{3.2}$$

almost surely, where D is the diameter of T with respect to ρ *and K is an absolute constant.* □

Proposition 3.3 (Fernique [2]). *Let* $(S, \mathscr{B}, \nu)$ *be a measurable function from* $\Omega \times S$ *to* $\mathbb{R}$. *Suppose* $\sup_{s \in S} \sigma_P(Z(s, \cdot)) < \infty$ *then* $\sigma_\nu(Z(\cdot, \omega))$ *is exponentially integrable.* □

Summing up the above theorems and proposition, we can prove the exponential integrability of a γ-sub-Gaussian vector as follows.

Theorem 3.4. *Let X be a B-valued random variable. Suppose that there exists a B-valued Gaussian vector G such that*

$$\mathbb{E}[e^{\langle y, X \rangle}] \leq \mathbb{E}[e^{\langle y, G \rangle}] \qquad (= e^{\frac{1}{2}\mathbb{E}[\langle y, G \rangle^2]})$$

for any $y \in B^*$. *Then, X is exponentially integrable.*

Proof. Since B is separable, there exists a countable subset T of B^* such that $0 \in T$ and

$$\sup_{t \in T} \langle t, x \rangle = \|x\|$$

for any $x \in B$. We consider two processes on T defined by

$$X(y) \rightleftarrows \langle y, X \rangle, \qquad G(y) \rightleftarrows \langle y, G \rangle, \qquad y \in T.$$

and two pseudo-metrics on T defined by

$$d(t, s) = \mathbb{E}[|G(t) - G(s)|^2]^{\frac{1}{2}},$$

$$\rho(t, s) = \tau(X(t) - X(s)), \qquad t, s \in T.$$

Then, by Theorem 3.1 there exists a probability measure μ on $(T, \mathscr{B}_d)$ which satisfies the condition (3.1) since $G(t)$ is bounded on T almost surely, where $\mathscr{B}_d$ is the Borel σ-algebra of the pseudo metric space (T, d).

Since $\rho(\cdot, \cdot)$ is a continuous function on $(T, d)^2$, then $B(t, \varepsilon) \in \mathscr{B}_d$ for any $t \in T$ and $\varepsilon > 0$, and by the hypothesis we have

$$\tau(X(t) - X(s)) \leqq \tau(G(t) - G(s)) = \mathbb{E}[|G(t) - G(s)|^2]^{\frac{1}{2}}.$$

Therefore, we have

$$\sup_{t \in T} \int_0^\infty \left(\log \left(1 + \frac{1}{\mu(B_\rho(t, \varepsilon))} \right) \right)^{\frac{1}{2}} d\varepsilon < \infty.$$

and by (3.2)

$$\sup_{t \in T} |X(t)| \leqq K\, Y(\omega) \sup_{t \in T} \int_0^{D/2} \left(\log \left(1 + \frac{1}{\mu(B_\rho(t, \varepsilon))} \right) \right)^{\frac{1}{2}} d\varepsilon,$$

where $Y(\omega) = \sigma_{\mu \times \mu}\left(1_{\{\rho(t,s) \neq 0\}} \dfrac{X(t) - X(s)}{\rho(t, s)} \right)$. Since τ and σ_P are equivalent, there exists $M > 0$ such that $\sigma_p(Z) \leqq M \tau(Z)$ for any $Z \in \mathscr{G}_0(\mathbb{R})$. Then, by the definition of ρ we have

$$\sigma_P \left(\frac{X(t) - X(s)}{\rho(t, s)} \right) \leqq M \tau \left(\frac{X(t) - X(s)}{\rho(t, s)} \right) = M$$

if $\rho(t, s) \neq 0$. Therefore,

$$\sup_{t, s \in T} \sigma_P \left(1_{\{\rho(t,s) \neq 0\}} \frac{X(t) - X(s)}{\rho(t, s)} \right) \leqq M < \infty.$$

Then, by Theorem 3.3 $Y(\omega)$ is exponentially integrable, and this implies that $\|X\| = \sup_t |X(t)|$ is exponentially integrable. □

§4. L_p-valued sub-Gaussian vectors

In this section we prove that an L_p-valued sub-Gaussian random vector is exponentially integrable. At first, we shall give two types of approximations for an L_p-function.

Proposition 4.1. *Let $(T, \mathscr{B}, \mu)$ be a σ-finite measure space with countably generated σ-algebra $\mathscr{B}$. Then,*

(1) There exists a sequence of countable partitions $\{\{A_{i,j}\}_{i\in\mathbb{N}}\}_{j\in\mathbb{N}}$ of T such that

$$\mu(A_{i,j})<\infty \text{ for any } i \text{ and } j,$$

$$\sigma(\{A_{i,j}\}_{i,j\in\mathbb{N}})=\mathscr{B},$$

$$\{A_{i,j+1}\}_{i\in\mathbb{N}} \text{ is a refinement of } \{A_{i,j}\}_{i\in\mathbb{N}} \text{ for any } j\in\mathbb{N}.$$

(2) There exists a sequence $\{n(j)\}$ of positive integers, such that, for any $f\in L_p(T)$,

$$\Phi_j(f)=\sum_{i=1}^{n(j)}\frac{1}{\mu(A_{i,j})}\int_{A_{i,j}} f(t)\,d\mu(t)\,1_{A_{i,j}}(\cdot) \tag{4.1}$$

converges to f in $L_p(T)$ as $j\to\infty$.

Remark. $\{A_{i,j}\}$ and $\{n(j)\}$ do not depend on the choice of f, and $1_{A_{i,j}}\in(L_p)^*$ for any i, j.

Proof. Since μ is σ-finite, there exists an increasing sequence $\{T_n\}$ of $\mathscr{B}$-measurable sets such that $\mu(T_n)<\infty$ for any n, and $\cup T_n=T$. Then, for any L_p-function f, $\{1_{T_n}f\}$ is an L_p-approximating sequence. Therefore, it is enough to prove the proposition in the case where μ is a finite measure, and without loss of generality, we may assume that μ is a probability measure.

Since $\mathscr{B}$ is countably generated, there exists a countable sub-family $\{B_n\}_{n\in\mathbb{N}}$ of $\mathscr{B}$ such that

$$\sigma(\{B_n\}_{n\in\mathbb{N}})=\mathscr{B}.$$

For each $j\in\mathbb{N}$, there exists a finite partition $\{A_{i,j}\}_{i=1}^{n(j)}$ such that

$$\sigma(\{A_{i,j}\}_{i=1}^{n(j)})=\sigma(\{B_n\}_{n\leq j}).$$

Then, it clearly satisfies the condition (1), and by the definition of $\Phi_j(f)$, we have

$$\Phi_j(f)=\mathbb{E}_\mu[f\,|\,\sigma(\{A_{i,j}\}_{i=1}^{n(j)})]$$

for any $f\in L_p(T)$, where $\mathbb{E}_\mu[f\,|\,\mathscr{H}]$ is the conditional expectation of f with respect to the sub-σ-algebra $\mathscr{H}$ of $\mathscr{B}$. Therefore, from the martingale convergence theorem $\Phi_j(f)$ converges to f in $L_p(T)$ and also μ-almost surely as $j\to\infty$. □

Proposition 4.2. *Let $(T, \mathscr{B}, \mu)$ be a measure space which satisfies the hypothesis of Proposition* 4.1, *and $X = \{X(t)\}_{t \in T}$ be a measurable random process on T. Suppose that, for $p \geqq 1$,*

$$\int |X(t)|^p \, d\mu(t) < \infty \qquad \text{a.s.,} \tag{4.2}$$

$$\mathbb{E}[|X(t)|^p] < \infty \qquad \mu\text{-a.e.}$$

Then, there exists a sequence of sub-families $\{\{A_{i,j}\}_{i=1}^{n(j)}\}_{j \in \mathbb{N}}$ of $\mathscr{B}$, where $\mu(A_{i,j}) < \infty$ for any i, j, and a sequence of subsets $\{\{A_{i,j}\}_{i=1}^{n(j)}\}_{j \in \mathbb{N}}$ of T such that

$$X_j(t) = \sum_{i=1}^{n(j)} X(t_{i,j}) \, 1_{A_{i,j}}(t)$$

almost surely converges to $X(t)$ in $L_p(T)$.

Proof. As in the proof of Proposition 4.1, we may assume $\mu(T) < \infty$. We define a pseudo-metric ρ on T by

$$\rho(t, s) \rightleftharpoons \mathbb{E}[|X(t) - X(s)|^p]^{1/p},$$

and let $B_\rho(t, \varepsilon)$ be the closed ρ-ball with center t and radius ε.

By (4.2), the process X is a $L_p(T)$-valued random vector. Then, using the same method as in the proof of Proposition 4.1, we can obtain that $\{\Phi_j(X)\}$ converges to X $\mu \times P$-almost everywhere, where $\{\Phi_j(X)\}$ is the approximating sequence defined by (4.1). Define

$$T_1 \rightleftharpoons \{t \colon \Phi_j(X)_t \to X(t) \text{ a.s.}\},$$

then $\mu(T_1^c) = 0$. By the definition of $\Phi_j(X)$, for every $t \in T_1$ there exists a sequence $\{y_j\}$ in $(L_p)^*$ such that $\langle y_j, X(\cdot)\rangle$ converges to $X(t)$ almost surely, so that $X(t)$ is $\sigma(X)$-measurable, where $\sigma(X)$ is the sub-σ-algebra of $\mathscr{F}$ generated by the $L_p(T)$-valued random vector X. On the other hand, since $\mathscr{B}$ is countably generated, the Banach space $L_p(T)$ is separable and $\sigma(X)$ is also countably generated. Therefore $(T_1, \rho) \cong \{X(t) \colon t \in T_1\} \subset L_p(\Omega, \sigma(X), P)$ is a separable pseudo metric space.

Let $\{s_k\}$ be a countable dense subset of (T_1, ρ), then by (4.2), for any $j \in \mathbb{N}$, there exists $n(j) \in \mathbb{N}$ such that,

$$P\left(\int_{(\bigcup_{k \leqq n(j)} B(s_k, 1/j^2))^c} |X(t)|^p \, d\mu(t) > \frac{1}{j}\right) < \frac{1}{j}.$$

For any $j \in \mathbb{N}$ and $i \leqq n(j)$ we define

$$A_{i,j} \rightleftharpoons B(s_i, 1/j^2) \setminus \left(\bigcup_{k=1}^{i-1} B(s_k, 1/j^2)\right),$$

$$t_{i,j} \rightleftharpoons s_i,$$

$$X_j(t) = \sum_{i=1}^{n(j)} X(t_{i,j}) \, 1_{A_{i,j}}(t).$$

Then,

$$P\left(\int |X_j(t)-X(t)|^p\, d\mu(t) > \frac{2}{j}\right)$$

$$\leqq P\left(\sum_{i=1}^{n(j)} \int_{A_{i,j}} |X_j(t)-X(t)|^p\, d\mu(t) > \frac{1}{j}\right)$$

$$+P\left(\int_{\bigcap_{i\leqq n(j)} A_{i,j}^c} |X(t)|^p\, d\mu(t) > \frac{1}{j}\right)$$

$$\leqq j \times \sum_{i=1}^{n(j)} \int_{A_{i,j}} \mathbb{E}[|X_j(t)-X(t)|^p]\, d\mu(t) + \frac{1}{j}$$

$$\leqq j^{-2p+1} \sum_{i=1}^{n(j)} \mu(A_{i,j}) + \frac{1}{j} \leqq \frac{(\mu(T)+1)}{j} \to 0 \quad \text{as} \quad j\to\infty.$$

Therefore, the sequence $\{X_j\}$ converges to X in probability, and a subsequence converges almost surely. □

Then, we can show the following theorems.

Theorem 4.3. *Let X be an $L_p(T, \mathscr{B}, \mu)$-valued sub-Gaussian vector $(1\leqq p<\infty)$, where $(T, \mathscr{B}, \mu)$ is a σ-finite measure space with countably generated σ-algebra $\mathscr{B}$. Then X is exponentially integrable.*

Proof. Let $\{A_{i,j}\}$ be the sequence in $\mathscr{B}$ obtained by Proposition 4.1. We may assume that $1_{A_{i,j}} \in (L_p)^*$ for any i, j. Put

$$Y_j(\cdot) \rightleftarrows \sum_{i=1}^{n(j)} \frac{1}{\mu(A_{i,j})} \int_{A_{i,j}} X(t)\, d\mu(t)\, 1_{A_{i,j}}(\cdot).$$

Then, by Proposition 4.1, $\{Y_j\}$ converges to X almost surely in the space L_p. Therefore, by Proposition 2.2, it is enough to show the following inequality for some $C>0$.

$$\tilde{\tau}(\|Y_j\|) \leqq C\mathbb{E}[\|Y_j\|] < \infty, \tag{4.3}$$

for every $j\in\mathbb{N}$, where $\|Y_j\| \rightleftarrows \int_T |Y_j(t)|^p\, d\mu(t)\}^{1/p}$.

It follows from the definition of sub-Gaussian vector, Remark 2 after Definition 2.1 and Proposition 2.3 (b) that there exists $C_1>0$ such that,

$$\tilde{\tau}\left(\int_{A_{i,j}} X(t)\, d\mu(t)\right) \leqq C_1 \mathbb{E}\left[\left| \int_{A_{i,j}} X(t)\, d\mu(t)\right|\right]. \tag{4.4}$$

for any $t\in T$. Therefore by the definition of Y_j we have

$$\mathbb{E}[\|Y_j\|] \leqq \mathbb{E}\left[\sum_{i=1}^{n(j)} \{\left| \int_{A_{i,j}} X(t)\, d\mu(t)\right| \mu(A_{i,j})^{1/p}\}\right] < \infty,$$

and, since $L_p(T)$ is separable ($\mathscr{B}$ is countably generated), the function $\omega \to |X(\cdot, \omega)|$ is Bochner integrable with respect to the measure P as an $L_p(T)$-valued measurable function (Yosida [15] p. 132). Then, we have

$$\begin{aligned}(\int_T \mathbb{E}[|Y_j(t)|]^p\, d\mu(t))^{1/p} &= \| \int_\Omega |Y_j|\, dP \| \\ &\leqq \int_\Omega \| \, |Y_j| \, \| \, dP \eqqcolon M_j < \infty.\end{aligned}$$

From (4.4) it follows that

$$\int \tilde{\tau}(Y_j(t))^p\, d\mu(t) \leqq \int C_1^p\, \mathbb{E}[|Y_j|]^p\, d\mu(t) \leqq (C_1 M_j)^p,$$

and by the definition of $\tilde{\tau}$, we obtain

$$\int \alpha_n^p\, \mathbb{E}[|Y_j(t)|^{2n}]^{p/2n}\, d\mu(t) \leqq (C_1 M_j)^p. \tag{4.5}$$

Put $n_0 = \min\{n\colon 2n/p \geqq 1\}$, fix any $n \geqq n_0$ and put

$$r \coloneqq 2n/p \geqq 1.$$

Then, (4.5) implies

$$\int \mathbb{E}[Z(t)^r]^{1/r}\, d\mu(t) \leqq \alpha_n^{-p}(C_1 M_j)^p,$$

where $Z(t) = |Y_j(t)|^p$, and the function $t \to Z(t)$ is Bochner integrable with respect to the measure μ as an $L_r(\Omega)$-valued measurable function, since it takes only finite vectors

$$\left\{\frac{1}{\mu(A_{i,j})} \int_{A_{i,j}} X(t)\right\}_{i=1}^{n(j)} \subset L_r(\Omega).$$

Therefore, we have,

$$\begin{aligned}\mathbb{E}[(\int Z(t)\, d\mu(t))^r]^{1/r} &= \| \int Z(t)\, d\mu(t) \|_{L_r(\Omega)} \\ &\leqq \int \|Z(t)\|_{L_r(\Omega)}\, d\mu(t) \\ &\leqq \alpha_n^{-p}(C_1 M_j)^p.\end{aligned}$$

Hence,

$$\alpha_n \mathbb{E}[(\int |Y_j(t)|^p\, d\mu(t))^{2n/p}]^{1/2n} \leqq C_1 M_j$$

for any $n \geqq n_0$, and since the L_{2m}-norm is weaker than the L_{2n_0}-norm for any $m \leqq n_0$, there exists $C > 0$ such that

$$\tilde{\tau}(\|Y_j\|) \leqq C M_j = C\mathbb{E}[\|Y_j\|] < \infty,$$

where C is depend only on p and C_1. This implies the inequality (4.3) and complete the proof. □

By the inequality (4.3) and Proposition 2.2 we have:

Corollary. *Let $\mathscr{A}$ be a family of $L_p(T)$-valued sub-Gaussian vectors ($p \geqq 1$), where $T=(T, \mathscr{B}, \mu)$ is the measure space which satisfies the hypothesis of Theorem 4.3. We assume that there exists $C>0$ such that any element $X \in \mathscr{A}$ satisfies*

$$\mathbb{E}[e_{\langle y, x\rangle}] \leqq e^{\frac{C}{2}\mathbb{E}[\langle y, X\rangle^2]} < \infty$$

for any $y \in B^*$. Then, there exists $C_1 = C_1(C, p) > 0$ such that

$$\tilde{\tau}(X) \leqq C_1\, \mathbb{E}[\|X\|] < \infty \quad \text{for any } X \in \mathscr{A}. \quad \square$$

Using Proposition 4.2 instead of Proposition 4.1, we have:

Theorem 4.4. *Let $(T, \mathscr{B}, \mu)$ be the measure space which satisfies the hypothesis of Theorem 4.3, and $X(t)$ be a measurable random process on T. Suppose that there exist $C>0$ such that*

$$\tilde{\tau}(X(t)) \leqq C\mathbb{E}[|X(t)|^2]^{\frac{1}{2}} < \infty \quad \text{for any } t \in T,$$

$$\int |X(t)|^p \, d\mu(t) < \infty \quad \text{a.s.}$$

Then, $X(t)$ is exponentially integrable as an $L_p(T)$-valued random vector. $\square$

It follows from Theorem 4.3, that every sub-Gaussian vector in a real separable Hilbert space is exponentially integrable. Moreover, using the above corollary we can prove that every $L_p(T \to H)$-valued sub-Gaussian vector is exponentially integrable, where $L_p(T \to H)$ is the space of all L_p-functions from T to a real separable Hilbert space H.

Theorem 4.5. *Let $X = X(\cdot)$ be an $L_p(T \to H)$-valued random variable, and assume that there exists $C>0$ such that*

$$\mathbb{E}[e^{\int (h, X(t))\, y(t)\, d\mu(t)}] \leqq e^{\frac{C}{2}\mathbb{E}[|\int (h, X(t))\, y(t)\, d\mu(t)|^2]} < \infty$$

for each $(h, X(\cdot))$, $h \in H$ and any $y \in (L_p(T \to \mathbb{R}))^*$, where $(\cdot, \cdot)$ denotes the inner product of H. Then, X is exponentially integrable. $\square$

The proof of this theorem is parallel to that of Theorem 4.3.

Next, we show that every L_p-valued sub-Gaussian vector is also a γ-sub-Gaussian vector.

Lemma 4.6. *Under the same hypothesis with Theorem 4.3, there exists an L_p-valued Gaussian vector G such that,*

$$\mathbb{E}[\langle y, X\rangle^2] = \mathbb{E}[\langle y, G\rangle^2] \quad \text{for any } y \in (L^p)^*.$$

Proof. By the same way as in the proof of Theorem 4.3 we can prove that there exists $C>0$ such that

$$\tau(X(t)) \leqq C\mathbb{E}[|X(t)|^2]^{\frac{1}{2}} < \infty$$

for almost every $t \in T$. We define a Gaussian process G, as follows.

$$\mathbb{E}[G(t)\,G(s)] = \begin{cases} \mathbb{E}[X(t)\,X(s)] & \text{if } X(t), X(s) \in L_2(\Omega). \\ 0 & \text{otherwise.} \end{cases}$$

Then, since $X(t)$ and $G(t)$ satisfy the hypothesis of Proposition 2.3, there exist positive constants $C_1, C_2 > 0$ such that,

$$\mathbb{E}[|G(t)|^p] = C_1\,\mathbb{E}[|G(t)|^2]^{p/2},$$

$$\mathbb{E}[|X(t)|^2]^{p/2} \leqq C_2\,\mathbb{E}[|X(t)|^p],$$

for almost every $t \in T$.

The process G is continuous in probability with respect to the pseudo metric ρ on T defined by

$$\rho(t, s) \rightleftarrows \mathbb{E}[|X(t) - X(s)|^p]^{1/p} \qquad (t, s \in T).$$

Therefore, there exists a (T, ρ)-Borel measurable version of G, and this will be denoted by G also. Since the (T, ρ)-Borel σ-algebra is a sub-σ-algebra of $\mathscr{B}$, $G(t)$ is $\mathscr{B}$-measurable almost surely.

By the above arguments we have

$$\begin{aligned} \mathbb{E}[\textstyle\int |G(t)|^p\,d\mu(t)] &= \textstyle\int \mathbb{E}[|G(t)|^p]\,d\mu(t) \\ &= C_1 \textstyle\int \mathbb{E}[|G(t)|^2]^{p/2}\,d\mu(t) \\ &= C_1 \textstyle\int \mathbb{E}[|X(t)|^2]^{p/2}\,d\mu(t) \\ &\leqq C_1\,C_2 \textstyle\int \mathbb{E}[|X(t)|^p]\,d\mu(t) < \infty. \end{aligned}$$

This implies that $G(\cdot) \in L_p(T)$ almost surely. By Theorem 4.3 we have $\mathbb{E}[\|X(\cdot)\|^2] < \infty$, where $\|\cdot\|$ is the L_p-norm on (T, μ). Therefore,

$$\begin{aligned} &\mathbb{E}[\textstyle\iint |X(t)\,X(s)\,y(t)\,y(s)|\,d\mu(t)\,d\mu(s)] \\ &\leqq \|y\|_{L_q}^2\,\mathbb{E}[\|X(t)\|_{L_p}^2] < \infty. \end{aligned}$$

By Fubini's Theorem and the definition of G,

$$\begin{aligned} \mathbb{E}[\langle y, X\rangle^2] &= \mathbb{E}[\textstyle\iint X(t)\,X(s)\,y(t)\,y(s)\,d\mu(t)\,d\mu(s)] \\ &= \textstyle\iint \mathbb{E}[X(t)\,X(s)]\,y(t)\,y(s)\,d\mu(t)\,d\mu(s) \\ &= \textstyle\iint \mathbb{E}[G(t)\,G(s)]\,y(t)\,y(s)\,d\mu(t)\,d\mu(s) \\ &= \mathbb{E}[\langle y, G\rangle^2]. \quad \square \end{aligned}$$

§5. Relations between the two definitions of sub-Gaussian vectors

In general, the two definitions of sub-Gaussian vectors are not comparable. In this section, we shall give illuminating examples. At first, we give an example of a γ-sub-Gaussian vector which is not a sub-Gaussian vector in our definition.

Proposition 5.1 (see Kahane [5], p. 20). *Let* $\{u_n\}$ *be a sequence in B, and* $\{X_n\}$, $\{Y_n\}$ *be sequences of independent symmetric random variables such that*

$$\sum X_n u_n, \sum Y_n u_n \text{ converge a.s.},$$
$$|X_n| \leqq |Y_n| \text{ a.s. for any } n \in \mathbb{N}.$$

Then for any convex increasing function $\varphi(\cdot)$ *on* $[0, \infty)$, *we have*

$$\mathbb{E}[\varphi(\|\sum X_n u_n\|)] \leqq \mathbb{E}[\varphi(\|\sum Y_n u_n\|)]. \quad \square$$

Example 5.1. Let $\{e_n\}$ be an orthonormal system in an infinite dimensional Hilbert space H with inner product $(\cdot, \cdot)$ and norm $|\cdot|$, $\{\gamma_n\}$ denote a standard Gaussian sequence (i.i.d. $\gamma_1 \sim N(0, 1)$) and $\{a_n\}$ be a ℓ_2-sequence such that $a_n \neq 0$ for any $n \in \mathbb{N}$. Put

$$X_n = \gamma_n 1_{\{|\gamma_n| \geqq n\}}.$$

Then,

(1) $X = \sum X_n a_n e_n$ and $\Gamma = \sum \gamma_n a_n e_n$ converge almost surely.

(2) $\mathbb{E}[e^{(y, X)}] \leqq \mathbb{E}[e^{(y, \Gamma)}]$ for any $y \in H$.

(3) $\sigma_p(X_n) = \sigma_P\left(\frac{1}{a_n}(e_n, X)\right) \geqq \sqrt{2}$.

Since $\mathbb{E}[X_n^2]$ converges to 0 and σ_P is equivalent to τ, we have

$$\lim_{n \to \infty} \frac{\tau((e_n, X))}{\mathbb{E}[(e_n, X)^2]^{\frac{1}{2}}} = \infty.$$

Therefore, X is not a sub-Gaussian vector, but a γ-sub-Gaussian vector.

Proof of (2) *and* (3). (2) By Proposition 5.1 and the definition of X_n we have

$$\mathbb{E}[\cosh(y, X)] \leqq \mathbb{E}[\cosh(y, \Gamma)].$$

Since X and Γ are symmetric, we obtain

$$\mathbb{E}[e^{(y, X)}] \leqq \mathbb{E}[e^{(y, \Gamma)}]$$

for any $y \in H$. $\square$

(3) By the definition of X_n,

$$\mathbb{E}[e^{\frac{1}{2} X_n^2}] \geqq \frac{2}{\sqrt{2\pi}} \int_n^\infty e^{\frac{1}{2}x^2} e^{-\frac{1}{2}x^2} dx = \infty > 2.$$

Therefore, $\tau_\sigma(X_n) \geqq \sqrt{2}$ for any $n \in \mathbb{N}$. $\square$

Next, we shall give an example of a sub-Gaussian vector which is not γ-sub-Gaussian.

Proposition 5.2. *Let* (,) *be a non-negative definite symmetric bilinear form of* $B^* \times B^*$, $|\cdot|$ *denote the semi-norm determined by* (,) *and G be a B-valued Gaussian vector which satisfies*

$$|y|^2 \leqq \mathbb{E}[\langle y, G\rangle^2]$$

for any $y \in B^*$. *Then, there exists a B-valued Gaussian vector* $\tilde{G}$ *such that, for any* $y \in B^*$,

$$|y|^2 = \mathbb{E}[\langle y, \tilde{G}\rangle^2].$$

Proof. We shall prove the proposition for the case where $(\cdot, \cdot)$ is positive definite. Put

$$(y, z)_G \rightleftarrows \mathbb{E}[\langle y, G\rangle \langle z, G\rangle], \qquad |y|_G \rightleftarrows (y, y)_G^{\frac{1}{2}}$$

for any $y, z \in B^*$, and let H_G be the $|\ |_G$-completion of B^*. Put

$$\mu(A) \rightleftarrows P(G \in A)$$

for any Borel set A in B. Then (B, H_G, μ) is an abstract Wiener space. Let ι be the identity map from $(B^*, |\ |_G)$ to $(B^*, |\ |)$. Then it can be extended to a continuous map from H_G to H, where H is the $|\cdot|$-completion of B^*. In this sense H in a subset of H_G, hence we can assume that H is included by B. Let φ be an isomorphism from H_G. Then, $\|\cdot\|$ is a measurable semi-norm (see Kuo [8], p. 59) on H if and only if $\|\varphi\iota\cdot\|$ is a measurable semi-norm on H_G, and since $\varphi\iota$ is a bounded operator on H_G, $\|\varphi\iota\cdot\|$ is a measurable semi-norm on H (see Kuo [8], p. 62). Therefore, there exists a Gaussian measure ν on B such that

$$(y, z) = \int_B \langle y, x\rangle \langle z, x\rangle \, d\nu(x)$$

for any $y, z \in B^*$. □

Theorem 5.3. *Let* $\{\varepsilon_n\}$ *be a Rademacher sequence (i.i.d.* $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = \frac{1}{2}$.*),* $\{\gamma_n\}$ *be a standard Gaussian sequence and* $\{u_n\}$ *be a sequence in B. Suppose that the sum* $R = \sum \varepsilon_n u_n$ *converges almost surely, and that there exists a B-valued Gaussian vector G such that*

$$\mathbb{E}[e^{\langle y, R\rangle}] \leqq \mathbb{E}[e^{\langle y, G\rangle}]$$

for any $y \in B^*$. *Then, the sum* $\sum \gamma_n u_n$ *converges almost surely.*

Proof. From the hypothesis

$$\begin{aligned}\mathbb{E}[\langle y, R\rangle^2] &\leqq \tau(\langle y, R\rangle)^2 \\ &\leqq \tau(\langle y, G\rangle)^2 = \mathbb{E}[\langle y, G\rangle^2].\end{aligned}$$

Then, by Proposition 5.2 there exists a B-valued Gaussian vector $\tilde{G}$ such that

$$\mathbb{E}[\langle y, \tilde{G}\rangle^2] = \mathbb{E}[\langle y, R\rangle^2]$$

for any $y \in B^*$. Put

$$G_N \rightleftarrows \sum_{n=1}^{N} \gamma_n u_n.$$

Then,

$$\mathbb{E}[e^{i\langle y, G_N\rangle}] = e^{-\frac{1}{2}\sum_{n=1}^{N}\langle y, u_n\rangle^2}$$
$$\to e^{-\frac{1}{2}\sum_{n=1}^{\infty}\langle y, u_n\rangle^2} \quad \text{as} \quad N\to\infty$$
$$= \mathbb{E}[e^{i\langle y, \tilde{G}\rangle}].$$

By Ito-Nisio's theorem [4], $\sum \gamma_n u_n$ converges almost surely. □

If a Gaussian series $\sum \gamma_n u_n$ converges almost surely, then the Rademacher series $\sum \varepsilon_n u_n$ also converges almost surely with the same sequence $\{u_n\} \subset B$ (Jain and Marcus [11]). But the converse is not true in general.

Theorem 5.4 (Maurey, Pisier [13]). *The following two conditions are equivalent.*

(1) *$\sum \varepsilon_n u_n$ converges almost surely if and only if $\sum \gamma_n u_n$ converges almost surely for any sequence $\{u_n\}$ in B.*

(2) *B has finite cotype, that is, there exists $p<\infty$ such that, if $\sum \varepsilon_n u_n$ converges almost surely then $\sum \|u_n\|^p < \infty$ for any sequence $\{u_n\}$ in B.* □

A Rademacher series is always exponentially integrable, if it converges almost surely.

Theorem 5.5 (Kwapien [9]). *Let $\{\varepsilon_n\}$ be a Rademacher sequence and $\{u_n\}$ be a sequence in B. Suppose that $R = \sum \varepsilon_n u_n$ converges almost surely, then*

$$\mathbb{E}[e^{\alpha\|R\|^2}] < \infty$$

for any $\alpha > 0$. □

Summing up the above theorems we have:

Example 5.2. Let $\{\varepsilon_n\}$ be a Rademacher sequence and $\{\gamma_n\}$ be a normal sequence. If B does not have finite cotype, there exists a sequence $\{u_n\}$ in B such that

(1) $R = \sum \varepsilon_n u_n$ converges almost surely, but $\sum \gamma_n u_n$ does not converge almost surely.

(2) *R is a sub-Gaussian vector*, but it is not γ-sub-Gaussian.

(3) R is exponentially integrable.

Proof of the first part of (2). Let Y and Z be independent real sub-Gaussian random variables. Then,

$$\tau(Y+Z)^2 \leqq \tau(Y)^2 + \tau(Z)^2.$$

Therefore,

$$\tau(\langle y, \sum \varepsilon_n u_n\rangle)^2 = \tau(\sum \langle y, u_n\rangle \varepsilon_n)^2$$
$$\leqq \sum \tau(\langle y, u_n\rangle \varepsilon_n)^2$$
$$= \sum \langle y, u_n\rangle^2 = \mathbb{E}[\langle y, \sum \varepsilon_n u_n\rangle^2]. \quad \square$$

Remark. Jain and Marcus [12], (p. 228) gave a concrete example of above $\{u_n\}_{n\in\mathbb{N}}$ as a sequence of vectors in $C[0, 1]$.

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References

1. Buldygin, V.V., Kozachenko, Yu.V.: Sub-Gaussian random variables. Ukrainan Math. J. **32**, 483–489 (1980)
2. Fernique, X.: Régularité des trajectoires des fonctions aléatoires gaussiennes. (Lect. Notes Math., vol. 480) Berlin Heidelberg New York: Springer 1974
3. Heinkel, B.: Mesures majorantes et théorème de la limite centrale dans $C(S)$. Z. Wahrscheinlichkeitstheor. Verw. Geb. **38**, 339–351 (1977)
4. Ito, K., Nisio, M.: On the convergence of sums of independent Banach space valued random variables. Nagoya Math. J. **47**, 15–28 (1968)
5. Kahane, J.P.: Some random series of functions. 2nd ed.: Cambridge: Cambridge University Press 1985
6. Kahane, J.P.: Propriétés locales des fonctions à séries de Fourier aléatoires. Studia Math. **19**, 1–25 (1960)
7. Kadec, M.I., Pełczyński, A.: Bases, lacunary sequences and complemented subspaces in the space L_p. Studia Math. **21** (1962)
8. Kuo, H.H.: Gaussian measures in Banach spaces. (Lect. Notes Math., vol. 463) Berlin Heidelberg New York: Springer 1975
9. Kwapién, S.: Probability in Banach space. Oberwolfach (1975). (Lect. Notes Math., vol. 526, pp. 157–158) Berlin Heidelberg New York: Springer 1976
10. Jain, N.C., Marcus, M.B.: Continuity of subgaussian processes. In: Kuelbs, J (ed.) Probability on Banach spaces. vol. 4., pp. 81–196 New York: Dekker 1978
11. Jain, N.C., Marcus, M.B.: Integrability of infinite sums of independent vector-valued random variables. Trans. Am. Math. Soc. **212**, 1–36 (1975)
12. Jain, N.C., Marcus, M.B.: Central limit theorems for $C(S)$-valued random variables. J. Funct. Anal. **19**, 216–231 (1975)
13. Maurey, B., Pisier, G.: Séries de variables aléatoires vectorielles indépendantes at propriétés géométriques des espaces de Banach. Studia Math. **58**, 45–90 (1976)
14. Talagrand, M.: Regularity of Gaussian processes. Acta Math. **159**, 99–149 (1987)
15. Yosida, K.: Functional analysis (Grundlehren vol. 123) Berlin Heidelberg New York: Springer 1966