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Rates of convergence and optimal spectral bandwidth for long range dependence

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Summary. For a realization of length n from a covariance stationary discrete time process with spectral density which behaves like λ^{1-2H} as $\lambda \to 0+$ for $\frac{1}{2} < H < 1$ (apart from a slowly varying factor which may be of unknown form), we consider a discrete average of the periodogram across the frequencies $2\pi j/n, j = 1, \ldots, m$, where $m \to \infty$ and $m/n \to 0$ as $n \to \infty$. We study the rate of convergence of an analogue of the mean squared error of smooth spectral density estimates, and deduce an optimal choice of m.

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1 Introduction

This paper derives convergence rates and formulae for optimal bandwidths in nonparametric spectral analysis of time series with long range dependence. Let $\{x_t;\ t = 1, 2, \ldots\}$ be a covariance stationary stochastic process having power spectrum $f(\lambda)$, $-\pi < \lambda \leqq \pi$, and lag-j autocovariance $\gamma_j = E(x_1 - Ex_1)(x_{1+j} - Ex_1) = \int_{-\pi}^{\pi} \cos(j\lambda)\, f(\lambda)\, d\lambda$. On the basis of observations at $t = 1, \ldots, n$, define the periodogram and discretely averaged periodogram

$$(1.1) \qquad I(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} x_t e^{it\lambda} \right|^2, \hat{F}(\lambda) = \frac{2\pi}{n} \sum_{j=1}^{[\lambda n/2\pi]} I(\lambda_j),$$

where $\lambda_j = 2\pi j/n$ and $[\cdot]$ here means "integer part". Consider an integer-valued "bandwidth" sequence $m = m_n$ satisfying

Assumption 1 $m = m_n \to \infty$, $m = o(n)$, as $n \to \infty$.

When $0<f(0)<\infty$, which can be viewed as a symptom of weak dependence in x_t, $\hat{F}(\lambda_m)/\lambda_m$ is consistent for $f(0)$ under mild additional conditions. When $f(\lambda)$ is smooth at $\lambda=0$, formulae for "optimal" m asymptotically minimizing the mean squared error (MSE)

$$E\{\hat{F}(\lambda_m)/\lambda_m - f(0)\}^2 \tag{1.2}$$

are long-established (see e.g. Grenander and Rosenblatt [7, pp. 153–155]). Processes for which $f(0)=\infty$ are of increasing practical and theoretical interest; they are often said to exhibit "long range dependence". Many are covered by the assumption that f varies regularly at $\lambda=0$.

Assumption 2 As $\lambda \to 0+$,

$$f(\lambda) \sim g(\lambda) \overset{\text{def}}{=} L\left(\frac{1}{\lambda}\right)\lambda^{1-2H}, \quad \tfrac{1}{2}<H<1,$$

where "$\sim$" means that the ratio of left- and right-hand sides tends to 1, and $L(\lambda)$ is a function of slow variation at infinity, i.e.

$$\frac{L(t\lambda)}{L(\lambda)} \to 1, \quad \text{as} \quad \lambda \to \infty, \qquad \text{for all } t>0. \tag{1.3}$$

Assumption 2 covers many parametric models for long range dependence such as fractional autoregressive integrated moving average (ARIMA) and fractional noise models (see e.g. Fox and Taqqu [5]). Because it makes no parametric assumptions about medium- or short-run behaviour of x_t, it is of wide practical applicability.

In case Assumption 2 holds we introduce the scaled MSE

$$\hat{\mathrm{MSE}}_m = E\{B(\lambda_m)\}^2, \tag{1.4}$$

where $B(\lambda)=\hat{F}(\lambda)/G(\lambda)-1$ and $G(\lambda)=\int_0^\lambda g(\lambda)\,d\lambda$. The case $0<f(0)<\infty$ corresponds to $H=1/2$ and $L(\lambda)\equiv f(0)$, when (1.4) reduces to (1.2) divided by $f(0)^2$. For $1/2<H<1$, Robinson [12] gave conditions for consistency of $\hat{H}=1-\log\{\hat{F}(q\lambda_m)/\hat{F}(\lambda_m)\}/(2\log q)$ where $0<q<1$ (allowing L in Assumption 2 to be of unknown form). The choice of m is of interest here. Applying (2.7) below, $\hat{H}-H=\{B(\lambda_m)-B(q\lambda_m)\}/(2\log q)+O_p(B^2(\lambda_m)+B^2(q\lambda_m))+o(1)$, as $n\to\infty$, under Assumption 1. Results below indicate that the "O_p" term is suitably small and assuming that the "$o(1)$" term is suitably small (it is zero when L is constant) it follows that we can consider as an "asymptotic MSE" of $\hat{H}$, $E[\{B(\lambda_m)-B(q\lambda_m)\}/2\log q]^2$, whose study (including optimal bandwidth choice) will benefit from information about (1.4). Other uses of $\hat{F}$ in case $1/2<H<1$ were also explored in [12]. It is also possible that $\hat{F}(\lambda_m)$ will be computed in the hope of estimating a finite $f(0)$ in a situation where long range dependence cannot be entirely ruled out *a priori*, so we would like to know how the optimal m differs between the finite $f(0)$ case and Assumption 2. Study of (1.4) when $1/2<H<1$ also represents an extension of the classical analysis of (1.2) in the smooth spectral density case.

To describe the bias component in (1.4), strengthen Assumption 2 to

Assumption 3 For some $E_{\alpha H} \neq 0$ and $0<\alpha \leqq 2$,

$$\frac{f(\lambda)}{g(\lambda)}=1+E_{\alpha H}\lambda^{\alpha}+o(\lambda^{\alpha}), \quad \text{as} \quad \lambda \to 0+. \tag{1.5}$$

Assumption 3 is equivalent to taking $f(\lambda)=h(\lambda)\,g(\lambda)$ where $h(0)=1$ and h is in $\mathrm{Lip}(\alpha)$ for $0<\alpha\leqq 1$, or is differentiable with derivative in $\mathrm{Lip}(\alpha-1)$ for $1<\alpha\leqq 2$. When $0<f(0)<\infty$, it corresponds to the usual smoothness condition imposed on $f(\lambda)=h(\lambda)\,f(0)$ (the cases $\alpha=1$ and $\alpha=2$ are stressed in much of the literature). In general we allow $E_{\alpha H}$ to depend on H as well as α, because $h(\lambda)$ can depend on H, as in the fractional ARIMA case.

When $1/2<H<3/4$ we find that the leading terms in $\mathrm{M\hat{S}E}_m$ are fortunately invariant to the form of the slowly varying function L, as is the optimal bandwidth m, and that while they differ from corresponding quantities in the case $0<f(0)<\infty$, their rates are the same. For example, the optimal m is of form $A(H,\alpha)\,n^{2\alpha/(2\alpha+1)}$, so the rate is free of H, which affects only the scale factor $A(H,\alpha)$; the latter is an increasing function of H.

When $3/4<H<1$ a different type of result emerges, as might be expected because f is square integrable in a neighbourhood of the origin when $1/2<H<3/4$, but not when $3/4<H<1$. The rates of $\mathrm{M\hat{S}E}_m$ and the optimal m now differ from those when $0<f(0)<\infty$. For example, the optimal m is of form $A(H,\alpha)\,n^{\alpha/(\alpha+2-2H)}$. Thus the rate depends on H, tending to n as $H\to 1$, and suggesting that a strong degree of long range dependence calls for a substantially larger bandwidth. Another interesting aspect of the case $3/4<H<1$ is that the MSE and optimal bandwidth for $\hat{F}(\lambda_m)$ differ from those for the continuously averaged periodogram when the mean Ex_1 is known, namely

$$\tilde{F}(\lambda_m)=\int_0^{\lambda_m}\tilde{I}(\lambda)\,d\lambda, \tag{1.6}$$

where

$$\tilde{I}(\lambda)=\frac{1}{2\pi n}\left|\sum_{t=1}^{n}(x_t-Ex_1)\,e^{it\lambda}\right|^2.$$

Most experience in frequency domain time series analysis suggests that continuous and discrete averaging of periodograms makes no difference to basic asymptotic properties. This experience covers $\hat{F}$ and $\tilde{F}$ under weak dependence, averaged periodogram (c.f. $\hat{F}$) and weighted autocovariance (c.f. $\tilde{F}$) spectral density estimates (see e.g. Brillinger [2, Chap. 7], Hannan [9, Chap. 5]), as well as Gaussian estimates of parametric time series models under both weak and long range dependence (see e.g. Hannan [10], Fox and Taqqu [5], Dahlhaus [3], Giraitis and Surgailis [6]). When $3/4<H<1$, however, the optimal m for (1.6) (apart from a possible factor involving L) is of form $A'(H,\alpha)\,n^{\alpha/(\alpha+2-2H)}$ where $A'(H,\alpha)$ can be larger or smaller than $A(H,\alpha)$, and the minimized $\hat{\text{MSE}}_m$ can be larger or smaller than the minimized

$$\hat{\text{MSE}}_m = E\{\tilde{F}(\lambda_m)/G(\lambda_m)-1\}^2.$$

The regularity conditions for the case $1/2<H<3/4$ are strictly weaker than those for the case $3/4<H<1$. While we obtain optimal rates of increase of m as a function of n, all rate results for MSE and technical lemmas in the paper assume only the minimal Assumption 1 on m. Our results can be straightforwardly generalized to cover elaborations on $\hat{F}$ or $\tilde{F}$ which involve the sorts of non-uniform weighting or tapering which have often been found useful in spectral analysis (cf. Brillinger [2, Chapters 3 and 7], Hannan [9, Chapter 5], Dahlhaus [4], Zurbenko [16]), and to averaged periodograms around a given non-zero-frequency singularity in f.

We discuss the cases $1/2<H<3/4$ and $3/4<H<1$ separately in the next two sections. It is convenient to assume Gaussianity of x_t in Sect. 2 and 3, so that complications arising from non-zero fourth cumulants are avoided. In Sect. 4 we give a condition under which the results of these Sections are robust to non-Gaussianity.

We briefly mention some additional slightly related references. [13] derived limit distribution theory for discrete Fourier transforms of processes with long range dependence; [4] studied spectral analysis of processes with spectral peaks which increase in magnitude with sample size; [8] and [11] studied rates of convergence and optimal bandwidth in kernel probability density and derivative-of-probability-density estimates based on data with long range dependence.

2 MSE when $1/2<H<3/4$

Throughout the paper we make use of the following fundamental properties of slowly varying functions (see e.g. Bingham et al. [1, pp. 26, 27, 58]): for all $\beta>0$, as $\lambda\to 0+$:

$$\text{(2.1)}\quad \begin{aligned} &\text{(i)}\ \sup_{0\le\mu\le\lambda} L\left(\frac{1}{\mu}\right)\mu^{\beta} \sim L\left(\frac{1}{\lambda}\right)\lambda^{\beta},\\ &\text{(ii)}\ \sup_{\mu\ge\lambda} L\left(\frac{1}{\mu}\right)\mu^{-\beta} \sim L\left(\frac{1}{\lambda}\right)\lambda^{-\beta},\end{aligned}$$

$$\text{(2.2)}\quad \begin{aligned} &\text{(i)}\ \int_0^{\lambda} L\left(\frac{1}{\mu}\right)\mu^{\beta-1}\,d\mu \sim \frac{1}{\beta} L\left(\frac{1}{\lambda}\right)\lambda^{\beta},\\ &\text{(ii)}\ \int_{\lambda}^{A} L\left(\frac{1}{\mu}\right)\mu^{-\beta-1}\,d\mu \sim \frac{1}{\beta} L\left(\frac{1}{\lambda}\right)\lambda^{-\beta},\end{aligned}$$

where (2.2) (ii) assumes $L(\lambda)$ is locally bounded on $[1/A, \infty)$.

We introduce two further assumptions.

Assumption 4 For any $\delta \in (0, 1)$, $D \in (1, \infty)$, as $\lambda \to 0+$

$$\sup_{-D\lambda \leqq \mu \leqq \delta\lambda} \frac{|f(\lambda)-f(\lambda-\mu)|}{|\mu|\, g(|\mu|)} = O\left(\frac{1}{\lambda}\right). \tag{2.3}$$

Assumption 5 x_t is a Gaussian process.

A sufficient condition for Assumption 4 is that $f(\lambda)$ is differentiable in a neighbourhood $(0, \varepsilon)$ of the origin, with derivative $f'(\lambda)$ satisfying

$$f'(\lambda) = O\left(\frac{g(\lambda)}{\lambda}\right), \quad \text{as} \quad \lambda \to 0+, \tag{2.4}$$

because the left hand side of (2.3) is then

$$O\left(\frac{1}{g(\lambda)} \sup_{(1-\delta)\lambda \leqq \mu \leqq (1+D)\lambda} |f'(\mu)|\right) = O\left(\frac{1}{\lambda}\right), \quad \text{as} \quad \lambda \to 0+,$$

noting that for all $t>0$ $g(\lambda)/g(t\lambda) \to t^{2H-1}$ as $\lambda \to 0+$ from (1.3). [3], [5] and [6] employed conditions similar to (2.4) in their study of asymptotic inference in parametric models for long range dependence, the most popular of which, fractional ARIMA and fractional noise models, satisfy this condition. Another sufficient condition for Assumption 4 will be presented in Sect. 3. Assumption 5 could be relaxed to the requirement that x_t have zero fourth-order cumulants and cross-cumulants.

Introduce the Dirichlet and Fejér kernels $D(\lambda) = \sum_{t=1}^{n} e^{it\lambda}$, $K(\lambda) = (2\pi n)^{-1} |D(\lambda)|^2$, respectively. We repeatedly use the properties (see e.g. Zygmund [17, pp. 49–51, 88])

$$|D(\lambda)| \leqq 2n/(1+nu), 0 \leqq \lambda \leqq \pi; \int_{-\pi}^{\pi} K(\lambda)\, d\lambda = 1; K(\lambda) \text{ is even.} \tag{2.5}$$

Theorem 1 *Under Assumptions* 1, 3, 4 *and* 5 *and* $1/2 < H < 3/4$, *as* $n \to \infty$

$$\widehat{\mathrm{MSE}}_m \sim 4(1-H)^2 \left[\frac{1}{(3-4H)m} + \left\{\frac{E_{\alpha H}}{2-2H+\alpha}\right\}^2 \left(\frac{2\pi m}{n}\right)^{2\alpha}\right].$$

Theorem 1 holds for all m satisfying Assumption 1, but when the m^{-1} and $(m/n)^{2\alpha}$ terms do not balance the smaller one, as $n \to \infty$, is not necessarily the second term in an asymptotic expansion, without a suitable extension of Assumption 1.

An identical result holds for $\tilde{F}(\lambda_m)$ and also for

$$\bar{F}(\lambda_m) = \int_0^{\lambda_m} \bar{I}(\lambda)\, d\lambda,$$

where

$$\bar{I}(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} (x_t - \bar{x})\, e^{it\lambda} \right|^2$$

and $\bar{x} = n^{-1}(x_1 + \ldots + x_n)$. The proofs are very similar to that of Theorem 1, indeed somewhat simpler because Lemma 4 below is not required. $\tilde{F}(\lambda_m)$ can only be computed when Ex_1 is known, and Monte Carlo simulations suggest that $\bar{F}(\lambda_m)$ is inferior to $\hat{F}(\lambda_m)$ for small or moderate-sized n, owing apparently to the slow convergence of $\bar{x}$ when $H > 1/2$.
A trivial consequence of Theorem 1 is:

Corollary 1 *A bandwidth* $\hat{m}$ *which minimizes* $\mathrm{M\hat{S}E}_m$ *as* $n \to \infty$ *is*

$$\hat{m} = \left\{ \frac{(2-2H+\alpha)^2}{2\alpha E_{\alpha H}^2 (2\pi)^{2\alpha}(3-4H)} \right\}^{\frac{1}{2\alpha+1}} n^{\frac{2\alpha}{2\alpha+1}}.$$

Thus the minimized $\mathrm{M\hat{S}E}_m$ converges to zero at rate $n^{-2\alpha/(2\alpha+1)}$. The factor $(2-2H+\alpha)^2/(3-4H)$ increases in H for given α, tending to infinity as $H \to 3/4$. On the other hand, putting $H = 1/2$ and $\alpha = 2$ in the formula gives $\hat{m} = \{9/4 E_{2,1/2}(2\pi)^4\}^{1/5} n^{4/5}$, which corresponds, as expected, to the optimal bandwidth formula in Hannan [9, p. 286] for the Daniell estimate when f has bounded second derivative at $\lambda \equiv 0$ (Hannan's M is $n/2m$).

Proof of Theorem 1

$$B(\lambda) = \frac{\hat{F}(\lambda) - E\hat{F}(\lambda)}{G(\lambda)} + \frac{E\hat{F}(\lambda) - E\tilde{F}(\lambda)}{G(\lambda)} + \frac{E\tilde{F}(\lambda) - F(\lambda)}{G(\lambda)} + \left\{ \frac{F(\lambda)}{G(\lambda)} - 1 \right\}, \tag{2.6}$$

where $F(\lambda) = \int_0^{\lambda} f(\lambda)\, d\lambda$, $0 < \lambda \leq \pi$. Thus

$$\mathrm{M\hat{S}E}_m - \frac{V(\hat{F}(\lambda_m))}{G(\lambda_m)^2} - \left\{ \frac{F(\lambda_m)}{G(\lambda_m)} - 1 \right\}^2 = O(r(\lambda_m)),$$

where, by elementary inequalities,

$$r(\lambda)=\frac{1}{G(\lambda)^2}[|E\hat{F}(\lambda)-E\tilde{F}(\lambda)|\{|E\hat{F}(\lambda)-E\tilde{F}(\lambda)|+|F(\lambda)-G(\lambda)|\}$$
$$+|E\tilde{F}(\lambda)-F(\lambda)|\{|E\tilde{F}(\lambda)-F(\lambda)|+|F(\lambda)-G(\lambda)|\}].$$

From (2.2) (i)

$$G(\lambda)\sim L\left(\frac{1}{\lambda}\right)\frac{\lambda^{2(1-H)}}{2(1-H)}, \quad \text{as} \quad \lambda\to 0+, \tag{2.7}$$

so it suffices to prove that as $n\to\infty$

$$\frac{F(\lambda_m)}{G(\lambda_m)}-1\sim E_{\alpha H}\frac{2(1-H)\lambda_m^\alpha}{2(1-H)+\alpha}, \tag{2.8}$$

$$r(\lambda_m)=o\left(\frac{1}{m}\right), \tag{2.9}$$

$$V(\hat{F}(\lambda_m))\sim\frac{1}{m}L^2\left(\frac{1}{\lambda_m}\right)\frac{\lambda_m^{4-4H}}{3-4H}. \tag{2.10}$$

We establish these properties via a series of lemmas.
First (2.8) is proved by

Lemma 1 *Under Assumptions 2 and 3, as* $\lambda\to 0+$

$$\frac{F(\lambda)}{G(\lambda)}-1\sim E_{\alpha H}\frac{2(1-H)\lambda^\alpha}{2(1-H)+\alpha}.$$

Proof. From (1.5),

$$F(\lambda)-G(\lambda)-E_{\alpha H}\int_0^\lambda L\left(\frac{1}{\mu}\right)\mu^{1-2H+\alpha}d\mu=o(\lambda^\alpha G(\lambda)),$$

then apply (2.7) and another consequence of (2.2)(i),

$$\int_0^\lambda L\left(\frac{1}{\mu}\right)\mu^{1-2H+\alpha}d\mu\sim L\left(\frac{1}{\lambda}\right)\frac{\lambda^{2(1-H)+\alpha}}{2(1-H)+\alpha}, \quad \text{as} \quad \lambda\to 0+. \quad \square$$

Now (2.9) is a consequence of Assumption 1, (2.7), (2.8) and two further results to be established, namely

$$E\tilde{F}(\lambda_m)-F(\lambda_m)=O\left(\frac{g(\lambda_m)}{n}\right), \tag{2.11}$$

$$E\{\hat{F}(\lambda_m)-\tilde{F}(\lambda_m)\}=O\left(L\left(\frac{1}{\lambda_m}\right)m^\eta n^{2(H-1)}\right) \tag{2.12}$$

as $n \to \infty$, for $0 < \eta < 3/2 - 2H$.

To derive (2.11), we introduce first a representation for the bias $E\tilde{F}(\lambda) - F(\lambda)$.

Lemma 2 *For all* $\lambda \in (0, \pi]$,

$$E\tilde{F}(\lambda) - F(\lambda) = \int_{-\pi}^{\pi} K(\mu)\{J_\lambda(\mu) - F(\lambda)\}\, d\mu,$$

where

$$\begin{aligned} J_\lambda(\mu) &= 2F(\pi) - F(2\pi + \mu - \lambda), && -\pi < \mu \leqq \lambda - \pi, \\ &= F(\lambda - \mu), && \lambda - \pi < \mu \leqq \lambda, \\ &= -F(\mu - \lambda), && \lambda \leqq \mu \leqq \pi. \end{aligned}$$

Proof. Direct calculation gives

$$E\{(\tilde{F}(\lambda)\} = \int_{-\pi}^{0} K(\mu)\{J_\lambda(\mu) - F(\mu)\}\, d\mu + \int_{0}^{\pi} K(\mu)\{J_\lambda(\mu) + F(\mu)\}\, d\mu$$

where $F(-\mu) = F(\mu)$, $\mu > 0$. Apply evenness of K and (2.5). □

Lemma 3 *Under Assumptions 2 and 4, for all n,*

$$n[E\{\tilde{F}(\lambda)\} - F(\lambda)] = O(g(\lambda)), \quad \text{as} \quad \lambda \to 0+. \tag{2.13}$$

Proof. On application of Lemma 2, the left-hand side of (2.13) is dominated in absolute value by $A_1 + \ldots + A_7$, where

$$A_1 = n\left|\int_{-\lambda/2}^{\lambda/2} K(\mu)\{F(\lambda - \mu) - F(\lambda)\}\, d\mu\right|,$$

$$A_2 = 2F(\lambda)\, n \int_{\lambda/2}^{\pi} K(\mu)\, d\mu,$$

$$A_3 = n \int_{\lambda/2}^{\lambda} K(\mu)\, F(\lambda - \mu)\, d\mu,$$

$$A_4 = n \int_{\lambda}^{\varepsilon} K(\mu)\, F(\mu - \lambda)\, d\mu,$$

$$A_5 = n \int_{-\varepsilon/2}^{\lambda/2} K(\mu)\, F(\lambda - \mu)\, d\mu,$$

$$A_6 = n \int_{\varepsilon}^{\pi} K(\mu)\, J_\lambda(\mu)\, d\mu,$$

$$A_7 = n \int_{-\pi}^{-\varepsilon/2} K(\mu)\, J_\lambda(\mu)\, d\mu,$$

for $\lambda<\varepsilon<\pi-\lambda$. Now

$$F(\lambda)-F(\lambda-\mu)-\mu f(\lambda)=\int_{\lambda-\mu}^{\lambda}\{f(\theta)-f(\lambda)\}\,d\theta,\qquad |\mu|<1/2\lambda,$$

so by evenness of K, (2.5) and Assumption 4

$$\begin{aligned}|A_1|&\leqq n\int_{-\lambda/2}^{\lambda/2}K(\mu)|\mu|\max_{|\theta|\leqq|\mu|}|f(\lambda)-f(\lambda-\theta)|\,d\mu\\&=O\left(\frac{n}{\lambda}\int_{-\lambda/2}^{\lambda/2}K(\mu)\,\mu^2 g(|\mu|)\,d\mu\right)\\&=O\left(\frac{G(\lambda)}{\lambda}\right)=O(g(\lambda))\end{aligned}$$

as $\lambda\to 0+$. Throughout, let C be a generic finite, positive constant. Because F is increasing and $F(\lambda)=O(G(\lambda))$ as $\lambda\to 0+$,

$$A_3\leqq A_2=O\left(G(\lambda)\int_{\lambda/2}^{\pi}\frac{d\mu}{\mu^2}\right)=O(g(\lambda)),$$

$$\begin{aligned}A_4+A_5&=O\left(\int_{\lambda}^{\varepsilon}\frac{F(\mu)}{\mu^2}\,d\mu+\int_{\lambda/2}^{\varepsilon/2}\frac{F(3\mu)}{\mu^2}\right)\\&=O\left(\int_{\lambda/2}^{\infty}\frac{g(\mu)}{\mu}\,d\mu\right)=O(g(\lambda))\end{aligned}$$

after choosing ε so small that $F(\mu)\leqq CG(\mu)\leqq CL(1/\mu)\,\mu^{2(1-H)})$ for $\mu<2\varepsilon$, and $L(\lambda)$ is locally bounded on $[1/\varepsilon,\infty]$ (see e.g. [1, p. 13]. Finally

$$A_6+A_7=O\left(\int_{\varepsilon/2}^{\infty}\frac{d\mu}{\mu^2}\right)=O(\varepsilon^{-1}).\quad\square$$

Thus (2.11) is proved. Now consider (2.12), the discrepancy between the expectations of the discretely and continuously-averaged periodograms.

Lemma 4 *Under Assumptions 1, 2 and 4, as $n\to\infty$, for any $\eta>0$,*

$$E\{\tilde F(\lambda_m)-\hat F(\lambda_m)\}=O\left[L\left(\frac{1}{\lambda_m}\right)m^{\eta}n^{2(H-1)}\right].\tag{2.14}$$

Proof. Let $S_j=[\lambda_{j-1},\lambda_j]$, $j=1,\ldots,m$, and define

$$P_j(\eta,\theta)=\sup_{\lambda\in S_j}\int_{\eta}^{\theta}K(\mu)f(\lambda-\mu)\,d\mu.$$

Note that $I(\lambda_j) \equiv \tilde{I}(\lambda_j)$ for $j=1, \ldots, m<n/2$. Choose ε so small that $f(\lambda) \leqq Cg(\lambda)$ and the left side of (2.3) is bounded by C/λ for $0<\lambda<\varepsilon$, then n so large that $2\lambda_m<\varepsilon$. The left side of (2.14) is dominated by

$$\begin{aligned}(2.15)\qquad &\frac{4\pi}{n}\sum_{j=1}^{m}\{P_j(\varepsilon,\pi)+P_j(3\lambda_j/2,\varepsilon)+P_j(\lambda_j/2,3\lambda_j/2)\\ &+P_j(-\varepsilon,-\lambda_j/2)+P_j(-\pi,-\varepsilon)\}\\ &+\frac{4\pi}{n}\sum_{j=1}^{2}P_j(-\lambda_j/2,\lambda_j/2)\\ &+\sum_{j=3}^{m}\int_{S_j}\int_{-\lambda_j/2}^{\lambda_j/2}K(\mu)|f(\lambda_j-\mu)-f(\lambda-\mu)|\,d\mu\,d\lambda.\end{aligned}$$

$P_j(\varepsilon,\pi)+P_j(-\pi,-\varepsilon)$ is easily seen to be $O(n^{-1})$ from (2.5) and $Ex_1^2<\infty$, whereas, as $n\to\infty$

$$\begin{aligned}P_j(3\lambda_j/2,\varepsilon) &\leqq 4\sup_{\lambda\in S_j}\sup_{3\lambda_j/2\leqq\mu\leqq\varepsilon} g(\mu-\lambda)\int_{\lambda_j}^{\pi}\frac{d\mu}{n\mu^2}\\ &=O\left(\frac{g(\lambda_j)}{j}\right)\end{aligned}$$

with the same bound for $P_j(-\varepsilon,-\lambda_j/2)$, and

$$P_j(\lambda_j/2,3\lambda_j/2)=O\left(\frac{F(\lambda_j)}{n\lambda_j^2}\right)=O\left(\frac{g(\lambda_j)}{j}\right).$$

The contribution of the above terms to (2.14) is of order n^{-1} times

$$\begin{aligned}\sum_{j=1}^{m} g(\lambda_j)/j &= O\left(\sup_{1\leqq j\leqq m}\left\{L\left(\frac{1}{\lambda_j}\right)\lambda_j^{\eta}\right\}\sum_{j=1}^{\infty}\lambda_j^{1-2H-\eta}j^{-1}\right)\\ &=O\left(m^{\eta}n^{2H-1}L\left(\frac{1}{\lambda_m}\right)\right),\end{aligned}$$

as $n\to\infty$. For $j=1,2$,

$$P_j(-\lambda_j/2,\lambda_j/2)\leqq 2n\int_0^{2\lambda_j}f(\lambda)\,d\lambda=O(L(n)\,n^{2H-1})$$

as $n \to \infty$. For $j \geqq 3$, the double integral in (2.15) is bounded by $2\pi/n$ times

$$\sup_{|u| \leqq \lambda_j/2} \sup_{\lambda \in S_j} \frac{|f(\lambda_j - u) - f(\lambda - u)|}{(\lambda_j - \lambda)\, g(\lambda_j - \lambda)} \sup_{\lambda \in S_j} (\lambda_j - \lambda)\, g(\lambda_j - \lambda)$$

$$= O\left(\frac{L(n)\, n^{2H-1}}{j}\right)$$

as $n \to \infty$, from Assumption 4, (2.1)(i) and (2.5). But

$$L(n)\, n^{2H-1} \sum_{j=1}^{m} j^{-1} = O(L(n)\, n^{2H-1} \log m)$$

$$= O\left(L\left(\frac{1}{\lambda_m}\right) m^{\eta} n^{2H-1}\right)$$

as $n \to \infty$, because $L(n) = O(m^{\eta} L(1/\lambda_m))$ from (2.1)(i). □

The proof of (2.12), and thus (2.9), is complete. It remains to prove (2.10). For this purpose it is useful to first establish the following result, which is well known with m replaced by n (see e.g. [17, p. 67]); we need m rather than n on the right-hand side of (2.16) below in order to avoid any strengthening of Assumption 1 in Theorem 1.

Lemma 5 *Under Assumption 1,*

$$\int_0^{\lambda_m} |D(u)|\, du \sim \frac{2}{\pi} \log m, \quad \text{as} \quad n \to \infty. \tag{2.16}$$

Proof. The left side of (2.16) is

$$\int_0^{\lambda_m} \left|\frac{\sin(nu/2)}{\sin(u/2)}\right| du = 2 \int_0^{\lambda_m} \left|\frac{\sin(nu/2)}{u}\right| du + O(1),$$

because $\operatorname{cosec}(u) - u^{-1}$ is bounded on $(0, 1/2\pi)$. Now

$$\int_0^{\lambda_m} \left|\frac{\sin(nu/2)}{u}\right| du = \int_0^{\lambda_1} \sum_{j=0}^{m-1} \left|\frac{\sin(n(u+\lambda_j)/2)}{u+\lambda_j}\right| du$$

$$= \int_0^{\pi} \frac{\sin u}{u}\, du$$

$$+ \int_0^{\lambda_1} \sin(nu/2) \left\{\sum_{j=1}^{m-1} \frac{1}{u+\lambda_j}\right\} du.$$

The term in braces is lower- and upper-bounded by $\sum_{j=2}^{m} (j\lambda_1)^{-1}$ and $\sum_{j=1}^{m-1} (j\lambda_1)^{-1}$, and thus equals $(\log m + O(1))/\lambda_1$, whereas $\int_0^{\pi} \sin(u)/u\, du < \infty$, $\int_0^{\lambda_1} \sin(nu/2)\, du = 4/n$. □

Now define $R(\lambda, \theta) = \int_{-\pi}^{\pi} D(u)\, D(\lambda+\theta-u)\{f(\lambda-u)-f(\lambda)\}\, du$. The following technical lemma will be helpful.

Lemma 6 *Under Assumptions 1, 2 and 4,*

$$\overline{\lim_{n\to\infty}} (\log m)^{-1} \max_{1 \leqq j \leqq m} \frac{|R(\lambda_j, \pm\lambda_j)|}{g(\lambda_j)/\lambda_j} < \infty, \tag{2.17}$$

$$\overline{\lim_{n\to\infty}} (\log m)^{-1} \max_{j<k \leqq m} \frac{|R(\lambda_j, \lambda_k)| + |R(\lambda_k, \lambda_j)|}{g(\lambda_j)/\lambda_k} < \infty, \tag{2.18}$$

$$\overline{\lim_{n\to\infty}} (\log m)^{-1} \max_{k<j \leqq m} \frac{|R(\lambda_j, -\lambda_k)|}{g(\lambda_k)/(\lambda_j-\lambda_k)} < \infty, \tag{2.19}$$

$$\overline{\lim_{n\to\infty}} (\log m)^{-1} \max_{k<j \leqq 2k \leqq m} \frac{|R(\lambda_j, -\lambda_k)|}{g(\lambda_j-\lambda_k)/\lambda_k} < \infty, \tag{2.20}$$

$$\overline{\lim_{n\to\infty}} (\log m)^{-1} \max_{j<k \leqq m} \frac{|R(\lambda_j, -\lambda_k)|}{g(\lambda_j)/(\lambda_k-\lambda_j)} < \infty, \tag{2.21}$$

$$\overline{\lim_{n\to\infty}} (\log m)^{-1} \max_{j<k \leqq 2j \leqq m} \frac{|R(\lambda_j, -\lambda_k)|}{g(\lambda_k-\lambda_j)/\lambda_j} < \infty. \tag{2.22}$$

Proof. Consider first (2.19) and (2.20), so $j > k$. Define

$$S(a, b) = \int_a^b D(u)\, D(\lambda_j-\lambda_k-u)\{f(\lambda_j-u)-f(\lambda_j)\}\, du,$$

$$T_j(a, b) = \sup_{a \leqq u \leqq b} \frac{|f(\lambda_j)-f(\lambda_j-u)|}{|u|\, g(|u|)}.$$

Choose $\varepsilon > 3\lambda_m$ as in the proof of Lemma 4. Using (2.5), integrability of f, Assumptions 1, 2 and 4, and Lemma 5, as $n \to \infty$:

$$S(-\pi, -\varepsilon) \leqq \frac{4}{\varepsilon^2} \int_{\varepsilon}^{\pi} \{f(\lambda_j+u)+f(\lambda_j)\}\, du = O(1+g(\lambda_j));$$

$$S(-\varepsilon, -\lambda_j) \leqq 4 \sup_{\lambda_j \leqq u \leqq \varepsilon} \{f(\lambda_j-u)+f(\lambda_j)\} \int_{-\varepsilon}^{-\lambda_j} \frac{du}{|u|(\lambda_j-\lambda_k-u)}$$

$$= O\left(g(\lambda_j) \int_{\lambda_j}^{\infty} \frac{du}{u^2}\right) = O\left(\frac{g(\lambda_j)}{\lambda_j}\right);$$

$$S(-\lambda_j, (\lambda_j-\lambda_k/2)) = O\left(\frac{1}{\lambda_j-\lambda_k} \int_{-\lambda_j}^{(\lambda_j-\lambda_k)/2} \frac{|f(\lambda_j)-f(\lambda_j-u)|}{|u|}\, du\right)$$

$$= O\left(\frac{1}{\lambda_j-\lambda_k}\, T_j(-\lambda_j, \lambda_j/2) \int_0^{\lambda_j} g(u)\, du\right) = O\left(\frac{g(\lambda_j)}{\lambda_j-\lambda_k}\right),$$

or, for $j \leqq 2k$,

$$S(-\lambda_j, (\lambda_k-\lambda_j/)2) = O\,[T_j(-\lambda_j, (\lambda_j-\lambda_k)/2)\, g(\lambda_j-\lambda_k) \log m]$$
$$= O\left(\frac{g(\lambda_j-\lambda_k)}{\lambda_j} \log m\right)$$

and

$$S((\lambda_k-\lambda_j)/2, (\lambda_j-\lambda_k)/2)$$
$$= O\left(\frac{1}{\lambda_j-\lambda_k} T_j(-\lambda_j/2, \lambda_j/2) \int_0^{\lambda_j-\lambda_k} g(u)\, du\right)$$
$$= O\left(\frac{g(\lambda_j-\lambda_k)}{\lambda_j}\right);$$
$$S((\lambda_j-\lambda_k)/2, \lambda_j-\lambda_k/2)$$
$$= O\left(\frac{1}{\lambda_j-\lambda_k} \sup_{(\lambda_j-\lambda_k)/2 \leqq u \leqq \lambda_j-\lambda_k/2} \{f(\lambda_j-u)+f(\lambda_j)\} \log m\right)$$
$$= O\left(\frac{g(\lambda_k)}{\lambda_j-\lambda_k} \log m\right),$$

or, for $k \geqq j/2$,

$$S((\lambda_j-\lambda_k)/2), \lambda_j-\lambda_k/2) = O(T_j(0, \lambda_j-\lambda_k/2) \log m)$$
$$= O\left(\frac{g(\lambda_j-\lambda_k)}{\lambda_k} \log m\right);$$
$$S(\lambda_j-\lambda_k/2, \lambda_j+2\lambda_k) = O\left(\frac{1}{(\lambda_j-\lambda_k/2)\,\lambda_k} \int_{\lambda_j-\lambda_k/2}^{\lambda_k+2\lambda_k} \{f(\lambda_j-u)+f(\lambda_j)\}\, du\right)$$
$$= O\left(\frac{g(\lambda_j)}{\lambda_j}\right);$$
$$S(\lambda_j+2\lambda_k, 3\lambda_j) = O\left(\frac{1}{\lambda_j}\{\sup_{\lambda_j+2\lambda_k \leqq u \leqq 3\lambda_j} \{f(\lambda_j-u)+f(\lambda_j)\}\} \log m\right)$$
$$= O\left(\frac{g(\lambda_k)}{\lambda_j} \log m\right);$$
$$S(3\lambda_j, \varepsilon) = O\left(\sup_{3\lambda_j \leqq u \leqq \varepsilon} \{f(\lambda_j-u)+f(\lambda_j)\} \int_{3\lambda_j}^{\varepsilon} \frac{du}{u(u+\lambda_k-\lambda_j)}\right)$$
$$= O\left(\frac{g(\lambda_j)}{\lambda_j}\right);$$
$$S(\varepsilon, \pi) = O\left(\frac{1}{\varepsilon^2} \int_{-\pi}^{\pi} \{f(\lambda_j-u)+f(\lambda_j)\}\, du\right]$$
$$= O(1+g(\lambda_j)).$$

Thus (2.19) and (2.20) are proved. Now consider (2.21) and (2.22). Split the range of integration at the points $-\varepsilon$, $-2\lambda_k$, $(\lambda_j-\lambda_k)/2$, $\lambda_j/2$, $2\lambda_j$, $2\lambda_k$, ε. The details are then much the same as before and we only discuss the contributions from $(-2\lambda_k, (\lambda_j-\lambda_k)/2)$ and $((\lambda_j-\lambda_k)/2, \lambda_j/2)$. We have, as $n\to\infty$,

$$S(-2\lambda_k, (\lambda_j-\lambda_k)/2)=O\left(\frac{1}{\lambda_k-\lambda_j}\sup_{-2\lambda_k<u\leqq(\lambda_j-\lambda_k)/2}\{f(\lambda_j-u)+f(\lambda_j)\}\log m\right)$$
$$=O\left(\frac{g(\lambda_j)}{\lambda_k-\lambda_j}\log m\right),$$

or, for $j\geqq k/2$

$$S(-2\lambda_k, (\lambda_j-\lambda_k)/2)=O(T_j(-2\lambda_j, (\lambda_j-\lambda_k)/2)\, g(\lambda_k-\lambda_j)$$
$$=O\left(\frac{g(\lambda_k-\lambda_j)}{\lambda_j}\log m\right);$$
$$S((\lambda_j-\lambda_k)/2, \lambda_j/2)=O\left(\frac{1}{\lambda_k-\lambda_j}\sup_{(\lambda_j-\lambda_k)/2\leqq u\leqq\lambda_j/2}\{f(\lambda_j)+f(\lambda_j-u)\}\log m\right)$$
$$=O\left(\frac{g(\lambda_j)}{\lambda_k-\lambda_j}\log m\right),$$

or, for $j\geqq 1/2k$,

$$S((\lambda_j-\lambda_k)/2, (\lambda_k-\lambda_j)/2)$$
$$=O\left(\frac{1}{\lambda_k-\lambda_j}T_j((\lambda_j-\lambda_k)/2, (\lambda_k-\lambda_j)/2)\int_{(\lambda_j-\lambda_k)/2}^{(\lambda_k-\lambda_j)/2} g(u)\,du\right)$$
$$=O\left(\frac{g(\lambda_k-\lambda_j)}{\lambda_j}\right)$$

and

$$S((\lambda_k-\lambda_j)/2, \lambda_j/2)=O\left(T_j((\lambda_k-\lambda_j)/2, \lambda_j/2)\, g(\lambda_k-\lambda_j)\log m\right)$$
$$=O\left(\frac{g(\lambda_k-\lambda_j)}{\lambda_j}\log m\right).$$

The remaining contributions are of order $g(\lambda_j)(\log m)/\lambda_k$, to give (2.21) and (2.22). The proof for $R(\lambda_j, -\lambda_j)$ in (2.17) is simpler and rather similar to that of Lemma 3; we omit the details but mention that one can split $[-\pi, \pi]$ at $-\varepsilon$, $-\lambda_j/2$, $\lambda_j/2$, $2\lambda_j, \varepsilon$. For $j, k\geqq 1$, $R(\lambda_j, \lambda_k)$ is easier to handle than $R(\lambda_j, -\lambda_k)$, (2.17) for $R(\lambda_j, \lambda_j)$ and (2.18) resulting without any need to distinguish $1/2j\leqq k<j$ or $1/2k\leqq j<k$ or to use Assumption 4. The details repeat parts of the proof for $R(\lambda_j, -\lambda_k), j>k$, above so we merely hint that the reader partition $[-\pi, \pi]$ at $-\varepsilon$, $-\lambda_j$, $\lambda_j/2$, $\lambda_j-\lambda_k/2$, $\lambda_j+\lambda_k/2$, $3\lambda_j$, ε for $j>k$; at $-\varepsilon$, $-2\lambda_k$, $\lambda_j/2$, $3\lambda_j/2$, $\lambda_j+\lambda_k/2$, $3\lambda_k$, ε for $j<k$; and at $-\varepsilon$, $-\lambda_j/2$, $\lambda_j/2$, $3\lambda_j/2$, $3\lambda_j$, ε for $j=$k. □

We are now able to establish (2.10).

Lemma 7 *Under Assumptions* 1, 2, 4 *and* 5 *and* $1/2<H<3/4$, *as* $n\to\infty$

$$V(\hat{F}(\lambda_m)) \sim \frac{1}{m} L^2\left(\frac{1}{\lambda_m}\right) \frac{\lambda_m^{4-4H}}{3-4H}.$$

Proof. By Assumption 5

$$V(\hat{F}(\lambda_m)) = \frac{1}{n^4} \sum\sum_{j,k=1}^{m} \{Q(\lambda_j, -\lambda_k)\, Q(\lambda_k, -\lambda_j) + Q(\lambda_k, \lambda_j)\, Q(-\lambda_j, -\lambda_k)\},$$

where $Q(\lambda, \theta) = \int_{-\pi}^{\pi} D(u)\, D(\lambda+\theta-u)\, f(\lambda-u)\, du$. Because $\int_{-\pi}^{\pi} D(u)\, D(\lambda-u)\, du$ $=2\pi D(\lambda)$ and $D(\lambda_j-\lambda_k)=n, j=k$, mod$(n)$; $=0$, otherwise, it follows that for $j,k=1,\dots,m$, $Q(\lambda_j, -\lambda_k) = R(\lambda_j, -\lambda_k) + 2\pi n \delta_{jk} f(\lambda_j)$, $Q(\lambda_j, \lambda_k) = R(\lambda_j, \lambda_k)$, $Q(-\lambda_j, -\lambda_k) = R(-\lambda_j, -\lambda_k)$, where δ_{jk} is the Kronecker delta. Thus $V(\hat{F}(\lambda_m))$ is

$$\begin{aligned}(2.23)\quad & \frac{4\pi^2}{n^2} \sum_{j=1}^{m} f(\lambda_j)^2 + \frac{4\pi}{n^3} \sum_{j=1}^{m} f(\lambda_j)\, R(\lambda_j, -\lambda_j) \\ & + \frac{1}{n^4} \sum_{j=1}^{m} \{R(\lambda_j, -\lambda_j)^2 + |R(\lambda_j, \lambda_j)|^2\} \\ & + \frac{1}{n^4} \sum_{j=1}^{m} \sum_{k>j} \{R(\lambda_j, -\lambda_k)\, R(\lambda_k, -\lambda_j) + R(\lambda_k, \lambda_j)\, \overline{R(\lambda_j, \lambda_k)}\},\end{aligned}$$

noting that $R(-\lambda, -\theta) = \overline{R(\lambda, \theta)}$. It follows from Proposition 1 of [12] that under Assumptions 1 and 2 and using (2.2)(i),

$$\frac{4\pi^2}{n^2} \sum_{j=1}^{m} f(\lambda_j)^2 \sim \frac{2\pi}{n} \int_0^{\lambda_m} f(\lambda)^2\, d\lambda \sim \frac{2\pi}{n} L^2\left(\frac{1}{\lambda_m}\right) \frac{\lambda_m^{3-4H}}{3-4H}$$

as $n\to\infty$. Set $\eta=(3-4H)/2$. By Assumptions 1, 2 and (2.1)(i) and Lemma 6, as $n\to\infty$ the second term in (2.23) is

$$\begin{aligned}(2.24)\quad & O\left(\frac{\log m}{n^3} \sum_{j=1}^{m} \frac{g(\lambda_j)^2}{\lambda_j}\right) \\ & = O\left(\frac{\log m}{n^{4-4H-\eta}} \left\{\sup_{1\le j\le m} L^2\left(\frac{1}{\lambda_j}\right) \lambda_j^{\eta}\right\} \sum_{j=1}^{\infty} j^{1-4H-\eta}\right) \\ & = O\left(\frac{(\log m)\, m^{\eta}}{n^{4-4H}} L^2\left(\frac{1}{\lambda_m}\right)\right) = O\left((\log m)\, m^{4H-4+\eta} L^2\left(\frac{1}{\lambda_m}\right) \lambda_m^{4-4H}\right).\end{aligned}$$

Proceeding similarly, as $n \to \infty$ the third term in (2.23) is

$$O\left(\frac{(\log m)^2}{n^4} \sum_{j=1}^{m} \frac{g(\lambda_j)^2}{\lambda_j^2}\right) = O\left(\frac{(\log m)^2 m^n}{n^{4-4H}} L^2\left(\frac{1}{\lambda_m}\right)\right), \tag{2.25}$$

while

$$\begin{aligned}
&\frac{1}{n^4} \sum_{j=1}^{m} \sum_{k>j} R(\lambda_j, -\lambda_k) R(\lambda_k, -\lambda_j) \qquad (2.26)\\
&= O\left(\frac{(\log m)^2}{n^4} \sum_{j=1}^{m} \left\{ \sum_{k=j+1}^{\min(2j,m)} \frac{g^2(\lambda_k - \lambda_j)}{\lambda_j^2} \right.\right.\\
&\quad \left.\left. + \sum_{k=2j+1}^{m} \frac{g^2(\lambda_j)}{(\lambda_k - \lambda_j)^2} \right\}\right)\\
&= O\left[\frac{(\log m)^2}{n^2} \sum_{j=1}^{m} \left\{ \frac{1}{j^2} \sum_{k=1}^{j} g^2(\lambda_k) + g^2(\lambda_j) \sum_{k=j}^{\infty} \frac{1}{k^2} \right\}\right]\\
&= O\left[\frac{(\log m)^2}{n^2} \sum_{j=1}^{m} g^2(\lambda_j) \sum_{k=j}^{m} \frac{1}{k^2}\right]\\
&= O\left[\frac{(\log m)^2}{n^3} \sum_{j=1}^{m} \frac{g^2(\lambda_j)}{\lambda_j}\right],
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{n^4} \sum_{j=1}^{m} \sum_{k>j} R(\lambda_k, \lambda_j) \overline{R(\lambda_j, \lambda_k)} \qquad (2.27)\\
&= O\left(\frac{(\log m)^2}{n^4} \sum_{j=1}^{m} g(\lambda_j)^2 \sum_{k>j} \lambda_k^{-2}\right)\\
&= O\left(\frac{(\log m)^2}{n^3} \sum_{j=1}^{m} \frac{g(\lambda_j)^2}{\lambda_j}\right),
\end{aligned}$$

and it is easily seen that (2.24)–(2.27) are $o(m^{-1} L^2(1/\lambda_m) \lambda_m^{4-4H})$. □

3 MSE when $3/4 < h < 1$

When $3/4 < H < 1$, $f(\lambda)$ is no longer square-integrable on a neighbourhood of the origin and different results and theory apply. We introduce

Assumption 6 The γ_j are quasi-monotonically convergent to zero, that is, $\gamma_j \to 0$ as $j \to \infty$ and there exists $J < \infty$ and $B < \infty$ such that for all $j \geqq J$

$$\gamma_{j+1} < \gamma_j \left(1 + \frac{B}{j}\right).$$

Assumption 6 is implied if the γ_j are eventually monotonically decreasing. Under Assumption 6, it follows from Yong [15, Theorem III-12] that Assumption 2 is equivalent to

$$\gamma_j \sim D_H L(j) j^{2H-2}, \quad \text{as} \quad j \to \infty, \tag{3.1}$$

where $D_H = 2\Gamma(2(1-H))\cos((1-H)\pi)$, so that, for example, γ_j is eventually positive. Assumption 6 implies also that for large enough J

$$|\gamma_j - \gamma_{j+1}| \leqq \gamma_j - \gamma_{j+1} + \frac{2B\gamma_j}{j}, \quad \text{all} \quad j \geqq J, \tag{3.2}$$

and thus

$$\sum_{j=J}^{\infty} |\gamma_j - \gamma_{j+1}| \leqq \gamma_J + 2B \sum_{j=J}^{\infty} \frac{\gamma_j}{j} = O(\gamma_J) \tag{3.3}$$

as $J \to \infty$. Assumption 6 is strictly stronger than Assumption 4, as the following Lemma indicates. The lemma is doubtless known, but we have failed to locate a reference.

Lemma 8 *Under Assumptions 2 and 6,*

$$\min(\lambda, |\lambda-\mu|)|f(\lambda) - f(\lambda-\mu)| = O(|\mu|\, g(|\mu|)), \quad \textit{as} \quad |\mu| \to 0,$$

uniformly in $\lambda \in (0, \pi)$.

Proof. For $\lambda \leqq 2\mu$ the result is obvious. For $2\mu < \lambda < \pi$, $\pi|f(\lambda) - f(\lambda-\mu)|$ is bounded by

$$\left|\sum_{j=1}^{p} \gamma_j\{\cos(j\lambda) - \cos(j(\lambda-\mu))\}\right| + \left|\sum_{j=p+1}^{\infty} \gamma_j \cos(j\lambda)\right| + \left|\sum_{j=p+1}^{\infty} \gamma_j \cos(j(\lambda-u))\right|. \tag{3.4}$$

The first term of (3.4) is twice

$$\begin{aligned} &\left|\sum_{j=1}^{p} \gamma_j \sin(j(\lambda-\mu/2)) \sin(j\mu/2)\right| \\ &= \left|\sum_{j=1}^{p-1} \{\gamma_{j+1} \sin((j+1)\mu/2) - \gamma_j(\sin(j\mu/2))\} \sum_{\ell=1}^{j} \sin(\ell(\lambda-\mu/2)) \right. \\ &\quad \left. - \gamma_p \sin\left(\frac{p\mu}{2}\right) \sum_{\ell=1}^{p} \sin(\ell(\lambda-\mu/2))\right|, \end{aligned} \tag{3.5}$$

by summation by parts, where the factor in braces is bounded in absolute value by $|\mu|(j|\gamma_j-\gamma_{j+1}|+|\gamma_{j+1}|)$. Thus (3.5) is

$$O\left(\frac{|\mu|}{|\lambda-\mu/2|}\sum_{j=1}^{p}\{j|\gamma_j-\gamma_{j+1}|+|\gamma_{j+1}|\}\right)=O\left(\frac{|\mu|}{|\lambda-\mu/2|}L(p)\,p^{2H-1}\right)$$

as $p\to\infty$, using Lemma I-16 of [15], (2.5), (3.1) and (3.2), where the order is uniform in $\lambda\in(0,\pi)$. On the other hand, using (3.3),

$$\sum_{j=p+1}^{\infty}\gamma_j\cos(j\lambda)=\sum_{j=p+1}^{\infty}(\gamma_j-\gamma_{j+1})\sum_{\ell=p+1}^{j}\cos(\ell\lambda)$$
$$=O\left(\frac{|\gamma_p|}{\lambda}\right)=O\left(\frac{1}{\lambda}L(p)\,p^{2H-2}\right)$$

as $p\to\infty$, uniformly in $\lambda\in(0,\pi)$, and likewise $\sum_{j=p+1}^{\infty}\gamma_j\cos(j(\lambda-\mu)) = O(L(p)\,p^{2H-2}/|\lambda-\mu|)$. Now choose $p\sim|\mu|^{-1}$. □

In view of Lemma 8, Assumption 6 implies a global restriction on $f(\lambda)$ beyond the mere integrability imposed in Sect. 2: $f(\lambda)$ now satisfies an approximate $\mathrm{Lip}(2-2H)$ condition outside a neighbourhood of the origin, thereby ruling out long-memory behaviour at non-zero frequencies.

Theorem 2 *Under Assumptions 1, 2, 3, 5 and 6, and $3/4<H<1$, as $n\to\infty$*

$$\widehat{\mathrm{MSE}}_m\sim A_1\frac{L(n)^2}{L\left(\frac{1}{\lambda_m}\right)^2}(2\pi m)^{4H-4}+A_2\frac{L(n)}{L\left(\frac{1}{\lambda_m}\right)}\frac{(2\pi m)^{2H-2+\alpha}}{n^{\alpha}}+A_3\left(\frac{2\pi m}{n}\right)^{2\alpha},\tag{3.6}$$

where

$$A_1=2D_H^2(1-H)^2\left\{\frac{1}{(4H-3)(2H-1)}+\frac{1}{2H^2(2H-1)^2}-\frac{1}{H^2(4H-1)}-\frac{4\Gamma(2H-1)^2}{\Gamma(4H)}\right\},$$

$$A_2=-\frac{4D_H\,E_{\alpha H}(1-H)^2}{H(2H-1)(2-2H+\alpha)},\qquad A_3=\frac{4E_{\alpha H}^2(1-H)^2}{(2-2H+\alpha)^2}.$$

Corollary 2 *A bandwidth* $\hat{m}$ *which minimizes* $\hat{\mathrm{MSE}}_m$ *as* $n \to \infty$ *is*

$$(3.7)\quad \left(\frac{L\left(\frac{1}{\lambda_{\hat{m}}}\right)}{L(n)}\right)^{\frac{1}{2-2H+\alpha}} \cdot \hat{m} \sim \frac{n^{\frac{\alpha}{2-2H+\alpha}}}{2\pi}\left(\frac{D_H(2-2H+\alpha)}{4\alpha}\right.$$
$$\cdot\left[\frac{2H-2+\alpha}{E_{\alpha H}(2H-1)}+\frac{1}{|E_{\alpha H}|}\left\{\frac{(2-2H+\alpha)^2}{H^2(2H-1)^2}\right.\right.$$
$$+16\alpha(1-H)\left(\frac{1}{(4H-3)(2H-1)}-\frac{1}{H^2(4H-1)}\right.$$
$$\left.\left.\left.\left.-\frac{4\Gamma^2(2H-1)}{\Gamma(4H)}\right)\right\}^{1/2}\right]\right)^{\frac{1}{2-2H+\alpha}}$$

When $L(n)/L(1/\lambda_m) \to 1$ as $n \to \infty$, the formulae in Theorem 2 and Corollary 2 simplify, and indicate that $\hat{\mathrm{MSE}}_m$ is of the form $B_1 m^{4H-4} + B_2 m^{2H-2+\alpha} n^{-\alpha} + B_3 m^{2\alpha} n^{-2\alpha}$, that minimized $\hat{\mathrm{MSE}}_m$ converges to zero at rate $n^{(4H-4)\alpha/(2-2H+\alpha)}$, and that $\hat{m} \sim B_4 n^{\alpha/(2-2H+\alpha)}$, for constant $B_1, \ldots, B_4$. $L(n)/L(1/\lambda_m) \to 1$ as $n \to \infty$ if, for example, L is asymptotically constant, or $L(\lambda) = |\log \lambda|$ and $m \sim n^{\psi}, 0 < \psi < 1$ (but not $m \sim n/\log n$).

By way of contrast the corresponding results for the continuously averaged periodogram (1.6) (which assumes $\mathrm{E}x_1$ is given) are:

Theorem 3 *Under Assumptions 1, 2, 3, 5 and 6, and* $3/4 < H < 1$, *as* $n \to \infty$

$$\tilde{\mathrm{MSE}}_m \sim A_1' \frac{L(n)^2 (2\pi n)^{4H-4}}{L\left(\frac{1}{\lambda_m}\right)^2} + A_3\left(\frac{2\pi m}{n}\right)^{2\alpha},$$

where

$$A_1' = \frac{2D_H^2(1-H)^2}{(4H-3)(2H-1)}.$$

Corollary 3 *A bandwidth* $\tilde{m}$ *which minimizes* $\tilde{\mathrm{MSE}}_m$ *as* $n \to \infty$ *satisfies*

$$(3.8)\quad \left(\frac{L\left[\frac{1}{\lambda_{\tilde{m}}}\right]}{L(n)}\right)^{\frac{1}{2-2H+\alpha}} \cdot \tilde{m} = \frac{n^{\frac{\alpha}{2-2H+\alpha}}}{2\pi} \cdot \left[\frac{D_H^2}{E_{\alpha H}^2}\frac{(2-2H+\alpha)^2(1-H)}{\alpha(4H-3)(2H-1)}\right]^{\frac{1}{4-4H+2\alpha}}.$$

It is easily seen that $A_1 < A_1'$, that a sufficient condition for $\hat{\mathrm{MSE}}_m < \tilde{\mathrm{MSE}}_m$ is $E_{\alpha H} < 0$, and that a sufficient condition for $\hat{m} < \tilde{m}$ is $(2H-2+\alpha)E_{\alpha H} < 0$.

The rates of minimized $\tilde{\mathrm{MSE}}_m$ and $\tilde{m}$ are the same as those of minimized $\hat{\mathrm{MSE}}_m$ and $\hat{m}$ respectively.

Because the proofs of Theorem 3 and Corollary 3 are similar to, but simpler than, those of Theorem 2 and Corollary 2, we give only the latter.

Proof of Theorem 2 Commencing again from (2.6), Lemmas 1 and 3, and Lemmas 9 and 10 below, give

$$\hat{\mathrm{MSE}}_m \sim G(\lambda_m)^{-2}[V(\hat{F}(\lambda_m)) + \{E\tilde{F}(\lambda_m) - E\hat{F}(\lambda_m)\}^2 \\ -2\{E\tilde{F}(\lambda_m) - E\hat{F}(\lambda_m)\}\{F(\lambda_m) - G(\lambda_m)\} + \{F(\lambda_m) - G(\lambda_m)\}^2]$$

and thence (3.6). □

Lemma 4 is not useful when $H > 3/4$. In this case the effect of replacing $E\tilde{F}(\lambda_m)$ by $E\hat{F}(\lambda_m)$ affects $\hat{\mathrm{MSE}}_m$ non-negligibly, and we require the more delicate

Lemma 9 *Under Assumptions 1, 2 and 6, as $n \to \infty$*

$$E\{\tilde{F}(\lambda_m) - \hat{F}(\lambda_m)\} \sim V(\bar{x})/2 \sim \frac{D_H L(n)\, n^{2H-2}}{2H(2H-1)}. \tag{3.9}$$

Proof. For $m < n$, $\tilde{I}(\lambda_j) = I(\lambda_j), j = 1, \ldots, m$, and $\hat{F}(\lambda_m) = (2\pi/n) \sum_{j=1}^{m} \tilde{I}(\lambda_j)$. Now $\int_0^{2\pi} \tilde{I}(\lambda)\, d\lambda = (2\pi/n) \sum_{j=1}^{n} \tilde{I}(\lambda_j)$, and because $\tilde{I}(\lambda) = \tilde{I}(2\pi - \lambda)$,

$$\int_0^{2\pi} \tilde{I}(\lambda)\, d\lambda = 2 \int_0^{\pi} \tilde{I}(\lambda)\, d\lambda,$$

$$\frac{2\pi}{n} \sum_{j=1}^{n} \tilde{I}(\lambda_j) = \frac{4\pi}{n} \sum_{j=1}^{(n-1)/2} \tilde{I}(\lambda_j) + (\bar{x} - Ex_1)^2, \qquad n \quad \text{odd},$$

$$= \frac{4\pi}{n} \sum_{j=1}^{n/2} \tilde{I}(\lambda_j) - \frac{2\pi}{n} \tilde{I}(\pi) + (\bar{x} - Ex_1)^2, \quad n \quad \text{even}.$$

It follows that, with S_j defined as in Lemma 4's proof, the left side of (3.9) is

$$V(\bar{x})/2 - \int_{\pi(1-\frac{1}{n})}^{\pi} E\tilde{I}(\lambda)\, d\lambda + \sum_{j=m+1}^{(n-1)/2} \int_{S_j} E\{\tilde{I}(\lambda_j) - \tilde{I}(\lambda)\}\, d\lambda, \quad n \quad \text{odd}, \tag{3.10}$$

$$V(\bar{x})/2 - \frac{\pi}{n} E\tilde{I}(\pi) + \sum_{j=m+1}^{n/2} \int_{S_j} E\{\tilde{I}(\lambda_j) - \tilde{I}(\lambda)\}\, d\lambda, \quad n \quad \text{even}. \tag{3.11}$$

By Lemma 3.1 of Taqqu [14], $V(\bar{x}) \sim D_H L(n) n^{2H-2}/H(2H-1)$, as $n \to \infty$. Because $E\tilde{I}(\lambda)$ is the Cesaro sum of the Fourier series of $f(\lambda)$, which is bounded outside a neighbourhood of the origin in view of Lemma 8, we have $\sup_{\pi(1-n^{-1}) \leqq \lambda \leqq \pi} E\tilde{I}(\lambda) < \infty$, and thus the middle terms in (3.10) and (3.11) are both $O(n^{-1})$. It remains to show that the final terms of (3.10) and (3.11) are negligible relative to $V(\bar{x})$, and it suffices to consider the one in (3.11), which can be rewritten as the negative of

$$\frac{1}{\pi} \sum_{\ell=1}^{n-1} \gamma_\ell \left(1-\frac{\ell}{n}\right) \left\{ \int_{\lambda_m}^{\pi} \cos(\ell\lambda)\, d\lambda - \frac{2\pi}{n} \sum_{j=m+1}^{n/2} \cos(\ell\lambda_j) \right\}.$$

The factor in braces is

$$\text{(3.12)} \qquad \frac{\pi \sin((m+\lambda_\ell/2)}{n \sin(\lambda_\ell/2)} - \frac{\sin(m\lambda_\ell)}{\ell} - \frac{\pi}{n}(-1)^\ell$$

$$= \frac{2\sin(\lambda_\ell/4)\cos((m+1/4)\lambda_\ell}{\ell}$$

$$\text{(3.13)} \qquad + \frac{\pi \sin((m+1/2)\lambda_\ell)}{n\ell \sin(\lambda_\ell/2)} \left\{ \ell - \frac{n}{\pi} \sin(\lambda_\ell/2) \right\}$$

$$\text{(3.14)} \qquad - \frac{\pi}{n}(-1)^\ell.$$

The contribution of each of the three terms on the right will be estimated separately. First consider the one in (3.12). By summation by parts,

$$\text{(3.15)} \qquad \sum_{\ell=1}^{n-1} \frac{\gamma_\ell}{\ell}\left(1-\frac{\ell}{n}\right) \sin(\lambda_\ell/4)\cos((m+1/4)\lambda_\ell)$$

$$= \sum_{\ell=1}^{n-2} \Delta_\ell P_\ell + \frac{\gamma_{n-1}}{n(n-1)} P_{n-1},$$

where

$$\Delta_\ell = \frac{\gamma_\ell}{\ell}\left(1-\frac{\ell}{n}\right) - \frac{\gamma_{\ell+1}}{\ell+1}\left(1-\frac{\ell+1}{\mathrm{n}}\right),$$

$$P_\ell = \sum_{k=1}^{\ell} \sin(\lambda_k/4)\cos((m+1/4)\lambda_k).$$

Now $P_\ell = O(\min(\ell/m, \ell^2/n))$ uniformly, because $|\sin x| \leqq |x|$ implies $P_\ell = O(\ell^2/n)$ and summation by parts gives

$$P_\ell = \sum_{k=1}^{\ell-1} (\sin(\lambda_k/4) - \sin(\lambda_{k+1}/4)) \sum_{s=1}^{k} \cos((m+1/4)\lambda_s)$$

$$+ \sin(\lambda_\ell/4) \sum_{s=1}^{\ell} \cos((m+1/4)\lambda_s) = O\left(\frac{\ell}{m}\right)$$

using also $|\sin(x+y)-\sin x|\leqq|y|$ and (2.5). For all ℓ, $|\Delta_\ell|<\infty$. For n and then $\ell<n-1$ chosen large enough, it is easily seen that $\Delta_\ell=O(|\gamma_\ell|\ell^2)$. Thus for n and then $J<n/m$ large enough, (3.15) is

$$O\left(\frac{1}{n}+\frac{1}{n}\sum_{\ell=J+1}^{[n/m]}|\gamma_\ell|+\frac{1}{m}\sum_{\ell=[n/m]}^{n-1}\left|\frac{\gamma_\ell}{\ell}\right|+\frac{|\gamma_n|}{nm}\right)$$

$$=O\left(\frac{1}{n}\sum_{\ell=1}^{[n/m]}L(\ell)\,\ell^{2H-2}+\frac{1}{m}\sum_{\ell=[n/m]}^{\infty}L(\ell)\,\ell^{2H-3}\right)$$

$$=O\left(\frac{1}{n}L\left(\frac{n}{m}\right)\left(\frac{n}{m}\right)^{2H-1}\right)=o(L(n)\,n^{2H-2}),$$

because of Assumptions 1 and 2, because of Lemma 1–16 of [15], and because (2.1)(ii) implies $L(n/m)\,m^{1-2H}=o(L(n))$. To deal with the contribution from (3.13), note that

$$\sum_{K=1}^{\ell}\sin((m+1/2)\,\lambda_k)\left\{k-\frac{n}{\pi}\sin(\lambda_k/2)\right\}$$

$$=\sum_{k=1}^{\ell-1}\left\{\frac{n}{\pi}(\sin(\lambda_{k+1}/2)-\sin(\lambda_k/2))-1\right\}\sum_{S=1}^{k}\sin((m+1/2)\,\lambda_s)$$

$$-\left(\ell-\frac{n}{\pi}\sin(\lambda_\ell/2)\right)\sum_{s=1}^{\ell}\sin((m+1/2)\,\lambda_s)$$

$$=O\left(\frac{\ell^3}{n^2}\cdot\frac{n}{m}\right)=O\left(\frac{\ell^3}{nm}\right)$$

uniformly for $\ell\leqq n-1$, because of (2.5), $|\sin x-x|\leqq x^3/6$ and $|\sin(x+y)-\sin x-y|\leqq(x+y)^2y$ for $x,y>0$. Now

$$\frac{\gamma_\ell}{\ell\sin(\lambda_\ell/2)}\left(1-\frac{\ell}{n}\right)-\frac{\gamma_{\ell+1}}{(\ell+1)\sin(\lambda_{\ell+1}/2)}\left(1-\frac{\ell+1}{n}\right)$$

$$\text{(3.16)}\quad\left\{\begin{aligned}&=\frac{1-\ell/n}{\ell\sin(\lambda_\ell/2)}(\gamma_\ell-\gamma_{\ell+1})+\frac{\gamma_{\ell+1}}{\ell}\left(\frac{1-\ell/n}{\sin(\lambda_\ell/2)}-\frac{1-(\ell+1)/n}{\sin(\lambda_{\ell+1}/2)}\right)\\&\quad+\frac{\gamma_{\ell+1}(1-(\ell+1)/n)}{\sin(\lambda_{\ell+1}/2)}\left(\frac{1}{\ell}-\frac{1}{\ell+1}\right).\end{aligned}\right.$$

Treating the cases $\ell\leqq n/2$ and $\ell>n/2$ separately and using $(\sin x)/x\geqq 2/\pi$ for $0<x\leqq\pi/2$ and $\sin x=\sin(\pi-x)$ gives

$$\left|\frac{1-\ell/n}{\sin(\lambda_\ell/2)}\right|\leqq\frac{n}{2\ell},\qquad 1\leqq\ell\leqq n-2,$$

and using also $|\sin x - \sin(x+y)| \leqq |y|$ and $|(d/dx)(x/\sin x)| \leqq x^3/3(\sin x)^2 \leqq \pi^2 x/12$ for $0<x\leqq \pi/2$ gives, after some calculation,

$$\left|\frac{1-\ell/n}{\sin(\lambda_\ell/2)} - \frac{1-(\ell+1)/n}{\sin(\lambda_{\ell+1}/2)}\right| \leqq \frac{\pi n}{2\ell^2}, \qquad 1 \leqq \ell \leqq n-2.$$

It follows that (3.16) is bounded in absolute value by $3(n/\ell^2)(|\gamma_\ell - \gamma_{\ell+1}| + |\gamma_{\ell+1}|/\ell)$. Applying Assumption 6, for J sufficiently large

$$\frac{\pi}{n}\sum_{\ell=1}^{n-1} \gamma_\ell \left(1-\frac{\ell}{n}\right) \frac{\sin((m+)\,\lambda_\ell/2)}{\ell \sin(\lambda_\ell/2)} \left\{\ell - \frac{n}{\pi}\sin(\lambda_\ell/2)\right\}$$

$$= O\left(\frac{1}{n} + \frac{1}{nm}\sum_{\ell=J+1}^{n-1} |\gamma_\ell| + \frac{|\gamma_n|}{m}\right)$$

$$= O\left(L\left(\frac{n}{m}\right)\frac{L(n)\,n^{2H-2}}{m}\right) = o(L(n)\,n^{2H-2}),$$

as $n \to \infty$. Finally the contribution from (3.14) is

$$\frac{\pi}{n}\left\{\sum_{\ell=1}^{n-2} (\gamma_\ell - \gamma_{\ell+1}) \sum_{k=1}^{\ell} \left(1-\frac{k}{n}\right)(-1)^k + O(1)\right\}$$

$$= O\left(\frac{1}{n}\sum_{\ell=1}^{n-2} |\gamma_\ell - \gamma_{\ell+1}|\left(1+\frac{\ell}{n}\right)\right) = O\left(\frac{1}{n}\right). \qquad \square$$

Lemma 10 *Under Assumptions 1, 2 and 6 and* $3/4 < H < 1$, *as* $n \to \infty$

$$V(\hat{F}(\lambda_m)) \sim V\left[\frac{1}{2n}\sum_{t=1}^{n} (x_t - \bar{x})^2\right] \tag{3.17}$$

$$\sim D_H^2 \frac{L(n)^2}{n^{4-4H}} \left\{\frac{1}{(4H-3)(4H-2)} - \frac{1}{2H^2(4H-1)} - \frac{2\Gamma(2H-1)^2}{\Gamma(4H)}\right\}. \tag{3.18}$$

Proof. From the first part of the proof of Lemma 9,

$$\hat{F}(\lambda_m) = \frac{1}{2n}\sum_{t=1}^{n} (x_t - \bar{x})^2 - \frac{2\pi}{n}\sum_{j=m+1}^{(n-1)/2} \tilde{I}(\lambda_j), \quad n \text{ odd}$$

$$= \frac{1}{2n}\sum_{t=1}^{n} (x_t - \bar{x})^2 - \frac{2\pi}{n}\sum_{j=m+1}^{n/2} \tilde{I}(\lambda_j) + \frac{\pi}{n}\tilde{I}(\pi), \quad n \text{ even}.$$

By Assumption 5

$$V\left(\frac{1}{2n}\sum_{t=1}^{n}(x_t-\bar{x})^2\right)=\frac{1}{2n^2}\left\{\sum\sum_{t,s=1}^{n}\gamma_{t-s}^2-\frac{2}{n}\sum_{t=1}^{n}\left(\sum_{s=1}^{n}\gamma_{t-s}\right)^2\right.$$

$$\left.+\frac{1}{n^2}\left(\sum\sum_{t,s=1}^{n}\gamma_{t-s}\right)^2\right\}.$$

As $n\to\infty$

$$\frac{1}{n^3}\sum_{t=1}^{n}\left(\sum_{s=1}^{n}\gamma_{t-s}\right)^2\sim D_H^2\,L(n)^2\,n^{4H-4}\int_0^1\left\{\int_0^1|x-y|^{2H-2}dy\right\}^2dx$$

$$\sim\frac{2D_H^2}{(2H-1)^2}\left\{\frac{1}{4H-1}+\frac{\Gamma(2H)^2}{\Gamma(4H)}\right\}L(n)^2n^{4H-4},$$

and applying also Lemma 3.1 of [14], after rearrangement the right side of (3.17) is seen to be approximated by (3.18). By Assumptions 2 and 6, $(\pi/n)\,\tilde{I}(\pi)$ has variance

$$\frac{1}{2n^4}\left\{\sum_{u=1-n}^{n-1}(n-|u|)(-1)^u\gamma_u\right\}^2=O\left(\frac{1}{n^4}\left\{n\sum_{1}^{n}|\gamma_u-\gamma_{u+1}|+\sum_{1}^{n}|\gamma_u|\right\}^2\right)$$

as $n\to\infty$, and it then suffices to show that, for n even,

$$V\left(\frac{2\pi}{n}\sum_{j=m+1}^{n/2}\tilde{I}(\lambda_j)\right)=\frac{1}{n^4}\sum\sum_{j,k=m+1}^{n/2}U_{jk}$$

$$=o(L(n)^2n^{4H-4})\quad\text{as}\quad n\to\infty,$$

where $U_{jk}=W_{j,k-j}\,W_{k,j-k}+W_{j,-j-k}\,W_{k,j+k}$, with

$$W_{j,k}=\sum_{u=1-n}^{n-1}\gamma_u\,e^{iu\lambda_j}S_k(u),$$

$$S_k(u)=\sum_{t=\max(1,u+1)}^{\min(n,u+n)}e^{it\lambda_k}.$$

For $0<k\leqq 1/2n$, summation by parts gives

$$|W_{k,0}|\leqq n\gamma_0+2n\sum_{1}^{p-1}|\gamma_u| \tag{3.19}$$

$$+\frac{4n}{\lambda_k}\sum_{p}^{n-2}|\gamma_u-\gamma_{u+1}|+\frac{4n}{\lambda_k}|\gamma_{n-1}|$$

for any integer $p \in [2, n-2]$. We can pick $\varepsilon > 0$ such that for $1 \leqq k \leqq \varepsilon n$, $L(n) \leqq \mathrm{CL}\,(n/k)\,k^{2-2H}$ by (2.1)(i), and

$$\sum_{1}^{[n/k]} |\gamma_u| \leqq CL\left(\frac{n}{k}\right)\left(\frac{n}{k}\right)^{2H-1},$$

$$\sum_{[n/k]}^{n} |\gamma_u - \gamma_{u+1}| \leqq CL\left(\frac{n}{k}\right)\left(\frac{n}{k}\right)^{2H-2},$$

by Assumptions 2 and 6, (2.2)(i), (2.2)(ii), and [15, p. 30]. It follows that uniformly in such k and sufficiently large n, (3.19)$\leqq \mathrm{Cng}(\lambda_k)$. On the other hand (3.19)$\leqq \mathrm{Cn}$ by Assumptions 2 and 6 uniformly in $\varepsilon n \leqq k \leqq n$. It follows that $\overline{\ell i m}_{n \to \infty} \max_{m < k \leqq n/2} \{|W_{k,0}|/(ng(\lambda_k))\} < \infty$. For $0 < j, k \leqq n/2$ and $j \neq k$,

$$|W_{k,j-k}| \leqq \frac{2}{|\lambda_j - \lambda_k|} \sum_{1-p}^{p-1} |\gamma_u| + \sum_{p}^{n-2} |\gamma_u - \gamma_{u+1}|\,|R_{jk}(p,u)|$$
$$+ |\gamma_{n-1}|\,|R_{jk}(p, n-1)|,$$

where

$$R_{jk}(p,u) = \sum_{v=p}^{u} \left\{ e^{iv\lambda_k} \sum_{v+1}^{n} e^{it(\lambda_j - \lambda_k)} + e^{-iv\lambda_k} \sum_{1}^{n-v} e^{it(\lambda_j - \lambda_k)} \right\}$$
$$= (1 - e^{i(\lambda_j - \lambda_k)})^{-1} \sum_{v=p}^{u} [e^{i(\lambda_j - \lambda_k)} \{(e^{iv\lambda_j} - e^{iv\lambda_k}) + (e^{-iv\lambda_k} - e^{-iv\lambda_j})\}],$$

so that $|R_{jk}(p,u)| \leqq 4|\lambda_j - \lambda_k|^{-1}(\lambda_j^{-1} + \lambda_k^{-1})$. By an argument similar to that for $j = k$,

$$\lim_{n \to \infty} \max_{\substack{m < j,k \leqq n/2 \\ j \neq k}} \{|W_{k,j-k}|\,|\lambda_j - \lambda_k| / g((\lambda_j^{-1} + \lambda_k^{-1})^{-1})\} < \infty.$$

Clearly the same results hold for $W_{j,k-j}$, so

$$\sum\sum_{j,k=m+1}^{n/2} W_{j,k-j} W_{k,j-k}$$
$$= O\left(n^2 \sum_{m+1}^{n/2} g^2(\lambda_j) + \sum\sum_{\substack{m+1 \\ j<k}}^{n/2} g^2((\lambda_j^{-1} + \lambda_k^{-1})^{-1}) \frac{1}{|\lambda_j - \lambda_k|^2}\right)$$
$$= O\left(L^2\left(\frac{1}{\lambda_m}\right) n^{4H} \left\{ \sum_{m}^{\infty} j^{2-4H} + \sum_{m}^{\infty} j^{2-4H} \sum_{k>j} (j-k)^2 \right\}\right)$$
$$= O\left(L^2\left(\frac{1}{\lambda_m}\right) n^{4H} m^{3-4H}\right) = o(L^2(n)\, n^{4H}) \quad \text{as} \quad n \to \infty$$

for $3/4<H<1$, applying (2.1)(ii). Next note that $W_{-k,n}=W_{-k,0}=\bar{W}_{k,0}$, whereas $W_{-k,n/2}=0$. By proceeding much as before we can deduce that

$$\lim_{n\to\infty} \max_{\substack{m<j,k\leq n/2\\ j,k\neq n/2,k}} \{|W_{-k,j+k}| |\sin((\lambda_j+\lambda_k/2))|/g((\lambda_j^{-1}+\lambda_k^{-1})^{-1})\}<\infty,$$

with the same result for $W_{j,-j-k}$. Now $1/|\sin((\lambda_j+\lambda_k/2))|$ is of order $1/(\lambda_j+\lambda_k)$ for $j+k<n/2$, and of order $1/(2\pi-\lambda_j-\lambda_k)$ for $n/2<j+k<n$. By proceeding in much the same way as before we deduce that

$$\sum_{j,k=m+1}^{n/2}\sum W_{j,-j-k}W_{-k,j+k}=o(L^2(n)\,n^{4H}) \quad \text{as} \quad n\to\infty. \quad \square$$

Proof of Corollary 2 $L(\lambda)\sim L_1(\lambda)$ as $\lambda\to\infty$ for a slowly varying function $L_1(\lambda)$ which is differentiable for all sufficiently large λ (see [1, p. 14]). Thus for large enough n the right hand side of (3.6) has dominant term with derivative

$$(3.20) \quad 4(H-1)\,A_1\frac{L(n)^2}{L_1\left(\frac{n}{r}\right)^2}r^{4H-5} +(2H-2+\alpha)\,A_2\frac{L(n)}{L_1\left(\frac{n}{r}\right)}\frac{r^{2H-3+\alpha}}{n^\alpha}+\frac{2\alpha A_3\,r^{2\alpha-1}}{n^{2\alpha}}$$

$$(3.21) \quad +2A_1\,nL(n)^2\frac{L_1'\left(\frac{n}{r}\right)}{L_1^3\left(\frac{n}{r}\right)}r^{4H-6}+A_2\,nL(n)\frac{L_1'\left(\frac{n}{r}\right)}{L_1^2\left(\frac{n}{r}\right)}\frac{r^{2H-4+\alpha}}{n^\alpha},$$

where $r=2\pi m$ and $L_1'(\lambda)=(d/d\lambda)\,L_1(\lambda)$. But $\lambda L_1'(\lambda)/L(\lambda)\to 0$ as $\lambda\to\infty$ [1, p. 14], so that (3.21) is dominated by (3.20) as $n\to\infty$. Thus we can take $\hat{r}=2\pi\hat{m}$ to be a zero of

$$\frac{2\alpha A_3}{n^{2\alpha}}\left\{\frac{L\left(\frac{n}{r}\right)}{L(n)}r^{2-2H+\alpha}\right\}^2+\frac{(2H-2+\alpha)\,A_2}{n^\alpha}\left\{\frac{L\left(\frac{n}{r}\right)}{L(n)}r^{2-2H+\alpha}\right\}+4(H-1)\,A_1,$$

and the remainder of the proof is routine. $\square$

4 Robustness to non-Gaussianity

When x_t is non-Gaussian, the MSE contains an additional term depending on fourth cumulants and lagged fourth cumulants. Nevertheless this term may make a negligible contribution to the MSE, in which case the results of Sect. 2 and 3 continue to hold. Three different sorts of analysis of term are possible. The first recognizes that long range dependence can simultaneously be present in second cumulants but absent from fourth cumulants, as is true in the Gaussian case. "Weak dependence" conditions on fourth cumulants similar to those of [2, Chap. 2], [9, Chap. 5] in the smooth spectrum case can thus be employed, though the rate of convergence required will be different. The second approach would allow for long memory behaviour in lagged cumulants or cumulant spectra. A definitive treatment of this case would likely be complicated and lengthy. The third type of approach, which we employ, generalizes the Gaussian assumption in such a way that x_t satisfies a relatively general form of linear process. The moving average weights influence not only second cumulant structure but also fourth cumulant structure. We allow the moving average weights to decay in a manner consistent with the long range dependence in power spectra or autocovariances discussed in Sects. 2 and 3, and this implies a form of long range dependence in second and fourth cumulants.

Assumption 7 $x_t = Ex_1 + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j}$, where, for $t=0, \pm 1, \ldots$, ε_t satisfies

$$E(\varepsilon_t)=0,\ V(\varepsilon_t)=1,\ E(\varepsilon_t\,\varepsilon_s)=0,\ t\neq s,$$
$$cum(\varepsilon_r, \varepsilon_t, \varepsilon_s, \varepsilon_u)=\kappa_u,\ r=t=s=u,$$
$$=0, \text{ otherwise},$$

and

$$\max_{-\infty<u<\infty} |\kappa_u| < \infty,$$

while the θ_j satisfy

$$|\theta_j - \theta_{j+1}| \leq \frac{B|\theta_j|}{j} \tag{4.1}$$

for all $j > J$ and some $J < \infty$, $B < \infty$, and

$$\theta_j \sim \varphi_j \quad \text{or} \quad \theta_j \sim -\varphi_j \quad \text{as} \quad j \to \infty, \tag{4.2}$$

where

$$\varphi_j = \left(\frac{2}{\pi}\right)^{1/2} \Gamma(3/2-H)\cos((1-H)\pi)\, L^{1/2}(j)\, j^{H-3/2},\ 1/2<H<1.$$

(4.1) of Assumption 7 is stronger than quasi-monotonic convergence of the θ_j to zero (cf. Assumption 4 and (3.2)).

Theorem 4 *Under Assumptions 1, 3 and 7 the results of Theorems 1–3 hold.*

Corollary 4 *The results of Corollaries 1–3 hold.*

We need only give the

Proof of Theorem 4 We first need

Lemma 11 *Assumption 7 implies Assumptions 2 and 6.*

Proof. Assumption 7 indicates that $f(\lambda)$ can be written $f(\lambda)=|\theta(\lambda)|^2/2\pi$, where $\theta(\lambda)=\sum_{j=0}^{\infty}\theta_j e^{ij\lambda}$. Assumption 7 parts (4.1) and (4.2) with Theorems III-11 and III-12 of [15] give

$$\theta(\lambda)\sim\pm\left(\frac{\pi}{2}\right)^{1/2}\cos((1-H)\pi)L^{1/2}\left(\frac{1}{\lambda}\right)\lambda^{1/2-H}\left\{\sec\left(\left(\frac{3}{4}-\frac{H}{2}\right)\pi\right)\right.$$
$$\left.+i\operatorname{cosec}\left(\left(\frac{3}{4}-\frac{H}{2}\right)\pi\right)\right\}$$

as $\lambda\to 0+$, noting that (4.1) and (4.2) also imply the θ_j have bounded variation, that is $\sum_{j=0}^{\infty}|\theta_j-\theta_{j+1}|<\infty$. Thus Assumptions 2 is implied. For $s>0$, $\gamma_s=\sum_{u=0}^{\infty}\theta_u\theta_{u+s}$. As $s\to\infty$

$$\sum_{u=0}^{s}\theta_u(\theta_{u+s+1}-\theta_{u+s})=O\left(\frac{\varphi_s}{s}\sum_{u=1}^{s}|\theta_u|\right)=O\left(\frac{|\gamma_s|}{s}\right),$$

$$\sum_{u=s+1}^{\infty}\theta_u(\theta_{u+s+1}-\theta_{u+s})=O\left(\frac{1}{s}\sum_{u=s+1}^{\infty}|\theta_u\theta_{u+s}|\right)$$
$$=O\left(\frac{1}{s}\sum_{u=s}^{\infty}\varphi_u^2\right)=O\left(\frac{|\gamma_s|}{s}\right),$$

using Lemma I-16 of [15], Assumption 7 and (2.2). Thus the γ_s are quasi-monotonically convergent to zero and Assumption 6 is implied. □

Lemma 8 has already established that Assumption 6 implies Assumption 4. Thus Theorems 1 and 2 continue to hold with $\widehat{\mathrm{MSE}}_m$ replaced by $\widehat{\mathrm{MSE}}_m-\hat{K}_m/G(\lambda_m)^2$, where

$$\hat{K}_m=\frac{1}{n^4}\sum_{j,k=1}^{m}\sum\sum\sum_{q,r,s,t=1}^{n}\sum\operatorname{cum}\{x_q,x_r,x_s,x_t\}\,e^{i(q-r)\lambda_j-i(s-t)\lambda_k}\tag{4.3}$$

can be non-zero due to the absence of Assumption 5. Likewise Theorem 3 continues to hold with $\widetilde{\mathrm{MSE}}_m$ replaced by $\widetilde{\mathrm{MSE}}_m-\tilde{K}_m/G(\lambda_m)^2$, where $\tilde{K}_m$ has integrals in place of the sums over j and k in (4.3). The proof is completed by the following Lemma, and an analogous result for $\tilde{K}_m$ whose statement and very similar proof we omit. □

Lemma 12 *Under Assumptions 1 and 7,*

$$\hat{K}_m = O\left(\frac{G(\lambda_m)^2}{n}\right), \quad as \quad n \to \infty.$$

Proof. By the first part of Assumption 7,

$$\hat{K}_m = \frac{1}{n^4} \sum_{j,k=1}^{m}\sum \sum_{q,r,s,t=1}^{n}\sum\sum\sum \sum_{u=-\infty}^{n} \kappa_u \theta_{q-u} \theta_{r-u} \theta_{s-u} \theta_{t-u} e^{i(q-r)\lambda_j - i(s-t)\lambda_k},$$

with the convention that $\theta_t = 0$, $t < 0$. Thus

$$(4.4) \qquad |\hat{K}_m| \leq \{\max_{-\infty < u < \infty} |\kappa_u|\} \frac{1}{n^4} \sum_{u=-n}^{\infty} \left\{ \sum_{j=1}^{m} \left| \sum_{t=1}^{n} \theta_{t+u} e^{it\lambda_j} \right|^2 \right\}^2.$$

First consider $u \in [-n, -1]$. Then

$$\sum_{t=1}^{n} \theta_{t+u} e^{it\lambda} = e^{-iu\lambda} \sum_{t=0}^{n-1} \theta_t d_{tu}(\lambda),$$

where

$$d_{tu}(\lambda) = e^{it\lambda}, t \leqq n+u; = 0, t > n+u.$$

Then from (2.5), for $\lambda \in (0, \pi)$

$$(4.5) \qquad \left| \sum_{t=1}^{n} \theta_t d_{tu}(\lambda) \right| \leqq \sum_{t=0}^{[1/\lambda]} |\theta_t| + \frac{2}{\lambda} \left\{ \sum_{[1/\lambda]+1}^{n-2} |\theta_t - \theta_{t+1}| + |\theta_{n-1}| \right\}.$$

It easily follows from Assumptions 1 and 7 that

$$\lim_{n \to \infty} \max_{-n \leqq u \leqq -1} \max_{1 \leqq j \leqq m} \frac{\left| \sum_{t=0}^{n-1} \theta_t d_{tu}(\lambda_j) \right|}{g^{1/2}(\lambda_j)} < \infty,$$

and thus

$$\sum_{u=-n}^{-1} \left\{ \sum_{j=1}^{m} \left| \sum_{t=1}^{n} \theta_{t+u} e^{it\lambda_j} \right|^2 \right\}^2 = O\left(n \left\{ \sum_{j=1}^{m} g(\lambda_j) \right\}^2 \right) = O(n^3 G(\lambda_m)^2),$$

as $n \to \infty$. Next, for $u \in [0, n-1]$,

$$\sum_{t=1}^{n} \theta_{t+u} e^{it\lambda}$$
$$= e^{-iu\lambda} \left\{ \theta(\lambda) - \sum_{t=n+u+1}^{\infty} \theta_t e^{it\lambda} - \sum_{t=0}^{u} \theta_t e^{it\lambda} \right\}.$$

It is easily seen that $\left| \sum_{t=0}^{u} \theta_t e^{it\lambda} \right|$ has the bound in (4.5), and so

$$\sum_{u=0}^{n-1} \left\{ \sum_{j=1}^{m} \left| \sum_{t=0}^{u} \theta_t e^{it\lambda_j} \right|^2 \right\}^2 = O(n^3 G(\lambda_m)^2), \quad \text{as} \quad n \to \infty.$$

Next,

$$\sum_{u=0}^{n-1} \left\{ \sum_{j=1}^{m} |\theta(\lambda_j)|^2 \right\}^2 = O\left(n \left\{ \sum_{j=1}^{m} g(\lambda_j) \right\}^2 \right) = O(n^3 G(\lambda_m)), \quad \text{as} \quad n \to \infty,$$

and because

$$\left| \sum_{t=n+u+1}^{\infty} \theta_t e^{it\lambda} \right| \leqq \frac{2}{\lambda} \sum_{t=n+u+1}^{\infty} |\theta_t - \theta_{t+1}|,$$

it follows that as $n \to \infty$

$$\sum_{u=0}^{n-1} \left\{ \sum_{j=1}^{m} \left| \sum_{t=n+u+1}^{\infty} \theta_t e^{it\lambda_j} \right|^2 \right\}^2 = O\left(n \theta_n^4 \left(\sum_{j=1}^{m} \lambda_j^{-2} \right)^2 \right)$$
$$= O(L^2(n) n^{4H-1}) = o(n^3 G^2(\lambda_m)),$$

using (2.1)(i). Finally the contribution to the bound in (4.4) from $u \geqq n$ is

$$\sum_{u=n}^{\infty} \left\{ \sum_{j=1}^{n} \left| \sum_{t=1}^{n} \theta_{t+u} e^{it\lambda_j} \right|^2 \right\}^2$$
$$\leqq 4 \sum_{u=n}^{\infty} \left\{ \sum_{j=1}^{m} \lambda_j^{-2} \left(\sum_{t=1}^{n-1} |\theta_{t+u} - \theta_{t+u+1}| + |\theta_{n+u}| \right)^2 \right\}^2$$
$$= O\left(n^4 \sum_{u=n}^{\infty} \theta_u^4 \right) = O(L^2(n) n^{4H-1}), \quad \text{as} \quad n \to \infty. \quad \square$$

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