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Sobolev spaces of Banach-valued functions associated with a Markov process

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Summary. We discuss Sobolev spaces of Banach-valued functions. They are extensions of Sobolev spaces of scalar functions. We use a gamma transform of a semigroup associated with a Markov process. A typical example is the Ornstein-Uhlenbeck process on the Wiener space.

Mathematics Subject Classification: 60H, 60J

1 Introduction

Sobolev spaces on the Wiener space are well-developed and play an important role in the Malliavin calculus. In the Malliavin calculus, we treated only Hilbert-valued functions.

Recently, some researchers have begun to discuss Banach-valued functions in the framework of the Malliavin calculus. For example, Ren [9] proved that a solution to a stochastic differential equation has continuous paths quasi surely, (i.e. except for a set of zero capacity). Further Feyel-de La Pradelle [3] introduced a Sobolev space of Banach-valued functions on the Wiener space by using derivatives (see also Malliavin-Nualart [8]).

In the paper we attempt to define a Sobolev space in a different way. Our method is based on the Markov property of the semigroup. So we can develop it for a general Markov process. We use the fact that the semigroup acts on not only scalar valued functions but also Banach-valued functions. Moreover it defines a strongly continuous contraction semigroup on the space of all Banach-valued Borel functions u such that $\|u\|^p$ is integrable.

The organization of this paper is as follows. In Sect. 2, we define the Sobolev spaces. The gamma transformation is used in an essential way.

Next we show the existence of a quasi-continuous modification in Sect. 3. Lastly, we discuss some examples in Sect. 4. We show that the solution to a stochastic differential equation has a quasi-continuous modification.

2 Sobolev spaces of Banach valued functions

Let X be a separable metric space. We denote the Borel σ-field by $\mathscr{B}(X)$. Let m be a Borel probability measure on X. Suppose that a homogeneous Markov process on X is given. We denote its transition probability by $p(t, x, dy)$. So $p(t, x, dy)$ satisfies following conditions:

(2.1) $$p(0, x, dy) = \delta_x(dy)$$

(2.2) $$p(t+s, x, dy) = \int_X p(t, x, dz)\, p(s, z, dy)$$

(2.3) $$p(t, x, X) \leqq 1.$$

Here δ_x denotes the Dirac measure at x.

Then the semigroup $\{T_t\}$ is defined by

(2.4) $$T_t f(x) = \int_X f(y)\, p(t, x, dy).$$

The above expression is well-defined for $f \in \mathscr{B}_b(X)$, where $\mathscr{B}_b(X)$ is the set of all bounded Borel functions on X. We assume that, for all $t \geqq 0$,

(2.5) $$\int_X T_t f(x)\, m(dx) \leqq \int_X f(x)\, m(dx) \qquad f \in \mathscr{B}_b(X), f \geqq 0.$$

By this inequality, it is easy to extend T_t to $L^p(X, m)$. Now we can define a contraction semigroup on $L^p(X, m)$ by (2.4).

Further we assume that

(A.1) $\{T_t\}$ forms a strongly continuous semigroup on $L^p(X, m)$.

The above semigroup acts on scalar functions, but we want to consider Banach valued functions. Let B be a real separable Banach space. We denote by $L^p(X, m; B)$ the space of all B-valued measurable functions u satisfying

(2.6) $$\|u\|_p := \{\int_X \|u(x)\|_B^p\, m(dx)\}^{1/p} < \infty.$$

As usual, we identify two functions which are equal to each other m-a.e. For $u \in L^p(X, m; B)$, we set

(2.7) $$T_t u(x) = \int_X u(y)\, p(t, x, dy).$$

Here the integral on the right is the Bochner integral. By noting that

$$\begin{aligned}\|T_t u\|_p^p &\leqq \int_X \{\int_X \|u(y)\|_B\, p(t,x,dy)\}^p\, m(dx)\\ &\leqq \int_X \int_X \|u(y)\|_B^p\, p(t,x,dy)\, m(dx)\\ &\leqq \int_X \|u(x)\|_B^p\, m(dx)\\ &= \|u\|_p^p\end{aligned}$$

$T_t u(x)$ can be defined for m-a.e. x, and $\{T_t\}$ is a contraction semigroup. Further $\{T_t\}$ is strongly continuous. To see this, recall that functions u of the form

$$u=\sum_{i=1}^{n} f_i\, b_i \qquad f_i\in L^p(X,m),\ b_i\in B \tag{2.8}$$

are dense in $L^p(X,m;B)$. It is easy to see that $\lim_{t\downarrow 0}\|T_t u-u\|_p=0$ for u as in (2.8), since $\{T_t\}$ is strongly continuous on $L^p(X,m)$. Then it easily follows that $\{T_t\}$ is strongly continuous on $L^p(X,m;B)$.

Let us denote the generator by A. Now for $r\geqq 0$ we can define $(1-A)^{-r/2}$ by the following gamma transformation:

$$(1-A)^{-r/2}=\frac{1}{\Gamma(r/2)}\int_0^\infty t^{r/2-1}\, e^{-t}\, T_t\, dt. \tag{2.9}$$

$(1-A)^{-r/2}$ is a contraction operator, since $\{T_t\}$ is a contraction semigroup. We set

$$\mathscr{F}_{r,p}(B):=(1-A)^{-r/2}(L^p(X,m;B)). \tag{2.10}$$

We call $\mathscr{F}_{r,p}(B)$ the *Sobolev space* with degree r and integrability index p.

We have the following proposition, by the same proof as in [7, Proposition 2.3].

Proposition 2.1 *$(1-A)^{-r/2}$ is injective and $\mathscr{F}_{r,p}(B)$ is dense in $L^p(X,m;B)$.*

For $v=(1-A)^{-r/2}u$, $u\in L^p(X,m;B)$, define a norm $\|\cdot\|_{r,p}$ by

$$\|v\|_{r,p}:=\|u\|_p. \tag{2.11}$$

Then $(\mathscr{F}_{r,p}(B), \|\cdot\|_{r,p})$ forms a Banach space. In particular, the case of $B=\mathbf{R}$ has been studied by many people, and we can use a general theory of (r,p)-capacity. We simply denote $\mathscr{F}_{r,p}(\mathbf{R})$ by $\mathscr{F}_{r,p}$.

Next proposition gives a criterion whether v is in $\mathscr{F}_{r,p}(B)$, by reduction to the scalar case.

Proposition 2.2 *Take any $v \in L^p(X, m; B)$. Then $v \in \mathscr{F}_{r,p}(B)$ if and only if there exists a $u \in L^p(X, m; B)$ such that for any $\varphi \in B^*$,*

$$\langle \varphi, v \rangle \in \mathscr{F}_{r,p} \quad \text{and} \quad (1-A)^{r/2} \langle \varphi, v \rangle = \langle \varphi, u \rangle. \tag{2.12}$$

Moreover, under the above condition, $(1-A)^{r/2} v = u$.

Proof. The necessity is trivial. We prove the sufficiency. It is enough to show $(1-A)^{-r/2} u = v$. Since B is separable, we can take a countable family $\{\varphi_n\} \subseteq B^*$ such that $x=0$ if and only if $\langle x, \varphi_i \rangle = 0$, $i=1, 2, \ldots$. We claim that for all i,

$$\langle \varphi_i, (1-A)^{-r/2} u \rangle = \langle \varphi_i, v \rangle \qquad m\text{-a.e.}$$

To see this, using the assumption, we have

$$\begin{aligned}\langle \varphi_i, (1-A)^{-r/2} u - v \rangle &= \left\langle \varphi_i, \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t u \, dt \right\rangle - \langle \varphi_i, v \rangle \\ &= \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} T_t \langle \varphi_i, u \rangle \, dt - \langle \varphi_i, v \rangle \\ &= (1-A)^{-r/2} \langle \varphi_i, u \rangle - \langle \varphi_i, v \rangle \\ &= 0\end{aligned}$$

as desired. □

For a Banach space B and $d \in \mathbf{N}$, $C([0, 1]^d \to B)$ (the set of all B-valued contionuous functions on $[0, 1]^d$) is again a Banach space with a norm

$$\|\xi\|_C = \sup_{t \in [0,1]^d} \|\xi_t\|_B .$$

The $C([0, 1]^d \to B)$-valued function ξ on X can be regarded as a family of B-valued functions with a parameter $t \in [0, 1]^d$: $\xi = (\xi_t)_{t \in [0,1]^d}$. Then the following criterion is useful for applications.

Corollary 2.3 *Take any $\xi \in L^p(X, m; C([0, 1]^d \to B))$. Then $\xi \in \mathscr{F}_{r,p}(C([0, 1]^d \to B))$ if and only if there exists an $\eta \in L^p(X, m; C([0, 1]^d \to B))$, such that for any $t \in [0, 1]^d$,*

$$\xi_t \in \mathscr{F}_{r,p}(B) \quad \text{and} \quad (1-A)^{r/2} \xi_t = \eta_t . \tag{2.13}$$

Moreover, under the above condition, $(1-A)^{r/2} \xi = \eta$.

Proof. The proof is similar to that of Proposition 2.2. We show the sufficiency. By the assumption, for any $t\in[0,1]^d$,

$$((1-A)^{-r/2}\xi)_t-\eta_t=(1-A)^{-r/2}\xi_t-\eta_t=0 \qquad m\text{-a.e.}$$

Hence by the continuity,

$$(1-A)^{-r/2}\xi=\eta \qquad m\text{-a.e.},$$

as required. □

We use the above corollary as follows: If $((1-A)^{r/2}\xi_t)_{t\in[0,1]^d}$ admits a version (η_t) which is continuous in t m-a.e. and

$$\int_X \sup_{t\in[0,1]^d} \|\eta_t\|_B^p\, m(dx)<\infty,$$

then $\xi=(\xi_t)_t\in\mathscr{F}_{r,p}(C([0,1]^d\to B))$.

3 Quasi-continuous modifications

In this section we discuss the connection to the (r,p)-capacity. In particular, we show the existence of a quasi-continuous modification of an element of $\mathscr{F}_{r,p}(B)$.

First a quick review of the (r,p)-capacity. For details, see [5, 3] and [7]. For $[0,\infty]$-valued lower semicontinuous (l.s.c.) functions h, define $C_{r,p}(h)$ by

$$C_{r,p}(h):=\inf\{\|u\|_{r,p}^p;\ u\in\mathscr{F}_{r,p},\ u\geqq h,\ m\text{-a.e.}\},$$

and for an arbitrary $[-\infty,\infty]$-valued function f (not assumed to be measurable),

$$C_{r,p}(f):=\inf\{C_{r,p}(h);\ h \text{ is l.s.c. and } h(x)\geqq|f(x)|,\ \forall x\in X\}.$$

Here and in the sequel we use the convention $\inf\phi=\infty$.

Then the following properties hold. For any functions $f, f_1, f_2, \ldots$, and $\lambda\geqq 0$,

(3.1) $$C_{r,p}(\lambda f)=\lambda^p C_{r,p}(f),$$

(3.2) $$|f_1(x)|\leqq|f_2(x)| \quad \forall x\in X \Rightarrow C_{r,p}(f_1)\leqq C_{r,p}(f_2),$$

(3.3) $$C_{r,p}(\sup_n |f_n|)\leqq\sum_n C_{r,p}(f_n),$$

(3.4) $$C_{r,p}(\sum_n f_n)^{1/p}\leqq\sum_n C_{r,p}(f_n)^{1/p},$$

(3.5) $$C_{r,p}(\{x\in X; f(x)\geqq\lambda\})\leqq\frac{1}{\lambda^p}C_{r,p}(f).$$

For a set B, we define

$$C_{r,p}(B)=C_{r,p}(1_B).$$

Here 1_B denotes the indicator function of B.

We say that a property holds quasi-everywhere (q.e.), if it holds except on a set of capacity 0. To get a regularity of capacity, we assume the following:

(A.2) $\mathscr{F}_{r,p}\cap C_b(X)$ is dense in $\mathscr{F}_{r,p}$ and $1\in\mathscr{F}_{r,p}$.

Then the capacity satisfies

$$0\leqq f_1\leqq f_n\leqq\dots\uparrow f\Rightarrow C_{r,p}(f)=\sup_n C_{r,p}(f_n). \tag{3.6}$$

Further, for any function $f\in\mathscr{F}_{r,p}$, there exists a quasi-continuous function g such that $f=g$ m-a.e. Here a function g is said to be quasi-continuous, if there exists a sequence of increasing closed sets $\{F_n\}$, such that $C_{r,p}(X\setminus F_n)\to 0$ and $g|_{F_n}$ is continuous. The above quasi-continuous function g is called a quasi-continuous modification of f. We usually denote it by $\tilde{f}$.

Now we proceed to B-valued functions.

Proposition 3.1 *$\mathscr{F}_{r,p}(B)\cap C_b(X\to B)$ is dense in $\mathscr{F}_{r,p}(B)$.*

Proof. Since functions u of the form

$$u=\sum_{i=1}^n f_i b_i,\quad f_i\in L^p(X,m),\ b_i\in B$$

are dense in $L^p(X,m;B)$, the functions

$$(1-A)^{-r/2}u=\sum_{i=1}^n(1-A)^{-r/2}f_i b_i$$

are dense in $\mathscr{F}_{r,p}(B)$. We may assume that $\|b_i\|_B\leqq 1$. By the assumption (A.2), for any $\varepsilon>0$, there exist $g_i\in\mathscr{F}_{r,p}\cap C_b(X)$ such that

$$\|(1-A)^{-r/2}f_i-g_i\|_{r,p}\leqq\varepsilon/n.$$

Therefore we have

$$\begin{aligned}\left\|\sum_{i=1}^n(1-A)^{-r/2}f_i b_i-\sum_{i=1}^n g_i b_i\right\|_{r,p}&=\left\|\sum_{i=1}^n f_i b_i-\sum_{i=1}^n(1-A)^{r/2}g_i b_i\right\|_p\\&\leqq\sum_{i=1}^n\|(f_i-(1-A)^{r/2}g_i)b_i\|_p\\&\leqq\sum_{i=1}^n\|f_i-(1-A)^{r/2}g_i\|_p\\&\leqq\sum_{i=1}^n\|(1-A)^{-r/2}f_i-g_i\|_{r,p}\\&\leqq\varepsilon.\end{aligned}$$

It is clear that $\sum_i g_i b_i \in \mathscr{F}_{r,p}(B) \cap C_b(X \to B)$ and the assertion follows. □

Next we show that any element of $\mathscr{F}_{r,p}(B)$ admits a quasi-continuous modification. Before proving this, we recall that $f \in \mathscr{F}_{r,p}$ satisfies

$$C_{r,p}(\tilde{f}) \leqq \|f\|_{r,p}^p,$$

(see, e.g. [7, Proposition 5.2]). We will extend this inequality to B-valued functions. First we have the following:

Proposition 3.2 *For any $v \in \mathscr{F}_{r,p}(B) \cap C_b(X \to B)$,*

$$C_{r,p}(\|v\|_B) \leqq \|v\|_{r,p}^p.$$

Proof. From the definition, there exists a $u \in L^p(X, m; B)$, such that $v = (1-A)^{-r/2} u$, i.e.,

$$v(x) = \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} \int_X u(y)\, p(t, x, dy)\, dt.$$

Hence,

$$\|v(x)\|_B \leqq \frac{1}{\Gamma(r/2)} \int_0^\infty t^{r/2-1} e^{-t} \int_X \|u(y)\|_B\, p(t, x, dy)\, dt$$

which implies

$$\|v\|_B \leqq (1-A)^{-r/2} \|u\|_B \qquad m\text{-a.e.}$$

On the other hand, since $\|u\|_B \in L^p(X, m)$, $(1-A)^{-r/2} \|u\|_B \in \mathscr{F}_{r,p}$, and therefore $(1-A)^{-r/2} \|u\|_B$ admits a quasi-continuous modification, which we denote by $((1-A)^{-r/2} \|u\|_B)^\sim$. We have

$$\|v\|_B \leqq ((1-A)^{-r/2} \|u\|_B)^\sim \quad \text{m-a.e.}$$

The above functions are both quasi-continuous, and we have

$$\|v\|_B \leqq ((1-A)^{-r/2} \|u\|_B)^\sim \quad \text{q.e.}$$

Thus we obtain

$$\begin{aligned} C_{r,p}(\|v\|_B) &\leqq C_{r,p}(((1-A)^{-r/2} \|u\|_B)^\sim) \\ &\leqq \|(1-A)^{-r/2} \|u\|_B\|_{r,p}^p \\ &= \| \|u\|_B \|_p^p \\ &= \|u\|_p^p \\ &= \|v\|_{r,p}^p \end{aligned}$$

as desired. □

Now we can state the main theorem of this section.

Theorem 3.3 *Any $v \in \mathscr{F}_{r,p}(B)$ admits a quasi-continuous modification. Further, denoting it by $\tilde{v}$, we have*

$$C_{r,p}(\|\tilde{v}\|_B) \leqq \|v\|_{r,p}^p. \tag{3.7}$$

Proof. Take $\{v_n\} \subseteq \mathscr{F}_{r,p}(B) \cap C_b(X \to B)$ such that $\|v - v_n\|_{r,p} \to 0$. Then by (3.5) and Proposition 3.2,

$$C_{r,p}(\|v_n - v_m\|_B \geqq \varepsilon) \leqq \frac{1}{\varepsilon^p} C_{r,p}(\|v_n - v_m\|_B) \leqq \frac{1}{\varepsilon^p} \|v_n - v_m\|_{r,p}^p.$$

Now by a standard argument (see e.g. [4]), we can take a subsequence $\{v_{n_j}\}$ such that $\{v_{n_j}\}$ converges q.e. and the limit is quasi continuous. The limit is the quasi-continuous modification of v.

To show the inequality (3.7), we note that there exists some $u \in L^p(X, m; B)$ such that $v = (1-A)^{-r/2} u$. Then

$$\|v\|_B \leqq (1-A)^{-r/2} \|u\|_B \quad \text{a.e.}$$

and hence

$$(\|v\|_B)^\sim \leqq ((1-A)^{-r/2} \|u\|_B)^\sim \quad \text{q.e.}$$

where $^\sim$ denotes the quasi-continuous modification. But $\|\tilde{v}\|_B = (\|v\|_B)^\sim$, and we have

$$\begin{aligned} C_{r,p}(\|\tilde{v}\|_B) &\leqq C_{r,p}(((1-A)^{-r/2} \|u\|_B)^\sim) \\ &\leqq \|(1-A)^{-r/2} \|u\|_B\|_{r,p}^p \\ &= \| \, \|u\|_B\|_p^p \\ &= \|u\|_p^p \\ &= \|v\|_{r,p}^p \end{aligned}$$

which is (3.7). □

By combining (3.7) with (3.5), we have the following Chebyshev type inequality:

$$C_{r,p}(\|\tilde{v}\| \geqq \lambda) \leqq \frac{1}{\lambda^p} \|v\|_{r,p}^p.$$

As an application, we give a capacity version of Kolmogorov's criterion for path continuity. It was proved by Ren [9] for real valued processes.

Theorem 3.4 *Let $\xi = (\xi_t)$ be a B-valued process with a parameter $t \in [0, 1]^d$. Suppose that $\xi_t \in \mathscr{F}_{r,p}(B)$ for any t. Further suppose that there exist constants $\beta > 0$, $c > 0$ such that*

$$\|\xi_t - \xi_s\|_{r,p}^p \leqq c\,|t-s|^{d+\beta}. \tag{3.8}$$

Then ξ admits a quasi-continuous modification $\tilde{\xi}$ as a $C([0,1]^d \to B)$-valued function, such that

$$C_{r,p}\left(\sup_{s\neq t} \frac{\|\tilde{\xi}_t - \tilde{\xi}_s\|_B}{|t-s|^\gamma}\right) < \infty \tag{3.9}$$

for every $0<\gamma<\beta/p$. In particular, the paths of $(\tilde{\xi}_t)$ are Hölder continuous of order γ q.e.

Proof. The proof is just a repetition of the measure case. But we give a proof for completeness.

From the assumption we have

$$\begin{aligned} E[\|(1-A)^{r/2}\xi_t-(1-A)^{r/2}\xi_s\|_B^p] &= \|(1-A)^{r/2}\xi_t-(1-A)^{r/2}\xi_s\|_p^p \\ &= \|\xi_t-\xi_s\|_{r,p}^p \\ &\leqq c\,|t-s|^{d+\beta}. \end{aligned}$$

Here E denotes integration with respect to m. Noting that $(1-A)^{-r/2}$ is Markovian, we have similarly

$$\begin{aligned} E[\|\xi_t-\xi_s\|_B^p] &= E[\|(1-A)^{-r/2}((1-A)^{r/2}\xi_t-(1-A)^{r/2}\xi_s)\|_B^p] \\ &\leqq E[\|(1-A)^{r/2}\xi_t-(1-A)^{r/2}\xi_s\|_B^p] \\ &\leqq c\,|t-s|^{d+\beta}. \end{aligned}$$

Hence, by Kolmogorov's criterion, we may assume that (ξ_t) and $((1-A)^{r/2}\xi_t)$ are continuous in t m-a.e. and

$$E[\sup_{t\in[0,1]^d}\|\xi_t\|_B^p]<\infty,$$

$$E[\sup_{t\in[0,1]^d}\|(1-A)^{r/2}\xi_t\|_B^p]<\infty.$$

By Corollary 2.3, we obtain $\xi\in\mathscr{F}_{r,p}(C([0,1]^d\to B))$. Then, by Theorem 3.3, ξ admits a quasi continuous modification $\tilde{\xi}$ as a $C([0,1]^d\to B)$-valued function.

Now we borrow the argument of [11, Theorem I.2.1]. As a convention, we use the norm of the parameter $t\in[0,1]^d$ defined by $|t|=\max_i|t_i|$. Let D_m be the set of all points in $[0,1]^d$ whose components are all equal to $i\,2^{-m}$ for some integer i, and set $D=\bigcup_m D_m$. Let further Δ_m be the set of pairs $(s,t)\in D_m\times D_m$ such that $|t-s|=2^{-m}$. Finally set $K_i=\max_{(s,t)\in\Delta_m}\|\xi_t-\xi_s\|_B$. Then by the assumption, there exists a constant $J>0$ such that

$$\begin{aligned}C_{r,p}(K_i)&\leqq\sum_{(s,t)\in\Delta_i}C_{r,p}(\|\xi_s-\xi_t\|_B)\\&\leqq\sum_{(s,t)\in\Delta_i}\|\xi_s-\xi_t\|_{r,p}^p\\&\leqq J\,2^{-i\beta}.\end{aligned}$$

For a point s in D, there exists an increasing sequence (s_m) such that $s_m\in D_m$ and $s_m\leqq s$ and $s_m=s$ from some m on. Moreover we may choose a similar sequence (t_m) for $t\in D$. If $|s-t|\leqq 2^{-m}$, then either $s_m=t_m$ or $(s_m,t_m)\in\Delta_m$, and in either case

$$\xi_s-\xi_t=\sum_{i=m}^{\infty}(\xi_{s_{i+1}}-\xi_{s_i})+\xi_{s_m}-\xi_{t_m}+\sum_{i=m}^{\infty}(\xi_{t_i}-\xi_{t_{i+1}}).$$

It follows that

$$\|\xi_s-\xi_t\|_B\leqq K_m+2\sum_{i=m+1}^{\infty}K_i\leqq 2\sum_{i=m}^{\infty}K_i.$$

Consequently, setting $M_\gamma=\sup\{\|\xi_s-\xi_t\|_B/|t-s|^\gamma;\ s,t\in D,\ s\neq t\}$,

$$\begin{aligned}M_\gamma&\leqq\sup_m\{2^{(m+1)\gamma}\sup_{2^{-m-1}<|t-s|\leqq 2^{-m}}\|\xi_t-\xi_s\|_B;\ s,t\in D,\ s\neq t\}\\&\leqq\sup_m\left\{2\cdot 2^{(m+1)\gamma}\sum_{i=m}^{\infty}K_i\right\}\\&\leqq 2^{\gamma+1}\sum_{i=0}^{\infty}2^{i\gamma}K_i.\end{aligned}$$

Now by (3.4), we get with $J'=2^{\gamma+1}J$,

$$C_{r,p}(M_\gamma)^{1/p}\leqq J'\sum_{i=0}^{\infty}2^{i\gamma}C_{r,p}(K_i)^{1/p}\leqq J'\sum_{i=0}^{\infty}2^{i(\gamma-\beta/p)}<\infty$$

which shows (3.9). □

4 The Ornstein-Uhlenbeck semigroup on Wiener space

In this section, we discuss the Ornstein-Uhlenbeck semigroup on the classical Wiener space as a typical example. Let (W^d,P^W) be the d-dimensional Wiener space, i.e., $W^d=C_0([0,1]\to\mathbf{R}^d)$ is the set of all $\mathbf{R}^d$-valued continuous paths on the interval [0, 1] starting at 0, and P^W is the Wiener measure. As usual, W^d is a Banach space with the supremum norm, and we denote an element of W^d by w. Let $\mathscr{F}$ be the Borel σ-field on W^d, and define a filtration $(\mathscr{F}_t)$ by

$$\mathscr{F}_t=\sigma\{w_s;\ s\leqq t\}.$$

The Ornstein-Uhlenbeck semigroup $\{T_t\}$ is defined as follows:

$$T_t f(x)=\int_{W^d} f(e^{-t}x+\sqrt{1-e^{-2t}}\,y)\,P^W(dy).$$

In this case, we denote the associated Sobolev space by $W^{r,p}(B)$ in place of $\mathscr{F}_{r,p}(B)$. If B is a Hilbert space, $W^{r,p}(B)$ is already defined in the Malliavin calculus and agrees with our definition.

We take $C([0,1]\to\mathbf{R}^n)$ as our Banach space B. Let $X=(X_t)$ be an $\mathbf{R}^d$-valued process with a parameter $t\in[0,1]$. We assume that (X_t) is continuous in t a.e. and further that

$$E[\sup_{0\leqq t\leqq 1}|X_t|^p]<\infty.$$

This means that $X\in L^p(W^d, P^W; C([0,1]\to\mathbf{R}^d))$. Then Corollary 2.3 states that, if $\{(1-L)^{r/2}X_t\}_{t\in[0,1]}$ has a version which is continuous in t P^W-a.e. and

$$E[\sup_{0\leqq t\leqq 1}|(1-L)^{r/2}X_t|^p]<\infty,$$

then $X=(X_t)\in W^{r,p}(C([0,1]\to\mathbf{R}^n))$. Moreover, applying Theorem 3.3, we have that (X_t) admits a quasi-continuous modification $(\tilde{X}_t)$ as a $C([0,1]\to\mathbf{R}^n)$-valued function with

$$C_{r,p}(\sup_{0\leqq t\leqq 1}|\tilde{X}_t|)\leqq E[\sup_{0\leqq t\leqq 1}|(1-L)^{r/2}X_t|^p]$$

or for $\lambda>0$,

$$C_{r,p}(\sup_{0\leqq t\leqq 1}|\tilde{X}_t|\geqq\lambda)\leqq\frac{1}{\lambda^p}E[\sup_{0\leqq t\leqq 1}|(1-L)^{r/2}X_t|^p].$$

Next, let us consider a martingale. Suppose that (M_t) is an $(\mathscr{F}_t)$-martingale and $M_1\in W^{r,p}$. Here we take $p>1$. Then (M_t) has a quasi-continuous modification $(\tilde{M}_t)$ as a $C([0,1]\to\mathbf{R})$-valued function, and

$$C_{r,p}(\sup_{0\leqq t\leqq 1}|\tilde{M}_t|\geqq\lambda)\leqq\frac{1}{\lambda^p}\left(\frac{p}{p-1}\right)^p\|M_1\|_{r,p}^p.$$

To see this, it is enough to notice that

$$(1-L)^{r/2} M_t=(1-L)^{r/2} E[M_1|\mathscr{F}_t]=E[(1-L)^{r/2} M_1|\mathscr{F}_t]$$

which implies that $((1-L)^{r/2} M_t)$ has a continuous version with

$$E[\sup_{0\leq t\leq 1} |(1-L)^{r/2} M_t|^p]<\infty.$$

The above Doob type inequality was remarked first in [10] in the framework of smooth martingales.

Now we proceed to stochastic differential equations. We consider the following stochastic differential equation:

$$\begin{aligned} dX_t^i&=a_\alpha^i(X_t)\circ dw_t^\alpha+a_0^i(X_t)\,dt, \\ X_0^i&=x^i. \end{aligned} \tag{4.1}$$

Here $a_\alpha=(a_\alpha^i)$, $a_0=(a_0^i)\in C_b^\infty(\mathbf{R}^d\to\mathbf{R}^n\otimes\mathbf{R}^d)$, where C_b^∞ denotes the space of all bounded C^∞-functions with bounded derivatives of all order. Further, following the custom, we omit the summation sign for repeated indices. Then the existence and the uniqueness of the solution to (4.1) are well-known. We show the following:

Theorem 4.1 *The solution* (X_t) *to* (4.1) *belongs to* $W^{r,p}(C([0,1]\to\mathbf{R}^n))$ *and therefore admits a quasi-continuous modification as a* $C([0,1]\to\mathbf{R}^n)$*-valued function.*

Proof. We will prove that, for any $m\in\mathbf{N}$, $(L^m X_t)_{t\in[0,1]}$ has a continuous version with

$$E[\sup_{0\leq t\leq 1} |L^m X_t|^p]<\infty, \tag{4.2}$$

by showing that $(L^m X_t)_{t\in[0,1]}$ satisfies a stochastic differential equation.

In the case of $m=1$, it is well-known that

$$\begin{aligned} dLX_t^i=&\{\partial_j a_\alpha^i(X_t)\,LX_t^j+\partial_{kl}^2 a_\alpha^i(X_t)\,\sigma_t^{kl}-a_\alpha^i(X_t)\}\circ dw_t^\alpha \\ &+\{\partial_j a_0^i(X_t)\,LX_t^j+\partial_{kl}^2 a_0^i(X_t)\,\sigma_t^{kl}\}\,dt. \end{aligned} \tag{4.3}$$

Here $\partial_j=\dfrac{\partial}{\partial x^j}$ and $\sigma_t^{ij}:=(DX_t^i, DX_t^j)_{H^*}$, DX_t^i being the H-derivative of X_t^i. This matrix $\sigma_t=(\sigma_t^{ij})$ is called the Malliavin covariance matrix of X_t. σ_t satisfies the following stochastic differential equation (see, e.g. [12]).

$$\begin{aligned} d\sigma_t^{ij}=&\{\partial_k a_\alpha^i(X_t)\,\sigma_t^{kj}+\sigma_t^{ik}\,\partial_k a_\alpha^j(X_t)\}\circ dw_t^\alpha \\ &+\{\partial_k a_0^i(X_t)\,\sigma_t^{kj}+\sigma_t^{ik}\partial_k a_0^j(X_t)+a_\alpha^i(X_t)\,a_\alpha^j(X_t)\}\,dt. \end{aligned} \tag{4.4}$$

For simplicity, we write the above equation in matrix notation as follows:

$$d\sigma_t = \{\partial a_\alpha(X_t)\,\sigma_t + \sigma_t\,\partial a_\alpha(X_t)\} \circ d w_t^\alpha$$
$$+ \{\partial a_0(X_t)\,\sigma_t + \sigma_t \partial a_0(X_t) + a_\alpha(X_t) \otimes a_\alpha(X_t)\}\, dt,$$

$$dLX_t = \{\partial a_\alpha(X_t)\, LX_t + \mathrm{tr}(\partial^2 a_\alpha(X_t)\,\sigma_t) - a_\alpha(X_t)\} \circ d w_t^\alpha$$
$$+ \{\partial a_0(X_t)\, LX_t + \mathrm{tr}(\partial^2 a_0(X_t)\,\sigma_t)\}\, dt.$$

By the uniqueness of the solution, (LX_t) is uniquely determined and continuous in t a.e. Further we have for any $p \geqq 1$,

$$E[\sup_{0 \leqq t \leqq 1} |LX_t|^p] < \infty.$$

This proves the first order case.

Next we show the second order case. Set $X_t^{(1,1)} = \sigma_t$, $X_t^{(1,2)} = LX_t$. Here we regard all vectors as column vectors. Then the above equation can be written in the following form:

$$\text{(4.5)}\quad dX_t^{(1,1)} = \{E_\alpha^{(1,1)}(X_t)\, X_t^{(1,1)} + G_\alpha^{(1,1)}(X_t)\} \circ d w_t^\alpha$$
$$+ \{E_0^{(1,1)}(X_t)\, X_t^{(1,1)} + G_0^{(1,1)}(X_t)\}\, dt,$$

$$dX_t^{(1,2)} = \{E_\alpha^{(1,2)}(X_t)\, X_t^{(1,2)} + F_\alpha^{(1,2)}(X_t)\, X_t^{(1,1)} + G_\alpha^{(1,2)}(X_t)\} \circ d w_t^\alpha$$
$$+ \{E_0^{(1,2)}(X_t)\, X_t^{(1,2)} + F_0^{(1,2)}(X_t)\, X_t^{(1,1)} + G_0^{(1,2)}(X_t)\}\, dt.$$

Here $E_\alpha^{(1,k)}$, $F_\alpha^{(1,k)}$, $\alpha = 0, 1, \ldots, d$, $k = 1, 2$, are matrix valued functions, and $G_\alpha^{(1,k)}$ are vector valued functions. Further $E_\alpha^{(1,k)} \in C_b^\infty$ and $F_\alpha^{(1,k)}$, $G_\alpha^{(1,k)} \in C_\uparrow^\infty$, where $C_\uparrow^\infty$ denotes the space of functions whose derivatives of all order have polynomial growth. Finally we set $X_t^{(1)} = \begin{bmatrix} X_t^{(1,1)} \\ X_t^{(1,2)} \end{bmatrix}$.

To prove (4.2), we show that the processes

$$X_t^{(2,1)} = (DX_t^{(1,1)}, DX_t)_{H^*},$$

$$X_t^{(2,2)} = (DX_t^{(1,1)}, DX_t^{(1,1)})_{H^*},$$

$$X_t^{(2,3)} = (DX_t^{(1,2)}, DX_t)_{H^*},$$

$$X_t^{(2,4)} = (DX_t^{(1,2)}, DX_t^{(1,1)})_{H^*},$$

$$X_t^{(2,5)} = (DX_t^{(1,2)}, DX_t^{(1,2)})_{H^*},$$

$$X_t^{(2,6)} = LX_t^{(1,1)},$$

$$X_t^{(2,7)} = LX_t^{(1,2)},$$

satisfy the following stochastic differential equations

$$
\begin{aligned}
(4.6)\qquad dX_t^{(2,1)} &= \{E_\alpha^{(2,1)}(X_t)\,X_t^{(2,1)} + G_\alpha^{(2,1)}(X_t, X_t^{(1)})\} \circ dw_t^\alpha \\
&\quad + \{E_0^{(2,1)}(X_t)\,X_t^{(2,1)} + G_0^{(2,1)}(X_t, X_t^{(1)})\}\,dt, \\
dX_t^{(2,2)} &= \{E_\alpha^{(2,2)}(X_t)\,X_t^{(2,2)} + F_\alpha^{(2,2)}(X_t, X_t^{(1)})\,X_t^{(2,1)} \\
&\quad + G_\alpha^{(2,2)}(X_t, X_t^{(1)})\} \circ dw_t^\alpha \\
&\quad + \{E_0^{(2,2)}(X_t, X_t^{(1)})\,X_t^{(2,2)} + F_0^{(2,2)}(X_t, X_t^{(1)})\,X_t^{(2,1)} \\
&\quad + G_0^{(2,2)}(X_t, X_t^{(1)})\}\,dt, \\
&\ \vdots \\
dX_t^{(2,7)} &= \{E_\alpha^{(2,7)}(X_t)\,X_t^{(2,7)} + F_\alpha^{(2,7)}(X_t, X_t^{(1)}) \begin{bmatrix} X_t^{(2,1)} \\ \vdots \\ X_t^{(2,6)} \end{bmatrix} \\
&\quad + G_\alpha^{(2,7)}(X_t, X_t^{(1)})\} \circ dw_t^\alpha \\
&\quad + \{E_0^{(2,7)}(X_t, X_t^{(1)})\,X_t^{(2,7)} + F_0^{(2,7)}(X_t, X_t^{(1)}) \begin{bmatrix} X_t^{(2,1)} \\ \vdots \\ X_t^{(2,6)} \end{bmatrix} \\
&\quad + G_0^{(2,7)}(X_t, X_t^{(1)})\}\,dt.
\end{aligned}
$$

Here $E_\alpha^{(2,k)} \in C_b^\infty$, $F_\alpha^{(2,k)}$, $G_\alpha^{(2,k)} \in C_\uparrow^\infty$.

First we investigate $X_t^{(2,1)} = (DX_t^{(1,1)}, DX_t)_{H^*}$. Recalling (4.4), we have

$$
\begin{aligned}
& d(DX_t^{(1,1)}, DX_t)_{H^*} \\
&\quad = [\{\partial E_\alpha^{(1,1)}(X_t)\,X_t^{(1,1)} + \partial G_\alpha^{(1,1)}(X_t)\}(DX_t, DX_t)_{H^*} \\
&\qquad + E_\alpha^{(1,1)}(X_t)(DX_t^{(1,1)}, DX_t)_{H^*} + (DX_t^{(1,1)}, DX_t)_{H^*}\,\partial a_\alpha(X_t)] \circ dw_t^\alpha \\
&\qquad + [\{\partial E_0^{(1,1)}(X_t)\,X_t^{(1,1)} + \partial G_0^{(1,1)}(X_t)\}(DX_t, DX_t)_{H^*} \\
&\qquad + E_0^{(1,1)}(X_t)(DX_t^{(1,1)}, DX_t)_{H^*} + (DX_t^{(1,1)}, DX_t)_{H^*}\,\partial a_0(X_t) \\
&\qquad + (E_\alpha^{(1,1)}(X_t)\,X_t^{(1,1)} + G_\alpha^{(1,1)}(X_t)) \otimes a_\alpha\}\,dt.
\end{aligned}
$$

The above equation is linear in $(DX_t^{(1,1)}, DX_t)_{H^*}$ and its coefficients depend only on X_t and belong to C_b^∞. In addition, $X_t^{(2,k)}$, $k = 2, \ldots, 7$ do not appear. Similarly we have

$$
\begin{aligned}
& d(DX_t^{(1,1)}, DX_t^{(1,1)})_{H^*} \\
&\quad = [\{\partial E_\alpha^{(1,1)}(X_t)\,X_t^{(1,1)} + \partial G_\alpha^{(1,1)}(X_t)\}(DX_t, DX_t^{(1,1)})_{H^*} \\
&\qquad + E_\alpha^{(1,1)}(X_t)(DX_t^{(1,1)}, DX_t^{(1,1)})_{H^*} \\
&\qquad + (DX_t^{(1,1)}, DX_t)_{H^*}\{\partial E_\alpha^{(1,1)}(X_t)\,X_t^{(1,1)} + \partial G_\alpha^{(1,1)}(X_t)\} \\
&\qquad + (DX_t^{(1,1)}, DX_t^{(1,1)})_{H^*}\,E_\alpha^{(1,1)}(X_t)] \circ dw_t^\alpha \\
&\qquad + [\{\partial E_0^{(1,1)}(X_t)\,X_t^{(1,1)} + \partial G_0^{(1,1)}(X_t)\}(DX_t, DX_t^{(1,1)})_{H^*} \\
&\qquad + E_0^{(1,1)}(X_t)(DX_t^{(1,1)}, DX_t^{(1,1)})_{H^*} \\
&\qquad + (DX_t^{(1,1)}, DX_t)_{H^*}\{\partial E_0^{(1,1)}(X_t)\,X_t^{(1,1)} + \partial G_0^{(1,1)}(X_t)\} \\
&\qquad + (DX_t^{(1,1)}, DX_t^{(1,1)})_{H^*}\,E_0^{(1,1)}(X_t) \\
&\qquad + (E_\alpha^{(1,1)}(X_t)\,X_t^{(1,1)} + G_\alpha^{(1,1)}(X_t)) \otimes (E_\alpha^{(1,1)}(X_t)\,X_t^{(1,1)} + G_\alpha^{(1,1)}(X_t))]\,dt
\end{aligned}
$$

and

$$\begin{aligned}
&d(DX_t^{(1,2)}, DX_t)_{H^*} \\
&\quad=[\{\partial E_\alpha^{(1,2)}(X_t)\, X_t^{(1,2)}+\partial F_\alpha^{(1,2)}(X_t)\, X_t^{(1,1)}+\partial G_\alpha^{(1,2)}(X_t)\}(DX_t, DX_t)_{H^*} \\
&\qquad+E_\alpha^{(1,2)}(X_t)(DX_t^{(1,2)}, DX_t)_{H^*}+F_\alpha^{(1,2)}(X_t)(DX_t^{(1,1)}, DX_t)_{H^*} \\
&\qquad+(DX_t^{(1,2)}, DX_t)_{H^*}\, \partial a_\alpha(X_t)] \circ d w_t^\alpha \\
&\qquad+[\{\partial E_0^{(1,2)}(X_t)\, X_t^{(1,2)}+\partial F_0^{(1,2)}(X_t)\, X_t^{(1,1)}+\partial G_0^{(1,2)}(X_t)\}(DX_t, DX_t)_{H^*} \\
&\qquad+E_0^{(1,2)}(X_t)(DX_t^{(1,2)}, DX_t)_{H^*}+F_0^{(1,2)}(X_t)(DX_t^{(1,1)}, DX_t)_{H^*} \\
&\qquad+(DX_t^{(1,2)}, DX_t)_{H^*}\, \partial a_0(X_t) \\
&\qquad+(E_\alpha^{(1,2)}(X_t)\, X_t^{(1,2)}+F_\alpha^{(1,2)}(X_t)\, X_t^{(1,1)}+G_\alpha^{(1,2)}) \otimes a_\alpha(X_t))]\, dt.
\end{aligned}$$

We can get similar stochastic differential equations for $(DX_t^{(1,2)}, DX_t^{(1,1)})_{H^*}$ and $(DX_t^{(1,2)}, DX_t^{(1,2)})_{H^*}$.

Further we have

$$\begin{aligned}
dLX_t^{(1,1)}=&\{L(E_\alpha^{(1,1)}(X_t))\, X_t^{(1,1)}+E_\alpha^{(1,1)}(X_t)\, LX_t^{(1,1)} \\
&+\partial E_\alpha^{(1,1)}(X_t)(DX_t, DX_t^{(1,1)})_{H^*}+L(G_\alpha^{(1,1)}(X_t)) \\
&-\{E_\alpha^{(1,1)}(X_t)\, X_t^{(1,1)}+G_\alpha^{(1,1)}(X_t)\} \circ d w_t^\alpha \\
&+\{L(E_0^{(1,1)}(X_t))\, X_t^{(1,1)}+E_0^{(1,1)}(X_t)\, LX_t^{(1,1)} \\
&+\partial E_0^{(1,1)}(X_t)(DX_t, DX_t^{(1,1)})_{H^*}+L(G_0^{(1,1)}(X_t))\}\, dt
\end{aligned}$$

and

$$\begin{aligned}
dLX_t^{(1,2)}=&\{L(E_\alpha^{(1,2)}(X_t))\, X_t^{(1,2)}+E_\alpha^{(1,2)}(X_t)\, LX_t^{(1,2)} \\
&+\partial E_\alpha^{(1,2)}(X_t)(DX_t, DX_t^{(1,2)})_{H^*} \\
&+L(F_\alpha^{(1,2)}(X_t))\, X_t^{(1,1)}+F_\alpha^{(1,2)}(X_t)\, LX_t^{(1,1)} \\
&+\partial F_\alpha^{(1,2)}(X_t)(DX_t, DX_t^{(1,1)})_{H^*}+L(G_\alpha^{(1,2)}(X_t)) \\
&-E_\alpha^{(1,2)}(X_t)\, X_t^{(1,2)}-F_\alpha^{(1,2)}(X_t)\, X_t^{(1,1)}-G_\alpha^{(1,2)}(X_t)\} \circ d w_t^\alpha \\
&+\{L(E_0^{(1,2)}(X_t))\, X_t^{(1,2)}+E_0^{(1,2)}(X_t)\, LX_t^{(1,2)} \\
&+\partial E_0^{(1,2)}(X_t)(DX_t, DX_t^{(1,2)})_{H^*} \\
&+L(F_0^{(1,2)}(X_t))\, X_t^{(1,1)}+F_0^{(1,2)}(X_t)\, LX_t^{(1,1)} \\
&+\partial F_0^{(1,2)}(X_t)(DX_t, DX_t^{(1,1)})_{H^*}+L(G_0^{(1,2)}(X_t))\}\, dt.
\end{aligned}$$

Thus we have (4.4). The Eq. (4.4) can be solved recursively, and we have for $k=1, \ldots, 7$,

$$E[\sup_{0 \leqq t \leqq 1} |X_t^{(2,k)}|^p]<\infty.$$

In particular, noting that $X_t^{(2,7)}=L^2 X_t$, we have proved the second order case.

In the third order case, we have a system of stochastic differential equations of the form:

$$dX_t^{(3,k)} = \{E_\alpha^{(3,k)}(X_t)\, X_t^{(3,k)} + F_\alpha^{(3,k)}(X_t, X_t^{(1)}, X_t^{(2)}) \begin{bmatrix} X_t^{(3,1)} \\ \vdots \\ X_t^{(3,k-1)} \end{bmatrix}$$

$$+ G_\alpha^{(3,k)}(X_t, X_t^{(1)}, X_t^{(2)})\} \circ d w_t^\alpha$$

$$+ \{E_0^{(3,k)}(X_t)\, X_t^{(3,k)} + F_0^{(3,k)}(X_t, X_t^{(1)}, X_t^{(2)}) \begin{bmatrix} X_t^{(3,1)} \\ \vdots \\ X_t^{(3,k-1)} \end{bmatrix}$$

$$+ G_0^{(3,k)}(X_t, X_t^{(1)}, X_t^{(2)})\}\, dt.$$

Here $X_t^{(2)} = \begin{bmatrix} X_t^{(2,1)} \\ \vdots \\ X_t^{(2,7)} \end{bmatrix}$ and $E_\alpha^{(3,k)} \in C_b^\infty$, $F_\alpha^{(3,k)}$, $G_\alpha^{(3,k)} \in C_\uparrow^\infty$.

By induction we can see that (4.2) holds for any m. □

Finally we remark that, by the above proof, we can prove the quasi-everywhere existence of stochastic flows.

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References

1. Albeverio, S., Fukushima, M., Hansen, W., Ma, Z.M., Röckner, M.: An invariance result for capacities on Wiener space. J. Funct. Anal. **106**, 35–49 (1992)
2. Feyel, D., La Pradelle, A. de: Espaces de Sobolev gaussiens. Ann. Inst. Fourier, Grenoble **39**, 875–908 (1989)
3. Feyel, D., La Pradelle, A. de: Capacités gaussiennes. Ann. Inst. Fourier **41**, 49–76 (1991)
4. Fukushima, M.: Dirichlet forms and Markov Processes. North Holland Kodansha Amsterdam Tokyo, 1980
5. Fukushima, M., Kaneko, H.: On (r, p)-capacities for general Markovian semigroups. In: Albeverio, S. (ed.) Infinite dimensional analysis and stochastic processes, pp. 41–47. Pitman, 1985
6. Kaneko, H.: On (r, p)-capacities for Markov processes. Osaka J. Math. **23**, 325–336 (1986)
7. Kazumi, T., Shigekawa, I.: Measures of finite (r, p)-energy and potentials on a separable metric space. Séminaire de Probabilités 26 (to appear)
8. Malliavin, P., Nualart, D.: Quasi sure analysis of stochastic flows and Banach space valued smooth functionals on the Wiener space. J. Funct. Anal. **112**, 287–317 (1993)
9. Ren, J.: Analyse quasi-sûre des équations différentielles stochastiques. Bull. Sci. Math., II. sér. **114**, 187–213 (1990)
10. Ren, J.: On smooth martingales (Preprint)
11. Revuz, D., Yor, M.: Continuous martingales and Brownian motion. Berlin Heidelberg New York: Springer 1991

12. Stroock, D.: Some applications of stochastic calculus to partial differential equations. École d'été de Saint-Flour, XI, 1981. (Lect. Notes Math., vol. 976, pp. 268–382) Berlin Heidelberg New York: Springer 1983
13. Sugita, H.: Positive generalized Wiener functions and potential theory over abstract Wiener spaces. Osaka J. Math. **25**, 665–696 (1988)
14. Takeda, M.: (r, p)-capacity on the Wiener space and properties of Brownian motion. Z. Wahrscheinlichkeitstheor. Verw. Geb. **68**, 149–162 (1984)