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Conditioned super-Brownian motion

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Summary. We investigate classes of conditioned super-Brownian motions, namely H-transforms P^H with non-negative finitely-based space-time harmonic functions $H(t, \mu)$. We prove that P^H is the unique solution of a martingale problem with interaction and is a weak limit of a sequence of rescaled interacting branching Brownian motions. We identify the limit behaviour of H-transforms with functions $H(t, \mu) = h(t, \mu(1))$ depending only on the total mass $\mu(1)$. Using the Palm measures of the super-Brownian motion we describe for an additive space-time harmonic function $H(t, \mu) = \int h(t, x)\, \mu(dx)$ the H-transform P^H as a conditioned super-Brownian motion in which an immortal particle moves like an h-transform of Brownian motion.

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1 Introduction

Measure-valued processes, in particular superprocesses, have been extensively studied in recent years. For a survey of the relevant facts and literature we refer to the lecture notes of D.A. Dawson [D]. In this paper we concentrate on a class of conditioned super-Brownian motions defined as H-transforms in the sense of J.L. Doob for certain space-time harmonic functions H of the super-Brownian motion. If the function H is not *multiplicative* this conditioning leads us to processes which do not belong to the class of measure-valued branching processes defined in [W], because the mean of the branching law of an H-transform depends on the whole population. As a special case of a Girsanov transformation, H-transforms are superprocesses with interaction considered in [D, 10.1.1, 7.2.2].

1. Our basic datum is the distribution P of the critical super-Brownian motion with intensity 2α starting from a finite measure μ_0 on $\mathbb{R}^d$, i.e., the superprocess connected with the equation

$$\frac{\partial}{\partial t} u = \tfrac{1}{2} \Delta u - \alpha u^2, \tag{1.1}$$

cf. (1.15) below.

We consider P as a probability measure on the space D_M of cadlag paths in the space M of finite measures over $\mathbb{R}^d$ equiped with the weak topology. Let $X = (X_t)_{t \geqq 0}$ denote the coordinate process in D_M, $(\mathscr{F}_t)_{t \geqq 0}$ the right-continuous filtration generated by X, and $\mathscr{F} = \bigvee_{t \geqq 0} \mathscr{F}_t$.

We call a non-negative function H on $[0, \infty) \times M$ space-time harmonic for P iff the process

$$(H(t, X_t))_{t \geqq 0}$$

is a martingale under P with $H(0, \mu_0) = 1$. According to [Me] a space-time harmonic function H allows us to construct the H-transform P^H as a probability measure on D_M such that for every $t \geqq 0$

$$dP^H|_{\mathscr{F}_t} = H(t, X_t)\, dP|_{\mathscr{F}_t}. \tag{1.2}$$

The investigation of some classes of H-transforms is the aim of our paper.

2. In order to state the first theorem we introduce some notation

If f is a measurable function on $\mathbb{R}^d$ and $\mu \in M$ we define $\mu(f) := \int f(x)\, \mu(dx)$ and $e_f(\mu) := \exp(-\mu(f))$. For a time-dependent function $f(t, \cdot)$ we consequently write

$$\mu(f(t)) = \int_{\mathbb{R}^d} f(t, x)\, \mu(dx).$$

The vector $(\mu(f_1(t)), \ldots, \mu(f_m(t)))$ is denoted by $\vec{\mu}(\vec{f}(t))$. We define the directional derivative F' of a function F on M by

$$F'_x(\mu) := \lim_{\varepsilon \downarrow 0} \frac{F(\mu + \varepsilon \delta_x) - F(\mu)}{\varepsilon} \tag{1.3}$$

provided the limit exists for every $x \in \mathbb{R}^d$, $\mu \in M$.

All space-time harmonic functions in this paper belong to the class $\mathscr{F}\mathscr{B}$ of finitely based functions F defined as follows:

$$F(t, \mu) = \phi(t, \mu(f_1(t)), \ldots, \mu(f_m(t))) \tag{1.4}$$

where

$$\phi \in C^{1,2}([0, \infty) \times \mathbb{R}^m), \quad f_i \in C^{1,2}([0, \infty) \times \mathbb{R}^d), \quad 1 \leqq i \leqq m, \ m \in \mathbb{N}.$$

We are now able to state our first main result:

Theorem A *Let the function $H \in \mathscr{F}\mathscr{B}$ be space-time harmonic for the super-Brownian motion. The measure P^H is the unique measure on D_M such that for every positive $f \in C_0^2$ the process $M^H[e_f]$ defined by*

$$(1.5)\qquad M_t^H[e_f] := e_f(X_t) - e_f(X_0) - \int_0^t X_s\left(\alpha f^2 - \frac{1}{2}\Delta f - 2\alpha \frac{H'(s, X_s)}{H(s, X_s)} \cdot f\right) e_f(X_s)\, ds$$

is a local martingale and such that $P^H[X_0 = \mu_0]$.

The main difficulty in the proof is caused by the zeros of H. We resolve this problem by localization with the stopping times

$$(1.6)\qquad T_n := \inf\left\{t > 0 \mid H(t, X_t) \notin \left[\frac{1}{n}, n\right]\right\}.$$

3. We interpret the coefficient

$$(1.7)\qquad \frac{H'_x(s, \mu)}{H(s, \mu)}$$

as the interacting branching of a "masspoint" at place x, time s induced by the configuration μ. This interpretation is justified by the approximation result. Theorem B, below.

4. The only H-transform of super-Brownian motion already considered in the literature is the process conditioned on non-extinction [EP1, RR1]. As S.N. Evans and E. Perkins remarked this is the H-transform with $H(s, \mu) = \mu(1)$. This space-time harmonic function belongs to two subclasses of $\mathscr{F}\mathscr{B}$. It is an additive function and it is a function to the total mass $\mu(1)$, see 5. below. This process is described in [RR1] as a process with interacting immigration as a generalization of the processes considered in [KW]. In our general setting, we give an intrinsic characterization in terms of interacting branching.

5. The following examples will be analysed:

(i) Additive H-transforms, where $H(s, \mu) = \mu(h(s))$, cf. 7. below and Sect. 3.
(ii) H-transforms depending only on the total mass $H(s, \mu) = \eta(s, \mu(1))$ and products of those functions with additive functions, i.e., $H(s, \mu) = \mu(h(s))\ \eta_0(s, \mu(1))$, cf. Sect. 2.2.2 (iii), (iv), and Sect. 4.
(iii) A multiplicative space-time harmonic function $H(s, \mu) = e^{-\mu(h(s))}$ yields a non-critical super-Brownian motion as its H-transform, (cf. Sect. 2.2.2 (v) and e.g. [W, ER, Dy2]).

6. Our results should be viewed as a first step towards the probabilistic Martin boundary of super-Brownian motion. In [EP2] it is proved, that if the transition function $\mathscr{P}(t, x, dy)$ of a Markov process satisfies

$$(1.8)\qquad \int \mathscr{P}(t, x, \cdot)\, \mu_1(dx) \ll \int \mathscr{P}(t + h, x, \cdot)\, \mu_2(dx)$$

then the transition function $P(t, v, d\mu)$ of the superprocess over this Markov process satisfies

$$P(t, \mu_1, \cdot) \ll P(t+h, \mu_2, \cdot). \tag{1.9}$$

The Brownian motion on $\mathbb{R}^d$ has property (1.8). Therefore, according to the results of E.B Dynkin in [Dyl, Sect. 10] there exists a Martin kernel of the super-Brownian motion w.r.t. the reference measure $P(t, \mu_0, \cdot)$. Additionally all extremal space-time harmonic functions H^e arise as limits of the Martin kernel. Every H-transform with $H = \int_E H^e m^H(de)$ can be viewed as the super-Brownian motion conditioned to have the exit distribution m^H at the Martin boundary.

We will give a detailed description of the conditioning for additive H-transforms (cf. Theorem 3.10 in Sect. 3.4) and for H-transforms depending on the total mass (cf. Sect. 2.2.2 (iii)).

7. Consider now an *additive H-transform* P^H. The n-th approximation P^n of the super-Brownian motion is a rescaled critical branching Brownian motion. The critical branching implies that the function H is also space-time harmonic for every P^n (cf. Sect. 3.1). We prove that the sequence $\{P^{n,H}\}_{n\in\mathbb{N}}$ of the corresponding H-transforms converges weakly to the H-transform of the super-Brownian motion (Theorem 3.2, Sect. 3.2).

Subsequently we compute the Laplace functional of the transition function (Theorem 3.6, Sect. 3.3). This formula connects additive H-transforms with the Palm distribution of the super-Brownian motion and gives insight into the conditioning implicit in the H-transform (Theorem 3.10, Sect. 3.4). The additive H-transform can be viewed as the super-Brownian motion conditioned on the event, that there is an "immortal particle" ξ_t in the support of the super-Brownian motion such that ξ_t converges in the Martin topology as the h-transform of the Brownian motion. In addition, the Laplace functional enables us to prove a representation of an additive H-transform as a sum of a superprocess and an independent random measure generated by the immortal particle. This generalizes the representation of superprocesses conditioned on non-extinction in [E].

8. An H-transform is the limit of a sequence of interacting branching Brownian motions.

For fixed $n \in \mathbb{N}$, let $X^{1,n,H}$ denote the interacting (binary) branching Brownian motion with intensities

$$q_0^{(n)} = (s, x, \mu) = \beta^{(n)}(s, x, \mu)\, p_0^{(n)}(s, x, \mu) = n\alpha \cdot \left(1 - \frac{H'_x\left(s, \frac{\mu}{n}\right)}{nH\left(s, \frac{\mu}{n}\right)}\right), \tag{1.10}$$

$$q_2^{(n)}(s, x, \mu) = \beta^{(n)}(s, x, \mu)\, p_2^{(n)}(s, x, \mu) = n\alpha \cdot \left(1 + \frac{H'_x\left(s, \frac{\mu}{n}\right)}{nH\left(s, \frac{\mu}{n}\right)}\right),$$

and with mass 1 of each particle. The condition

$$(1.11)\qquad \inf_{x\in\operatorname{supp}(\mu)}\left(1\mp\frac{H'_x\left(s,\frac{\mu}{n}\right)}{nH\left(s,\frac{\mu}{n}\right)}\right)\geqq 0$$

for all point measures μ ensures that $q_i^{(n)}$, $i=0, 2$, are intensities. The construction of the processes $X^{1,n,H}$ is carried out in Sect. 4.1. It generalizes results of [MR, RR2] to unbounded and time-inhomogeneous intensities. Define the process $X^{n,H}$ on D_M by $X^{n,H}:=n^{-1}X^{1,n,H}$ and let $P^{n,H}$ be its distribution on D_M.

Theorem B *Let H be a space-time harmonic function of the super-Brownian motion satisfying* (1.11) *and*

$$(1.12)\qquad 0\leqq\mu\left(\frac{H'(s,\mu)}{H(s,\mu)}\right)\leqq\mu(1)\,K_1+K_0<\infty \qquad \text{for all } \mu\in M$$

with K_0, $K_1\in[0,\infty)$. *Then the sequence of the rescaled interacting branching Brownian motions* $\{P^{n,H}\}_{n\in\mathbb{N}}$ *with branching intensities* (1.10) *converges weakly in the space of probability measures over* D_M *towards the H-transform of the super-Brownian motion.*

This result justifies the interpretation of the additional drift in the H-transform of the super-Brownian motion as the interaction in the branching behaviour. By mixing, we can generalize Theorem B slightly.

Lemma 1.1 *Let* $H(s,\mu)=\int_E H^e(s,\mu)\,m(de)$ *be a mixture of space-time harmonic functions* H^e *with some finite measure m such that there exist approximating particle systems* $\{P^{n,H^e}\}_n$ *for every* $e\in E$ *with*

$$(1.13)\qquad P^{n,H^e}\Rightarrow P^{H^e}\quad \text{on}\quad D_M.$$

Then P^H *is the weak limit of the sequence* $\{P^{n,H}\}_n$ *with*

$$(1.14)\qquad P^{n,H}:=\int_E P^{n,H^e}\,m(de).$$

Proof. Use bounded convergence. □

Remarks. (i) Additive space-time harmonic functions satisfy the assumption of Theorem B, cf. Sect. 4.

(ii) If there exists $K_1<\infty$ s.t.

$$0\leqq\frac{H'_x(s,\mu)}{H(s,\mu)}\leqq K_1\,\forall s\in[0,\infty),\qquad x\in\mathbb{R}^d,\ \mu\in M$$

(bounded interaction in the mean of the branching law), then the convergence result is contained in [MR].

9. Let us recall a definition and some martingale properties of the super-Brownian motion

Proposition 1.2 *Let* $\alpha > 0$.
a. There is an M-valued strong Markov process with cadlag paths and transition function $P_t(\mu, d\nu)$ *whose Laplace functional satisfies*

$$\int e_f(\nu)\, P_t(\mu, d\nu) = \exp(-\mu(V_t f)) \tag{1.15}$$

for every positive function $f \in C_0^2$. *The function* $V_t f$ *is the unique solution of* (1.1) *with* $V_0 f = f$.

This process is called the super-Brownian motion.

b. Let X be a cadlag M-valued process with $X_0 = \mu_0$, *adapted to a right continuous filtration. Then the following properties are equivalent:*

1. For all positive $f \in C_0$ *and* $T > 0$ *the process*

$$\{e^{-X_t(V_{T-t} f)}\}_{0 \leq t \leq T}$$

is a martingale.

2. For all positive $f \in C_0^2$ *the process* $M[e_f]$ *defined by* $M_t[e_f] :=$

$$e_f(X_t) - e_f(X_0) + \int_0^t e_f(X_s)\, X_s(\tfrac{1}{2}\Delta f - \alpha f^2)\, ds \tag{1.16}$$

is a martingale.

3. For all $f \in C_0^2$ *the process* $M(f)$ *defined by*

$$M_t(f) := X_t(f) - X_0(f) - \int_0^t X_s(\tfrac{1}{2}\Delta f)\, ds$$

is a continuous local martingale with increasing process

$$d\langle M(f)\rangle_s = 2\alpha X_s(f^2)\, ds.$$

4. For every finitely based function F having base functions $f_i \in C_b^2$ *the process* $M[F]$ *defined by*

$$\begin{aligned} M_t[F] &:= F(t, X_t) - F(0, X_0) \\ &\quad - \int_0^t X_s\left(\left(\frac{1}{2}\Delta + \frac{\partial}{\partial s}\right) F'\right) + \alpha X_s(F'') + \frac{\partial}{\partial s}\phi(s, \vec{X}_s(\vec{f}(s)))\, ds \end{aligned} \tag{1.17}$$

is a local martingale.

The super-Brownian motion is the unique solution of one and hence of all the above equivalent "martingale problems".

c. For the super-Brownian motion X starting from a measure with a compact support the process defined by (1.17) *is a local martingale for every* $F \in \mathcal{F}\mathcal{B}$.

Proof. a. and b. are standard (cf. e.g. [R, I, D, MR]).

Assertion c. follows because the support of the super-Brownian motion is compact [DIP, Theorem 1.2]. □

Remarks. (i) We will need the extension of the martingale problem to unbounded base functions in Sect. 2.1 in order to state a necessary condition of space-time harmonicity in terms of the operator associated with the martingale problem.

(ii) All results are given in terms of (super-) Brownian motion. But there are of course generalizations to other superprocesses under sufficient regular conditions. In [O2] the result of Theorem A is even extended to H-transforms of all measure-valued diffusions defined as in [D, 7.1], including the Fleming-Viot process, and to functions H which belong only *locally* to $\mathcal{F}\mathcal{B}$.

2 Proof of Theorem A

Proposition 1.2c yields the operator $\mathcal{A}$ in (2.1) below, associated with the martingale problem of the super-Brownian motion and thereby a necessary condition for space-time harmonicity of functions in $\mathcal{F}\mathcal{B}$.

2.1 Space-time harmonic functions of the super-Brownian motion

Let the mapping $\mathcal{A}$ on $\mathcal{F}\mathcal{B}$ be defined by

$$(2.1)\quad \mathcal{A}\mathcal{F}(t,\mu):=\mu\left(\left(\frac{1}{2}\Delta+\frac{\partial}{\partial t}\right)F'+\alpha F''\right)+\frac{\partial}{\partial t}\phi(t,\mu(f(t)))$$

$$=\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\phi(t,\vec{\mu}(\vec{f}(t)))\,\mu\left(\frac{1}{2}\Delta f_i(t)+\frac{\partial}{\partial t}f_i(t)\right)$$

$$+\alpha\sum_{i,j=1}^{m}\frac{\partial^2}{\partial x_i\,\partial x_j}\phi(t,\vec{\mu}(\vec{f}(t)))\,\mu(f_i(t)\,f_j(t))+\frac{\partial}{\partial t}\phi(t,\vec{\mu}(\vec{f}(t))).$$

If $H\in\mathcal{F}\mathcal{B}$ is space-time harmonic then a.s. $\mathcal{A}H(t,X_t)=0$ for dt a.s.

2.2 H-transforms

We describe the H-transform of the super-Brownian motion with a space-time harmonic function $H\in\mathcal{F}\mathcal{B}$ by proving that P^H is uniquely determined as the solution of a martingale problem. An H-transform can be considered as a super-Brownian motion with interaction. For superprocesses which exhibit an interaction also in the variance of the branching law the question of uniqueness of the solution to the martingale problem is open, cf. [MR]. But because H-transforms have their interaction only in the mean of the branching law, the uniqueness follows by a suitable stopping argument from the Girsanov transformation for measure-valued processes proved by D.A. Dawson [D, 7.2.2, 10.1.2].

The interaction of an H-transform induced by state μ of the process is manifest in the non-criticality $H'_x(t,\mu)\, H^{-1}(t,\mu)$ of the branching law of a "mass-point" at x at time t.

2.2.1 Proof of Theorem A. In the setting of [D, 7.2] the domain of the martingale problem of the super-Brownian motion is $\{F(\mu)=\phi(\mu(f))\,|\,\phi\in C_b^\infty(\mathbb{R}),\, f\in D(\Delta)\}$. By Proposition 1.2.b.2 and because C_0^2 is a core of $(\Delta, D(\Delta))$ uniqueness of the solution of the martingale problem still holds if the domain is restricted to the exponential functions, i.e., $\phi(x)=e^{-x}$ and positive $f\in C_0^2$, cf. also [R, Theorem 1.3].

Hence we can apply Theorem 7.2.2 in [D]. We have to write the martingale $(H(t,X_t))_{t\geqq 0}$ as an exponential martingale. Up to the stopping time T_n defined in (1.6) we have the equation

$$\log H(t,X_t)=\int_0^t \frac{1}{H(s,X_s)}\, dH(s,X_s)-\frac{1}{2}\int_0^t \frac{1}{H^2(s,X_s)}\, d\langle H(s,X_s)\rangle. \tag{2.2}$$

By Itô's lemma and Proposition 1.2 we obtain

$$H(t,X_t)=M_t[H]=\int_0^t \int_{\mathbb{R}^d} H'_x(s,X_s)\, M(ds,dx), \tag{2.3}$$

where $M(ds,dx)$ is the martingale measure associated with the super-Brownian motion, which is an extension of the martingales $M(f)$, $f\in C_b^2$, cf. [D, 7.1.3], [MR]. By the covariation for martingale measures (or Proposition 1.2 applied to $H^2(t,\mu)$) it follows that

$$d\langle H(s,X_s)\rangle = 2\alpha X_s((H'(s,X_s))^2)\, ds. \tag{2.4}$$

Hence

$$H(t,X_t)=\exp\left(\int_0^t \int_{\mathbb{R}^d} \frac{H'_x(s,X_s)}{H(s,X_s)}\, M(ds,dx)-\frac{1}{2}\int_0^t X_s\left(2\alpha\left(\frac{H'(s,X_s)}{H(s,X_s)}\right)^2\right) ds\right). \tag{2.5}$$

It follows that the martingale problem associated with P^H is well-posed up to T_n, i.e., $P^H|_{\mathscr{F}_{T_n}}$ is the unique solution. But it is easy to see (use e.g. [El, Theorem 4.16] and that H is the density of P^H), that

$$P^H[\sup T_n=\infty]=1. \tag{2.6}$$

Uniqueness of the stopped martingale problem and (2.6) imply that for every solution P^* of the (global) martingale problem the random variable $\sup_n T_n$ is also P^* a.s. infinite. The assertion is now proved, because $\bigvee_n \mathscr{F}_{T_n-}=\mathscr{F}_{\infty-}=\mathscr{F}_\infty$, cf. [De, T31]. □

Remarks. (i) By Proposition 1.2.b.3. it is easy to see that P^H is even the unique measure s.t. $(M^H_{t\wedge T_n}[e_f])_{t\geqq 0}$ is only a local martingale for all $f\in C_0^2$ and $n\in\mathbb{N}$.

(ii) In the present case of H-transforms with $H\in\mathscr{F}\mathscr{B}$, Dawson's Girsanov transformation can be avoided by using ordinary stochastic calculus, Proposition 1.2

and Eq. (2.1). We derive from the Girsanov transformation for real-valued martingales that P^H is a solution of the martingale problem in Theorem A. Additionally we get the exponential martingale under every solution P^* and from this the local characteristics of $X(f)$. This yields the martingale property of every $M^H[F]$ defined by

$$(2.7)\quad M_t^H[F]:=F(t,X_t)-F(0,X_0)-\int_0^t\left[\frac{\partial}{\partial s}\phi(s,\vec{X}_s(\vec{f}(s)))\right.$$
$$\left.+X_s\left(\left(\frac{1}{2}\Delta+\frac{\partial}{\partial s}+2\alpha\frac{H'(s,X_s)}{H(s,X_s)}\right)F'(s,X_s)+\alpha F''(s,X_s)\right)\right]ds.$$

(This approch is in the spirit of [ER].) In particular, $H^{-1}(t, X_t)$ is a martingale up to T_n. Then proceed by "backward transformation" to prove the assertion of Theorem A.

2.2.2 Examples. We assume $H(0,\mu_0)=1$.

(i) *Non-extinction.* Let us define for every $t>0$ and every $A\in\mathscr{F}_t$ the measure P' by $P'[A]:=\lim_{r\to\infty} P[A\,|\,X_{t+r}(1)>0]$. Then it is shown in [EP1, RR1] that $P'=P^H$ with the space-time harmonic function $H(t,\mu)=\mu(1)$.

(ii) *Additive harmonic functions.* Because we consider critical super-Brownian motion, a function $H(t,\mu)=\mu(h(t))$ is space-time harmonic iff h is space-time harmonic for Brownian motion, which is equivalent to

$$\frac{\partial}{\partial t}h(t,x)+\frac{1}{2}\Delta h(t,x)=0.$$

The corresponding H-transform exhibits the non-criticality

$$\lim_{\varepsilon\downarrow 0}\frac{\mu(h(t))+\varepsilon h(t,x)-\mu(h(t))}{\varepsilon}\cdot\mu(h(t))^{-1}=h(t,x)\,\mu(h(t))^{-1}.$$

This H-transform is analysed in Sect. 3.

(iii) *Harmonic functions of the total mass.* Let $\alpha=1$. A function $H(t,\mu)=\eta(t,\mu(1))$ is space-time harmonic iff η is space-time harmonic for Feller's continuous-state branching process, which is equivalent to

$$\frac{\partial}{\partial t}\eta(t,x)+2x\cdot\frac{\partial^2}{\partial x^2}\eta(t,x)=0.$$

The corresponding H-transform causes the following additional drift in the branching behaviour:

$$\eta^{-1}(t,\mu(1))\cdot\frac{\eta(t,\mu(1)+\varepsilon)-\eta(t,\mu(1))}{\varepsilon}$$
$$\to\frac{\partial}{\partial x}\eta(t,x)\Big|_{x=\mu(1)}\eta^{-1}(t,\mu(1)).$$

Hence the total drift is the drift which an η-transform of Feller's continuous-state branching process causes, i.e., $2\alpha x \frac{\partial}{\partial x} \log \eta(s,x)|_{x=\mu(1)}$. In [O1] the Martin boundary of Feller's continuous-state branching process was identified as $[0,\infty) \cup \{\emptyset\}$. The extremal space-time harmonic functions η^e, $e \in [0,\infty) \cup \{\emptyset\}$ are:

$$\eta^\emptyset(s,x) = 1,$$
$$\eta^0(s,x) = x$$
$$\eta^c(s,x) = \sum_{k=1}^{\infty} \frac{(cx)^k}{k!(k-1)!} \left(\sum_{k=1}^{\infty} \frac{c^k}{k!(k-1)!} \right)^{-1} e^{-sc}, \qquad 0 < c < \infty.$$

The Martin boundary theory (cf. [Dy1, Fo1, Fo2]) yields the limit behaviour of the coordinate process in the *Martin topology*, i.e.,

$$P^{\eta^e}\left[\frac{X_t(1)}{t^2} \to e\right] = 1 \quad \text{if} \quad e \in (0,\infty),$$

$$P^{\eta^0}\left[X_t(1) > 0 \text{ for all } t \geqq 0 \text{ and } \frac{X_t(1)}{t^2} \to 0\right] = 1,$$

and

$$P^\emptyset[\text{There is a } t > 0 \text{ s.t. } X_t(1) = 0] = 1,$$

and also the interpretation of the H-transform as a conditioned process:

$$P^{H^e} = \lim_{\frac{a_t}{t^2} \to e} P[\cdot | X_t(1) = a_t > 0], \qquad e \in [0,\infty).$$

(iv) *Harmonic functions of product type.* Let $\alpha = 1$. A function $H(t,\mu) = \mu(h_1(t))\, h_2(t,\mu(1))$ is space-time harmonic iff h_1 and h_2 are space-time harmonic for Brownian motion and Feller's continuous-state branching process conditioned on non-extinction, respectively. In particular,

$$h_2(t,x) = x^{-1}\eta(t,x)$$

where η is space-time harmonic for Feller's continuous-state branching process and of course (cf. [O1])

$$\frac{\partial}{\partial t} h_2(t,x) + 2x \frac{\partial^2}{\partial x^2} h_2(t,x) + 4 \frac{\partial}{\partial x} h_2(t,x) = 0.$$

Such a function H produces in its H-transform the interaction term

$$2\alpha \left(\frac{h_1(t,x)}{\mu(h_1(t))} + \frac{\partial}{\partial x} \log h_2(t,\mu(1)) \right) \tag{2.8}$$

at position x, time t which depends on the state μ of the process.

(v) *Multiplicative harmonic functions.* A function $H(t,\mu)=\exp(-\mu(h(t))$ is space-time harmonic if h satisfies

$$\frac{\partial}{\partial t}h(t,x)+\frac{1}{2}\Delta h(t,x)=\alpha h^2(t,x).$$

The additional total drift is $2\alpha(-\mu(h(t)\,F'))$. This H-transform keeps the branching property and does not entail any additional interaction. It belongs to the class of non-critical superprocesses considered by several authors, cf. e.g. [W] or [ER].

2.2.3 Martingale problem of the H-transform on D_{M_0}. In order to use convenient tightness criteria in Sect. 3 and 4 we have to consider the martingale problem on a slightly changed state space. Let $\dot{\mathbb{R}}^d:=\mathbb{R}^d\cup\{\infty\}$ denote the one point compactification of $\mathbb{R}^d$. We extend every $f\in C_0$ to $\tilde{f}\in C(\dot{\mathbb{R}}^d)$ by $\tilde{f}(\infty)=0$. Let $M_0:=M(\dot{\mathbb{R}}^d)$ denote the space of finite measures on $\dot{\mathbb{R}}^d$, cf. [I]. Every element $\mu\in M$ is viewed as an element $\tilde{\mu}\in M_0$ by $\tilde{\mu}=\mu(\cdot\setminus\{\infty\})$. According to a result of Iscoe [I] the super-Brownian motion does not charge the point at infinity. In the proof of Theorem A we only use test functions f vanishing at infinity. Therefore these arguments are independent of the fact whether we view X as a D_M or a D_{M_0} valued random variable. Hence it is obvious that the only probability measure on D_{M_0} such that for all positive $f\in C_0^2$ the process

$$M_t[e_{\tilde{f}}]:=e_{\tilde{f}}(X_t)-e_{\tilde{f}}(X_0)+\int_0^t e_{\tilde{f}}(X_s)\,X_s\left(\frac{1}{2}\tilde{\Delta}f+2\alpha\frac{H'(s,X_s)}{H(s,X_s)}\tilde{f}-\alpha\tilde{f}^2\right)ds$$

is a local martingale, is the distribution of $\tilde{X}$ under P^H. In particular, the solution $\tilde{P}$ of this martingale problem on D_{M_0} satisfies $\tilde{P}[D_M]=1$.

3 Additive H-transforms

First we prove that an additive H-transform of the super-Brownian motion is the weak limit of additive H-transforms of the approximating branching Brownian motion. Therefore we summarize some properties of H-transforms of branching Brownian motion.

3.1 H-transform of the n-th Approximation

Let the operator $\mathscr{A}^n$ be defined on $\mathscr{F}\mathscr{B}$ by

$$\begin{aligned}\mathscr{A}^n F(s,\mu):=&\frac{\partial}{\partial s}\phi(s,\tilde{\mu}(\tilde{f}(s)))\\&+\sum_{k=1}^m\frac{\partial}{\partial x_k}\phi(s,\tilde{\mu}(\tilde{f}(s)))\,\mu\left(\frac{1}{2}\Delta f_k(s)+\frac{\partial}{\partial s}f_k(s)\right)\\&+\frac{1}{2n}\sum_{k,l=1}^m\frac{\partial^2}{\partial x_k\,\partial x_l}\phi(s,\tilde{\mu}(\tilde{f}(s)))\cdot\mu(\nabla f_l\cdot\nabla f_k)\\&+\alpha n^2\mu\left(F\left(s,\left(\mu-\frac{1}{n}\delta\right)\right)+F\left(s,\left(\mu+\frac{1}{n}\delta\right)\right)-2F(s,\mu)\right).\end{aligned}$$

The distribution P^n on D_M of the rescaled branching Brownian motion is the unique probability measure on D_M such that

$$F(t, X_t) - F(0, X_0) - \int_0^t \mathscr{A}^n F(s, X_s)\, ds$$

is a local martingale for every $F \in \mathscr{F}\mathscr{B}$, cf. e.g. [R].

Since a space-time harmonic function $H^n \in \mathscr{F}\mathscr{B}$ satisfies P^n a.s. that $\mathscr{A}^n H^n(s, X_s) = 0$ ds a.s., and since $\mathscr{A}^n$ exhibits for $G(\mu) = \psi(\vec{\mu}(\vec{g}))$, $F(\mu) = \phi(\vec{\mu}(\vec{f})) \in \mathscr{F}\mathscr{B}$ the product rule

$$\begin{aligned}\mathscr{A}^n FG(\mu) = {} & F(\mu) \cdot \mathscr{A}^n G(\mu) + G(\mu) \cdot \mathscr{A}^n F(\mu) \\ & + \frac{1}{2n} \sum_{i,j} \frac{\partial \phi}{\partial x_i}(\vec{\mu}(\vec{f})) \frac{\partial \psi}{\partial x_j}(\vec{\mu}(\vec{g}))\, \mu(\nabla g_j \cdot \nabla f_i) \\ & + n^2 \alpha \mu \left(\left(F\left(\mu - \frac{1}{n}\delta\right) - F(\mu) \right) \left(G\left(\mu - \frac{1}{n}\delta\right) - G(\mu) \right) \right. \\ & \left. + \left(F\left(\mu + \frac{1}{n}\delta\right) - F(\mu) \right) \left(G\left(\mu + \frac{1}{n}\delta\right) - G(\mu) \right) \right),\end{aligned}$$

we easily derive the following

Proposition 3.1 *Let P^{n,H^n} be the distribution of an H^n-transform of the n-th approximation of the super-Brownian motion, $H^n = \eta^n(\vec{\mu}(\vec{h}^n(s))) \in \mathscr{F}\mathscr{B}$. Then for every function $F \in \mathscr{F}\mathscr{B}$ the process $M^{n,H^n}[F]$ defined by*

$$M_t^{n,H^n}[F] := F(t, X_t) - F(0, X_0) - \int_0^t \mathscr{A}^{n,H^n} F(s, X_s)\, ds \tag{3.1}$$

is a local martingale under P^{n,H^n}, where

$$\begin{aligned}\mathscr{A}^{n,H^n} F(s, \mu) := {} & \mathscr{A}^n F(s, \mu) + \frac{1}{2n} \sum_{i,j} \frac{\partial \phi}{\partial x_i}(s, \vec{\mu}(\vec{f}(s))) \frac{\frac{\partial \eta^n}{\partial x_j}(s, \vec{\mu}(\vec{h}^n(s)))}{H^n(s, \mu)}\, \mu(\nabla h_j^n(s)\, \nabla f_i) \\ & + n^2 \alpha \mu \left(\left(F\left(\mu - \frac{1}{n}\delta\right) - F(\mu) \right) \frac{H^n\left(s, \mu - \frac{1}{n}\delta\right) - H^n(s, \mu)}{H^n(s, \mu)} \right. \\ & \left. + \left(F\left(\mu + \frac{1}{n}\delta\right) - F(\mu) \right) \frac{H^n\left(s, \mu + \frac{1}{n}\delta\right) - H^n(s, \mu)}{H^n(s, \mu)} \right).\end{aligned}$$

Examples. (i) Since under P^n the intensity measure of X_t is also determined by $E^n[X_t(f)] = \mu_0(T_t f)$, every space-time harmonic function of the Brownian motion gives an additive space-time harmonic function of the branching Brownian motion at every approximation level. This is important in Theorem 3.2 below.

(ii) In the case $n=1$, $\alpha=1$ all space-time harmonic functions $\eta^1(s,\mu(1))$ of the total mass are computed in [O1].

3.2 Convergence theorem for additive H-transforms

We now state and prove the convergence theorem for additive H-transforms. For the proof we use the method of [MR, RR2], but the approximating H-transforms do not satisfy the assumption of boundedness of the mean of the reproduction law and of the drift coefficient in the one-particle motion in [MR, RR2], because $\frac{h(s,x)}{\mu(h(s))}$ is not uniformly bounded in $x\in\mathbb{R}^d$, $\mu\in M$. Hence, in particular, we have to control the drift of the motion of an individual particle. Therefore we first consider extremal space-time harmonic functions $h(t,x) = h^a(t,x) = e^{ax-\frac{1}{2}a^2 t}$.

Theorem 3.2 *Let h be an extremal space-time harmonic function of the Brownian motion. Let* $H(s,\mu):=\mu(h(s))$ *be the corresponding additive space-time harmonic function of the super-Brownian motion and of the approximating branching Brownian motions. Let* $P^{n,H}$ *be the distribution of the additive H-transform of the rescaled branching Brownian motion on* D_M*. Then the sequence* $\{P^{n,H}\}_n$ *converges weakly to the additive H-transform of the super-Brownian motion* P^H.

By the integral representation of space-time harmonic functions and by bounded convergence we obtain

Corollary 3.3 *Let h be a space-time harmonic function of the Brownian motion,* $P^{n,H}$ *the corresponding additive H-transform of the approximating branching Brownian motion and* P^H *the additive H-transform of the super-Brownian motion. Then the sequence* $\{P^{n,H}\}_n$ *converges weakly to* P^H.

In the following subsections we prove that $P^{n,H}[\tilde{X}\in\cdot]\Rightarrow P^H[\tilde{X}\in\cdot]$ as measures on D_{M_0}. But, because by Subsection 2.2.3 $P^{n,H}[\tilde{X}\in D_M]=P^H[\tilde{X}\in D_M]=1$, it follows (e.g. by Corollary 3.2 in [EK, Chap. 3]) that $P^{n,H}$ converges also weakly to P^H as measures on D_M; cf. also "main remark" in [MR]. We will therefore denote $P^{n,H}[\tilde{X}\in\cdot]$ and $P^H[\tilde{X}\in\cdot]$ also by $P^{n,H}$ and P^H and the coordinate process on D_{M_0} by X.

3.2.1 The process of the total mass and the exponential martingale. Since the total mass process of an additive H-transform is a critical branching process conditioned on survival we have the following result (cf. e.g. [KW]).

Lemma 3.4 *Let* $P^{n,H}$ *be the distribution of the additive H-transform, not neccessarily with an extremal* h^a*, of the rescaled branching Brownian motion* $X^{n,\alpha}$ *on* D_{M_0} *with* $P^{n,H}[X_0=\mu_0]=1$*. Then the coordinate process X on* D_M *satisfies the following equations*

$$E^{n,H}[X_t(1)\,|\,X_r(1)]=2\alpha(t-r)+X_r(1) \quad \textit{and} \tag{3.2}$$

$$E^{n,H}[X_t^2(1)]\leqq\mu_0(1)+4\alpha t(\alpha t+1). \tag{3.3}$$

The main tool for the convergence result is the 'exponential martingale'.

Lemma 3.5 *Consider the situation in Lemma 3.4 and assume in addition that* $h(s,x)=h^a(s,x)=e^{a\cdot x-(s/2)a^2}$ *is an extremal space-time harmonic function of Brownian motion. Then the process* $M^{n,H}[e_f]$ *is a martingale for* $f\in C_b^2$ *under* $P^{n,H}$.

Proof. The random variable $\mathscr{A}^{n,H} e_f(s, X_s)$ is bounded by

$$X_s(1)(\|\tfrac{1}{2}\Delta f\|_\infty + (2n)^{-1}\|\nabla f\|_\infty^2 + \alpha n^2 \|e^{\frac{1}{n}f} + e^{-\frac{1}{n}f} - 2\|_\infty)$$
$$+(2n)^{-1} \sup_i \left\|\frac{\partial}{\partial x_i} f\right\|_\infty \sum_{i=1}^d |a_i| + n\alpha(\|1 - e^{n^{-1}f}\|_\infty + \|1 - e^{-n^{-1}f}\|_\infty).$$

For a localizing sequence $\{S_m\}_m$ of stopping times and a stopping time T bounded by $N \in \mathbb{N}$ this estimate yields

$$|M^{n,H}_{T \wedge S_m}[e_f]| \leqq 2 + N c_1 \sup_{s \leqq N} X_s(1) + N c_2$$

with two reals c_1 and c_2. Lemma 3.4 and the submartingale inequality prove that $\{M^{n,H}_{T \wedge S_m}[e_f]\}_m$ is a uniformly integrable family of random variables. Hence the assertion is proved. □

3.2.2 *Weak convergence of* $\{P^{n,H}\}_n$

Step 1. Tightness.

It is well-known, cf. e.g. [D, 3.7], that a sequence of measures $\{P^n\}_n \subset \mathscr{P}(D_{M_0})$ is tight if the one dimensional projections $\{P^n \circ X(f)\}_n$ are tight for every $f \in C_c^2 \cup \{1\}$. For tightness of the projections $X(f)$ we apply the criterion in [EK, Theorem 8.6, Chap. 3].

Since $E^{n,H}[X_t(1)] = \mu_0(1) + 2\alpha t$ is independent of n the first condition in [EK, Theorem 8.6, Chap. 3] is easily proved by the Chebyshev inequality. In order to prove the second condition in [EK, Theorem 8.6, Chap. 3] we choose for every n a sequence of stopping times $\{S^n_m\}_m$ which localizes $M[F]$ as well with $\phi(x) = x$ as with $\phi(x) = x^2$. By the representation of the semimartingales $X(f)$ and $X^2(f)$ the following inequality is obtained for $t \geqq 0$, $0 \leqq u \leqq \delta \leqq 1$:

$$E^{n,H}[(X_{(t+u)\wedge S^n_m}(f) - X_{t \wedge S^n_m}(f))^2 | \mathscr{F}_t]$$
$$= E^{n,H}[X^2_{(t+u)\wedge S^n_m}(f) - 2X_{t\wedge S^n_m}(f)\, X_{(t+u)\wedge S^n_m}(f) + X^2_{t\wedge S^n_m}(f) | \mathscr{F}_t]$$
$$= E^{n,H}\Big[X^2_{(t+u)\wedge S^n_m}(f) - X^2_{t\wedge S^n_m}(f) - 2X_{t\wedge S^n_m}(f)$$
$$\cdot \int_{t\wedge S^n_m}^{(t+u)\wedge S^n_m} \left(X_s\left(\frac{1}{2}\Delta f\right) + \frac{1}{2n} X_s\left(\frac{\nabla h^a(s)\, \nabla f}{X_s(h^a(s))}\right) + 2\alpha X_s\left(\frac{h^a(s)}{X_s(h^a(s))} f\right)\right) ds \Big| \mathscr{F}_t\Big]$$
$$= E^{n,H}\Bigg[\int_{t\wedge S^n_m}^{(t+u)\wedge S^n_m} \Bigg(2X_s\left(\frac{1}{2}\Delta f\right) X_s(f) + \frac{1}{n} X_s\left(|\nabla f|^2 + f \cdot X_s\left(\frac{\nabla h^a(s)\, \nabla f}{X_s(h^a(s))}\right)\right)$$
$$+ 4\alpha X_s(f)\, X_s\left(\frac{h^a(s)}{X_s(h^a(s))} f\right) + 2\alpha X_s(f^2) - 2X_{t\wedge S^n_m}(f)\, X_s(\tfrac{1}{2}\Delta f)$$
$$- X_{t\wedge S^n_m}(f) \frac{1}{n} X_s\left(\frac{\nabla h^a(s)\, \nabla f}{X_s(h^a(s))}\right) - X_{t\wedge S^n_m}(f)\, 4\alpha X_s\left(\frac{h^a(s)}{X_s(h^a(s))} f\right)\Bigg) ds \Bigg| \mathscr{F}_t\Bigg]$$
$$\leqq \delta E^{n,H}\Bigg[\left(\sup_{0 \leqq s \leqq t+1} X_s(1)\right)^2 \cdot 4\|\tfrac{1}{2}\Delta f\|_\infty \cdot \|f\|_\infty$$
$$+ \sup_{0 \leqq s \leqq t+1} X_s(1) \left(\frac{2}{n}\|\nabla f^2\|_\infty + \frac{2}{n}\|f\|_\infty \sum_{i=1}^d a_i \left\|\frac{\partial}{\partial x_i} f\right\|_\infty + \alpha 2\|f\|_\infty^2\right) \Bigg| \mathscr{F}_t\Bigg].$$

Letting $m \to \infty$ we have by Fatou's lemma

$$(3.4)\quad E^{n,H}[(X_{(t+u)}(f)-X_t(f))^2|\mathscr{F}_t] \leqq \delta E^{n,H}[(\sup_{0\leqq s\leqq t+1} X_s(1))^2 \cdot 4\|\tfrac{1}{2}\Delta f\|_\infty \cdot \|f\|_\infty$$
$$+\sup_{0\leqq s\leqq t+1} X_s(1)\left(\frac{2}{n}\|\nabla f^2\|_\infty + \frac{2}{n}\|f\|_\infty \sum_{i=1}^d a_i \left\|\frac{\partial}{\partial x_i} f\right\|_\infty + \alpha 2\|f\|_\infty^2\right)\Bigg|\mathscr{F}_t\Bigg].$$

Hence it remains to show that the expression

$$(3.5)\qquad \sup_n E^{n,H}[(\sup_{0\leqq s\leqq t+1} X_s(1))^2]$$

is finite. Since $X(1)$ is a non-negative submartingale under every P^n, Lemma 3.4 and the strong maximal L^2-inequality complete the proof of tightness.

Step 2. Identification of the limit point.

Let $P^{n_k,H} \Rightarrow P^\infty$. We prove that under P^∞ the process $M^H[e_f]$ defined by

$$M_t^H[e_F] := e_f(X_t) - e_f(X_0) + \int_0^t X_s\left(\frac{1}{2}\Delta f + 2\alpha \frac{h^a(s)}{X_s(h^a(s))} f - \alpha f^2\right) e_f(X_s)\, ds$$

is a martingale for every positive $f \in C_0^2$.

Let $\mathscr{A}^n$ be the mapping corresponding to the particle system X^n and $\mathscr{A}$ the mapping corresponding to the H-transform of the super-Brownian motion:

$$\mathscr{A}^n e_f(s,\mu) := -\mu\left(\frac{1}{2}\Delta f + \frac{\nabla h(s)\cdot \nabla f}{n\mu(h(s))}\right) e_f(\mu) + (1/2n)\,\mu(|\nabla f|^2)\, e_f(\mu)$$
$$+ e_f(\mu)\,\alpha n^2 \mu\left(\left(1+\frac{h(s)}{n\mu(h(s))}\right) e^{-\frac{1}{n}f} + \left(1-\frac{h(s)}{n\mu(h(s))}\right) e^{+\frac{1}{n}f} - 2\right);$$

$$\mathscr{A} e_f(s,\mu) := -\mu\left(\frac{1}{2}\Delta f + 2\alpha \frac{h(s)}{\mu(h(s))} f - \alpha f^2\right) e_f(\mu).$$

It sufficies to show that for $0 < s_1 < \ldots < s_l < s < t$, $G \in C_b\left(\bigtimes_{i=1}^l M\right)$

$$E^\infty\left[\left(e_f(X_t) - e_f(X_s) - \int_s^t \mathscr{A} e_f(\sigma, X_\sigma)\, d\sigma\right) G(X_{s_1}, \ldots, X_{s_l})\right] = 0.$$

By weak convergence

$$\lim_{k\to\infty} E^{n_k,H}[(e_f(X_t) - e_f(X_s))\, G(X_{s_1}, \ldots, X_{s_l})]$$
$$= E^\infty[(e_f(X_t) - e_f(X_s))\, G(X_{s_1}, \ldots, X_{s_l})].$$

Since $P^{n_k,H}$ solves the martingale problem in Lemma 3.5, it remains to show that

$$\left| E^{\infty}\left[\int_s^t \mathscr{A} e_f(\sigma, X_\sigma)\, d\sigma\, G(X_{s_1}, \ldots, X_{s_l})\right] - E^{n_k,H}\left[\int_s^t \mathscr{A}^{n_k} e_f(\sigma, X_\sigma)\, d\sigma\, G(X_{s_1}, \ldots, X_{s_l})\right]\right|$$

tends to 0 as $k \to \infty$. But this expression is dominated by

$$E^{n_k,H}\left[\left|\int_s^t \mathscr{A} e_f(\sigma, X_\sigma) - \mathscr{A}^{n_k} e_f(\sigma, X_\sigma)\, d\sigma\right| |G(X_{s_1}, \ldots, X_{s_l})|\right]$$
$$+\left| E^{\infty}\left[\int_s^t \mathscr{A} e_f(\sigma, X_\sigma)\, d\sigma\, G(X_{s_1}, \ldots, X_{s_l})\right] - E^{n_k,H}\left[\int_s^t \mathscr{A} e_f(\sigma, X_\sigma)\, d\sigma\, G(X_{s_1}, \ldots, X_{s_l})\right]\right|.$$

By a Taylor expansion it is easy to prove that the first term is bounded by

$$\frac{1}{n}((t-s)\, c_1 + c_2 \sup_{r \leqq t} E^{n_k,H}[X_r(1)]), \qquad c_1, c_2 < \infty,$$

and converges to 0 by Lemma 3.4. The function $\mathscr{A} e_f$ is not bounded in μ but continuous and $\mathscr{A} e_f(s, X_s)$ is dominated by the function $c_3\, X_s(1)$, $c_3 < \infty$. By Lemma 3.4 the latter function is bounded in $L^2(P^{n,H})$ uniformly in n and in $s \leqq t$. Hence, by uniform integrability and weak convergence the second term tends also to 0. □

Remark. This approximation differs from the approximation in Theorem B, because the particles are also subject to an interacting drift; namely the H-transform of a branching Brownian motion with individual mass 1 has

$$\frac{1}{2}\Delta f + \frac{1}{\mu(h(s))} \nabla h(s) \cdot \nabla f \tag{3.6}$$

as the generator of the motion and

$$p_{0,2}^H(\mu, s, x) = 1/2\left(1 - \frac{h(s, x)}{\mu(h(s))}\right), \quad 1/2\left(1 + \frac{h(s, x)}{\mu(h(s))}\right) \text{resp.} \tag{3.7}$$

as the branching law. The new drift in the motion shows that the drift caused by an h-transform of the Brownian motion is distributed among all living particles, as easily seen by considering $h = h^a$. In the limit, however, the interacting drift of the one particle motion disappears by scaling.

For $h = 1$ the branching law $p_{0,2}^H = 1/2(1 - 1/\mu(1))$, $1/2(1 + 1/\mu(1))$, resp., corresponds to the branching law of a Galton-Watson process conditioned on non-extinction. For general h, at branching times the particles are weighted with

h and are subject to the same reproduction as in the case $h=1$. In particular, this shows very nicely, that an H-transform moves to the subset of the state space where H takes greater values.

3.3 *Laplace functional of additive H-transforms*

We set $\alpha=\frac{1}{2}$.

Theorem 3.6 *Let P^H be the H-transform of the super-Brownian motion. Then for every $f\in C_0^2(\mathbb{R}^d)$, $r\leqq t$, we have*

$$(3.8)\qquad E^H[e_f(X_t)\,|\,X_r=\mu] = \frac{1}{\mu(h(r))}\,\mu\left(h(r,\cdot)\,\mathscr{P}^h_{r,\cdot}\left[\exp\left(-\int_r^t V_t^s f(\xi_s)\,ds\right)\right]\right) e^{-\mu(V_t^r f)},$$

where ξ is the coordinate process on $C([0,\infty),\mathbb{R}^d)$ and $\mathscr{P}^h_{r,x}$ denotes the h-transform of the Brownian motion in $\mathbb{R}^d$ starting in x at time r. The operator V_t^r is defined for $r<t$ as the time-inhomogeneous generalization of the non-linear semigroup associated with the super-Brownian motion via its Laplace functional, cf. [Dy2, D]:

$$(3.9)\qquad \frac{\partial}{\partial r} V_t^r f = -\frac{1}{2}\Delta V_t^r f + \frac{1}{2}(V_t^r f)^2,\; V_t^t f = f.$$

Proof.

$$(3.10)\qquad E^H[e_f(X_t)\,|\,X_r=\mu] = \frac{1}{\mu(h(r))}\,E[X_t(h(t))\,e_f(X_t)\,|\,X_r=\mu].$$

The second factor on the right-hand sight is computed by differentiation as $\mu(U_t^r f)\,e^{-\mu(V_t^r f)}$, where

$$(3.11)\qquad U_t^r f := \frac{\partial}{\partial\lambda} V_t^r(f+\lambda h(t))\Big|_{\lambda=0}$$

is the solution of

$$\frac{\partial}{\partial r} U_t^r f = -\frac{1}{2}\Delta U_t^r f + (V_t^r f)\cdot U_t^r f, \qquad U_t^t f = h.$$

The Feynman-Kac formula [Fr, p. 148] yields

$$(3.12)\qquad U_t^r f(x) = \mathscr{P}^1_{r,x}\left[h(t,\xi_t)\exp\left(-\int_r^t V_t^s f(\xi_s)\,ds\right)\right].$$

(Since $E[X_t(h(t))\,e^{-X_t(f)}\,|\,X_r=\mu] \leqq E[X_t(h(t))\,|\,X_r=\mu] = \mu(h(r)) < \infty$, we may interchance differentiation and integration.) $\square$

By differentiation, we get

Corollary 3.7 *The intensity measure of X_t under P^H is computed as follows:*

$$E^H[X_t(f)|X_r=\mu]$$
$$=\frac{1}{\mu(h(r))}\int_r^t \mu(\mathscr{P}_{r,\cdot}[h(s,\xi_s)\mathscr{P}_{s,\xi_s}[f(\xi_t)]])\,ds+\mu(\mathscr{P}_{r,\cdot}[f(\xi_t)]).$$

3.4 Conditioning and Palm measures

Now we want to express the Laplace functional of an additive H-transform by the Palm measures of the super-Brownian motion. The Palm measures $(Q_y)_{y\in\mathbb{R}^d}$ of a random measure Q over $\mathbb{R}^d$ are defined as the components of the desintegration of the Campbell measure C_Q, i.e.,

$$C_Q[A,B]=\int_A v(B)\,Q[dv]=\int_B\int_A Q_y[dv]\;C_Q[M,dy],$$

$A\in\mathscr{B}(M)$, $B\in\mathscr{B}(\mathbb{R}^d)$, [K, Sect. 10]. The measure $C_Q[M,dy]$ on $\mathbb{R}^d$ is the intensity measure of the random measure Q.

Lemma 3.8 *Let $v\in M$, f, $g(t,\cdot)\in C_0^2$, $x\in\mathbb{R}^d$ and $0<r<t$. The Campbell measure $C_{r,t,v}$ of $P[X_t\in\cdot|X_r=v]$, the distribution of the super-Brownian motion at time t starting in v at time r, is characterized by*

$$\int_{M\times\mathbb{R}^d} g(t,x)\,e_f(\mu)\,C_{r,t,v}[d\mu,dx]$$
$$=\exp(-v(V_r^t f))\int_{\mathbb{R}^d}\mathscr{P}_{r,x}\Big[g(t,\xi_t)\exp\Big(-\int_r^t V_t^s f(\xi_s)\,dr\Big)\Big]\,v(dx).$$

For $v=\delta_x$, let $((R_{r,t,x})_y)_{y\in\mathbb{R}^d}$ be the collection of the corresponding Palm measures on M. These have the Laplace functionals

$$(R_{r,t,x})_y[e_f]=e^{-V_r^t f(x)}\mathscr{P}_{r,x}\Big[\exp\Big(\int_r^t -V_t^s f(\xi_s)\,ds\Big)\Big|\xi_t=y\Big].$$

Proof. The formula for the Campbell measures follows by differentiation as in the proof of Theorem 3.6. The Laplace functional of the Palm measure is e.g. given in [D, 11.6].

Lemma 3.8 combined with Theorem 3.6 yields

Corollary 3.9 *Let P^H be the additive H-transform of super-Brownian motion with a space-time harmonic function h of Brownian motion. Let $v_{x,r,t}^h$ be the distribution of ξ_t under $\mathscr{P}_{r,x}^h$, the h-transform of Brownian motion,* i.e., *$v_{x,r,t}^h$ is the normal distribution $N(x+a,t-r)$ if $h=h^a$ is extremal. Then*

$$(3.13)\qquad E^H[e_f(X_t)|X_r=\delta_x]=\int_{\mathbb{R}^d}(R_{r,t,x})_y[e_f]\,v_{x,r,t}^h(dy).$$

The Palm measure at y can be viewed as the distribution of the random measure given that y is in the support of X, cf. e.g. [K, Sect. 10.2]. Hence Corollary

3.9 suggests that P^H is the super-Brownian motion conditioned on the event "there exists a moving particle ξ_t in the support of the super-Brownian motion such that $\frac{\xi_t}{t} \to a$ as $t \to \infty$, i.e., $\xi_t \to a$ in the Martin topology of Brownian motion".

This suggestion is justified in the following Theorem 3.10. We use the embedding of the Campbell measure of super-Brownian motion in a dynamical structure, which is based on a representation of conditioned superprocesses by S.N. Evans [E]. Theorem 3.10 generalizes this representation to all additive H-transforms.

According to [E] there exists a $M \times \mathbb{R}^d$-valued Markov process (Y_t, ζ_t) with transition function Π defined by

(3.14)
$$\int_{M \times \mathbb{R}^d} [e_f(\mu)\, g(y)]\, \Pi(r,(v,x),t,(d\mu, dy)) = e^{-v(V_t^r f)} \mathscr{P}_{r,x}\left[\exp\left(-\int_r^t V_t^s f(\xi_s)\, ds\right) g(\xi_t)\right]$$

for $f, g \in C_0^2$. The process (Y_t) is a super-Brownian motion conditioned on survival and is distributed as a sum $Y_t = Y_t' + X_t'$, where X' is a super-Brownian motion independent of $(Y_t', \zeta_t)_{t \geqq 0}$. For a fixed path $(\xi_s) = (z_s)$, the random measure Y_t' has the Laplace functional

(3.15)
$$\exp\left(-\int_r^t V_t^s f(z_s)\, ds\right).$$

This is the Laplace functional of the random measure

$$\int_r^t \int_M v N_z(ds, dv),$$

where N_z is a Poisson random measure with intensity measure

$$[s_1, s_2) \times A \to \int_{s_1}^{s_2} D_t^s(z(s), A)\, ds \qquad r < s_1 < s_2 < t.$$

The measure D_t^s is connected with super-Brownian motion by the representation of a superprocess as an infinitely divisible measure:

$$P[e_f(X_t) \mid X_r = \mu] = \exp(-\mu(\int (1 - e_f(v))\, D_t^r(\cdot, dv))).$$

The process Y' is therefore interpreted as an "immortal particle that moves around as a Brownian motion and throws off pieces of mass, which then proceed to evolve as super-Brownian motions". A similar description is given in [CRW, GRW] for the Palm measures of branching diffusions.

With this preparation we can state the dynamical version of Corollary 3.9.

Theorem 3.10 *Let (Y_t, ζ_t) be the continuous Markov process on $M \times \mathbb{R}^d$ with transition function Π starting from $(\delta_0, 0)$. Let Q be the distribution of this process and let h^a be an extremal space-time harmonic function of the Brownian motion.*

1. The function $\mathbb{H}(s,(\mu, x)):=h^a(s,x)$ *is space-time harmonic for* Q *and the* $\mathbb{H}$*-transform* $Q^{\mathbb{H}}$ *of* Q *is the limit of*

$$Q[\cdot\,|\,\zeta_t=x_t] \quad as \quad t\to\infty \quad and \quad \frac{x_t}{t}\to a.$$

2. The measure P^H *arises as the limit of*

$$(Q[\cdot\,|\,\zeta_t=x_t])\circ p^t \quad as \quad t\to\infty \quad and \quad \frac{x_t}{t}\to a$$

where $p^t((\mu_s)_{s\le t},(x_s)_{s\le t}):=(\mu_s)_{s\le t}$ *is the projection onto the first coordinate.*

3. The measure P^H *is also the distribution of the projection of* $Q^{\mathbb{H}}$ *onto the first coordinate. The measure* $Q^{\mathbb{H}}$ *is the distribution of a Markov process* $(X'_t+Y'_t, \zeta_t)_{t\ge 0}$ *where the process* X' *is a super-Brownian motion and the distribution* $\hat{Q}$ *of* (Y'_t, ζ_t) *is independent of* X' *and determined by*

$$\int e_f(\mu)\, g(\mu)\, \hat{Q}(d\mu, dy)=\mathscr{P}^{h^a}_{0,0}\left[\exp\left(-\int_0^t V_t^r f(\xi_r)\, dr\right) g(\xi_t)\right]$$

for $f, g\in C_0^2$. *The random measure* Y' *can be viewed as an immortal particle moving as an* h^a*-transform of the Brownian motion,* i.e., *a Brownian motion with drift* a. *This immortal particle produces at random times mass which evolves as a super-Brownian motion. The measure* Y'_t *is the sum of all these super-Brownian motions at time* t.

Proof. Setting $f=0$ and $g(t,x)=h^a(t,x)$ in (3.14) we derive the harmonicity of $\mathbb{H}$. Next we show that

$$Q[\cdot\,|\,\xi_t]=Q^{\xi_t,t}[\cdot],$$

where $Q^{x,t}$ is the distribution of a Markov process on the time interval $[0,t]$ with transition function $\Pi^{x,t}$ which is determined by

$$\int_{M\times\mathbb{R}^d} e_f(\mu_s)\, g(x_s)\, \Pi^{x,t}(r,(\mu_r, x_r), s, (d\mu_s, dx_s)) = \exp(-\mu_r(V_s^r f))\, \mathscr{P}^{x,t}_{r,x_r}\left[\exp\left(-\int_r^s V_s^r f(\xi_\tau)\, d\tau\right) g(\xi_s)\right], \tag{3.16}$$

$0\le r<s<t$. The measure $\mathscr{P}^{x,t}_{r,x_r}$ is the distribution of the Brownian bridge from x_r to x in the time interval $[r,t]$. We introduce the notation $\hat{\mu}:=(\mu, x)$ for an element in $M\times\mathbb{R}^d$. For notational simplicity we set $F_i(\hat{\mu})=e_{f_i}(\mu)\, g_i(x)$, $i=1, 2$ and show for $s_1<s_2<t$:

$$Q[Q^{\xi_t,t}[F_1(\hat{X}_{s_1})\, F_2(\hat{X}_{s_2})]\, g(\xi_t)]=Q[F_1(\hat{X}_{s_1})\, F_2(\hat{X}_{s_2})\, g(\xi_t)]. \tag{3.17}$$

The Markov property and (3.16) implies that the left-hand sight of (3.17) is equal to

(3.18)
$$\exp(-V_{s_1}^0(f_1+V_{s_2}^{s_1}f_2))\cdot\mathscr{P}_{0,x}\Big[g_1(\xi_{s_1})\,g_2(\xi_{s_2})\cdot\exp\Big(-\int_0^{s_1}V_{s_1}^{\tau}(f_1+V_{s_2}^{s_1}f_2)(\xi_\tau)\,d\tau-\int_{s_1}^{s_2}V_{s_2}^{\tau}f_2(\xi_\tau)\Big)g(\xi_t)\Big],$$

which equals the right-hand sight of (3.17) by definition of Q and Π. Considering the finite dimensional distributions of $Q^{t,x}$ we see by the usual arguments in the Martin boundary theory of Brownian motion (cf. [Fo1, Fo2]), that Q^{t_n,x_n} converges iff $t_n\to\infty$ and $\frac{x_n}{t_n}\to a$ as $n\to\infty$, for some $a\in\mathbb{R}^d$. The limit exhibits the transition function Π^a determined by

(3.19)
$$\int_{M\times\mathbb{R}^d} e_f(\mu_s)\,g(x_s)\,\Pi^a(r,(\mu_r,x_r),(d\mu_s,dx_s))=\exp(-\mu_r(V_s^r f))\,\mathscr{P}_{r,x_r}^{h^a}\Big[\exp\Big(-\int_r^s V_s^{\tau}f(\xi_\tau)\,d\tau\Big)g(\xi_s)\Big].$$

The measure $\mathscr{P}^{h^a}$ is the h^a-transform of Brownian motion. The tightness of $\{Q^{t_n,x_n}\}_n$ follows by a slight modification of criteria given in [EK, Chap. 3] from the investigation of

$$Q^{t_n,x_n}[(e_f(Y_t)\,g(\zeta_t)-e_f(Y_s)\,g(\zeta_s))^2\,|\,\mathscr{F}_s].$$

This proves assertion 1. Assertions 2 and 3 follow in the same way as in [E, Theorem 2.7] by the definition of Π.

Remark. Theorem 3.10 for superprocesses has the following analogue for the critical binary branching Brownian motion $\mathbb{P}$. The Campbell measure can be identified as (cf. [CRW, (2.1), (2.2)])

(3.20)
$$\mathbb{E}_{r,\mu}[X_t(g)\,e_f(X_t)]=\mu\Big(\mathscr{P}_{r,\cdot}\Big[g(\xi_t)\,e^{-f(\xi_t)}\exp\Big(\int_r^t V_t^s(1-e^{-f})(\xi_s)\,ds\Big)\Big]\mathbb{E}_{r,\mu-\delta}[e_f(X_t)]\Big).$$

This yields for the corresponding Palm measure that

(3.21)
$$(\mathbb{R}_{r,t,\delta_x})_y[e_f]=\mathscr{P}_{r,x}\Big[\exp\Big(\int_r^t V_t^s(1-e^{-f})(\xi_s)\,ds\Big)\Big|\,\xi_t=y\Big]e^{-f(y)}.$$

Formulas (3.20) and (3.21) imply for an additive H-transform $\mathbb{P}^H$ that

(3.22)
$$\mathbb{E}_{r,\delta_x}^H[e_f(X_t)]=\frac{1}{h(r,x)}\int_{\mathbb{R}^d}(\mathbb{R}_{r,t,\delta_x})_y[e_f]\,v_{r,t,x}^h(dy).$$

In analogy to (3.14) let us define the transition function Ψ by

$$\int_{M\times\mathbb{R}^d} [e_f(\mu)\, g(y)]\, \Psi(r,(v,x),t,(d\mu,dy)) \tag{3.23}$$

$$=\mathscr{P}_{r,x}\left[g(\xi_t)\, e^{-f(\xi_t)}\exp\left(\int_r^t V_t^s(1-e^{-f})(\xi_s)\,ds\right)\right]\mathbb{E}_{r,v-\delta_x}[e_f(X_t)].$$

Thereby, a process is given which is associated with a critical binary branching Brownian motion in the same way as Q is associated with super-Brownian motion.

4 Approximation of H-transforms

As announced in the introduction we deal with H-transforms satisfying conditions (1.2) and (1.11). In Sect. 3 we chose the approximating particle systems as H-transforms of the rescaled branching Brownian motion. In the general case this is not possible because there is no obvious relation between space-time harmonic functions of branching Brownian motion and those of super-Brownian motion. Instead we plug in the additional term $\frac{H'_x(s,\mu)}{H(s,\mu)}$ in the branching law of the branching Brownian motion. Because additive H-transforms also satisfy (1.12) and (1.11) we have a second sequence of approximating particle systems.

4.1 Construction of interacting branching Brownian motion

Definition. An interacting branching Brownian motion X^I with interacting branching intensities $\{\beta(s,x,\mu)\, p_i(s,x,\mu)\}_{i=0,2}$ is a point measure process derived from a particle system. Every particle diffuses as a Brownian motion, dies with death rate $\beta(s,x,\mu)$, and gives then rise to 2 or 0 offsprings with probability $p_2(s,x,\mu)$, $p_0(s,x,\mu)$, respectively. It has the defining property that for all $F\in\mathscr{F}\mathscr{B}$ the processes

$$\left(F(t,X_t^I)-F(0,X_0^I)-\int_0^t \mathscr{A}^I F(s,X_s^I)\,ds\right)_{t\geqq 0} \tag{4.1}$$

are local martingales, where

$$\begin{aligned}\mathscr{A}^1 F(s,\mu) := &\frac{\partial}{\partial s}\phi(s,\vec{\mu}(\vec{f}(s)))\\ &+\sum_{k=1}^m \frac{\partial}{\partial x_k}\phi(s,\vec{\mu}(\vec{f}(s)))\,\mu\left(\frac{1}{2}\Delta f_k(s)+\frac{\partial}{\partial s}f_k(s)\right)\\ &+\frac{1}{2}\sum_{k=1}^m \frac{\partial^2}{\partial x_k\,\partial x_k}\phi(s,\vec{\mu}(\vec{f}(s)))\,\mu(|\nabla f_k|^2)\\ &+\mu(\beta(s,\mu)\,p_2(s,\mu)(F(s,(\mu+\delta))-F(s,\mu))\\ &\cdot\,\beta(s,\mu)\,p_0(s,\mu)(F(s,(\mu-\delta))-F(s,\mu))).\end{aligned} \tag{4.2}$$

Existence. Given the sequence of interacting branching random walks the proof of the existence of an interacting branching Brownian motion is the same as in [RR 2], where an interacting branching random walk is constructed for time-homogeneous and bounded parameters.

For our generalization to unbounded and time-inhomogeneous parameters we use the theory of multivariate point processes (cf. J. Jacod [J]). Let

$$Af(x) := \gamma \int_{\mathbb{R}^d} (f(y) - f(x))\, \pi(x; dy) \tag{4.3}$$

be the generator of a jump process in $\mathbb{R}^d$.

Let $\tilde{\Omega} = \{\tilde{\omega} = \{t_m, \mu_m)_{m\in\mathbb{N}}$ with $0 \leqq t_1 < t_2 < \ldots$, and $\mu_m \in M_0\}$. We define the predictable random measure v^p on $\tilde{\Omega}$ by

$$\begin{aligned} v^p(\{t_m, \mu_m\}_{m\in\mathbb{N}}; ds, dv) := X_{s-}\,&(\gamma \textstyle\int \delta_{\delta_y - \delta}(dv)\, \pi(\cdot\,; dy) \\ &+ \beta p_0(s, x, X_{s-})\, \delta_{-\delta}(dv) + \beta p_2(s, x, X_{s-})\, \delta_{+\delta}(dv))\, ds, \end{aligned}$$

with $X_s(\tilde{\omega}) := X_s(\{t_m, \mu_m\}_{m\in\mathbb{N}}) := \sum_{t_n \leqq s} \mu_n$. Let $v(\cdot,\cdot) = \sum_n \delta_{t_n, \mu_n}(\cdot,\cdot)$ be the canonical random measure on $\tilde{\Omega}$. There exists a probability measure $\tilde{P}$ on $\tilde{\Omega}$ such that the stopped process $(W*(v - v^p))^{t_n}$ is a uniformly integrable martingale for every n and every predictable $W(\tilde{\omega}, s, \mu)$(– "$*$" indicates integration –). For $F \in \mathscr{F}\mathscr{B}$ we consider the predictable function $W(\tilde{\omega}, s, \mu) = F(X_{s-} + \mu) - F(X_{s-})$. Then the process M^{IBRW} [F] defined by

$$\begin{aligned} M^{\mathrm{IBRW}}[F]_t := {} & F(X_t) - F(X_0) \\ & - \int_0^t ds\, \gamma \int_{\mathbb{R}^d} X_s((F(X_s + \delta_y - \delta) - F(X_s)\, \pi(\cdot, dy)) \\ & + X_s(\beta p_0(s, X_s)\, F(X_s - \delta) + \beta p_2(s, X_s)\, F(X_s + \delta) - F(X_s)) \end{aligned} \tag{4.4}$$

is a local martingale until $T := \sup_n t_n$.

For $\beta p_{0,2}(s, \mu) = n\alpha \left(1 \mp \dfrac{H'\left(s, \frac{\mu}{n}\right)}{nH\left(s, \frac{\mu}{n}\right)}\right)$ it is obvious by the construction of $\tilde{P}$ in [J] and by assumption (1.12) that

$$\tilde{P}[T \leqq s] \leqq P[Y_s = \infty] = 0, \tag{4.5}$$

where $Y = (Y_s)_{s \geqq 0}$ is a one dimensional Poisson point process with intensity $Y_s(K_1 + 1 + \gamma) + K_0$ (we add a new particle at every jump, regardless of whether it is a "branching" or a "walking" of the process). Therefore $\tilde{P}[T = \infty] = 1$. Hence, if we define P as the distribution of X under $\tilde{P}$ on D_M, we get an interacting branching random walk.

4.2 Properties of the rescaled process, Proof of Theorem B

The process $\{X^{1,n,H}\}_n$ are defined in the introduction as the interacting branching Brownian motion starting from $\mu_0 \in M$, each particle having mass 1, and with branching intensities

$$(4.6) \qquad n\alpha \cdot \left(1 \mp \frac{H'_x\left(s, \frac{\mu}{n}\right)}{nH\left(s, \frac{\mu}{n}\right)}\right).$$

We consider the sequence $\{P^{n,H}\}_n$, where $P^{n,H}$ is the distribution of $X^n := n^{-1} X^{1,n,H}$.

The proof of Theorem B is a rerun of the proof of Theorem 3.2. Therefore, we only put together the properties of the rescaled process which we used there.

Lemma 4.1 (i) *For all $F \in C^2, f \in C_0^2 \cup \{1\}$ the processes $M^{H,n}[F]$ defined by*

$$(4.7) \quad M_t^{H,n}[F] := F(X_t(f)) - F(X_0(f)) - \int_0^t \left[X_s\left(\frac{1}{2}\Delta f\right) \frac{\partial}{\partial x} F(X_s(f)) + \frac{1}{2n} X_s(|\nabla f|^2) \frac{\partial^2}{\partial x^2} F(X_s(f)) + n^2 \alpha X_s\left(\left(1 - \frac{H'(s, X_s)}{nH(s, X_s)}\right)\left(F\left(X_s(f) - \frac{f}{n}\right) - F(X_s(f))\right) + \left(1 + \frac{H'(s, X_s)}{nH(s, X_s)}\right)\left(F\left(X_s(f) + \frac{f}{n}\right) - F(X_s(f))\right)\right)\right] ds$$

are local martingales.

(ii)

$$(4.8) \qquad \sup_{t \le t_0} \sup_n E^{n,H}[X_t(1)] \le (\mu_0(1) + K_0\, t_0\, 2\alpha)\, e^{2\alpha K_1 t_0} < \infty.$$

$$(4.9) \quad \sup_{t \le t_0} \sup_n E^{n,H}[X_t^2(1)] \le (\mu_0^2(1) + t_0(2\alpha + 4\alpha K_0)(\mu_0(1) + t_0\, 2\alpha\, K_0)\, e^{2\alpha K_1 t_0})\, e^{4\alpha K_1 t_0} < \infty.$$

(iii) *The process $M[e_f]$ is a martingale for every $f \in C_b^2$ and the process of the total mass $X(1)$ is a submartingale.*

Proof. Assertion (i) follows from the martingale properties of X^n. Assertion (ii) follows by (i) and assumption (1.12). Assertion (iii) is a consequence of (i) and (ii). □

Example. H-transform of product type. The functions $H(s, \mu) = \mu(h_1(s))\, h_2(s, \mu(1))$ considered in Subsection 2.2.2 (iv) satisfy the condition (1.12) if $\tilde{h}^c = h_2$ is an extremal space-time harmonic function of the Feller process conditioned on non-extinction, because

$$(4.10)\qquad \tilde{h}^c(s,x)=\sum_{k=1}^{\infty}\frac{c^k x^{k-1}}{k!(k-1)!}\left(\sum_{k=1}^{\infty}\frac{c^k}{k!(k-1)!}\right)^{-1}e^{-sc}$$

with $c\in(0,\infty)$, $\tilde{h}^0(s,x)=1$ or $\tilde{h}^{\emptyset}(s,x)=x^{-1}$, cf. [O1] and (2.8).

The term $1-\dfrac{H'_x\left(s,\frac{\mu}{n}\right)}{nH\left(s,\frac{\mu}{n}\right)}$ could be negative (for example if $h_1=1,\mu(1)=1$) and hence is not an intensity. If $\mu(1)>1$ then this term is an intensity. This reflects that for superprocesses the probability of extinction tends to 1 as $\mu(1)$ to 0, whereas for the particle systems the maximal probability of extinction in the next branch in $\frac{1}{2}$ and is attained iff $\mu(1)=1$.

If $h_2=1$ (4.6) equals $1-\dfrac{h_1(s,x)}{\mu(h_1(s))}$ which satisfies (1.12). Hence we have additionally to additive H-transforms of branching Brownian motion a second sequence of approximating particle systems for additive H-transforms. The only difference is the fact that in the approximation with additive H-transforms of branching Brownian motion there is also an interaction in the drift of the one particle motion which, however, disappears by scaling, cf. Remark in Sect. 3.2.2.

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