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# Brownian motion on the continuum tree

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**Summary.** We construct Brownian motion on a continuum tree, a structure introduced as an asymptotic limit to certain families of finite trees. We approximate the Dirichlet form of Brownian motion on the continuum tree by adjoining one-dimensional Brownian excursions. We study the local times of the resulting diffusion. Using time-change methods, we find explicit expressions for certain hitting probabilities and the mean occupation density of the process.

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## 0 Introduction

Over the last several years, various authors have extended the theory of continuous-time Markov processes to inherently rough sets. Barlow and Perkins [8], Lindstrom [11], Hambly [10], and Barlow and Bass [5–7] have defined Brownian motions on various classes of fractal sets. A characteristic of all these works is that the Brownian motions in question must be defined through approximations by simpler processes, generally random walks on approximating lattices. As a consequence, it is difficult to describe functionals of the resulting Brownian motion explicitly.

In this paper, we approach constructing such Brownian motions from a different angle. Our underlying state space is a *continuum tree*; think of this a limit at infinity of a finite, graph-theoretical tree of segements in Euclidean space. A natural approach to constructing a Brownian motion on such a state space is to start from Brownian motion on the approximating finite trees, then let the finite trees converge to the continuum limit while suitably rescaling time. It turns out that this can be done rigorously through potential theoretic methods. The details comprise the bulk of this paper.

A standard property of graph-theoretic trees is that any two points $x, y$ are joined by a unique path $[\![x, y]\!]$, a property common to the real line. This property allows many standard calculations for linear Brownian motion to be transferred to Brownian motion in the tree setting. As a result, the hitting distribution and the occupation density for Brownian motion on the continuum tree may be described fairly explicitly. Standard results on Brownian local times also carry over directly to the tree.

Brownian motion on the continuum tree was proposed in Aldous [3]; a rigorous construction was not provided, although an intuitive argument for the existence of the process was sketched. This paper formalizes that argument carefully. Besides clarifying Aldous' argument, we feel that the argument presented here has a couple of other points of interest.

(i) Certain fractal subsets of Euclidean space have natural approximations by trees. The construction given here (with a slight modification) will define Brownian motion on such spaces. Many of these fractals do not admit easy approximating lattices.
(ii) In Aldous [2–4] models for random continuum trees are developed, together with applications to family trees of branching processes and random combinatorial structures. Brownian motions can be constructed on realizations of these random trees. These may be useful for studying random walk on finite or discrete random trees.
(iii) In [3, Sect. 6.3.1], Aldous remarks that local time at the root for Brownian motion on the continuum tree can be recovered from the superprocess constructed over the Ray–Knight diffusion. Other interesting relationships might exist between superprocesses and diffusions on continuum trees.

The remainder of this paper is organized as follows: In the first section we review the properties of continuum trees that we will need and explicitly define our Brownian motions. In the second section, we construct the Dirichlet space for our process and develop some of its potential-theoretic properties. In the third section, we study additive functionals of our processes, prove that our Brownian motion has jointly continuous local times, and describe the occupation density of our process explicitly.

## 1 A review of continuum trees

Let $(\mathscr{S},\mu)$ be a continuum tree, in the sense of Aldous [4, Sect. 2.3]. That is, $\mathscr{S}\subset \ell^1(N)$ is closed, contains 0, and for every $x,y\in\mathscr{S}$, there is a unique path $[\![x,y]\!]$ connecting $x$ to $y$, of length $d(x,y)=\|x-y\|_1$; $\mu$ is a measure corresponding to choosing a point "at random" from $\mathscr{S}$.

Taking 0 as the root of the tree and $x,y\in\mathscr{S}$, define the branch point $b=b(x,y)$ as the point $b$ in $[\![0,x]\!]\cap[\![0,y]\!]$ of maximal distance 0. Say that $x\in\mathscr{S}$ is in the *skeleton* of $\mathscr{S}$ if there exists $y\in\mathscr{S}$ such that $x\in[\![0,y[\![$. If $x$ is not in the skeleton of $\mathscr{S}$, we say that $x$ is a leaf. Following Aldous [4], we assume

(a) If $x_1,x_2,x_3\in\mathscr{S}$ are such that $b(x_1,x_2)=b(x_1,x_3)=b(x_2,x_3)=b$, then at least one of $x_1,x_2,x_3$ equals $b$.
(b) $\mu\{x : x \text{ is a leaf of } \mathscr{S}\}=1$.
(c) $\mu\{y : x\in[\![0,y]\!]\}>0$ for all $x$ in the skeleton of $\mathscr{S}$.

We also assume

(d) $\mathscr{S}$ is compact and the support of $\mu$ is $\mathscr{S}$. (In the terminology of Aldous [4], $(\mathscr{S}, \mu)$ is said to be "leaf-dense".)
(e) $\mathscr{S}$ has finite box-counting dimension $\alpha$.

For convenience, we assume that if $x = (x_1, x_2, x_3, \ldots) \in \mathscr{S}$ then $(x_1, 0, 0, \ldots)$, $(x_1, x_2, 0, \ldots), (x_1, x_2, x_3, 0, \ldots), (x_1, \ldots, x_j, 0, 0, \ldots) \in \mathscr{S}$ for all $j$. (Thus, for any $x \in \mathscr{S}, [\![0,x]\!] = [\![0,x]\!]_{sp}$, as defined in Aldous [2–4].) Let $\{k_1, k_2, \ldots\}$ be the standard unit vector basis for $\ell^1$. Let $s_n$ be the maximal segment in the direction $k_n$ contained in $\mathscr{S}$. Let $\mathscr{S}_\infty = \bigcup_1^\infty s_n$. By the assumption that all paths are "special", we can see that $\mathscr{S}_\infty$ contains the skeleton of $\mathscr{S}$. It is not hard to see that this also implies that $\mathscr{S}_n = \bigcup_1^n s_k$ is connected $n = 1, 2, \ldots$.

As $\mathscr{S}$ is compact, $D = \sup_{x,y \in \mathscr{S}} d(x, y) < \infty$. Let $\lambda_i$ be Lebesgue measure on $s_i$, and define a measure $m_\infty$ on $\mathscr{S}_\infty$ by taking $m_\infty(E) = \sum_{i=1}^\infty 2^{-i} \lambda_i(s_i \cap E)$ for Borel sets $E \subset \mathscr{L}^1$. As $\lambda_i(s_i) \leqq D$ for all $i, m_\infty(\mathscr{S}) < \infty$.

Call a graph-theoretic tree with exactly $k$ leaves labeled $1, \ldots, k$, such that all internal nodes have exactly two children a *proper k-tree.* If $\mathscr{S}_k = \bigcup_1^k s_j$ then assumption a. implies that $\mathscr{S}_k$ is a proper $k$-tree. Let $\mu_k$ be Lebesgue measure on $\mathscr{S}_k$, normalized so that $\mu_k(\mathscr{S}_k) = 1$. We assume
(f) $\mu_k$ converges weakly to $\mu$ as $k \to \infty$.

Explicit constructions of pairs $(\mathscr{S}, \mu)$ satisfying assumptions (a) – (f) appear in [2,3]. In particular, if $B_t$ is a Brownian excursion started at 0 and conditioned to end at time 1, then a function $f$ can be defined mapping $B_t$ into a pair $(\mathscr{S}, \mu)$ which satisfy (a) – (f) with Probability 1. See [3] for a heuristic explanation and [4, Sect. 2.6] for a detailed proof. Alternatively, some well-known fractal subsets of Euclidean space are quite similar to the continuum trees.
Let $\{X_t\}$ be an $\mathscr{S}$-valued stochastic process. We make the following:

**Definition 1** *$\{X_t\}$ is an $(\mathscr{S}, \mu)$-Brownian motion if*

(i) *$\{X_t\}$ has continuous sample paths.*
(ii) *$\{X_t\}$ is strong Markov.*
(iii) *$\{X_t\}$ is symmetric with respect to the invariant measure $\mu$.*
(iv) *For each path $[\![a,b]\!] \subset \mathscr{S}$ and each $x \in ]\!]a,b[\![$,*

$$P^x[T_a < T_b] = \frac{d(x,b)}{d(a,b)}, \tag{1}$$

*where $T_z = \inf\{t : X_t = z\}, z \in \mathscr{S}$.*
(v) *For $x, y \in \mathscr{S}$, let $m_{x,y}$ be the mean occupation measure for $X_t$ started at $x$ and run until it first hits $y$. Then,*

$$m_{x,y}(dz) = 2d(c(z; [\![x,y]\!]), y)\mu(dz), \tag{2}$$

*where $(c(z; [\![x,y]\!])$ is the point where $[\![y,x]\!]$ and $[\![y,z]\!]$ diverge.*

## 2 Constructing the Dirichlet space

Our objective is to construct an $(\mathscr{S}\mu)$-Brownian motion explicitly. To do this, we will define a Dirichlet form on the Hilbert space $L^2(\mathscr{S}, m_\infty)$. Let $\mathscr{F}C_0^\infty$ be the set of functions $f : \ell^1 \to R$ such that $f$ depends on a finite set of coordinates $x_1, \ldots x_n$ and is a $C_0^\infty$ function when restricted to these coordinates. Since $\mathscr{S}$ is compact and $m_\infty(\mathscr{S}) < \infty, \mathscr{F}C_0^\infty \subset L^2(\mathscr{S}, m_\infty)$. We define forms

$$\mathscr{E}_i(f,g) = \frac{1}{2}\int_{s_i} \frac{\partial f}{\partial k_i}\frac{\partial g}{\partial k_i} d\lambda_i, \quad i = 1,2,\ldots ,$$

$$\mathscr{E}_\infty(f,g) = \sum_i \mathscr{E}_i(f,g) ,$$

$$\mathscr{D}(\mathscr{E}_i) = \mathscr{D}(\mathscr{E}_\infty) = \mathscr{F}C_0^\infty .$$

$\partial f/\partial k_i$ is a partial derivative in the direction $k_i$, in the sense of Gâteau; see [1, p. 409, expression (3.3)]. So that the sign of $\mathscr{E}$ will be well-defined, we establish a convention that integrals over $s_i$ will always be directed outwards from the root. For convenience, we will write $\mathscr{D}(\mathscr{E}_\infty) = \mathscr{D}_\infty$

**Theorem 2** *$\mathscr{E}_\infty$ is a symmetric, regular, local, Markovian, closable form.*

*Proof.* $\mathscr{E}_\infty$ is a sum of the forms $\mathscr{E}_i$. A standard calculation shows that $\mathscr{E}_\infty$ is closable if the forms $\mathscr{E}_i$ are; see Robinson [14], 1.2.9. Note that $\mathscr{E}_i(f,g)$ is a "classical Dirichlet form" in the sense of Albeverio and Röckner [1]; Theorem 3 of that paper gives necessary and sufficient conditions for the closability of such a form. Examination of $\mathscr{E}_i$ shows that it satisfies these conditions trivially. Thus, each $\mathscr{E}_i$ is closable, hence so is $\mathscr{E}_\infty$.
Routine arguments show that each $\mathscr{E}_i$ is symmetric, Markovian and local. (See, for example, [9, Sect. 2.1]) It is easy to see that these properties will pass through the sum to the form $\mathscr{E}_\infty$.
To see that $\mathscr{E}_\infty$ is regular, observe that $\mathscr{F}C_0^\infty$ is an algebra of functions, whose restrictions to $\mathscr{S}$ separate points and vanish nowhere. Thus, $\mathscr{D}(\mathscr{E}_\infty)$ is uniformly dense in $C(\mathscr{S})$ by the Stone–Weierstrass theorem and $\mathscr{E}_\infty$ is regular. □

**Corollary 3** (i) *Let $\mathscr{E}$ be the closure of $\mathscr{E}_\infty$. Then $\mathscr{E}$ is a Dirichlet form.*
(ii) *There exists an $m_\infty$-symmetric $\mathscr{S}$-valued strong Markov process $X^0$, with continuous sample paths and Dirichlet form $\mathscr{E}$.*

*Proof.* $\mathscr{E}$ is a Dirichlet form, by definition. Since $\mathscr{S}$ is compact, Statement (ii) follows from Fukushima [9. Theorems 6.21 and 6.22]. □

Let $\mathscr{D}(\mathscr{E})$ be the domain of $\mathscr{E}$. For any $\alpha > 0$ and $f,g \in \mathscr{D}(\mathscr{E})$ let $\mathscr{E}_\alpha(f,g) = \mathscr{E}(f,g) + \alpha \int fg\, dm_\infty$. It is well-known that $\mathscr{E}_\alpha$ defines an inner product which makes $\mathscr{D}(\mathscr{E})$ a Hilbert space.

Our Brownian motion will ultimately be produced by a time-change of $X^0$. To show that a suitable time change is possible, we prove

**Lemma 4** (i) *Let $\{f_n\} \subset \mathscr{D}_\infty$, and suppose $\mathscr{E}_1(f_n - f, f_n - f) \to 0$. Then $\{f_n\}$ is uniformly equicontinuous and $f$ is uniformly continuous.*

(ii) *There exists a constant* $C < \infty$ *such that* $\sup_{x \in \mathscr{S}} f(x) \leqq C\sqrt{\mathscr{E}_1(f,f)}$ *for all* $f$ *in* $\mathscr{D}(\mathscr{E})$.
(iii) *All non-empty subsets of* $\mathscr{S}$ *have positive capacity.*

*Proof.* Let $p,q \in \mathscr{S}_\infty$; without loss of generality, suppose $p,q, \in \mathscr{S}_n$. Let $\gamma = \gamma_{pq}$ be the unique path in $\mathscr{S}_n$ connecting $p$ and $q$. If $f \in \mathscr{D}(\mathscr{E}_\infty)$, then by elementary calculus $f(p) - f(q) = \sum_i \int_{s_i} (\partial f/\partial k_i)\sigma_i(\gamma)\mathbf{1}_\gamma(t)\lambda_i(dt)$, where $\sigma_i(\gamma)$ is 1 if $\gamma$ is directed away from the root on $s_i$, $-1$ if $\gamma$ is directed towards the root, and 0 if $\gamma$ does not cross $s_i$. Then,

$$|f(p) - f(q)| \leqq \left(\sum_i \int_{s_i} \left(\frac{\partial f}{\partial k_i}\right)^2 d\lambda_i\right)^{1/2} \left(\sum_{s_i} \int_{s_i} \mathbf{1}_\gamma \, d\lambda_i\right)^{1/2}$$
$$= \sqrt{2}\mathscr{E}(f,f)^{1/2} \cdot d(p,q)^{1/2}.$$

Suppose $\mathscr{E}_1(f_n - f, f_n - f) \to 0$. Then $f_n \to f$ in $L^2(m_\infty)$. Furthermore, since $\mathscr{E}(f_n, f_n)$ is bounded $\{f_n\}$ is uniformly equicontinuous. As $\mathscr{S}$ is compact, we can choose a subsequence $f_{n_k}$ such that $f_{n_k}$ converges uniformly to some limit $\tilde{f}$. But then $f = \tilde{f}(m_\infty - \text{a.s.})$. So, we may take $f$ to be uniformly continuous. Observe that passage to the limit gives $|f(x) - f(y)| \leqq \sqrt{2}\mathscr{E}(f,f)^{1/2} d(x,y)^{1/2}$ for all $f \in \mathscr{D}$. An elementary inequality gives $\frac{1}{2}f(x)^2 \leqq |f(x) - f(y)|^2 + f(y)^2$ for $f \in \mathscr{D}, x, y \in \mathscr{S}$. Substituting the bound on $|f(x) - f(y)|$ and integrating both sides over $\mathscr{S}$ with respect to $y$ gives

$$\frac{1}{2}f(x)^2 \cdot m_\infty(\mathscr{S}) \leqq 2\mathscr{E}(f,f) \cdot Dm_\infty(\mathscr{S}) + \int_{\mathscr{S}} f^2(y) m_\infty(dy),$$

which gives (ii).

To prove (iii), observe that (ii) implies that all point masses $\{\delta_x\}$ are of finite energy integral. Measures of finite energy integral charge no polar sets, so all singletons have positive capacity. As capacities are increasing set functions, it follows that all non-empty sets have positive capacity. □

From this we get

**Corollary 5** *Let* $\nu$ *be a finite measure on* $\mathfrak{B}(\mathscr{S})$. *Then,*

(i) *There exists an additive functional* $A^\nu$ *satisfying the relation*

$$\lim_{t \to 0} \frac{1}{t} \int_{\mathscr{S}} h(x) E^x \int_0^t f(X_s) dA_s^\nu m_\infty(dx) = \int_{\mathscr{S}} h(x) f(x) \nu(dx),$$

*for any* $\gamma$*-excessive function* $h$ *and nonnegative measurable* $f$.
(ii) *Let* $\tau_t = \inf\{s > 0 : A_s^\nu > t\}$. *Then* $X_t^\nu = X^0(\tau_t)$ *is a* $\nu$*-symmetric Hunt process.*
(iii) *The Dirichlet form of* $X^\nu$ *is* $\mathscr{E}^\nu(f,f) = \mathscr{E}(H_\nu\gamma f, H_\nu\gamma f)$, *where* $H_\nu$ *is the first hitting operator for the support* $M_\nu$ *of* $\nu$, *and* $\gamma$ *denotes restriction to* $M_\nu$. *If* $f \in \mathscr{D}$, *then* $f \in \mathrm{Dom}(\mathscr{E}^\nu)$.

*Proof.* Statement (i) is Theorem 5.1.3 in [9]. Statement (ii) follows from Silverstein [16, Lemma 5.1]. To see (iii), first note that since all non-empty sets have positive capacity for $\mathscr{E}$, the support of $\nu$ and the fine support of $A^{\nu}$ are the same. If $f \in \mathscr{D}$, then $f$ is continuous, hence bounded on $\mathscr{S}$. Thus, $f \in L^2(\nu)$, so (iii) follows from, Silverstein [16, Theorem 8.4.]. □

In particular,

**Corollary 6** *For every $z \in \mathscr{S}$ there exists a local time $L_t^z$ for $X^0$ at $z$.*

Besides local times, we are interested in two other additive functionals, together with their associated time-changes. First, suppose $A = A^{\mu}$, with corresponding time-change $\tau$. Then $X_t = X^0(\tau_t)$ is a $\mu$-symmetric Hunt process with continuous sample paths. We restate this as

**Corollary 7** *$\{X_t\}$ satisfies conditions* (i), (ii), *and* (iii) *of Definition 1.*

Now let $x, y \in \mathscr{S}, x \neq y$, and let $c = d(x,y)$. Define $\gamma : [0,1] \to [\![x,y]\!]$ by taking $\gamma(t)$ to be the unique point $z$ on $[\![x,y]\!]$ such that $d(x,z)/c = t$. Then, evidently, $\gamma$ is continuous. Let $\lambda = \lambda_{[\![x,y]\!]}$ be the image of Lebesgue measure on [0,1] under $\gamma$, let $A^{\lambda}$ be the additive functional corresponding to $\lambda$ and let $\tau_t^{\lambda}$ be the time change induced by $A^{\lambda} \cdot X_t^{\lambda} = X^0(\tau_t^{\lambda})$ is then a $\lambda$-symmetric Markov process.

If $f \in \mathscr{D}(\mathscr{E})$, then $f \in L^2(\lambda)$. Thus $H_{\lambda} f \in L^2(\lambda)$, and it is not difficult to show that

$$\mathscr{E}^{\lambda}(f,f) = \mathscr{E}(H_{\lambda}\gamma f, H_{\lambda}\gamma f) = \sum_i \int_{s_i \cap [\![x,y]\!]} \left(\frac{\partial f}{\partial K_i}\right)^2 d\lambda_i \,.$$

**Proposition 8** *Let $(\mathscr{F}^{\lambda}, \mathscr{E}_1^{\lambda})$ denote the Dirichlet space associated with $X^{\lambda}$. Then $(\mathscr{F}^{\lambda}, \mathscr{E}_1^{\lambda})$ is isometric to the Dirichlet space for Brownian motion on* [0,1] *reflecting at the endpoints.*

*Proof.* If $x, y \in \mathscr{S}_{\infty}$ then it is easy to see that there exist points $0 = v_0 < v_1 < \cdots < v_k = 1$ such that $\gamma$ restricted to $(v_j, v_{j+1})$ is a $C^{\infty}$ mapping. For arbitrary points $x, y \in \mathscr{S}$, by letting $x_k, y_k$ be the closest points in $\mathscr{S}_k$ to $x, y$ respectively, we can find an increasing sequence $\{v_n\}_{n=-\infty}^{\infty}$ such that $\lim_{n \to -\infty} v_n = 0$, $\lim_{n \to \infty} v_n = 1$, and $\gamma$ is a $C^{\infty}$ mapping when restricted to $(v_n, v_{n+1})$. On [0,1], define

$$\mathscr{E}^*(f,g) = \frac{1}{2}\int_0^1 \frac{df}{dt}\frac{dg}{dt}\,dt, \quad f, g \in \mathscr{D}^* \,, \tag{3}$$

$$\begin{aligned}\mathscr{D}^* = \{ & f \in C[0,1] : f \text{ restricted to } (v_n, v_{n+1}) \text{ is } C^{\infty}, n = \ldots, -1, 0, 1, \ldots\,; \\ & f \text{ is constant on } [0, v_{-n}] \text{ and } [v_n, 1] \text{ for some } n\}\,,\end{aligned}$$

where the derivatives in (3) are taken in the sense of Schwarz distributions.

If $f \in \mathscr{D}_{\infty} \cap \mathscr{F}_{\lambda}$, then we can define $\tilde{f}$ on [0,1] by $\tilde{f}(t) = f(\gamma(t))$. Then, $\tilde{f} \in D^*$. Clearly,

$$\int_{[\![x,y]\!]} f^2(z)\lambda(dz) = \int_0^1 \tilde{f}^2(t)dt, \quad \mathscr{E}^{\lambda}(f,f) = \frac{1}{2}\int_0^1 \left(\frac{d\tilde{f}}{dt}\right)^2 dt \,.$$

Thus, $\gamma$ induces an isometry between $(\mathscr{F}^{\lambda}, \mathscr{E}_1^{\lambda})$ and the closure of $(\mathscr{D}^*, \mathscr{E}_1^*)$. To complete the proof of the proposition, we must show $\mathscr{E}^*$ is a closable form generating reflecting Brownian motion on [0,1].

Let $f \in C^{\infty}[0,1]$. Let $\varepsilon > 0$, and choose $\delta$ sufficiently small that $|f(x) - f(y)| < \varepsilon/2$ for $|x-y| < \delta$. Note that $\int_{v_{-n}}^{v_n} (df/dt)^2 dt \to \int_0^1 (df/dt)^2 dt$ as $n \to \infty$. Choose $n$ sufficiently large that

$$\frac{1}{2}\int_0^1 \left(\frac{df}{dt}\right)^2 dt - \frac{1}{2}\int_{v_{-n}}^{v_n} \left(\frac{df}{dt}\right)^2 dt < \varepsilon/2\,, \quad v_{-n}, 1 - v_n < \delta\,.$$

Define

$$f^*(t) = \begin{cases} f(t), & v_{-n} < t < v_n\,, \\ f(v_{-n}), & 0 \leqq t \leqq v_{-n}\,, \\ f(v_n), & v_n \leqq t \leqq 1\,. \end{cases}$$

Then $f^* \in \mathscr{D}^*$ and

$$\begin{aligned} \mathscr{E}_1^*(f - f^*, f - f^*) &= \frac{1}{2}\left(\int_0^{v_{-n}} + \int_{v_n}^1\right)\left(\frac{df}{dt}\right)^2 dt \\ &\quad + \int_0^{v_{-n}} (f(t) - f(v_{-n}))^2 dt + \int_{v_n}^1 (f(t) - f(v_n))^2 dt \\ &\leqq \varepsilon/2 + \delta\varepsilon^2/4 \\ &< \varepsilon\,. \end{aligned}$$

Thus, $f \in \bar{\mathscr{D}}^*$ for all $f \in C^{\infty}[0,1]$, and $\bar{\mathscr{E}}^*(f,f)$ is given by the usual Dirichlet integral. But this is the Dirichlet form for reflecting Bownnian motion on [0,1]. (See, for example, [9], Theorem 2.3.1, and the discussion following Theorem 2.3.2.) □

As a consequence, if $T_t^{\lambda}$ is the transition semigroup associated with $X^{\lambda}$ and $f$ is a continuous, real-valued function on $\mathscr{S}$ then $T_t^{\lambda} f(z) = T_t \tilde{f}(\gamma^{-1} z)$, where $T_t$ is the transition semigroup for reflecting Brownian motion on [0,1]. Thus, $\gamma^{-1}(X^0(\tau_t^{\lambda}))$ is a realization of reflecting Brownian motion on [0,1]. In particular, it follows that the hitting distribution and the occupation density of $X^{\lambda}$ for $a \in [\![z,w]\!] \subset [\![x,y]\!]$ are given by

$$P^a[T_z < T_w] = \frac{d(a,w)}{d(z,w)}, \qquad m_a(dz) = \begin{cases} 2\frac{d(a,x)d(z,y)}{d(x,y)}\lambda(dz), & z \in [\![a,y]\!]\,, \\ 2\frac{d(z,x)d(a,y)}{d(x,y)}\lambda(dz), & z \in [\![a,x]\!]\,, \end{cases}$$

by directly applying the usual formulas for Brownian motion.

**Corollary 9** $\{X_t\}$ *satisfies* (*iv*) *of Definition 1.*

*Proof.* The distribution of hitting places is unaffected by changes in time scale, so *iv.* follows from Proposition 8. □

## 3 Analysis of the mean occupation density

Let $\mathscr{T}_n = (\hat{t}; x_1, \ldots, x_{2n-1})$ be a finite proper $n$-tree, in the sense of Aldous [4]. Thus, $\hat{t}$ is a rooted tree, with $n$ leaves in the usual sense of graph theory. Let

0 denote the root of $\hat{t}$, assume the degree of the root 0 is 1, and assume that the degree of $v$ is 3 if $v$ is an internal vertex of $\hat{t}$. Let $e_1, \ldots, e_{2n-1}$ denote the edges of $\hat{t}$ and let $x_j$ be the length of $e_j$. For vertices $v_1, v_2$ in $\hat{t}$, let $[\![v_1, v_2]\!]$ be the path from $v_1$ to $v_2$, and define the distance

$$d_n(v_1, v_2) = \sum_{e_j \in [\![v_1, v_2]\!]} x_j \, .$$

It will be convenient to represent $\mathscr{T}_n$ as a connected set of line segments in $\mathbf{R}^n$. Suppose $v_1, \ldots, v_n$ are the leaves of $\hat{t}$. Let $S_1 = (t, 0, \ldots, 0), 0 \leqq t \leqq d_n(0, v_1)$. Let $y_1$ be the branch point of $[\![0, v_1]\!]$ and $[\![0, v_2]\!]$. Let $r_1 = d_n(0, y_1)$. Then, let $s_1 = (r_1, t, 0, \ldots, 0), 0 \leqq t \leqq d_n(y_1, v_2)$. Note that any vertex $w \in [\![0, v_1]\!] \cup [\![0, v_2]\!]$ corresponds to a point $(w_1, w_2, 0, \ldots, 0)$, where $0 \leqq w_1 \leqq d_n(0, v_1), 0 \leqq w_2 \leqq d_n(y_1, v_2)$. Continuing recursively, let $y_j$ be the branch point between $[\![0, v_{j+1}]\!]$ and $\bigcup_1^j [\![0, v_i]\!]$, and let $s_{j+1} = (r_1, \ldots, r_j, t, 0, \ldots, 0), 0 \leqq t \leqq d_n(y_j, v_{j+1})$. Henceforward, we will always identify $\mathscr{T}_n$ and its set representation in $\mathbf{R}^n$.

Let $m_n$ be linear Lebesgue measure on $\mathscr{T}_n$ normalized so that $m_n(\mathscr{T}_n) = 1$. On $L^2(\mathscr{T}_n, m_n)$, define

$$\mathscr{E}_0^n(f, g) = \frac{1}{2} \sum_{i=1}^n \int_{s_i} \frac{\partial f}{\partial k_i} \cdot \frac{\partial g}{\partial k_i} dm_n \, ,$$

$$\mathscr{D}(\mathscr{E}_0^n) = \{f \in C(\mathscr{T}_n)\} : f = \tilde{f}|_{\mathscr{T}_n}, \tilde{f} \in C_0^\infty(\mathbf{R}^n) \, ,$$

where $\partial g / \partial k_i$ is the usual partial derivative of $f$ in the direction $k_i$. As in Sect. 2, $\mathscr{E}_0^n$ is a symmetric, regular, local Markovian, closable form on $L^2(\mathscr{T}_n, m_n)$. Let $\mathscr{E}^n = \bar{\mathscr{E}}_0^n$ and let $\mathscr{D}_n = \mathscr{D}(\mathscr{E}_n)$. Then $\mathscr{E}^n$ is a Dirichlet form, and there is a $m_n$-symmetric strong Markov process $Y_t^n$ with continuous sample paths and Dirichlet form $\mathscr{E}^n$.

Let $f \in \mathscr{D}(\mathscr{E}_0^n)$ and suppose $x \in \mathscr{T}_n$ is not a vertex. Then some neighborhood of $x$ contains no vertices, and we can compute a derivative $f'(x)$ by difference quotients; paths outward from the root will have a positive direction. Then, $\mathscr{E}(f, f) = \frac{1}{2} \int_{\mathscr{T}_n} (f')^2 dm_n$. The same formula will hold for general $f \in \mathscr{D}_n$ if we take derivatives in the sense of Schwarz distributions.

**Lemma 10** *Let $x, y \in \mathscr{T}_n$ and let $T_y = \inf\{t : Y_t = y\}$. For any bounded Borel-measurable $h$,*

$$E^x \int_0^{T_y} h(Y_t) dt = \int_{\mathscr{T}_k} h(z) \cdot 2d(c(z; [\![x, y]\!]), y) m_n(dz) \, . \tag{4}$$

*Proof.* Without loss of generality, let $h$ be a positive continuous function and suppose $y = 0$. Write $T = T_0$, and let $Y_t^{n,0}$ be $y_t^n$ killed on hitting 0. From Dynkin's formula and Theorem 4.4.1 of Fukushima [9] we deduce that the Dirichlet form $\mathscr{E}^0$ corresponding to $Y_t^0$ has domain $\mathscr{D}(\mathscr{E}_n^0) = \{u \in \mathscr{D}_n : u(0) = 0\}$, with $\mathscr{E}_n^0(u, u) = \mathscr{E}_n(u, u)$ for $u \in \mathscr{D}_n(\mathscr{E}^0)$.

Let $g(x) = \int_{\mathscr{T}} h(z) \cdot 2d(c(z; [\![x, 0]\!]), 0) m_n(dz)$. If $x \in \mathscr{T}$ is not a vertex of $\mathscr{T}$, then we can differentiate $g$ along $\mathscr{T}$ by the standard argument for differentiating integrals on $\mathbf{R}^1$; it is not hard to see that for such $x, g'(x) = \int_{\mathscr{T}} 2 \cdot h(z) \mathbf{1}[x \in [\![z, 0]\!]] m_n(dz)$. An argument similar to the proof of Proposition 8 shows that $g \in \mathscr{D}(\mathscr{E}_n^0)$.

Let $u$ be the restriction to $\mathscr{T}$ of a $C^\infty(\mathbf{R}^n)$ function with $u(0) = 0$. Then

$$\mathscr{E}_n^0(u,g) = \frac{1}{2}\int_{\mathscr{T}} u'(x) \cdot \int_{\mathscr{T}} 2 \cdot h(z)\mathbf{1}[x \in [\![z,0]\!]]m(dz)m(dx)$$
$$= \int_{\mathscr{T}} h(z) \cdot u(z)m(dz)\,.$$

As $u$ is arbitrary, this shows that $2d(c(z;[\![x,0]\!]),0)$ is the Green's function for the infinitesimal generator of $Y_t^{n,0}$, by the Corollary to Theorem 1.3.1 in [9]. Formula (3) follows as a consequence. □

**Corollary 11** *Let $\mathscr{T}_n$ be a proper n-tree, let $x, y \in \mathscr{T}_n$, and $Y_t$ be a diffusion process on $\mathscr{T}_n$. Suppose $Y_t$ satisfies* (1) *of Definition 1, and suppose the speed measure $v$ of $Y_t$ is equivalent to $m_n$. If $T_y = \inf\{t : Y_t = y\}$, then for any Borel measurable $f$,*

$$E^x \int_0^{T_y} f(Y_t)dt = \int_{\mathscr{T}_n} f(z) \cdot 2d(c(z;[\![x,y]\!]),y)v(dy)\,.$$

*Proof.* Since $Y_t$ satisfies (1) of Defintion 1, $Y_t$ is a time change of Brownian motion on $\mathscr{T}_n$. Let $p$ be the density of $v$ with respect to $m$. Then the time change of Brownian motion giving $Y_t$ corresponds to the additive functional $\int_0^t p(Y_s)ds$. The corollary then follows from Lemma 10, after a change of variables in the integral. □

We next extend (2) to the process $X^0$ defined in Corollary 3.

**Lemma 12** *Let $x, y \in \mathscr{S}_\infty$, and let $T_y = \inf\{t : X_t^0 = y\}$. Then for any Borel measurable $f$,*

$$E^x \int_0^{T_y} f(X_t^0)dt = \int_{\mathscr{S}_\infty} f(z) \cdot 2d(c(z;[\![x,y]\!]),y)m_\infty(dy)\,.$$

*Remark.* Since $m_\infty(\mathscr{S}\backslash\mathscr{S}_\infty) = 0$, we can take the integral on the right-hand side over $\mathscr{S}$.

*Proof.* Recall that we can write $\mathscr{S}_\infty = \bigcup_{n=1}^\infty s_n$, where $\mathscr{S}_k = \bigcup_{n=1}^k s_n$ is a proper $k$-tree. Without loss of generality, suppose $x \in s_1$.

For $k = 1,2,\ldots,$ let $A_t^k = \int_0^t \mathbf{1}_{\mathscr{S}_k}(X_s^0)\,ds, \tau_t^k = \inf\{s : A_s^k > t\}$. The $\alpha$-potential for $A_t^k$ is $E^x \int_0^\infty e^{-\alpha t}\,dA_t^k = E^x \int_0^\infty e^{-\alpha t}\mathbf{1}_{\mathscr{S}_k}(X_t^0)\,dt = R_\alpha \mathbf{1}_{\mathscr{S}_k}(x)$. Since $\mathscr{E}_\alpha(R_\alpha \mathbf{1}_{\mathscr{S}_k}, u) = \int_{\mathscr{S}_k} u\,dm_\infty$, we see that $A_t^k$ and $\tau_t^k$ are, respectively, the additive functional and the time-change corresponding to the measure $m_\infty(\cdot \cap \mathscr{S}_k)$. By Corollary 5, $X^0(\tau_t^k)$ has Dirichlet form $\frac{1}{2}\sum_1^k \int_{s_i}(\partial f/\partial k_i)(\partial g/\partial k_i)dx$ and is $m_\infty(\cdot \cap \mathscr{S}_k)$-symmetric. Thus, $X^0(\tau_t^k)$ is a time-changed Brownian motion on $\mathscr{S}_k$, with speed measure $m_\infty(\circ \cap \mathscr{S}_k)$.

By Corollary 11, if $f$ is a positive measurable function on $\mathscr{S}_\infty$,

$$E^x \int_0^{T_y} f(X_s^0)\mathbf{1}_{\mathscr{S}_k}(X_s)ds = E^x \int_0^{T_y} f(X(\tau_s^k))ds$$
$$= \int_{\mathscr{S}_\infty} f(z)2d(c(z;[\![x,y]\!]),y)\mathbf{1}_{\mathscr{S}_k}(z)m_\infty(dz)\,. \qquad (5)$$

Let $k \to \infty$ and apply the monotone convergence theorem to both sides of (5) to prove the lemma. □

*Remark.* We can also state Corollary 12 in this fashion: The 0-potential density of $X_t^0$ killed on hitting $y$ is $2d(c(z;[\![x,y]\!]),y)$.

From Lemma 12 we have the following

**Corollary 13** *For* $x, y \in \mathscr{S}, E^x T_y + E^y T_x = 2.d(x,y) \cdot m_\infty(\mathscr{S})$.

*Proof.* Recall that $m_\infty(\mathscr{S}) < \infty$. Let $x, y \in \mathscr{S}_\infty$ and apply the preceding lemma twice with $f \equiv 1$ to get

$$E^x T_y = E^x \int_0^{T_y} 1 \cdot ds = \int_{\mathscr{S}} 1 \cdot 2d(c(z;[\![x,y]\!]),y) m_\infty(dz)\,,$$
$$E^y T_x = \int_{\mathscr{S}} 1 \cdot 2d(c(z;[\![x,y]\!]),x) m_\infty(dz)\,.$$

Thus,
$$E^x T_y + E^y T_x = \int_{\mathscr{S}} 2d(x,y) m_\infty(dz) = 2d(x,y) m_\infty(\mathscr{S}_0)\,.$$

For $x \in \mathscr{S} \backslash \mathscr{S}_\infty, y \in \mathscr{S}_\infty$, let $x_k$ be the nearest point to $x$ in $\mathscr{S}_k$. Then $x_k \in [\![x,y]\!]$ for all $k$ and $x_k \to x$ as $k \to \infty$. Let $T_k = T_{x_k}$. By sample path continuity, $T_k \uparrow T_x, (P^y - \text{a.s.})$. By the strong Markov property,

$$E^x \int_0^{T_y} \mathbf{1}_{\mathscr{S}_k}(X_s^0) ds = E^{x_k} \int_0^{T_y} \mathbf{1}_{\mathscr{S}_k}(X_s^0) ds = \int_{\mathscr{S}} \mathbf{1}_{\mathscr{S}_k}(z) 2d(c(z;[\![x,y]\!]),x) m_\infty(dz)\,.$$

Letting $k \to \infty$ gives

$$E^x \int_0^{T_y} \mathbf{1}_{\mathscr{S}_k}(X_s^0) ds \to E^x \int_0^{T_y} \mathbf{1}_{\mathscr{S}_\infty}(X_s^0) ds = E^x \int_0^{T_y} 1 \cdot ds\,,$$

$$\int_{\mathscr{S}} \mathbf{1}_{\mathscr{S}_k}(z) 2d(c(z;[\![x,y]\!]),x) m_\infty(dz) \to \int_{\mathscr{S}} 2d(c(z;[\![x,y]\!]),x) m_\infty(dz)\,.$$

On the other hand,

$$E^y T_x = \lim_{k\to\infty} E^y T_k = \lim_{k\to\infty} \int_{\mathscr{S}} 2d(c(z;[\![x_k,y]\!]),x_k) m_\infty(dz)$$
$$= \int_{\mathscr{S}} 1 \cdot 2d(c(z;[\![x,y]\!]),x) m_\infty(dz)\,.$$

Thus, $E^x T_y + E^y T_x = 2d(x,y) m_\infty(\mathscr{S}_0)$, as before. A similar argument gives $E^x T_y + E^y T_x = 2d(x,y) m_\infty(\mathscr{S}_0)$ for arbitrary $y$ in $\mathscr{S}$. □

*Remark.* If we think of $\mathscr{S}$ as an object of finite resistivity, then the resistance from $x$ to $y$ is a constant multiple of $d(x,y)$. Thus, Corollary 13 can be restated as saying that $E^x T_y + E^y T_x$ equals the resistance between $x$ and $y$. This result is well-known for reversible Markov chians on finite graphs, from the connection between random walks and electrical networks.

This estimate on the expected hitting times is extremely strong. In particular, it implies that the local times of $X^0$ are jointly continuous. We state this result as a separate theorem.

**Theorem 14** *The local times $\{L_t^z\}$ of $X^0$ may be chosen to jointly continuous in $(z,t)$.*

*Proof.* For $\alpha > 0$, let $U_\alpha^0$ be the resolvent operator for $X_t^0$. For $x \in \mathscr{S}$, let $U_\alpha^0(x,A) = U_\alpha \mathbf{1}_A(x)$ be the corresponding $\alpha$-potential measure. By Lemma 3, all non-empty subsets of $\mathscr{S}$ have positive capacity for $\mathscr{E}$. By Theorem 4.31 in Fukushima [9], this implies that $X^0$ has no non-empty exceptional sets. Thus, by Theorem 4.2.2 of Fukushima [9], $U_\alpha^0(x,A)$ has a density $u^\alpha(x,y)$ with respect to $m_\infty$.

Let $u^1(x,y)$ be the 1-potential density for $X_t^0$. Let $\{G(z), z \in \mathscr{S}\}$ be a Gaussian process with covariance function $u^1$, and define $\eta(x,y) = (E(G(x) - G(y))^2)^{1/2}$. By Theorem 1 of Marcus and Rosen [13], to show that $L_t^x$ is jointly continuous almost surely it suffices to show that $G(z)$ has continuous sample paths almost surely.
For $x,y \in \mathscr{S}$, let $\psi(x,y) = E^x e^{-T_y}$ and $h(x,y) = (1-\psi(x,y)\psi(y,x))^{1/2}$. An elementary estimate gives $\psi(x,y) \geqq 1 - E^x(T_y \wedge 1)$, so

$$h^2(x,y) \leqq E^x T_y + E^x T_y = 2m_\infty(\mathscr{S})d(x,y)\,.$$

By Marcus and Rosen [13, Lemma 9.4], there exist constants $0 < C_0 \leqq C_1 < \infty$ such that $C_0\eta(x,y) \leqq h(x,y) \leqq C_1\eta(x,y)$. Thus, for $x,y \in \mathscr{S}$ and $\lambda \in \mathbf{R}$,

$$E\exp(\lambda(G(x) - G(y))) \leqq \exp\left(\frac{\lambda^2}{2} \cdot \frac{m_\infty(\mathscr{S})}{C_1^2} \cdot d(x,y)\right)\,.$$

Let $N(\mathscr{S},\varepsilon)$ be the minimum number of balls of radius $\varepsilon$ in the metric $d(\cdot,\cdot)$ required to cover $\mathscr{S}$. Since the box-counting dimension of $\mathscr{S}$ is $\alpha < \infty, N(\mathscr{S},\varepsilon) < C_3(1/\varepsilon)^\alpha$ for a suitably chosen constant $C_3$. Thus

$$\int_0^1 (\log(N(\mathscr{S},u^2))^{1/2} du < (\log C_3)^{1/2} + \sqrt{2\alpha}\int_0^1 (\log(1/u))^{1/2} du < \infty\,.$$

By Marcus and Pisier [12, Theorem 3.1], this suffices to show that $G$ has continuous sample paths. (Note that the metric $\rho$ in the statement of Theorem 3.1 is $C\sqrt{d}$.) This proves the theorem. □

We have already remarked that for any finite measure $\nu$ on $\mathscr{S}$ we can define a $\nu$-symmetric Hunt process with continuous sample paths as a time-change of $X^0$. This time change corresponds to an additive functional $A^\nu$ with Revuz measure $\nu$. As the local times $\{L_t^z\}$ are jointly continuous we can express $A^\nu$ explicitly as

$$A_t^\nu = \int_{\mathscr{S}} L_t^z \nu(dz)\,,$$

by applying the representation theorem for additive functionals. (See, for example, Sharpe [15, Exercise 75.2]).

Recall that $\mathscr{S}_k = \bigcup_{n=1}^{k} s_n$ and $\mu_k$ is normalized Lebesgue measure on $\mathscr{S}_k$. By assumption $(f)$, $\mu_k \Rightarrow \mu$. If we let $A_t^k = \int_{\mathscr{S}} L_t^z \mu_k(dz)$ and $A_t = \int_{\mathscr{S}} L_t^z \mu(dz)$, then $A_t^k(\omega) \to A_t(\omega)$ for each $t$ and each sample path $\omega$ such that $\{L_t^z\}$ is jointly continuous.

We now state the main result of this section.

**Theorem 15** *$\{X_t\}$ satisfies formula* (2).

*Proof.* It suffices to show (2) for positive continuous functions $f$. Furthermore, without loss of generality we can take $y = 0$; henceforth, let $T = T_0$. For $k = 1, 2, \ldots$, take $A_t^k$ and $A_t$ as defined above and let $\tau_t^k$ and $\tau_t$ be the corresponding time-changes. Let $X_t^k = X^0(\tau_t^k)$. Then, by Lemma 10,

$$E^x\left(\int_0^t f(X_s^k)ds\right) = \int_{\mathscr{S}_k} f(z)\cdot 2d(c(z,[\![x,0]\!]),0)\mu_k(dz)$$
$$= \int_{\mathscr{S}} f(z)\cdot 2d(c(z,[\![x,0]\!]),0)\mu_k(dz)\,. \tag{6}$$

As $k \to \infty$, the right-hand side of (6) converges to $\int_{\mathscr{S}} f(z))2d(c(z,[\![x,0]\!]),0) \times \mu(dz)$.

Now consider the left-hand side of (6). For any $\omega$ such that $X_t^0(\omega)$ is continuous in $t$ and $L_t^z(\omega)$ is jointly continuous in $(z,t)$, we have

$$\int_0^T f(X_s^k)ds = \int_0^T f(X^0(\tau_s^k))ds = \int_0^T f(X_s^0)dA_s^k\,.$$

As $k \to \infty$, we have

$$\int_0^T f(X_s^0)dA_s^k \to \int_0^T f(X_s^0)dA_s = \int_0^T f(X_s)ds\,.$$

It remains to show that $\{\int_0^T f(X_s^0)dA_s^k\}$ converges in $L^1$. To do this, first note that $f$ is bounded on $\mathscr{S}$, so

$$\left|\int_0^T f(X_s^k)ds\right| \leqq \int_0^T |f(X_s^0)|dA_s^k \leqq \|f\|A_T^k\,.$$

We shall show that $E^x(A_T^k)^2$ is bounded. It will then follow that $E^x(\int_0^T f(X_s^0) \times dA_s^k)^2$ is bounded and $\{\int_0^T f(X_s^0)dA_s^k\}$ is uniformly integrable.

First, note that

$$E^x(A_t^k)^2 = E^x\left(\int_{\mathscr{S}} L_t^z\mu_k(dz)\right)^2 \leqq E^x\int_{\mathscr{S}} (L_t^z)^2\mu_k(dz)\,.$$

Local times are defined within a multiplicative constant, so we can always assume that $E^x\int_0^\infty e^{-t}dL_t^y = u^1(x,y)$, with $u^1$ as defined in the proof of Theorem 14. With this normalization, $E^x\int_0^T dL_t^y = E^xL_T^y = 2d(c(y,[\![x,0]\!]),0)$, by the Remark following the proof of Corollary 12.

Arguing as in the proof of expression (4.8) in [13], we have

$$E^x(L_T^y)^2 = 2E^x\int_0^T\int_r^T dL_s^y dL_t^y = 2E^x\int_0^T E^{X_r^0}\int_0^T d(\theta_r \circ L_s^y)dL_r^y = 2E^x L_T^y \cdot E^y L_T^Y ,$$

where $\theta$ is the usual shift operator for Markov processes. The second equality follows by the ordinary Markov property, and the fact that $L_r^y$ is supported by $\{s : X_s^0(\omega) = y\}$ gives the third equality.

Thus $E^x(A_T^k)^2 \leqq 8D^2$, and Theorem 15 follows. □

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