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Probability Theory and Related Fields

Spectra of graphs and fractal dimensions. I

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Summary. In this paper we consider the nearest neighbour Random Walk on infinite graphs. We discuss the connection between the two smallest eigenvalues of the Laplacian of the graph and the diffusion speed of the RW.

1. Introduction

In the recent literature on graphs a new trend is apparent: the study of how spectral properties of graphs and other graph properties determining the behaviour of the RW on the graph (cf. [MW, SW]). We consider this paper as a continuation of our previous (cf. [T]). There we studied the *fractal dimension* d, the *resistance dimension* d_Ω, and the *RW dimension* d_R. We found under some restriction on the graph that

$$d_\Omega \leqq d$$

and

$$d_R = d + 2 - d_\Omega$$

and consequently

$$d_R \geqq 2 .$$

To establish this inequality the graph was supposed to be "smooth". This condition is formulated in detail in [T]. At the end of the present paper we shall give an example of a "nonsmooth" graph.

The main problem of this paper comes from Domokos Szász. He suggested studying the "spectral gap", i.e. the second smallest eigenvalue (the first nonzero and nontrivial) of the Laplacian of the graph and conjectured that this eigenvalue tends to zero as fast as the diffusion expands.

In the recent literature of theoretical physics one can find a lot of papers on fractals and spectral dimensions. Here we mention only a few of them cf. [DABK, R, RT]. We refer separately to Mohar's and Woess's work [MW], which is an excellent overview of questions of spectra of graphs arising in connection with RW-s on graphs.

In the next section we give the necessary definitions and in Sect. 3 we present our results.

2. Basic notations

2.1 Geometry

Let $G=(V,E)$ be a *connected* infinite *locally finite* graph without loops with vertex set V and edge set E. Let $a\in V$ be a fixed vertex.

We use the convenient graph distance

$$d(x,y)=\min\{k:\exists\{x_i\}_{i=0}^{i=k}:x_0=x,\ x_n=y,\ \forall 0<i\leqq k(x_{i-1},x_i)\in E\}.$$

Let $B_N=B(a,N)$ and $S_N=S(a,N)$, be the *ball* and *sphere* centered in $a\in V$ with radius N, ($B_N, S_N\subset V$). Let d_x denote the degree of the vertex $x\in V$. More formally

$$B_N=\{y\in V: d(a,y)\leqq N\}$$

$$S_N=\{y\in V: d(a,y)=N\}$$

$$d_x=|\{y\in V: d(a,y)\leqq 1\}|.$$

We often use the subgraphs G_N of G given by

$$G_N=(B_N,E_N)$$

where

$$E_N=\{e=(x,y)\in E: x,y\in B_N\}.$$

The degree of vertex x in G_N will be denoted by $d_{x,N}$. Note that $d_{x,N}=d_x$ if x is an internal point of B_N. □

Throughout this paper we suppose that G is *locally finite*, i.e. d_x has a fixed upper bound D for all $x\in V$.

The adjacence matrix $A_N=(a_{x,y})x,y\in B_N$ is defined as $a_{x,y}=1$ if $(x,y)\in E$ and zero otherwise. □

Let D_N be the diagonal matrix of $d_{x,N}$-s for $x\in B_N$. Then Λ_N the Laplace operator of G_N can be defined by

$$\Lambda_N=D_N-A_N. \quad \square$$

Let $\vec{1}^*=(1,\dots,1)$ be the vector of ones of length $|b_N=B_N|$. It is obvious that $\Lambda_N\vec{1}=0\vec{1}$ i.e., $\lambda_1=0$. We shall study the second smallest eigenvalue $\lambda_{2,N}$ of Λ_N. We note that Λ_N is symmetric and has nonnegative entries so $\lambda_{2,N}$ is positive according to the Perron-Frobenius theorem and the spectral mapping theorem.

2.2 Exponents

We are dealing with monotone sequences tending to zero or to infinity. For such a sequence $\{a_i\}$ we define

$$LS(\{a_i\})=\limsup_{i\to\infty}\frac{\ln(a_i)}{\ln(i)}$$

and for a sequence $\{k_i\}$ of indices such that $k_i \to \infty$

$$\operatorname{Lim}(\{a_i\}, \{k_i\}) = \lim_{i\to\infty} \frac{\ln(a_{k_i})}{\ln(k_i)}$$

provided the latter *exists*. We say that an exponent a is exact if there is a sequence of indices $\{k_i\}$ for the sequence $\{a_i\}$ such that

$$a = \operatorname{Lim}(\{a_i\}, \{k_i\}) = LS(\{a_i\})$$

and there are constants $1 < Q \leqq Q' < \infty$ forming uniform bound as follows

$$Q \leqq \frac{a_{k_i}}{a_{k_{i-1}}} \leqq Q' \quad \text{if} \quad a_i \to \infty \tag{1}$$

and

$$Q \leqq \frac{a_{k_{i-1}}}{a_{k_i}} \leqq Q' \quad \text{if} \quad a_i \to 0. \quad \square \tag{2}$$

Remark 1. We have defined in [T] a similar condition for the sequence k_i; instead of (1) (or (2)) there we supposed that

$$Q < \frac{k_i}{k_{i-1}} < Q' \tag{3}$$

holds. It is important to note that along all series we reach the same limit as it follows from the condition (1), (2) or (3).

Using the above notations we can now define the fractal dimension

$$d = LS(\{b_N\})$$

and the exponent of $\lambda_{2,N}$

$$d_2 = LS(\{\lambda_{2,N}^{-1}\}). \quad \square$$

The *isoperimetric constant* of a graph is

$$i_G = \min_{\substack{B \subset V \\ |B| < |V|/2}} \frac{|E(B, V \backslash B)|}{|B|}$$

where $E(A, B)$ is the set of edges connecting the vertex sets A and B. $\square$

It is easy to prove that

$$\lambda_{2,N} \leqq \frac{n-1}{n} i_G$$

where $n = |V|$. So we have that $\lambda_{2,N} \leqq i_G$. One can find a lot of further results for λ_2 (cf. [AM, A, F]) and we mention the fundamental work of Cheeger ([C]).

2.3 The Random Walk

Our main object is the RW; $\{X_n\}_{n\in N}$, $X_n \in V$ and $X_0 = a$. The transition probability is determined by G. From any $x \in V$ the Random Walk can jump to nearest neighbour vertices and chooses between them with equal probability:

$$P(X_n = y | X_{n-1} = x) = P(x,y) = \frac{1}{d_x}.$$

for all $(x, y) \in E$ and for all $n \in N$. We write P to denote the transition probability matrix of the RW. Let Q_N be the restriction of P to B_N.

$$Q_N(x, y) = P(x, y)$$

on B_N. □

Let $D_{a,N}$ be the diagonal matrix of vertex degrees restricted to $B_{a,N}$

$$D_{a,N} = (d_{y,z})_{y,z \in B_N}$$

where $d_{y,z} = d_y$ if $y = z$ and 0 otherwise. (Not to confuse with D_N.) The *Laplace operator* restricted to B_N is defined as follows

$$\Delta_N = D_N - D_N Q_N. \quad \square$$

We believe that the spectrum of Δ_N determines the important properties of the RW. As one can find in the literature (cf. [KSK, DS]) this operator is the inverse of the fundamental matrix determined by the RW plays a central role in the potential theory of Markov chains. As was pointed out in [MW] p. 13, the first eigenvalue of the Laplace operator of the subgraph G_N does not necessarily converge to the first eigenvalue of the Laplace operator of G, but the spectrum of Δ_N tends to the spectrum of the Laplacian on G.

The smallest eigenvalue of Δ_N will be denoted by $\mu_{1,N}$. □

Let us observe that $\mu_{1,N} > 0$ for the same reason that $\lambda_{2,N} > 0$. Let T_N be the first hitting time of S_N if the RW is started from $a \in V$, and let $E(T_N)$ be its expectation. □

Using the notations introduced above we now define

$$d_R = LS(E(T_N))$$

to be the RW dimension and

$$d_1 = LS(\mu_{1,N}^{-1})$$

to be the exponent of $\mu_{1,N}$. □

2.4 The electric network model

One can consider the finite graph G_N as an electric network (cf. [DS, NW]). The edges are unit resistors and we apply a voltage between the poles $a \in V$ and S_N such that the effective current $i_{a,N}$ along the network is 1. Here we shortcut the vertices of S_N into a single one. Let us denote the effective resistance between a and S_N by

$$R_N = R(a, S_N).$$

Let us consider the voltage $v_x, x \in B_N$, generated by the unit current. Then

$$v_a = R_N$$

and

$$v_y = 0 \quad \text{for all } y \in S_N.$$

It can be proved that (cf. [DS], p. 52)

$$P(X_i \text{ reaches } a \text{ before } S_N | X_0 = x) = v_x / R_N,$$

and if u_x is the expected number of visits of X_i to x before it reaches S_N, then (cf. [DS], p. 50)

$$u_x = d_{x,N} v_x . \tag{4}$$

It is clear that,

$$E(T_N) = \sum_{x \in V} u_x = \sum_{x \in V} d_{x,N} v_x .$$

Now we can give the exact definition of resistance dimension via

$$d_\Omega = 2 - LS(R_N), \quad \text{if } R_N \to \infty$$

and

$$d_\Omega = 2 - LS(R(G) - R_N), \quad \text{if } R_N \to R(G) < \infty . \ \square$$

Remark 2. The exponent d_Ω is said to be exact if R_N (or $R(G) - R_N$) satisfies the condition (1) (or (2)).

For the sake of brevity we introduce

$$\delta = d + 2 - d_\Omega - d_R . \ \square$$

2.5 The conditions

We say that G is moderate if

1. G is locally finite and connected graph without loops
2. d, d_Ω are exact exponents with a common $\{k_i\}$,
3. for the sequence of spheres S_{k_i} there is a sequence of equipotential surfaces Γ_{k_i} such that Γ_{k_i} lies strictly between S_{k_i} and $S_{k_{i+1}}$. □

Remark 3. We say that a graph is smooth if 1.–3. hold but we suppose (3) instead of (1) or (2) (see Remark 1).

Our previous result [T] states that $\delta = 0$ if G is smooth. On the other hand it is easy to see that the proof of $\delta = 0$ in [T] works if we assume that G is moderate but we do not use this fact in the present context.

Condition 3. means in detail that for any finite subgraph G_{k_i} the potential surfaces lie between the geometric surfaces if S_{k_i} is on the zero potential level ($v_a = 1$), i.e. it is an equipotential surface itself.

3. Results

Theorem 1. *If G is moderate then*

$$2d_2 \geqq d_R + 2\delta .$$

In particular, if $\delta = 0$

$$2d_2 \geqq d_R . \ \square$$

Theorem 2. *If* $R_N \to \infty$ *as* $N \to \infty$ *then*

$$d_1 \geqq d_R - \delta .$$

In particular, if $\delta = 0$ *then*

$$d_1 \geqq d_R . \ \square$$

4. Proofs

Proof of Theorem 1

We start with

$$E(T_N)=\sum_{x\in B_N} u_x=\sum_{x\in B_N} d_{x,N}v_x=\sum_{(x,y)\in E_N}(v_x+v_y). \tag{5}$$

The r.h.s. of (5) can be written as

$$\sum_{(x,y)\in E_N}(v_x+v_y)\geqq\sum_{(x,y)\in E_N^+}(v_x+v_y)=2\sum_{(x,y)\in E_N^+}\frac{1}{v_x-v_y}\int_{v_y}^{v_x} pdp \tag{6}$$

where $E_N^+=\{(x,y)\in E_N|v_x>v_y\}$. The r.h.s. of (6) can be put into the form

$$\sum_{(x,y)\in E_N^+}\frac{1}{v_x-v_y}2\int_0^{R_N}\chi(p\in(v_y,v_x])pdp$$
$$=2\int_0^{R_N}\sum_{(x,y)\in E_N^+}\frac{1}{v_x-v_y}\chi(p\in(v_y,v_x])pdp. \tag{7}$$

Now we introduce the edge set cutting the equipotential surface of an arbitrary potential p defined as

$$\mathscr{A}_p=\{(x,y)\in E_N^+|p\in(v_y,v_x]\}$$

and write

$$a_p=|\mathscr{A}_p|$$

for its cardinality. The region containing the reference point a and surrounded by the equipotential surface with potential p is defined by

$$\mathscr{B}_p=\{x\in V|v_x\geqq pK,$$

and its volume by

$$\beta_p=|\mathscr{B}_p|.$$

In terms of electric networks we can speak of "equipotential surface" for an arbitrary potential p in our graph in the following sense. If $p\in(v_x,v_y]$, $(x,y)\in E$ we can say that on this edge there is an internal point w with potential p. In this sense we use the notation $\Gamma_p=\{w\in \mathrm{Ex}\,(0,1]|v_w=p\}$ for the equipotential surface (cf. [T]). The sum in (7) now can be estimated from below using the relation between the harmonic and arithmetic mean as follows

$$\sum_{(x,y)\in\mathscr{A}_p}\frac{1}{v_x-v_y}\geqq\frac{a_p^2}{\sum_{(x,y)\in\mathscr{A}_p}(v_x-v_y)}.$$

Now remember that an equipotential surface is a cutset of the graph and the current crossing it is the total current. So the sum in the last denominator is the total current which is 1. It is important to note that the inequality becomes equality if and only if all terms in the sum are equal, i.e. if and only if the potential differences are the same on all of the edges. This may happen for nice graphs, e.g. for hierarchical structures. The r.h.s. of (7) can now be estimated from below by

$$\int_0^{R_N} pa_p^2dp. \tag{8}$$

We can use the isoperimetric constant as a lower bound if the volume of the ball is smaller than the half of the total volume, so we introduce

$$q=\inf\left\{p:\beta_p<\frac{b_N}{2}\right\}. \tag{9}$$

Let us consider the sequence producing the exact exponents, choose $N=N_m$ and abbreviate the double subscripts like A_{N_k} by A_k. It follows from the property 3 of moderateness that there is a *fixed* integer k such that

$$B_{m-k-4}\subset\mathscr{B}_q\subset B_{m-k}. \tag{10}$$

Condition 3. on equipotential surfaces implies that there is an equipotential surface Γ, of potential $r>q$ such that

$$B_{m-k-9}\subset\mathscr{B}_r\subset B_{m-k-5}$$

and so

$$B_{m-k-9}>c\cdot b_m \tag{11}$$

where $c=(Q')^{k+9}$. Now we can use the relation between the isoperimetric constant and $\lambda_{2,N}$ given by

$$a_p\geqq\lambda_{2,m}\beta_p$$

for all $p\geqq q$. We get that

$$\int_0^{R_N} pa_p^2dp\geqq\int_q^r pa_p^2dp\geqq\lambda_{2,m}^2b_r^2\int_q^r pdp$$

$$\geqq\frac{c}{2}\lambda_{2,m}^2b_m^2(r^2-q^2)\geqq\frac{c}{2}\lambda_{2,m}^2b_m^2(r-q)^2. \tag{12}$$

In order to calculate $r-q$ remember the properties relating the equipotential surfaces and the geometric surfaces. Here

$$\begin{aligned} r-q&=R(\Gamma_r,S_m)-R(\Gamma_q,S_m)=[R(a,S_m)-R(\Gamma_q,S_m)]-[R(a,S_m)+R(\Gamma_r,S_m)]\\ &=R(a,\Gamma_q)-R(a,\Gamma_r) \end{aligned} \tag{13}$$

and from the separation property of the surfaces we get that the r.h.s. of (13) is greater than

$$R(a,S_{m-k-4})-R(a,S_{m-5}). \tag{14}$$

Now we have to separate the cases when $R(G)$ is finite or not. There is no essential difference between the proofs, but the finite case is a little more complex. We present the calculation for the case $R(G)<\infty$ and the other case will be left to the reader.

By the definition of d_Ω we can introduce ϱ_k, ε_k as

$$\varrho_k=R(G)-R(a,S_k)=N_k^{2-d_\Omega+\varepsilon_k}$$

where $\varepsilon_k\to0$ as $k\to\infty$. Then (14) will have the form

$$\begin{aligned} R(a,S_{m-k-4})-R(a,S_{m-k-5})&=\varrho_{m-k-5}-\varrho_{m-k-4}\\ &\geqq(Q-1)\varrho_{m-k-4}\geqq c\varrho_m \end{aligned} \tag{15}$$

where $c=Q^{k+4}(Q-1)$. Collecting our inequalities and combining the constants (which do not depend on m) into one C, we get from (12) that

$$E(T_m)\lambda_{2,m}^{-2} \geqq Cb_m^2 \varrho_m^2,$$

which leads to the desired estimate if R_m is bounded:

$$d_R \geqq -2d_2 + 2(2-d_\Omega) + 2d,$$

i.e.

$$2d_2 \geqq 2d + 2(2-d_\Omega) - d_R = d_R + 2\delta$$

and if $\delta = (d+2-d_\Omega) - d_R = 0$ then

$$2d_2 \geqq d_R.$$

Remark 4. It is instructive to consider the Sierpinski graph [DS], since the isoperimetric number for this graph is rather small: $\frac{4}{|b_N|}$. So

$$d_2 \geqq d = \frac{\ln(\bar{d}+1)}{\ln(2)}$$

but

$$d_R = \frac{\ln(\bar{d}+3)}{\ln(2)}$$

where $\bar{d}$ is the dimension of the space the graph is embedded in. This shows that $2d_2$ and d_R can be rather far from each other.

Proof of Theorem 2

By the definition of $\mu_{1,N}$

$$\mu_{1,N} = \min_\phi \frac{(\phi, \Delta_N \phi)}{(\phi, \phi)}.$$

Let us consider $\phi = v$ on B_N. One can easily check that

$$(v, \Delta_N v) = R_N.$$

On the other hand

$$(v,v) = \sum_{x \in B_N} v_x^2 \geqq \frac{\left(\sum_{x \in V} v_x\right)^2}{b_N} \geqq \frac{\left(\sum_{x \in V} u_x\right)^2}{D^2 b_N} \geqq \frac{1}{D^2} \frac{E^2(T_N)}{b_N}$$

and we get that

$$E(T_N)^2 \leqq D^2 R_N b_N \mu_{1,N}^{-1}.$$

By our assumption on the exponents it follows that

$$2d_R \leqq d + 2 - d_\Omega + d_1 = \delta + d_R + d_1$$

and if $\delta = 0$ we get our statement.

Remark 5. Let us consider Z^3 and Z^4, the three and four dimensional integer lattices, as separate graphs and connect their origins by an edge. It is easy to see that this graph is not smooth. One can easily prove this statement by observing that the potential surfaces with the same value on the two graphs are very far from each other.

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References

[A] Alon, N.: Eigenvalues and expanders. Combinatorica **6**,83–89 (1986)

[AM] Alon, N., Milman, V.D.: λ_1, Isoperimetric inequalities for graphs and superconcentrators. J. of Comb. Theory, Ser. B**38**, 73–88 (1985)

[C] Cheeger, J.: A lower bound for the smallest eigenvalue of Laplacian. In: Gunning, R.C. (ed.) Problems in analysis, pp. 195–199. Princeton: Princeton University Press 1970

[DS] Doyle, P., Snell, J.L.: Random walks and electric networks. Carus Math. Monogr. **22**, (1984)

[DABK] Domany, E., Alexander, S., Bensimon, D., Kadanoff, L.D.: Solution to the Schrödinger equation on some fractal lattices. Phys. Rev., Ser. B**28**, 3110–3123 (1983)

[F] Friedel, M.: Algebraic connectivity of graphs. Czech. Math. J. **98**, 298–305 (1973)

[KSK] Kemeny, J.G., Snell, J.L., Knapp, A.W.: Denumerable Markov chains. Princeton: Nostrand 1966

[MW] Mohar, B., Woess, W.: A survey on spectrum of infinite graphs. Università degli studi di Milano, Dipartimento di Matematica "F. Enriques", Quaderno n. 27/1988

[NW] Nash-Williams, C.S.J.A.: Random walks and electric current in networks. Proc. Camb. Phil. Soc. **55**, 181–194 (1959)

[R] Rammal, R.: Spectrum of harmonic excitations on fractals. J. Phys. **45**, 191–206 (1984)

[RT] Rammal, R., Toulouse, T.: Random walks on fractal structures and percolation clusters. J. Phys. Lett. **44**, 1-13-1-22 (1983)

[SW] Soardi, P.M., Woess, W.: Uniqueness of currents in infinite resistive networks. Università degli studi di Milano, Dipartimento di Matematica "F. Enriques", Quaderno n. 23/1988

[T] Telcs, A.: Random walks on graphs, electric networks and fractals. Probab. Th. Rel. Fields **82**, 435–449 (1989)