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Probability Theory and Related Fields

Asymptotics of the generating function for the volume of the Wiener sausage

Michiel van den Berg [1], **Erwin Bolthausen** [2]

[1] Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, U.K.
[2] Institut für Angewandte Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

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Summary. We consider the generating function $\mathbb{E}\exp(\lambda\,|C_\varepsilon(t)|)$ of the volume of the Wiener sausage $C_\varepsilon(t)$, which is the ε-neighbourhood of the Wiener path in the time interval $[0, t]$. For $\lambda<0$, the limiting behavior for $t\to\infty$, up to logarithmic equivalence, had been determined in a celebrated work of Donsker and Varadhan. For $\lambda>0$ it had been investigated by van den Berg and Tóth, but in contrast to the case $\lambda<0$, there is no simple expression for the exponential rate known. We determine the asymptotic behaviour of this rate for small and large λ.

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1 Introduction and results

Let P be the Wiener measure on the space Ω of continuous paths $w\colon [0,\infty)\to\mathbb{R}^d$ satisfying $w(0)=0$, whose generator is the Laplacian Δ (i.e. the covariance matrix of $w(t)$ is $2tI$, where I is the identity matrix). If $w\in\Omega$, $\varepsilon>0$, $t>0$ let

$$C_\varepsilon(t, w)=\{y\in\mathbb{R}^d\,|\inf_{0\leq s\leq t}|y-w(s)|<\varepsilon\} \tag{1.1}$$

be the Wiener sausage for w. We denote its measure by $|C_\varepsilon(t)|$. In [1] it was shown that for $\lambda>0$

$$S(\lambda,\varepsilon)=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}[\exp(\lambda\,|C_\varepsilon(t)|)] \tag{1.2}$$

exists and is finite. By scaling, one has

$$S(\lambda, \varepsilon)=\varepsilon^{-2} S(\varepsilon^d \lambda, 1) \tag{1.3}$$

so one needs only to consider $S(\lambda)=S(\lambda, 1)$. It was proved in [1] that for $d=1$, $S(\lambda)=\lambda^2$. Upper and lower bounds for $S(\lambda)$ were obtained for $d=2, 3, \ldots$ A closed form expression or a simple variational formula for $S(\lambda)$ probably does not exist. Here we derive the leading asymptotics of $S(\lambda)$ for $\lambda \to 0$ and for $\lambda \to \infty$.

The leading asymptotics of $S(\lambda)$ for small λ is directly connected with the behaviour of $\mathbb{E}[|C_1(t)|]$ for large t. It is well known [2] that

$$\lim_{t\to\infty}(\log t)\, \mathbb{E}[|C_1(t)|]/t=4\pi, \qquad d=2, \tag{1.4}$$

$$\lim_{t\to\infty} \mathbb{E}[|C_1(t)|]/t=C(d), \qquad d=3, 4, \ldots, \tag{1.5}$$

where $C(d)$ is the Newtonian capacity of the unit ball in $\mathbb{R}^d$:

$$C(d)=d(d-2)\,\omega_d, \tag{1.6}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$ given by

$$\omega_d=\pi^{d/2}(\Gamma((d+2)/2))^{-1}. \tag{1.7}$$

Our results are the following.

Theorem 1 *In dimension two*

$$\lim_{\lambda\downarrow 0}\left(\log\frac{1}{\lambda}\right) S(\lambda)/\lambda=4\pi, \tag{1.8}$$

and in three or more dimensions

$$\lim_{\lambda\downarrow 0} S(\lambda)/\lambda=C(d). \tag{1.9}$$

Theorem 2 *In any dimension*

$$\lim_{\lambda\to\infty} S(\lambda)/\lambda^2=(\omega_{d-1})^2. \tag{1.10}$$

2 Proof of the upper bound in Theorem 1

Let $p \geqq 1$. Then

$$|C_1(t)| \leqq |C_1(([t/p]+1)p)| \leqq \sum_{j=0}^{[t/p]} |C_1(p)\circ\theta_{jp}|, \tag{2.1}$$

where θ_s is the shift on path space:

$$(\theta_s(\omega))(t)=\omega(s+t). \tag{2.2}$$

By the Markov property

$$\mathbb{E}[e^{\lambda|C_1(t)|}]\leqq\{\mathbb{E}[e^{\lambda|C_1(p)|}]\}^{[t/p]+1}. \tag{2.3}$$

Therefore

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\lambda|C_1(t)|}]\leqq\frac{1}{p}\log\mathbb{E}[e^{\lambda|C_1(p)|}]. \tag{2.4}$$

Because $e^x\leqq 1+x+(x^2/2)\,e^x$ for $x\geqq 0$ we get

$$\mathbb{E}[e^{\lambda|C_1(p)|}]\leqq 1+\lambda\,\mathbb{E}[|C_1(p)|]+\frac{\lambda^2}{2}\,\mathbb{E}[|C_1(p)|^2\exp(\lambda|C_1(p)|)]. \tag{2.5}$$

Since $\log(1+x)\leqq x$ for $x\geqq 0$ we have by (2.4) and (2.5)

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}[e^{\lambda|C_1(t)|}]\leqq&\frac{\lambda}{p}\,\mathbb{E}[|C_1(p)|]\\&+\frac{\lambda^2}{2p}\,\mathbb{E}[|C_1(p)|^2\exp(\lambda|C_1(p)|)].\end{aligned} \tag{2.6}$$

By Cauchy-Schwarz

$$\begin{aligned}&\mathbb{E}[|C_1(p)|^2\exp(\lambda|C_1(p)|)]\\&\quad\leqq\{\mathbb{E}[|C_1(p)|^4]\,\mathbb{E}[\exp(2\lambda|C_1(p)|)]\}^{1/2}\\&\quad\leqq p^2\{\mathbb{E}[p^{-4}|C_1(p)|^4\exp(-2p^{-1}|C_1(p)|)\exp(2p^{-1}|C_1(p)|)]\\&\qquad\cdot\mathbb{E}[\exp(2\lambda|C_1(p)|)]\}^{1/2}\\&\quad\leqq(2p/e)^2\{\mathbb{E}[\exp(2p^{-1}|C_1(p)|)]\,\mathbb{E}[\exp(2\lambda|C_1(p)|)]\}^{1/2}\\&\quad\leqq(2p/e)^2\,\mathbb{E}[\exp(2p^{-1}|C_1(p)|)],\end{aligned} \tag{2.7}$$

provided $p^{-1}\geqq\lambda$. Since $p\geqq 1$ we have by Hölder's inequality

$$\mathbb{E}[\exp(2p^{-1}|C_1(p)|)]\leqq\{\mathbb{E}[\exp(2|C_1(p)|)]\}^{1/p}. \tag{2.8}$$

We put $\lambda=2$ and transpose t and p in (2.3) and subsequently put $t=1$. This gives

$$\mathbb{E}[\exp(2|C_1(p)|)]\leqq\{\mathbb{E}[\exp(2|C_1(1)|)]\}^{[p]+1}. \tag{2.9}$$

From (2.6)–(2.9) we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\lambda|C_1(t)|}]$$

$$\text{(2.10)} \qquad \leqq \frac{\lambda}{p} \mathbb{E}[|C_1(p)|] + \frac{2\lambda^2 p}{e^2} \{\mathbb{E}[e^{2|C_1(1)|}]\}^{([p]+1)/p}$$

$$\leqq \frac{\lambda}{p} \mathbb{E}[|C_1(p)|] + \lambda^2 p \{\mathbb{E}[e^{2|C_1(1)|}]\}^2.$$

Note that $\mathbb{E}[\exp(2|C_1(1)|)] < \infty$ by the results of Sect. 5 in [3]. Let $\lambda \in (0, 1/e)$ and choose

$$\text{(2.11)} \qquad p(\lambda) = \lambda^{-1}(\log \lambda^{-1})^{-2}.$$

Then $p(\lambda) \geqq 1$ and $(p(\lambda))^{-1} > \lambda$. Moreover $\lambda \to 0$ implies $p(\lambda) \to \infty$. Consider $d=2$. Then by (1.4)

$$\text{(2.12)} \qquad \frac{\lambda}{p(\lambda)} \mathbb{E}[|C_1(p(\lambda))|] = \frac{4\pi\lambda}{\log p(\lambda)}(1+o(1)) = \frac{4\pi\lambda}{\log \frac{1}{\lambda}}(1+o(1)),$$

and

$$\text{(2.13)} \qquad \lambda^2 p(\lambda) = \frac{\lambda}{\left(\log \frac{1}{\lambda}\right)^2}.$$

Consider $d \geqq 3$. Then by (1.5)

$$\text{(2.14)} \qquad \frac{\lambda}{p(\lambda)} \mathbb{E}[|C_1(p(\lambda))|] = C(d)\,\lambda(1+o(1)).$$

The upper bound in Theorem 1 follows from (2.10), (2.12)–(2.14).

3 Proof of the lower bound in Theorem 1

For dimension $d \geqq 3$, the lower bound follows directly from Jensen's inequality:

$$\text{(3.1)} \qquad \frac{1}{t} \log \mathbb{E}[\exp(\lambda|C_1(t)|)] \geqq \frac{\lambda}{t} \mathbb{E}[|C_1(t)|],$$

which implies $S(\lambda) \geqq C(d)\lambda$ for any $\lambda \geqq 0$ by (1.5).

The case $d=2$ is more subtle. We cannot apply (3.1), because $\mathbb{E}[|C_1(t)|]/t \to 0$ as $t \to \infty$ by (1.4). We consider the law P^ρ of the Brownian motion with drift $\rho>0$ in the direction of the positive x-axis: $P^\rho = P\varphi_\rho^{-1}$, where $\varphi_\rho: \Omega \to \Omega$ is defined by

$$(3.2) \qquad (\varphi_\rho(w))(t) = w(t) + \rho\, t\, e,$$

$e=(1,0)$. The restrictions P^t, $P^{\rho,t}$ to the σ-field generated by the evaluation mappings $w \to w(s)$ for $0 \leqq s \leqq t$, are mutually absolutely continuous, and by the Cameron-Martin formula

$$(3.3) \qquad \frac{dP^t}{dP^{\rho,t}}(w) = \exp(-\rho\langle w(t), e\rangle/2 + \rho^2 t/4).$$

Applying this and Jensen's inequality to the integration with respect to $P^{\rho,t}$, we get

$$(3.4) \qquad \begin{aligned} &\frac{1}{t}\log \mathbb{E}[\exp(\lambda\,|C_1(t)|)] \\ &= \frac{\rho^2}{4} + \frac{1}{t}\log \mathbb{E}^{\rho,t}[\exp(-\rho\langle w(t), e\rangle/2 + \lambda\,|C_1(t)|)] \\ &\geqq \frac{\rho^2}{4} - \frac{\rho}{2t}\,\mathbb{E}^\rho[\langle w(t), e\rangle] + \frac{\lambda}{t}\,\mathbb{E}^\rho[|C_1(t)|] \\ &= -\frac{\rho^2}{4} + \frac{\lambda}{t}\,\mathbb{E}^\rho[|C_1(t)|]. \end{aligned}$$

(3.4) implies that for all $\rho>0$

$$(3.5) \qquad S(\lambda) \geqq -\frac{\rho^2}{4} + \lambda \inf_{t>0} \frac{1}{t}\,\mathbb{E}^\rho[|C_1(t)|].$$

Lemma. *For all $t>0$ and $\rho \in (0,1]$*

$$(3.6) \qquad \frac{1}{t}\,\mathbb{E}^\rho[|C_1(t)|] \geqq \frac{4\pi}{28 - \log \rho^2}.$$

Implementing (3.6) in (3.4) and choosing $\rho = \lambda^\alpha$, with $\alpha > \frac{1}{2}$ and $\lambda \in (0,1)$ gives

$$(3.7) \qquad \liminf_{\lambda \downarrow 0} \left(\log \frac{1}{\lambda}\right) S(\lambda)/\lambda \geqq 2\pi/\alpha.$$

The lower bound for (1.8) follows since $\alpha > \frac{1}{2}$ is arbitrary.

Proof of the Lemma. For $y \in \mathbb{R}^2$, let P_y^ρ be the law of the Brownian motion with drift starting at y. We need an upper bound for $\int_0^\infty P_y^\rho(w(s) \in D)\,ds$ where D is the unit ball in $\mathbb{R}^2$ with center $(0, 0)$. Note that

$$(3.8) \qquad P_y^\rho(w(s) \in D) = \int_D \frac{1}{4\pi s} \exp(-|x-y-\rho s e|^2/(4s))\,dx.$$

We have

$$(3.9) \qquad \int_0^\infty P_y^\rho(w(s) \in D)\,ds \leqq 1 + \int_1^{1/\rho^2} P_y^\rho(w(s) \in D)\,ds + \int_{1/\rho^2}^\infty P_y^\rho(w(s) \in D)\,ds.$$

Since $|D| = \pi$, we see that

$$(3.10) \qquad P_y^\rho(w(s) \in D) \leqq \frac{1}{4s},$$

and hence

$$(3.11) \qquad \int_0^\infty P_y^\rho(w(s) \in D)\,ds \leqq 1 + \frac{1}{4} \log \frac{1}{\rho^2} + \int_{1/\rho^2}^\infty P_y^\rho(w(s) \in D)\,ds.$$

To estimate the last term in (3.11) we observe that for $x, y \in D$

$$(3.12) \qquad |x-y-\rho s e|^2 \geqq \tfrac{1}{2}\rho^2 s^2 - |x-y|^2 \geqq \tfrac{1}{2}\rho^2 s^2 - 4,$$

and therefore

$$(3.13) \qquad \int_{1/\rho^2}^\infty P_y^\rho(w(s) \in D)\,ds \leqq \int_{1/\rho^2}^\infty \frac{1}{4s} e^{-\frac{\rho^2 s}{8} + \frac{1}{s}}\,ds$$
$$\leqq \int_1^\infty \frac{e}{4s} e^{-s/8}\,ds \leqq 2e \leqq 6.$$

Collecting these estimates, we get from (3.11) and (3.13)

$$(3.14) \qquad \int_0^\infty P_y^\rho(w(s) \in D)\,ds \leqq 7 - \tfrac{1}{4} \log \rho^2.$$

Let $D_x = x + D$, $x \in \mathbb{R}^2$, and σ_x be the first entrance time into D_x:

$$(3.15) \qquad \sigma_x(w) = \inf\{t \geqq 0\colon w(t) \in D_x\}.$$

By Fubini's theorem

$$\pi t = \int_{\mathbb{R}^2} dx \int_0^t ds\, P^\rho(w(s) \in D_x)$$

(3.16)
$$\leqq \int_{\mathbb{R}^2} dx\, \mathbb{E}^\rho \left[1_{\{\sigma_x \leqq t\}} \int_{\sigma_x}^{\infty} 1_{\{w(s) \in D_x\}}\, ds \right]$$

$$= \int_{\mathbb{R}^2} dx\, \mathbb{E}^\rho \left[1_{\{\sigma_x \leqq t\}} \int_0^{\infty} ds\, P^\rho_{w(\sigma_x)}(w(s) \in D_x) \right],$$

where we have used the strong Markov property. By (3.14) we have

(3.17)
$$\int_0^{\infty} P^\rho_{w(\sigma_x)}(w(s) \in D_x)\, ds \leqq 7 - \tfrac{1}{4} \log \rho^2,$$

on $\{\sigma_x \leqq t\}$, and therefore

(3.18)
$$\pi t \leqq \left(7 - \frac{1}{4} \log \rho^2\right) \int_{\mathbb{R}^2} P^\rho(\sigma_x \leqq t)\, dx = \left(7 - \frac{1}{4} \log \rho^2\right) \mathbb{E}^\rho [|C_1(t)|].$$

This proves the Lemma.

4 Proof of Theorem 2

We refer to Sect. 2 in [1] for the proof in the case $d=1$. Let $d = 2, 3, \ldots$ It follows from Theorem 1 (ii, iii) in [1] that

(4.1)
$$\liminf_{\lambda \to \infty} \lambda^{-2} S(\lambda) \geqq \omega_{d-1}^2.$$

Define $h\colon \mathbb{R}^+ \to \mathbb{R}^+$ by

(4.2)
$$h(y) = 2\,\omega_{d-1} (1+y)^d \int_0^{y/(2+2y)} (1-z^2)^{(d-1)/2}\, dz,$$

and ϕ for $p > 0$ by

(4.3)
$$\phi(-p) = -\log \left\{ \pi^{-1/2} (\Gamma((d-1)/2))^{-1}\, \Gamma(d/2) \int_0^{\pi} e^{p^{1/2} \cos\theta} (\sin\theta)^{d-2}\, d\theta \right\}.$$

Note that ϕ is both continuous and strictly monotone. Then by (4.25) in [1]

$$S(\lambda) \leqq \inf_{y>0} \{-y^{-2}\phi^{-1}(-\lambda h(y))\}, \tag{4.4}$$

where $\phi^{-1}: \mathbb{R}^- \to \mathbb{R}^-$ is the inverse function of ϕ.

Lemma. *For* $p \geqq 0$

$$-\phi(-p) \leqq p^{1/2}, \tag{4.5}$$

and for $p \geqq 1$ *and some constant* $k(d)$, *depending on* d,

$$-\phi(-p) \geqq p^{1/2} - 4^{-1}(d-1)\log p - k(d). \tag{4.6}$$

Proof.

$$\begin{aligned}\int_0^\pi e^{p^{1/2}\cos\theta}(\sin\theta)^{d-2}\,d\theta &\leqq e^{p^{1/2}}\int_0^\pi (\sin\theta)^{d-2}\,d\theta \\ &= e^{p^{1/2}}\pi^{1/2}\,\Gamma((d-1)/2)(\Gamma(d/2))^{-1},\end{aligned} \tag{4.7}$$

and (4.5) follows from (4.3) and (4.7). Furthermore

$$\begin{aligned}\int_0^\pi e^{p^{1/2}\cos\theta}(\sin\theta)^{d-2}\,d\theta &\geqq \int_0^{\pi/2} e^{p^{1/2}\cos\theta}(\sin\theta)^{d-2}\,d\theta \\ &\geqq e^{p^{1/2}}\int_0^{\pi/2} e^{-p^{1/2}\theta^2/2}(\sin\theta)^{d-2}\,d\theta \\ &\geqq e^{p^{1/2}}\int_0^{\pi/2} e^{-p^{1/2}\theta^2/2}(2\theta/\pi)^{d-2}\,d\theta \\ &\geqq e^{p^{1/2}}\int_0^{p^{-1/4}} e^{-p^{1/2}\theta^2}(2\theta/\pi)^{d-2}\,d\theta \\ &\geqq e^{p^{1/2}-1}\,p^{(1-d)/4}(2/\pi)^{d-2}(d-1)^{-1},\end{aligned} \tag{4.8}$$

which proves (4.6) with a constant

$$k(d) = (d-2)\log(\pi/2) + \log(\pi(d-1)) + 1. \tag{4.9}$$

By (4.4) we have for $y = \lambda^{-1/2}$

$$\lambda^{-2}S(\lambda) \leqq -\lambda^{-1}\phi^{-1}(-\lambda h(\lambda^{-1/2})). \tag{4.10}$$

Define $x: \mathbb{R}^+ \to \mathbb{R}^+$ by

$$x(\lambda) = -\phi^{-1}(-\lambda h(\lambda^{-1/2})). \tag{4.11}$$

Then by (4.11)

$$-\phi(-x(\lambda))=\lambda h(\lambda^{-1/2}) \tag{4.12}$$

and by (4.10), (4.11) and (4.12)

$$\lambda^{-2} S(\lambda) \leqq x(\lambda)\{-\phi(-x(\lambda))\}^{-2} \lambda\{h(\lambda^{-1/2})\}^2. \tag{4.13}$$

Since

$$h(y) \geqq \omega_{d-1} 2^{(1-d)/2} y, \tag{4.14}$$

we have by (4.5), (4.12) and (4.14)

$$x(\lambda) \geqq \omega_{d-1}^2 2^{(1-d)} \lambda. \tag{4.15}$$

Hence $\lambda \to \infty$ implies $x(\lambda) \to \infty$ and by (4.13)

$$\limsup_{\lambda\to\infty} \lambda^{-2} S(\lambda) \leqq \limsup_{x\to\infty} x\{-\phi(-x)\}^{-2} \limsup_{\lambda\to\infty} \lambda\{h(\lambda^{-1/2})\}^2. \tag{4.16}$$

By (4.2)

$$\lim_{\lambda\to\infty} \lambda\{h(\lambda^{-1/2})\}^2 = \omega_{d-1}^2, \tag{4.17}$$

and by (4.5) and (4.6)

$$\lim_{x\to\infty} x\{-\phi(-x)\}^{-2} = 1, \tag{4.18}$$

so that

$$\limsup_{\lambda\to\infty} \lambda^{-2} S(\lambda) \leqq \omega_{d-1}^2, \tag{4.19}$$

which completes the proof of Theorem 2.

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