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A recurrence criterion for Markov processes of Ornstein-Uhlenbeck type

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Summary. A Markov process of Ornstein-Uhlenbeck type was introduced in [5] as a Markov process on R^d generated by a Lévy process generator plus a drift term $-\sum_{j=1}^{d}\sum_{k=1}^{d} Q_{jk}\, x_k \frac{\partial}{\partial x_j}$ with the matrix $Q=(Q_{jk})$ having eigenvalues with positive real parts. A criterion for positive recurrence of processes of this type was given by Sato-Yamazato [5]. This paper gives a criterion for null recurrence and transience by a integral condition involving the Lévy measure in the case of one dimension. Multi-dimensional cases are also discussed.

1. Introduction

Let $M_+(\mathbb{R}^d)$ be the totality of real $d\times d$ matrices whose all eigenvalues have positive real parts. For $Q\in M_+(\mathbb{R}^d)$ an Ornstein-Uhlenbeck process is defined as a diffusion process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ on $\mathbb{R}^d$, whose sample path is governed by the following equation:

$$X_t = x - \int_0^t QX_s\, ds + B_t, \tag{1.1}$$

where $x\in\mathbb{R}^d$, and B_t is a d-dimensional standard Brownian motion defined on a complete probability space $(\Omega,\mathscr{F},P)$. Then it is well-known that for every $x\in\mathbb{R}^d$ the distribution of X_t converges to a Gaussian distribution as $t\to\infty$.

Next let us consider a generalized version of the equation (1.1) taking account of a process with homogeneous and independent increments (which we call *a Lévy process*) in place of B_t in (1.1).

Let A_t be a Lévy process on $\mathbb{R}^d$ whose characteristic function is given by

$$E(e^{i\langle z, A_t\rangle}) = \exp t\,\varphi(z), \tag{1.2}$$

$$\varphi(z) = -\frac{\langle \alpha z, z\rangle}{2} + i\langle z, m\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\rho(dx) \tag{1.3}$$

where $\alpha=(\alpha_{ij})_{1\leqq i,j\leqq d}$ is a symmetric, non-negative definite and real matrix, $m\in\mathbb{R}^d$, and ρ is a measure on $\mathbb{R}^d$ satisfying that $\rho(\{0\})=0$ and the integrability condition

$$\int_{\mathbb{R}^d} \frac{|x|^2}{1+|x|^2}\rho(dx)<+\infty. \tag{1.4}$$

For given $Q\in M_+(\mathbb{R}^d)$ and A_t, let us consider the following equation:

$$X_t=x-\int_0^t QX_s\,ds+A_t. \tag{1.5}$$

As easily seen, the equation (1.5) has a unique solution, and the solution defines a standard Markov process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ taking values in $\mathbb{R}^d$, which we call *a process of Ornstein-Uhlenbeck type* (shortly, *a process of OU type*) following [5].

Such a class of Markov processes were introduced by Wolfe [8] in one dimensional case, and by Sato & Yamazato [5] in multidimensional case, which are paid attention by a fact that every operator-selfdecomposable distribution appears as a limit distribution of a process of OU types as $t\to\infty$. In particular, they proved the following results.

Theorem 1.0. ([5]) *Let $p_t(x,dy)$ be the transition probability of a process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ associated with a $Q\in M_+(\mathbb{R}^d)$ and* (1.5). *Suppose that*

$$\int_{|x|\geqq 1} \log|x|\,p(dx)<+\infty. \tag{1.6}$$

Then there is a Q-selfdecomposable distribution μ such that $p_t(x,dy)$ converges weakly to μ as $t\to\infty$ for every $x\in\mathbb{R}^d$.

Conversely, if the condition (1.6) fails, then

$$\lim_{t\to\infty}\sup_x p_t(x,B)=0 \qquad \text{for every bounded subset } B.$$

As a next stage it is an interesting problem to find a criterion for the process to the null recurrent and transient in terms of Lévy measure. Concerning this problem it is to be noted that in the final section of [5] some examples of one dimensional processes of OU type are given that are null recurrent and transient.

In the present paper we settle the recurrence problem in the one dimensional case, and we will also discuss it in certain multidimensional cases.

Throughout this paper we will adopt the following definition of recurrence and transience.

Definition. Let $(\Omega,\mathscr{F},\mathscr{F}_r,P_x,X_t)$ be a Markov process of OU type on $\mathbb{R}^d$ associated with a $Q\in M_+(\mathbb{R}^d)$ and (1.5). We say that the process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is *recurrent* if there exists an $a\in\mathbb{R}^d$ such that

$$P_a(\liminf_{t\to\infty}|X_t-a|=0)=1. \tag{1.7}$$

On the other hand we say that the process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is *transient* if

$$P_x(\lim_{t\to\infty}|X_t|=+\infty)=1 \qquad \text{for every } x\in\mathbb{R}^d. \tag{1.8}$$

We remark that for a process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$, it always holds that

$$P_x(\limsup_{t\to\infty}|X_t|=+\infty)=1 \qquad \text{for every } x\in\mathbb{R}^d, \tag{1.9}$$

unless A_t is a deterministic process, i.e. $A_t=bt$ for some b.

In one dimensional case the equation (1.5) turns to

$$X_t=x-\gamma\int_0^t X_s\,ds+A_t, \tag{1.10}$$

where $x\in\mathbb{R}$, γ is a positive constant, and A_t is a one dimensional Lévy process with the characteristic function

$$E(e^{izA_t})=\exp t\varphi(z), \tag{1.11}$$

$$\varphi(z)=-\frac{\alpha z^2}{2}+ibz+\int_{\mathbb{R}}(e^{izx}-1-iza(x))\,\rho(dx), \tag{1.12}$$

where $\alpha\geqq 0$ and b are real constants, $a(x)=x$ if $|x|\leqq 1$, $a(x)=0$ otherwise, and ρ is a measure on $\mathbb{R}$ satisfying that $\rho(\{0\})=0$ and the integrability condition

$$\int_{\mathbb{R}}\min\{|x|^2,1\}\,\rho(dx)<+\infty, \tag{1.13}$$

Let $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ be the process of OU type taking values in $\mathbb{R}$ associated with (1.10). We remark that if A_t is not deterministic, the Markov process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is irreducible in the whole space $\mathbb{R}$ except the following two cases:

$$\alpha=0,\qquad \rho(-\infty,0)=0,\quad\text{and}\quad 0\leqq c_+=\int_0^1 x\rho(dx)<+\infty, \tag{1.14}$$

$$\alpha=0,\qquad \rho(0,+\infty)=0,\quad\text{and}\quad 0\leqq c_-=\int_{-1}^0 |x|\,\rho(dx)<+\infty. \tag{1.15}$$

On the other hand, in the case of (1.14) (or (1.15)) the Markov process is irreducible in $((b-c_+)/\gamma,\ +\infty)$ (or $(-\infty,\ (b+c_-)/\gamma)$) since A_t has no downward (or upward) jump.

Our main result of this paper is the following.

Theorem 1.1. *A one dimensional process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ governed by the equation* (1.10) *is recurrent or transient according as*

$$\int_0^1 dz\frac{1}{z}\exp\left(-\int_z^1\frac{\lambda_\rho(y)}{\gamma y}\,dy\right)=+\infty \quad\text{or}\quad <+\infty, \tag{1.16}$$

where

$$\lambda_\rho(y) = \int_{|x| \geqq 1} (1 - e^{-y|x|})\, \rho(dx). \tag{1.17}$$

The proof of transience is based on Fourier analysis method, that is quite standard. On the other hand for the proof of recurrence we first prove it under the assumption that A_t is an increasing and pure jump Lévy process by making use of an explicit formula of Laplace functional of a hitting time distribution for any point. For general case it can be reduced to this case. Theorem 1.1 will be proved in section 3.

In section 4 we discuss several classes of multi-dimensional cases. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ be a Markov process of OU type on $\mathbb{R}^d$ associated with a $Q \in M_+(\mathbb{R}^d)$ and (1.5). Under some restrictive assumption on $Q \in M_+(\mathbb{R}^d)$ and the Lévy measure ρ, we obtain a recurrence criterion for the process of OU type $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$.

For instance we will prove

Theorem 4.3. *Suppose that*

$$Q = (Q_{jk})_{1 \leqq i, j \leqq d} \in M_+(\mathbb{R}^d) \tag{1.18}$$

is symmetric, and the Lévy measure ρ of A_t is rotation invariant, i.e. $\rho(O(E)) = \rho(E)$ for every $E \in \mathscr{B}(\mathbb{R}^d)$ and every orthogonal transformation O: $\mathbb{R}^d \to \mathbb{R}^d$. Then the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is recurrent if and only if

$$\int_0^1 \frac{dr}{r} \exp\left(-\int_r^1 \frac{\lambda_\rho^1(y)}{\gamma y}\, dy\right) = +\infty, \tag{1.19}$$

where γ is the minimum eigenvalue of Q, and

$$\lambda_\rho^1(y) = \int_{|x_1| \geqq 1} (1 - e^{-y|x_1|})\, \rho(dx). \tag{1.20}$$

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ be a process of OU types on $\mathbb{R}^d$ under the situation of Theorem 4.3. If the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is transient, then the integral of (1.19) is convergent, hence "one-dimensional transience" occurs. Namely, since for any eigenvector e associated with the minimum eigenvalue γ of Q, the one-dimensional projection process $\langle X_t, e\rangle$ is also a one-dimensional process of OU type associated with γ and the Lévy process $\langle A_t, e\rangle$ hence it follows from Theorem 1.1 that

$$P_x(\lim_{t \to \infty} |\langle X_t, e\rangle| = +\infty) = 1 \qquad \text{holds for every } x \in \mathbb{R}^d.$$

This phenomenon is a special character of the processes of OU type unlike Lévy processes.

2. Preliminaries

We begin with summarizing some preliminary facts following [5]. Let G be an integro-differential operator defined by

$$Gf(x)=\tfrac{1}{2}\sum_{j,k=1}^{d}\alpha_{jk}D_jD_kf(x)+\sum_{j=1}^{d}m_jD_jf(x)+\int_{\mathbb{R}^d}\Big(f(x+y)-f(x)$$
$$-\sum_{j=1}^{d}\frac{y_j}{1+|y|^2}D_jf(x)\Big)\rho(dy)-\sum_{j,k=1}^{d}Q_{jk}x_kD_jf(x), \tag{2.1}$$

where D_j stands for partial derivative in x_j, $m=(m_j)\in\mathbb{R}^d$, $Q=(Q_{jk})\in M_+(\mathbb{R}^d)$, $\alpha=(\alpha_{jk})$ is symmetric and nonnegative definite, and ρ is a measure on $\mathbb{R}^d$ satisfying that $\rho(\{0\})=0$ and

$$\int_{\mathbb{R}^d}\frac{|x|^2}{1+|x|^2}\rho(dx)<+\infty. \tag{2.2}$$

We consider G as acting on C^2 functions defined on $\mathbb{R}^d$ with compact supports. Let $C_0(\mathbb{R}^d)$ be the Banach space of continuous functions vanishing at infinity with the supremum norm. Then G has the smallest closed extention in the Banach space $C_0(\mathbb{R}^d)$, which is the infinitesimal generator of a Markov semigroup T_t. Furthermore, it is easy to see that for the solution X_t of the equation (1.5) with $X_0=x$,

$$T_tf(x)=E(f(X_t))\qquad \text{for every } f\in C_0(\mathbb{R}^d). \tag{2.3}$$

Thus, corresponding to the Markov semigroup T_t, we have a Markov process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ taking values in $\mathbb{R}^d$, which is called *a process of OU type* associated with $Q\in M_+(\mathbb{R}^d)$ and (1.5).

Lemma 2.1. (cf. [5], Theor. 3.1) *Let $p_t(x,\cdot)$ be the transition probability of the process of OU types $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ associated with $Q\in M_+(\mathbb{R}^d)$ and* (1.5).

Then the characteristic function of $p_t(x,\cdot)$ is

$$\hat{p}_t(x,z)=\exp\Big(i\langle x,e^{-tQ^*}z\rangle+\int_0^t\varphi(e^{-sQ^*}z)\,ds\Big), \tag{2.4}$$

where $\varphi(z)$ is given in (1.3) *and Q^* stands for the transposed matrix of Q.*

Lemma 2.2. *Let X_t be the solution of the equation* (1.5) *associated with a $Q\in M_+(\mathbb{R}^d)$, a Lévy process A_t, and the initial condition $X_0=x$. Then X_t has the following decomposition*:

$$X_t=Y_t+Z_t, \tag{2.4}$$

where Y_t and Z_t are two processes independent of each other such that (*i*) *Y_t is the solution of* (1.5) *with the Q and a Lévy process A_t' of pure jump type and $Y_0=x$, and* (*ii*) *Z_t is the solution of* (1.5) *associated with the Q and a Lévy process A_t'' such that $Z_0=0$, and the distribution of Z_t is convergent as $t\to\infty$.*

Proof. Use the Lévy-Ito decomposition to decompose A_t into

$$A_t = A'_t + A''_t \tag{2.5}$$

such that A'_t and A''_t are independent of each other, A'_t is a Lévy process of pure jump type, and A''_t is a Lévy process with the Lévy measure satisfying the condition (1.6). Let Y_t and Z_t be the solutions of (1.5) with the same Q, the initial conditions $Y_0 = x$ and $Z_0 = 0$, and with A'_t and A''_t in place of A_t in (1.5) respectively. Then by Theorem 1.0 it is obvious that Y_t and Z_t satisfies the requirements (i)–(iii).

Let $\alpha \geqq 0$. For a measurable subset B of $\mathbb{R}^d$, and a positive measurable function f on $\mathbb{R}^d$, set

$$R_\alpha(x, B) = \int_0^\infty e^{-\alpha t} p_t(x, B)\, dt, \quad \text{and} \quad R_\alpha f(x) = \int_0^\infty R_\alpha(x, dy) f(y). \tag{2.6}$$

Lemma 2.3. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ be the process of OU type associated with a $Q \in M_+(\mathbb{R}^d)$ and a Lévy process A_t. Then there is an $r_0 > 0$ such that for every $r \geqq r_0$ and every $\alpha > 0$*

$$\inf_{x \in B_r} R_\alpha(x, B_r) > 0, \tag{2.7}$$

where $B_r = \{x \in \mathbb{R}^d : |x| \leqq r\}$.

Proof. Let $x \in \mathbb{R}^d$ be fixed. Using the decomposition of $X_t = Y_t + Z_t$ in Lemma 2.2 together with Theorem 1.0, we will show that there are constants $t_0 > 0$ and $r_0 > 0$ satisfying that

$$\inf_{|x| \leqq r} P_x(|X_t| \leqq r) > 0 \qquad \text{for every } t > t_0 \quad \text{and} \quad r > r_0, \tag{2.8}$$

which yields (2.7). Clearly

$$P(Y_t = e^{-tQ} x) \geqq P(A'_s = 0 \text{ for all } s \in [0, t]) > 0 \tag{2.9}$$

for every $t \geqq 0$. $Q \in M_+(\mathbb{R}^d)$ implies that for some $a > 0$ and $b > 0$,

$$|e^{-tQ} x| \leqq a e^{-bt} |x| \qquad \text{holds for every } t > 0, \tag{2.10}$$

hence by (2.9) we have a $c_t > 0$ such that

$$P_x(|Y_t| \leqq |x|/2) > c_t \qquad \text{for every } x \in \mathbb{R}^d. \tag{2.11}$$

On the other hand, by Theorem 1.0, Z_t has a limit distribution as $t \to \infty$, from which and together which (2.11), (2.8) follows.

Next we present a criterion of recurrence and transience for the process of OU type in terms of the transition probability.

For a measurable subset B of $\mathbb{R}^d$, the hitting time of B is defined as usual; $\sigma_B = \inf\{t \geqq 0 : X_t \in B\}$ if $\{\cdot\} \neq \phi$, $\sigma_B = +\infty$ otherwise.

Theorem 2.4. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ be the process of OU type associated with* (1.5), *and let $p_t(x, dy)$ be its transition probability.* (i) *If for some $x \in \mathbb{R}^d$, it holds*

$$R_0(x, B) < +\infty \qquad \textit{for every bounded subset } B \textit{ of } \mathbb{R}^d, \tag{2.12}$$

then (2.12) *holds for every* $x \in \mathbb{R}^d$ *and the process* $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ *is transient.*

(ii) *If for some* $x \in \mathbb{R}^d$ *and* $a \in \mathbb{R}^d$, *it holds*

$$R_0(x, U) = +\infty \quad \text{for every open subset } U \text{ containing } a, \tag{2.13}$$

then the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ *is recurrent.*

(iii) *Furthermore, if the process* $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ *is recurrent, then there is an* $a \in \mathbb{R}^d$ *such that* (2.13) *holds for every* $x \in \mathbb{R}^d$.

Proof. We first claim that for any Lipschitz continuous function f on $\mathbb{R}^d$ there is a constant $C > 0$ satisfying that

$$|R_\alpha f(x) - R_\alpha f(y)| \leqq C|x-y| \quad \text{for every } x, y \in \mathbb{R}^d \text{ and } \alpha > 0. \tag{2.14}$$

Denote by X_t and Y_t the solutions of the equation (1.5) associated with the common Q and A_t and with $X_0 = x$ and $Y_0 = y$. Then, clearly $X_t - Y_t = e^{-tQ}(x-y)$, so using (2.10) we easily obtain (2.14). The first part of (i) is obvious by (2.14). For the latter half of (i) we use Lemma 2.3. By the strong Markov property

$$R_0(x, B_r) \geqq P_x(\sigma_{B_r} < +\infty) \inf_{|y| \leqq r} R_0(y, B_r). \tag{2.15}$$

Hence from (2.7), (2.12) and (2.15) it follows that for any sufficient large $r > 0$

$$\begin{aligned} \lim_{t \to \infty} P_x(\sigma_{B_r}(\theta_t) < +\infty) &= \lim_{t \to \infty} \int_{\mathbb{R}^d} p_t(x, dy) P_y(\sigma_{B_r} < +\infty) \\ &\leqq \text{const.} \lim_{t \to \infty} \int_{\mathbb{R}^d} p_t(x, dy) R_0(y, B_r) \\ &= 0, \end{aligned} \tag{2.16}$$

which implies $P_x(\lim_{t \to \infty} |X_t| = +\infty) = 1$ for every $x \in \mathbb{R}^d$.

(ii): Let U be a bounded open set containing a, and $f \geqq 0$ be a C^2 function supported in U with $f(a) > 0$. By the strong Markow property

$$R_\alpha f(x) = E_x(e^{-\alpha\sigma_U} R_\alpha f(X_{\sigma_U})), \tag{2.17}$$

hence

$$\begin{aligned} (1 - E_x(e^{-\alpha\sigma_U})) R_\alpha f(x) &= E_x(e^{-\alpha\sigma_U}(R_\alpha f(X_{\sigma_U}) - R_\alpha f(x))) \\ &\leqq C \sup_{y \in U} |R_\alpha f(y) - R_\alpha f(x)|. \end{aligned} \tag{2.18}$$

Using (2.13) and (2.14), we have

$$P_x(\sigma_U < +\infty) = \lim_{\alpha \to 0} E_x(e^{-\alpha\sigma_U}) = 1 \quad \text{for every } x \in \mathbb{R}^d. \tag{2.19}$$

From this it is easy to see

$$P_x(\liminf_{t \to \infty} |X_t - a| = 0) = 1 \quad \text{for every } x \in \mathbb{R}^d, \tag{2.20}$$

thus we conclude the recurrence of the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$.

(iii): Suppose that the process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is recurrent. If the conclusion of (iii) fails, then for every $a\in\mathbb{R}^d$, there exists a bounded open set U containing a such that $R_0(x, U)<+\infty$ for some x, but by (2.14) it holds for all x. This implies the condition (2.13), so the process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is transient, completing the proof of (iii).

Next we show that the definition of recurrence in this paper is equivalent to the usual recurrence notion.

Theorem 2.5. *Suppose that the Markov process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is recurrent, and that for a point $b\in\mathbb{R}^d$,*

$$P_x(\sigma_V<+\infty)>0 \quad \textit{for every open set } V \textit{ containing } b \textit{ and every } x\in\mathbb{R}^d. \tag{2.21}$$

Then it holds that

$$P_x(\liminf_{t\to\infty}|X_t-b|=0)=1 \quad \textit{for every } x\in\mathbb{R}^d. \tag{2.22}$$

Proof. By Theorem 2.4 (iii) there is an a such that (2.13) holds. Let V be an open set containing b. By (2.21) it holds that for every x there is an $r>0$ such that

$$\int_0^r p_s(x, V)\,ds>0. \tag{2.23}$$

But using the Feller property of T_t one can see that for every bounded open set U containing a, there is an $r>0$ such that

$$\inf_{x\in U}\int_0^r p_s(x, V)\,ds>0. \tag{2.24}$$

Hence by (2.13) and (2.24)

$$\begin{aligned} rR_0(x, V) &\geqq \int_0^\infty du(u\wedge r)\,p_u(x, V)=\int_0^\infty\int_0^r p_{s+t}(x, V)\,ds\,dt \\ &=\inf_{y\in U}\int_0^r p_s(y, V)\,ds\,R_0(x, U) \\ &=+\infty \qquad \text{for every open set } V \text{ containing } b. \end{aligned} \tag{2.25}$$

Accordingly by Theorem 2.4 (ii) the proof of Theorem 2.5 is complete.

Finally we present an elementary lemma which will be often used.

Lemma 2.6. *Let g: $[0, 1]\to[0, +\infty)$ be a contionuous function. Suppose that (a) g is a C^2 function on $(0, 1)$ satisfying $g(0)=0$, (b) for some $c_1>0$ and $c_2>0$, $c_1\leqq g'(u)\leqq c_2$ for every $0<u<1$, (c) $ug''(u)$ is bounded in $0<u<1$. Then*

(i) $\sup_{1\geqq r\geqq 1/u}\left|\int_r^1 \frac{\cos ug(y)}{y}\,dy\right|$ *is bounded in $u\geqq 1$.*

(ii) *Let ρ be a measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} (1\wedge|u|)\,\rho(du)<+\infty$.*

Then $\int_r^1 \frac{dy}{y} \int_{\mathbb{R}} \rho(du)(e^{-|u|g(y)} - \cos ug(y))$ *is bounded in* $0<r<1$.

Proof. For (i), use inegral by parts to get

$$\left|\int_r^1 \frac{\cos ug(y)}{y} dy\right|$$
$$\leqq \left|\frac{\sin ug(1)}{ug'(1)} - \frac{\sin ug(r)}{urg'(r)}\right| + \int_r^1 \frac{|yg''(y)+g'(y)|}{u(yg'(y))^2} dy,$$

which is bounded in $u \geqq 1 \geqq r$ with $ru \geqq 1$ by the assumption.
(ii):

$$\sup_{0<r<1} \left|\int_r^1 \frac{dy}{y} \int_{\mathbb{R}} \rho(du)(e^{-|u|g(y)} - \cos ug(y))\right|$$
$$\leqq \int_{\mathbb{R}} \rho(du) \int_0^{1 \wedge (1/|u|)} \frac{dy}{y} |e^{-|u|g(y)} - \cos ug(y)|$$
$$+ \int_{|u| \geqq 1} \rho(du) \sup_{1 \geqq r \geqq 1/|u|} \left|\int_r^1 \frac{e^{-|u|g(y)} - \cos ug(y)}{y} dy\right|.$$

Use the inequality: $|e^{-|x|} - \cos x| \leqq 2|x|$ to show the finiteness of the first term. Moreover, using (i), one can easily see that the second term also is finite.

3. Proof of Theorem 1.1

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ be the one dimensional process of OU type associated with the equation (1.10). For simplicity we will henceforce assume $\gamma=1$ since other cases can be reduced to this case by changing the other parameters. Let us denote by $p_t(x, \cdot)$ the transition probability of $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$. Then by Lemma 2.1 the characteristic function of $p_t(x, \cdot)$ is

$$\hat{p}_t(x, z) = \exp\left(ixze^{-t} + \int_0^t \varphi(e^{-s}z)\, ds\right), \tag{3.1}$$

where $\varphi(z)$ is the one of (1.12).

We first prove Theorem 1.1 under the following situation:

$$\alpha = \rho(-\infty, 0) = 0, \quad \text{and} \quad \int_0^1 x\rho(dx) = c < +\infty. \tag{3.2}$$

Then the Lévy process A_t has no downward jump, so that the Markov process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is irreducible if we take $(v, +\infty)$ as its state space, where $v=b-c$. Note that in this case G of (2.1) turns to

$$Gf(x)=(v-x)f'(x)+\int_0^\infty (f(x+y)-f(x))\,\rho(dy). \tag{3.3}$$

For the associated process of OU type $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$, denote by σ_a the hitting time of $\{a\}$, i.e. $\sigma_a=\inf\{t\geqq 0\,|\,X_t=a\}$ if $\{\cdot\}\neq\phi$, and $\sigma_a=+\infty$ otherwise. We use the following formula hitting time distribution due to Hadjiev [2].

Theorem 3.1. ([2]) *Assume the condition* (3.2). *Then for every* $\lambda>0$ *and every* $x\geqq a>v$,

$$E_x(e^{-\lambda\sigma_a})=\frac{\int_0^\infty dz\, z^{\lambda-1}\exp\left((v-x)z+\int_1^z \frac{\check{\rho}(y)}{y}\,dy\right)}{\int_0^\infty dz\, z^{\lambda-1}\exp\left((v-a)z+\int_1^z \frac{\check{\rho}(y)}{y}\,dy\right)}, \tag{3.4}$$

where

$$\check{\rho}(y)=\int_0^\infty (1-e^{-yx})\,\rho(dx). \tag{3.5}$$

In order to make the paper selfcontained we here give a simple proof of Theorem 3.1 in the present context.

Lemma 3.2. *Let* $g_\lambda(z)=z^{\lambda-1}\exp\left(\int_1^z \frac{\check{\rho}(y)}{y}\,dy\right)$ *for* $\lambda>0$, *and denote its Laplace transform by* $f_\lambda(x)=\int_0^\infty e^{(v-x)z}g_\lambda(z)\,dz$ *for* $x>v$.

Then $f_\lambda(x)$ *is a decreasing* C^1*-function on* $(v, +\infty)$ *vanishing at infinity, and satisfies the following equation*:

$$(v-x)f'(x)+\int_0^\infty (f(x+y)-f(x))\,\rho(dy)=\lambda f(x) \quad \text{for } x>v. \tag{3.6}$$

Proof. By (3.2) it follows that for every $\varepsilon>0$ we have a constant $C_\varepsilon>0$ such that

$$\check{\rho}(y)\leqq C_\varepsilon+\varepsilon y. \tag{3.7}$$

Using this one can easily see that $f_\lambda(x)$ is well-defined. Next show that

$$z g_\lambda'(z)=(\lambda-1+\check{\rho}(z))\,g_\lambda(z), \qquad z>0, \tag{3.8}$$

which implies that $f_\lambda(x)$ is a solution of the equation (3.6).

Proof of Theorem 3.1. Apply the Dynkin formula to get

$$E_x(e^{-\lambda(\sigma_a\wedge t)}f_\lambda(X_{\sigma_a\wedge t}))=f_\lambda(x)+E_x\left(\int_0^{\sigma_a\wedge t} e^{-\lambda s}(Gf_\lambda-\lambda f_\lambda)(X_s)\,ds\right)$$

$$=f_\lambda(x) \quad \text{for } x\geqq a, \tag{3.9}$$

here we used Lemma 3.2 and (3.3). Note that if $x>a$, then $X_t>a$ and $f_\lambda(X_t)\leqq f_\lambda(a)$ for any $t<\sigma_a$, because X_t has no downward jump. Letting $t\to\infty$ in (3.9), we obtain

$$E_x(e^{-\lambda\sigma_a})=\frac{f_\lambda(x)}{f_\lambda(a)} \quad \text{for } x\geqq a,$$

which yields (3.4).

Theorem 3.3. *Consider the process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ under the condition* (3.2). *Then the process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is recurrent or transient according as*

$$\int_0^1 dz\frac{1}{z}\exp\left(-\int_z^1\frac{\check{\rho}(y)}{y}dy\right)=+\infty \quad \textit{or} \quad <+\infty. \tag{3.10}$$

where $\check{\rho}(y)$ is the one defined in (3.5).

Proof. Let $a>v$. Using (3.7) one can easily see that

$$\int_1^\infty\frac{dz}{z}\exp\left((v-x)z+\int_1^z\frac{\check{\rho}(y)}{y}dy\right)<+\infty \qquad \text{for every } x\geqq a. \tag{3.11}$$

Furthermore, if the integral of (3.10) is divergent,

$$\lim_{\lambda\downarrow 0}\frac{\int_0^1 dz\,z^{\lambda-1}\exp\left((v-x)z+\int_1^z\frac{\check{\rho}(y)}{y}dy\right)}{\int_0^1 dz\,z^{\lambda-1}\exp\left((v-a)z+\int_1^z\frac{\check{\rho}(y)}{y}dy\right)}=1. \tag{3.12}$$

Hence by (3.4) we have that for every $a>v$

$$P_x(\sigma_a<+\infty)=\lim_{\lambda\downarrow 0}E_x(e^{-\lambda\sigma_a})=1 \qquad \text{for every } x\geqq a, \tag{3.13}$$

which implies

$$P_x(\liminf_{t\to\infty}|X_t-v|=0)=1 \qquad \text{for every } x\in\mathbb{R}. \tag{3.14}$$

Thus we see the recurrence of the Markov process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$.

Conversely if the integral of (3.10) is convergent, by (3.4)

$$P_x(\sigma_a<+\infty)=\frac{\int_0^\infty dz\, z^{-1}\exp\left((v-x)z+\int_1^z \frac{\check{\rho}(y)}{y}\,dy\right)}{\int_0^\infty dz\, z^{-1}\exp\left((v-a)z+\int_1^z \frac{\check{\rho}(y)}{y}\,dy\right)}<1 \tag{3.15}$$

if $x>a$, and $\lim_{x\to\infty} P_x(\sigma_a<+\infty)=0$.

Here note that (3.15) implies

$$\int_{|x|\geqq 1} \log|x|\,\rho(dx)=+\infty,$$

and by Theorem 1.0,

$$\lim_{t\to\infty} P_x(|X_t|\leqq b)=0 \qquad \text{for every } b>0 \text{ and } x\in\mathbb{R}. \tag{3.16}$$

Combining this with (3.15) we get

$$\begin{aligned} P_x(\liminf_{t\to\infty}|X_t|<+\infty) &= \lim_{a\to\infty}\lim_{t\to\infty} P_x(\sigma_a(\theta_t)<+\infty)\\ &= \lim_{a\to\infty}\lim_{t\to\infty} E_x(P_{X_t}(\sigma_a<+\infty))\\ &= 0 \qquad \text{for every } x\in\mathbb{R}. \end{aligned} \tag{3.17}$$

Therefore the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is transient.

Now let us return to the general case. Recall that the process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is governed by the infinitesimal generator G of (2.1) with $\gamma=1$. We first prove transience when the integral of (1.8) with $\gamma=1$ is convergent.

Lemma 3.4. *Suppose the following condition:*

$$\int_0^1 dz\frac{1}{z}\exp\left(-\int_z^1 \frac{\lambda_\rho(y)}{y}\,dy\right)<+\infty, \tag{3.18}$$

where

$$\lambda_\rho(y)=\int_{|x|\geqq 1}(1-e^{-y|x|})\,\rho(dx).$$

Then the process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is transient.

Proof. For $a>0$, let us define

$$h_a(x)=a-|x| \quad \text{if } |x|\leqq a, \quad \text{and} \quad h_a(x)=0 \quad \text{otherwise}.$$

For the proof of transience, by Theorem 2.4 (i), it suffices to show that

$$\int_0^\infty dt\, T_t h_a(x)<+\infty \qquad \text{for every } a>0 \quad \text{and} \quad x\in\mathbb{R}. \tag{3.19}$$

It is easy to calculate its Forier transform

$$\hat{h}_a(z)=\frac{2(1-\cos az)}{z^2}.$$

From the Parseval theorem and (3.1) it follows that

$$\begin{aligned} T_t h_a(x) &= \frac{1}{2\pi}\int \hat{p}_t(x,z)\,\hat{h}_a(z)\,dz \\ &\leqq \frac{1}{2\pi}\int |\hat{p}_t(x,z)|\,\hat{h}_a(z)\,dz \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \exp\left(\int_0^t \operatorname{Re}\varphi(ze^{-s})\,ds\right)\frac{1-\cos az}{z^2}\,dz, \end{aligned}$$

hence we have

$$\begin{aligned} \int_0 T_t h_a(x)\,dt &\leqq \frac{1}{\pi}\int_{-\infty}^{\infty} dz\left(\int_0^{\infty} dt\exp\left(\int_0^t \operatorname{Re}\varphi(ze^{-s})\,ds\right)\right)\frac{1-\cos az}{z^2} \\ &= \frac{2}{\pi}\int_0^{\infty} dz\left(\int_0^z dr\frac{1}{r}\exp\left(\int_r^z \frac{\operatorname{Re}\varphi(y)}{y}\,dy\right)\right)\frac{1-\cos az}{z^2} \\ &\leqq \frac{2}{\pi}\int_0^{\infty} dz\left(\int_0^z dr\frac{1}{r}\exp\left(-\int_r^z \frac{\hat{\rho}^*(y)}{y}\,dy\right)\right)\frac{1-\cos az}{z^2}, \end{aligned} \tag{3.20}$$

where

$$\hat{\rho}^*(y)=\int_{-\infty}^{\infty}(1-\cos yx)\,\rho(dx). \tag{3.21}$$

Assuming that

$$\int_0^1 dr\frac{1}{r}\exp\left(-\int_r^1 \frac{\hat{\rho}^*(y)}{y}\,dy\right)<+\infty, \tag{3.22}$$

one can check that

$$\int_0^1 dz\left(\int_0^z dr\frac{1}{r}\exp\left(-\int_r^z \frac{\hat{\rho}^*(y)}{y}\,dy\right)\right)\frac{1-\cos az}{z^2}<+\infty, \tag{3.23}$$

and

$$\int_1^{\infty} dz\left(\int_0^1 dr\frac{1}{r}\exp\left(-\int_r^z \frac{\hat{\rho}^*(y)}{y}\,dy\right)\right)\frac{1-\cos az}{z^2}<+\infty. \tag{3.24}$$

Also, obviously

$$\int_1^{\infty} dz\left(\int_1^z dr\frac{1}{r}\exp\left(-\int_r^z \frac{\hat{\rho}^*(y)}{y}\,dy\right)\right)\frac{1-\cos az}{z^2}<+\infty. \tag{3.25}$$

Hence (3.19) follows from (3.20) and (3.23) to (3.25) under the condition (3.22). However it is easy to see that (3.22) is equivalent to the condition (3.18) since it holds by a modification of Lemma 2.6 that

$$\left|\int_r^1 \frac{\hat{\rho}^*(y)-\lambda_\rho(y)}{y}\,dy\right| \quad \text{is bounded in } 0<r\leqq 1. \tag{3.26}$$

Therefore the proof of Lemma 3.4 is complete.

For the proof of recurrence we use the Lévy-Ito decomposition of Lévy processes, that is, an arbitrary Lévy process A_t is constructed by means of a Poisson point process and a standard Brownian motion as follows:

$$A_t=\sqrt{\alpha}\,B_t+bt+\int_0^{t+}\int_{|y|\geqq 1} yN_p(ds\,dy)+\int_0^{t+}\int_{|y|<1} y\tilde{N}_p(ds\,dy), \tag{3.27}$$

where N_p is a Poisson point process on $\mathbb{R}\setminus\{0\}$ with characteristic measure ρ, $\tilde{N}_p(ds\,dy)=N_p(ds\,dy)-ds\,\rho(dy)$, and B_t is a standard Brownian motion independent of N_p.

Let

$$A_t^1=\int_0^{t+}\int_{|y|\geqq 1} yN_p(ds\,dy), \tag{3.28}$$

and consider the process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x^1,X_t^1)$ governed by the following equation:

$$X_t^1=x-\int_0^t X_s^1\,ds+A_t^1. \tag{3.29}$$

Lemma 3.5. *Suppose that*

$$\int_0^1 dz\,\frac{1}{z}\exp\left(-\int_z^1 \frac{\lambda_\rho(y)}{y}\,dy\right)=+\infty. \tag{3.30}$$

Then the process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x^1,X_t^1)$ is recurrent.

Proof. It is easy to see that

$$|X_t^1|=|x|-\int_0^t |X_s^1|\,ds+\int_0^{t+}\int_{|y|\geqq 1} (|X_{s-}^1+y|-|X_{s-}^1|)N_p(ds\,dy). \tag{3.31}$$

Let Y_t be the solution of the equation

$$Y_t=|x|-\int_0^t Y_s\,ds+\bar{A}_t, \tag{3.32}$$

where

$$\bar{A}_t=\int_0^{t+}\int_{|y|\geqq 1} |y|\,N_p(ds\,dy).$$

Noting that

$$\int_{|y|\geqq 1} (|X^1_{s-}+y|-|X^1_{s-}|)\, N_p(ds\, dp) \leqq d\bar{A}_s \qquad \text{for every } s\geqq 0,$$

by (3.31) and (3.32) one can easily see that

$$|X^1_t| \leqq Y_t \qquad \text{for every } t\geqq 0, \quad \text{a.s.} \tag{3.33}$$

Since the Levy measure $\bar{\rho}$ of the Levy process $\bar{A}_t$ is given by $\bar{\rho}(E)=\rho((E\cup(-E))\cap[-1, 1]^c)$ for each E contained in $\mathbb{R}_+$, the integral (3.10) is divergent for $\bar{\rho}$. Accordingly, Y_t is recurrent, and by Theorem 2.5 we see

$$P_x(\liminf_{t\to\infty} |X^1_t|=0) \geqq P_x(\liminf_{t\to\infty} Y_t=0)=1 \qquad \text{for every } x,$$

which yields the recurrence of X^1_t.

Now we are in position to complete the proof of Theorem 1.1. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P^2_x, X^2_t)$ be the process of OU type associated with the following equation:

$$X^2_t = x - \int_0^t X^2_s\, ds + A_t - A^1_t. \tag{3.34}$$

Noting that $A_t - A^1_t$ is a Lévy process and its Lévy measure is the restriction of ρ on $[-1, 1]\setminus\{0\}$, we have by Theorem 1.0 that the distribution of X^2_t converges to a limit distribution as $t\to\infty$.

Let $p^1_t(x,dy)$ and $p^2_t(x,dy)$ be the transition probabilities of the Markov processes associated with X^1_t and X^2_t respectively. Since X^1_t and X^2_t are mutually independent, and $X_t = X^1_t + X^2_t$ is a solution of the equation

$$X_t = x_1 + x_2 - \int_0^t X_s\, ds + A_t, \tag{3.35}$$

X_t is equivalent to the Markov process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ starting at x_1+x_2, hence it follows that

$$p_t(x_1+x_2, (a, b)) = \int_{y+z\in(a,b)} p^1_t(x_1, dy)\, p^2_t(x_2, dz), \qquad a<b. \tag{3.36}$$

Since X^1_t is recurrent, by Theorem 2.4, there is a $c\in\mathbb{R}$ such that

$$\int_0^\infty p^1_t(x, (c-\varepsilon, c+\varepsilon))\, dt = +\infty \qquad \text{for every } \varepsilon>0 \text{ and } x\in\mathbb{R}. \tag{3.37}$$

By Theorem 1.0, $p_t^2(x, \cdot)$ converges to a distribution as $t \to \infty$, hence from this and (3.37) that there exists an $a \in \mathbb{R}$ such that

$$\int_0^\infty p_t(x, (a-\varepsilon, a+\varepsilon))\, dt = +\infty \quad \text{for every } \varepsilon > 0 \quad \text{and} \quad x \in \mathbb{R}. \tag{3.38}$$

Accordingly, by Theorem 2.4 (ii), the Markov process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is recurrent.

4. Multi-dimensional case

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ be the standard Markov process of OU type on $\mathbb{R}^d$ $(d \geqq 2)$ associated with a $Q \in M_+(\mathbb{R}^d)$ and the equation

$$X_t = x - \int_0^t Q X_s\, ds + A_t, \tag{1.5}$$

where A_t is a Lévy process on $\mathbb{R}^d$ whose characteristic function is given by (1.2) to (1.4).

In this section we discuss the recurrence problem of the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ on $\mathbb{R}^d$ under some restrictive situations. We obtain the following results.

Theorem 4.1. *Suppose that*

$$Q = (Q_{jk})_{1 \leqq i,j \leqq d} \quad \textit{with} \quad Q_{jk} = \gamma \delta_{jk} \tag{4.1}$$

for some $\gamma > 0$, and the Lévy measure ρ of A_t is symmetric, i.e. $\rho(-E) = \rho(E)$ for every $E \in \mathscr{B}(\mathbb{R}^d)$, where δ_{jk} stands for the Kronecker symbol. Then the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is recurrent if and only if for all $z \in \mathbb{R}^d$, $\langle X_t, z\rangle$ is a one dimensional recurrent process of OU type, which is equivalent to

$$\int_0^1 \frac{dr}{r} \exp\left(-\int_r^1 \frac{\lambda_\rho^z(y)}{\gamma y}\, dy\right) = +\infty \quad \textit{for all } z \in \mathbb{R}^d \tag{4.2}$$

$$\lambda_\rho^z(y) = \int_{|\langle x,z\rangle| \geqq 1} (1 - e^{-|\langle x,z\rangle y|})\, \rho(dx) \quad \textit{for } z \in \mathbb{R}^d. \tag{4.3}$$

Theorem 4.2. *Suppose that the Markov process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is a direct product process of one dimensional processes of OU type $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x^j, X_t^j)$ $(1 \leqq j \leqq d)$ such that each Lévy measure ρ_j of the associated Levy process A_t^j is symmetric.*

Then the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is recurrent or transient according as

$$\int_0^1 \frac{dr}{r} \exp\left(-\int_r^1 \sum_{j=1}^d \frac{\lambda_{\rho_j}(y)}{\gamma_j y}\, dy\right) = +\infty \quad \text{or} \quad < +\infty. \tag{4.4}$$

Theorem 4.3. *Suppose that*

$$Q=(Q_{jk})_{1\leqq i,j\leqq d}\in M_+(\mathbb{R}^d) \tag{4.5}$$

is symmetric, and the Lévy measure ρ of A_t is rotation invariant, i.e. $\rho(O(E))=\rho(E)$ for every $E\in\mathscr{B}(\mathbb{R}^d)$ and every orthogonal transformation $O\colon \mathbb{R}^d\to\mathbb{R}^d$.

Then the process $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is recurrent or transient according as

$$\int_0^1 \frac{dr}{r}\exp\left(-\int_r^1 \frac{\lambda_\rho^1(y)}{\gamma y}\,dy\right)=+\infty \quad \text{or} \quad <+\infty, \tag{4.6}$$

where γ is the minimum eigenvalue of Q, and $\lambda_\rho^1(y)$ is the one defined by (4.3) with $z=(1,0,\ldots,0)$.

We first prepare the following lemma which makes the situation simpler in the proofs of Theorems 4.1 to 4.3.

Lemma 4.4. *Let $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ and $(\Omega,\mathscr{F},\mathscr{F}_t,P_y,Y_t)$ be two processes of OU type on $\mathbb{R}^d$ associated with a common $Q\in M_+(\mathbb{R}^d)$ and Lévy processes A_t and A_t' respectively. Suppose that A_t' is of pure jump type whose Lévy measure ρ' is the restriction of ρ (the Lévy measure of A_t) onto $\{x\in\mathbb{R}^d\,|\,|x|\geqq 1\}$, i.e. $\rho'(E)=\rho(E\cap(|x|>1))$ for every $E\in\mathscr{B}(\mathbb{R}^d)$. Then*

(i) *if (Y_t,P_y) is recurrent, (X_t,P_x) also is recurrent.*
(ii) *Conversely, under an additional assumption that ρ is symmetric, if (Y_t,P_y) is transient, then (X_t,P_x) also is transient.*

Proof. (i) can be shown by the same argument as the final stage in the proof of Theorem 1.1. For (ii), let $p_t(x,dy)$ and $p_t'(x,dy)$ be the transition probabilities of (X_t,P_x) and (Y_t,P_y) respectively. By the symmetry of ρ and (2.4) it is clear that

$$|\hat{p}_t(x,z)|\leqq\hat{p}_t'(0,z) \quad \text{for every } x \text{ and } z\in\mathbb{R}^d. \tag{4.7}$$

Recall that for $a>0$,

$$h_a(u)=a-|u| \quad \text{for } |u|\leqq a, \quad \text{and} \quad h_a(u)=0 \quad \text{otherwise,}$$

and define a function $k_a(x)$ on $\mathbb{R}^d$ by

$$k_a(x)=\prod_{j=1}^d h_a(x_j) \qquad \text{for } x=(x_j)_{1\leqq j\leqq d}\in\mathbb{R}^d.$$

Then

$$\hat{k}_a(z)=\prod_{j=1}^d \hat{h}_a(z_j)=2^d\prod_{j=1}^d\frac{1-\cos az}{z_j^2}. \tag{4.8}$$

Hence by the transience of (Y_t,P_y) and (4.7) together with Parseval theorem on Fourier transform we have

$$\int_0^\infty\Big(\int_{\mathbb{R}^d} p_t(x,dy)\,k_a(y)\Big)\,dt\leqq\int_0^\infty\Big(\int_{\mathbb{R}^d} p_t'(0,dy)\,k_a(y)\Big)\,dt<+\infty \qquad \text{for all } a>0,$$

which verifies the assumption of Theorem 2.4. Therefore the process (X_t, P_x) is transient.

Proof of Theorem 4.1. Supposing first that the Markov process $(\Omega, \mathscr{F}, \mathscr{F}_r, P_x, X_t)$ is transient we will show the transience of a one dimensional projection process $\langle X_t, z\rangle$ for a.e. $z\in\mathbb{R}^d$. By Lemma 4.4 we may assume that A_t is a pure jump Lévy process with Levy measure ρ satisfying $\rho(|x|\leqq 1)=0$. By (4.1) and Lemma 2.1,

$$\hat{p}_t(0,z)=\exp\left(-\int_0^t \int_{\mathbb{R}^d} (1-\cos e^{-\gamma s}\langle z,x\rangle)\,\rho(dx)\,ds\right), \tag{4.9}$$

hence

$$T_t k_a(0)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} dz\,\hat{k}_a(z)\exp\left(-\int_0^t \int_{\mathbb{R}^d} (1-\cos e^{-\gamma s}\langle z,x\rangle)\,\rho(dx)\,ds\right). \tag{4.10}$$

Since the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is transient, by Theorem 2.4 $\int_0^\infty T_t k_a(0)\,dt < +\infty$, so that we have

$$\begin{aligned}
&\int_0^\infty T_t k_a(0)\,dt\\
&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} dz\,\hat{k}_a(z)\int_0^1 \frac{dr}{\gamma r}\exp\left(-\int_r^1 \frac{dy}{\gamma y}\int_{\mathbb{R}^d} (1-\cos y\langle z,x\rangle)\,\rho(dx)\right)\\
&<+\infty
\end{aligned} \tag{4.11}$$

For $z\in\mathbb{R}^d\backslash\{0\}$, define a bounded measure $\rho_z(du)$ on $\mathbb{R}$ by

$$\int_{\mathbb{R}} \rho_z(du) f(u)=\int_{\mathbb{R}^d} \rho(dx) f(\langle z,x\rangle) \quad \text{for every function } f \text{ defined on } \mathbb{R} \tag{4.12}$$

Then from (4.11) and $\hat{k}_a(z)=\prod_{j=1}^d \hat{h}_a(z_j)>0$ for a.e. $z\in\mathbb{R}^d$, it follows that

$$\int_0^1 \frac{dr}{r}\exp\left(-\int_r^1 \frac{dy}{\gamma y}\int_{\mathbb{R}} (1-\cos yu)\,\rho_z(du)\right)<+\infty \qquad \text{for a.e. } z\in\mathbb{R}^d, \tag{4.13}$$

which, by (3.26), is equivalent to

$$\int_0^1 \frac{dr}{r}\exp\left(-\int_r^1 \frac{dy}{\gamma y}\int_{|u|\geqq 1} (1-e^{-y|u|})\,\rho_z(du)\,dy\right)<+\infty \qquad \text{for a.e. } z. \tag{4.14}$$

Note that for every $z\in\mathbb{R}^d\backslash\{0\}$, $\langle X_t, z\rangle$ is a one dimensional process of OU type associated with γ and the Levy process $\langle A_t, z\rangle$ having the Lévy measure ρ_z. Accordingly, by virtue of Theorem 1.1 and (4.14), $\langle X_t, z\rangle$ is a transient process of OU type for a.e. $z\in\mathbb{R}^d$.

Conversely, if the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is recurrent, then clearly $\langle X_t, z\rangle$ also is recurrent for every $z \in R^d$, which yields (4.2) together with Theorem 1.1. Thus we complete the proof of Theorem 4.1.

Proof of Theorem 4.2. By Lemma 4.4 we may assume $\rho_j([-1, 1])=0$ for all $1 \leqq j \leqq d$. Suppose that the integral of (4.4) is convergent. Let us denote

$$\lambda_j^*(y) = \int_{|u| \geqq 1} (1-\cos yu)\, \rho_j(du) \qquad (1 \leqq j \leqq d). \tag{4.15}$$

By Lemma 2.1,

$$\begin{aligned}
\hat{p}_t(0, z) &= \exp\left(-\sum_{j=1}^{d} \int_0^t \left(\int_{\mathbb{R}} (1-\cos e^{-\gamma_j s} z_j u)\, \rho_j(du)\right) ds\right)\\
&= \exp\left(-\sum_{j=1}^{d} \int_{\mathbb{R}} \left(\int_{\exp(-\gamma_j t)}^{1} \frac{dy}{\gamma_j y} (1-\cos y z_j u)\right) \rho_j(du)\right),\\
&\leqq \exp\left(-\int_{\exp(-t\gamma)}^{1} \sum_{j=1}^{d} \frac{\lambda_j^*(y z_j)}{\gamma_j y}\, dy\right),
\end{aligned} \tag{4.16}$$

where $\gamma = \min_{1 \leqq j \leqq d} \gamma_j$.

Recall the functions $k_a(x)$ and $\hat{k}_a(z)$ in (4.8). By the Parseval theorem and (4.16),

$$\begin{aligned}
&\int_0^\infty T_t k_a(x)\, dt\\
&= \frac{1}{(2\pi)^d} \int_0^\infty \int_{\mathbb{R}^d} \hat{p}_t(x, z)\, \hat{k}_a(z)\, dz\, dt\\
&\leqq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{k}_a(z)\, dz \int_0^\infty dt \exp\left(-\int_{\exp(-t\gamma)}^{1} \sum_{j=1}^{d} \frac{\lambda_j^*(y z_j)}{\gamma_j y}\, dy\right)\\
&= \frac{1}{\gamma(2\pi)^d} \int_{\mathbb{R}^d} \hat{k}_a(z)\, dz \int_0^1 \frac{dr}{r} \exp\left(-\int_r^1 \sum_{j=1}^{d} \frac{\lambda_j^*(y z_j)}{\gamma_j y}\, dy\right).
\end{aligned} \tag{4.17}$$

Note by modifying Lemma 2.6 that

$$\int_r^1 \frac{\lambda_j^*(yz) - \lambda_{\rho_j}(y|z|)}{y}\, dy \quad \text{is bounded in } (r, z) \in (0, 1) \times \mathbb{R}. \tag{4.18}$$

Since $\lambda_{\rho_j}(y)$ is increasing in $y>0$, we see

$$\int_r^1 \frac{\lambda_{\rho_j}(y|z_j|)}{y}\, dy \geqq \int_r^1 \frac{\lambda_{\rho_j}(y)}{y}\, dy - \int_{|z_j| \wedge 1}^{1} \frac{\lambda_{\rho_j}(y)}{y}\, dy. \tag{4.19}$$

From (4.17), (4.18) and (4.19) it follows that

$$\int_0^\infty T_t k_a(x)\,dt \leqq \text{const.} \int_{\mathbb{R}^d} \hat{k}_a(z)\,dz \exp\left(\sum_{j=1}^d \int_{|z_j|\wedge 1}^1 \frac{\lambda_{\rho_j}(y)}{\gamma_j y}\,dy\right) \cdot \int_0^1 \frac{dr}{r} \exp\left(-\int_r^1 \sum_{j=1}^d \frac{\lambda_{\rho_j}(y)}{\gamma_j y}\,dy\right). \tag{4.20}$$

Noting that $\lambda_{\rho_j}(y) \to 0$ as $y \to 0$, we see easily

$$\int_{\mathbb{R}} \hat{h}_a(z_j) \exp\left(\int_{|z_j|\wedge 1}^1 \frac{\lambda_{\rho_j}(y)}{\gamma_j y}\,dy\right) < +\infty, \qquad (1 \leqq j \leqq d). \tag{4.21}$$

Thus we obtain

$$\int_0^\infty T_t k_a(x)\,dt < +\infty \qquad \text{for every } a>0 \quad \text{and} \quad x \in \mathbb{R}^d, \tag{4.22}$$

hence, by Theorem 2.4, the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is transient.

Conversely, suppose that the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is transient. Recalling that $\rho_j([-1, 1])=0$ for all $1 \leqq j \leqq d$, by Theorem 2.4, Lemma 2.1, and (4.15) we have

$$\int_0^\infty T_t k_a(0)\,dt = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{k}_a(z)\,dz \int_0^\infty dt \exp\left(-\int_0^t \sum_{j=1}^d \lambda_j^*(e^{-s\gamma_j} z_j)\,ds\right) < +\infty, \tag{4.23}$$

so that

$$\int_0^\infty dt \exp\left(-\int_0^t \sum_{j=1}^d \lambda_j^*(e^{-s\gamma_j} z_j)\,ds\right) < +\infty \qquad \text{for a.e. } z \in \mathbb{R}^d. \tag{4.24}$$

By the same changes of variables as (4.17), (4.24) turns to

$$\int_0^1 \frac{dr}{r} \exp\left(-\int_r^1 \sum_{j=1}^d \frac{\lambda_j^*(y z_j)}{\gamma_j y}\,dy\right) < +\infty \qquad \text{for a.e. } z \in \mathbb{R}^d. \tag{4.25}$$

Using (4.18) we have

$$\int_0^1 \frac{dr}{r} \exp\left(-\int_r^1 \sum_{j=1}^d \frac{\lambda_{\rho_j}(y z_j)}{\gamma_j y}\,dy\right) < +\infty \qquad \text{for a.e. } z \in \mathbb{R}^d. \tag{4.26}$$

Fix a $z=(z_j)_{1 \leqq j \leqq d}$ such that $z_j>0$ for all $1 \leqq j \leqq d$, and that (4.26) holds for z. Then it is easy to check that (4.26) is equivalent to the convergence of the integral (4.4). Therefore we have completed the proof of Theorem 4.2.

Proof of Theorem 4.3. By Lemma 4.4 we may assume that A_t is of pure jump type and the Lévy measure ρ satisfies $\rho(|x|\leqq 1)=0$. We can further assume that Q is diagonal; $Q_{jk}=\gamma_j\delta_{jk}$ with $\gamma_j>0$ $(1\leqq j\leqq d)$ by the symmetry of $Q\in M_+(\mathbb{R}^d)$ and the rotation invariance of ρ.

Suppose that the process of OU type $(\Omega,\mathscr{F},\mathscr{F}_t,P_x,X_t)$ is transient. Then by Theorem 2.4

$$\begin{aligned}
&\int_0^\infty dt\, T_t k_a(0)\\
&=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\hat{k}_a(z)\,dz\int_0^\infty dt\exp\left(-\int_0^t\int_{|x|\geqq 1}\left(1-\cos\sum_{j=1}^d e^{-\gamma_j s}z_j x_j\right)\rho(dx)\,ds\right)\\
&<+\infty,
\end{aligned}\tag{4.27}$$

from which it follows that for a.e. $z\in\mathbb{R}^d$

$$\int_0^\infty dt\exp\left(-\int_0^t\int_{|x|\geqq 1}\left(1-\cos\sum_{j=1}^d e^{-\gamma_j s}z_j x_j\right)\rho(dx)\,ds\right)<+\infty.\tag{4.28}$$

Recalling $\gamma=\min\limits_{1\leqq j\leqq d}\gamma_j$, use change of variables: $e^{-\gamma s}=y$ together with the rotation invariance of ρ. Then (4.28) turns to

$$\begin{aligned}
&\int_0^1\frac{dr}{r}\exp\left(-\int_r^1\frac{dy}{\gamma y}\int_{|x|\geqq 1}\left(1-\cos\sum_{j=1}^d y^{\alpha_j}z_j x_j\right)\rho(dx)\right)\\
&=\int_0^1\frac{dr}{r}\exp\left(-\int_r^1\frac{dy}{\gamma y}\int_{|x|\geqq 1}(1-\cos g^z(y)\,x_1)\,\rho(dx)\right)\\
&=\int_0^1\frac{dr}{r}\exp\left(-\int_r^1\frac{dy}{\gamma y}\int_{|x|\geqq 1}(1-\cos g^z(y)u)\,\rho_1(du)\right)\\
&<+\infty\qquad\text{for a.e. } z\in\mathbb{R}^d,
\end{aligned}\tag{4.29}$$

where $\alpha_j=\gamma_j/\gamma\geqq 1$, ρ_1 is a bounded measure on $\mathbb{R}$ defined by

$$\int_{\mathbb{R}^d} f(x_1)\,\rho(dx)=\int_{\mathbb{R}} f(u)\,\rho_1(du)\qquad\text{for every function } f \text{ on } \mathbb{R},$$

and

$$g^z(y)=\left(\sum_{j=1}^d y^{2\alpha_j}z_j^2\right)^{1/2}\qquad\text{for } z\in\mathbb{R}^d.\tag{4.30}$$

Taking a $z=(z_j)$ satisfying that $z_j>0$ $(1\leqq j\leqq d)$ and (4.29) holds for z, we can apply Lemma 2.6 for $g^z(y)$. Hence,

$$\int_0^1\frac{dr}{r}\exp\left(-\int_r^1\frac{dy}{\gamma y}\int_{\mathbb{R}}(1-e^{-|u|g^z(y)})\,\rho_1(du)\right)<+\infty.\tag{4.31}$$

Noting that ρ_1 is a bounded measure on $\mathbb{R}$, and that for some $c>0$, $g^z(y)\leqq cy$ holds for every $0<y\leqq 1$, we obtain

$$\int_0^1 \frac{dr}{r}\exp\left(-\int_r^1 \frac{dy}{\gamma y}\int_{\mathbb{R}} (1-e^{-c|u|y})\,\rho_1(du)\right)<+\infty, \tag{4.32}$$

which is equivalent to the convergence of the integral (4.6).

Conversely, if the integral (4.6) is convergent, then for any unit eigenvector e of Q associated with the eigenvalue γ, $\langle X_t, e\rangle$ is a one dimensional process of OU type associated with γ and the Lévy process $\langle A_t, e\rangle$ having the Lévy measure ρ_1. Therefore, by virtue of Theorem 1.1, we see

$$P_x(\lim_{t\to\infty} |\langle X_t, e\rangle| = +\infty) = 1.$$

This implies that the process $(\Omega, \mathscr{F}, \mathscr{F}_t, P_x, X_t)$ is transient, completing the proof of Theorem 4.3.

Finally we discuss an interesting example which is found in [5]. For $c>0$, let ρ^c be a bounded measure on $\mathbb{R}$ defined by

$$\begin{aligned} \rho^c(du) &= \frac{c}{u(\log u)^2}\,du \qquad \text{for } |u|\geqq 2,\\ &=0 \qquad\qquad\qquad\ \ \text{otherwise.}\end{aligned}$$

Let $(X_t^{(c)}, P_x)$ be a one dimensional process of OU type on $\mathbb{R}$ associated with $\gamma>0$ and a pure jump Lévy process A_t with the Lévy measure ρ^c. Then it is known in [5] that

(i) if $2c\leqq\gamma$, then the process is null recurrent, but
(ii) if $2c>\gamma$, then the process is transient.

Of course, we can show this by checking the condition (1.16) in our Theorem 1.1.

We next consider a direct product process $(\mathbb{X}_t = (X_t^1, X_t^2, \ldots, X_t^d), P_{\mathbb{X}})$ such that each component process X_t^j is a one dimensional process of OU type whose probability law is the same as $(X_t^{(c)}, P_x)$.

Then by Theorem 4.2 the d-dimensional process $(\mathbb{X}_t, P_{\mathbb{X}})$ is transient if and only if $2cd>\gamma$. Furthermore, noting that every one dimensional projection process is of OU type, we see by Theorem 1.1 that for $z\in\mathbb{R}^d\setminus\{0\}$ $P_{\mathbb{X}}(\lim_{t\to\infty} |\langle \mathbb{X}_t, z\rangle| = +\infty) = 1$ holds for every $x\in\mathbb{R}^d$ if and only if $\#\{1\leqq j\leqq d \mid z_j\neq 0\} > \gamma/2c$.

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