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Predictable projections for point process filtrations

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Summary. We consider a point process Φ with the Polish phase space $(\mathbf{X}, \mathscr{X})$ and a system of σ-fields $\mathscr{F}(x)$, $x \in \mathbf{X}$, generated by Φ on certain sets $\Gamma(x) \in \mathscr{X}$. We define predictability for random processes indexed by $\mathbf{X}$ and for random measures on $\mathbf{X}$ and prove the existence and uniqueness of predictable and dual predictable projections under a regularity condition on Φ. For $\mathbf{X} = \mathbb{R}^2_+$ and under monotonicity assumptions on the sets Γ_x we will identify the predictable projections of some simple processes as regular versions of certain martingales.

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1 Introduction

We consider a point process Φ on a locally compact second countable Hausdorff (LCCB) space $\mathbf{X}$ defined on the probability space $(\Omega, \mathscr{F}, P)$. Let $\{\mathscr{F}(x): x \in \mathbf{X}\}$ be a filtration of σ-fields which is generated by Φ on certain measurable sets $\Gamma_x \subseteq \mathbf{X}$, satisfying some properties which will be specified later. Under a regularity condition on Φ we will construct unique predictable projections of random processes with parameter space $\mathbf{X}$, where predictability is a property of measurability induced by the filtration. The corresponding dual projection leads to a unique representation of a random measure on $\mathbf{X}$ as a sum of a predictable random measure and a so-called martingale-like measure.

In such a general point process framework we are only aware of the results by van der Hoeven (1982, 1983), which are intimately related to the work by Papangelou (1974) and Kallenberg (1978, 1983) on conditional intensity measures. Van der Hoeven studied a Gibbs filtration, where $\Gamma_x = \mathbf{X} \setminus \{x\}$. But a historical review should begin with the classical case of stochastic processes with a real parameter (see Dellacherie and Meyer 1978, 1982), where one considers a right-continuous P-complete filtration $\{\mathscr{F}_t: t \in \mathbb{R}_+\}$. The predictable projection of a process $\{Z(t): t \in \mathbb{R}_+\}$ is a predictable version of $\{E[Z(t)|\mathscr{F}_{t-}]\}$. Uniqueness is guaranteed by the predictable section theorem.

The situation becomes more difficult in the case of a right-continuous filtration $\{\mathscr{F}_x: x \in \mathbb{R}^n_+\}$, $n \in \mathbb{N}$. Here right-continuity refers to the natural partial ordering $\leqq$ on $\mathbb{R}^n_+$. Predictability means measurability with respect to the σ-field generated by the sets $H \times (x, y]$, where $H \in \mathscr{F}_x$, $(x, y] = \{z: x < z \leqq y\}$ and the relation $<$ is defined componentwise. Since $(\mathbb{R}^n_+, \leqq)$ is not totally ordered for $n > 1$, the classical methods for $n = 1$ do not apply. To overcome these difficulties Cairoli and Walsh (1974) introduced a condition of conditional independence, called (F4), which unfortunately is very restrictive. Under (F4) Merzbach and Zakai (1980) (see also Doléans-Dade and Meyer, 1979) proved the existence and uniqueness of predictable projections and Merzbach and Nualart (1988) derived a martingale representation result. Now we consider the more special situation where $\mathscr{F}_x$ contains the information of a point process on $\mathbb{R}^n_+$ restricted to the set $(0, x]$ (0 is the zero vector). The corresponding proper choice of the sets Γ_x is $\Gamma_x = \{z: z < x\}$. For this case Buckdahn (1984) solved our problem without further assumptions using explicit expressions for martingales. He also considered the filtration under which all the individual points of the point process are stopping times. Other relevant references in this context are Mazziotto and Szpirglas (1985), Mazziotto and Merzbach (1988) and Arenas (1989). There are also partial results for other point process filtrations on $\mathbb{R}^2_+$. We refer to Dozzi (1981) for the case of (F4), to Al-Hussaini and Elliott (1985) for a one-point process and to Ivanoff and Merzbach (1990).

Now we return to a point process with a general LCCB phase space. Under a regularity condition (Σ), Papangelou (1974) introduced its conditional intensity measure by way of a limiting procedure. He already noticed that Murali Rao (1969) (see also Doléans, 1968) used a similar method for the construction of dual predictable projections in the classical case. Cairoli (1971) and Dozzi (1981) gave similar results on $\mathbb{R}^2_+$. Kallenberg (1978, 1983) was able to dispense with the regularity condition and introduced the conditional intensity measure (with respect to the point process) of an arbitrary random measure. Using a predictable section theorem based on stopping times, van der Hoeven (1982, 1983) succeeded in proving the existence and uniqueness of predictable projections. The predictable σ-field is generated by the sets $H \times B$, where B is a measurable and bounded subset of the phase space and $H \in \mathscr{F}$ is measurable with respect to the restriction $B^c \Phi$ of Φ to the complement B^c of B. The corresponding dual projection of a random measure coincides with the conditional intensity measure.

In the present paper we shall define the predictable σ-field on $\Omega \times \mathbf{X}$ to be generated by the mapping $(\omega, x) \mapsto (\Gamma_x \Phi(\omega), x)$. This notion of predictability is more general than in all the point process examples above. One can interpret $\Gamma_x \Phi$ as the history at point x. In the case of $\Gamma_x = \{x\}^c$, for example, we are back in the framework of van der Hoeven (1982). We have not been able to prove a predictable section theorem based on random points as in the latter paper. Therefore we define the predictable projection of a process $\{Y(x): x \in \mathbf{X}\}$ by a limiting procedure, yielding the result $E[Y(x) | \Gamma_x \Phi]$ at every point x. The dual predictable projection will turn out to be a direct generalization of the conditional intensity measure. The uniqueness will be proved with certain absolute continuity relations for Campbell measures on $\Omega \times \mathbf{X}$. However, we will need a regularity condition. This condition is satisfied by thinnings or by Gibbsian point processes and reduces to Papangelou's condition (Σ) in the corresponding special case.

We present our material as follows. After discussing the notion of predictability in Sect. 2 we will present the projection theorems in Sects. 3 and 4. For $\mathbf{X} = \mathbb{R}^2_+$ and under additional assumptions on the system $\{\Gamma_x: x \in \mathbf{X}\}$ we will demonstrate in Sect. 5 the existence of regular versions of certain martingales. This establishes the connection with the results in the literature cited above. The projection theorems will be proved in Sects. 6 and 7. The latter sections are also interesting in themselves, since we will show more general results without using a regularity condition on Φ. Roughly speaking, we will prove the existence and uniqueness of projections of random processes and random measures that vanish outside a certain regularity set defined by Φ and Γ.

2 Predictability

Consider the LCCB space $(\mathbf{X}, \mathscr{X})$ and let $\mathscr{S} \subseteq \mathscr{X}$ be a DC-semiring of bounded sets (we refer to Kallenberg, 1983, for the definitions not given here). Let $\{\mathscr{F}(B), \mathscr{F}(x): B \in \mathscr{X}, x \in \mathscr{S}\}$ be a *filtration* in the following sense:

(2.1) $$\mathscr{F}(B) \subseteq \mathscr{F}(A) \quad \text{if } A \subseteq B,\ A, B \in \mathscr{X},$$

(2.2) $$\sigma\Big(\bigcup_{n \geq 1} \mathscr{F}(B_n)\Big) = \mathscr{F}(x) \quad \text{if } B_n \downarrow \{x\},\ d(B_n) \downarrow 0,$$

where d is a metric on $\mathbf{X}$ generating $\mathscr{X}$ and $d(B)$ is the corresponding diameter of a set B.

We call the σ-field generated by the sets

$$H \times B, \qquad B \in \mathscr{S}, \qquad H \in \mathscr{F}(B),$$

the *predictable σ-field* $\mathscr{P}$ consisting of the predictable sets. (A more exact notation would refer to the filtration.) A function from $\Omega \times \mathbf{X}$ into $\mathbb{R}$ is called *predictable* if it is measurable with respect to $\mathscr{P}$.

We will not keep this level of generality and introduce more concrete filtrations generated by point processes. We start with a system $\Gamma = \{\Gamma_x, \Gamma_B\colon x \in \mathbf{X}, B \in \mathcal{S}\}$ of $\mathcal{X}$-measurable sets satisfying

$$\Gamma_A \supseteq \Gamma_B \quad \text{if } A \subseteq B,\ A, B \in \mathcal{X}, \tag{2.3}$$

$$\Gamma_{B_n} \uparrow \Gamma_x \quad \text{if } B_n \downarrow \{x\} \text{ and } d(B_n) \downarrow 0. \tag{2.4}$$

Let Φ be a point process on $\mathbf{X}$ satisfying

$$\Phi(\Gamma_x \setminus \Gamma_B) < \infty \quad \text{if } x \in B \in \mathcal{S} \tag{2.5}$$

and put

$$\mathcal{F}(B) := \sigma(\Gamma_B \Phi), \tag{2.6}$$

$$\mathcal{F}(x) := \sigma(\Gamma_x \Phi), \tag{2.7}$$

where $A\Phi$ denotes the restriction of Φ onto a set $A \in \mathcal{X}$. Then (2.1) and (2.2) are easy consequences of (2.3)–(2.5). Next we show how the classical examples fit into our model.

Example 2.1 Let $\mathbf{X} = (0, \infty)$ and take $\mathcal{S}$ as the set of all intervals $(a, b]$ with $a < b$. Put $\Gamma_x = (0, x)$ and $\Gamma_{(a,b]} = (0, a]$. Then predictability coincides with the classical definition (see Dellacherie and Meyer, 1978) given in terms of the internal filtration $\{\mathcal{F}_t^\Phi\colon t > 0\}$ generated by Φ. In that case we have $\mathcal{F}(t) = \mathcal{F}_{t-}^\Phi$. Of course this example can be generalized to multivariate (marked) point processes on the real half-line.

Example 2.2 For $\Gamma_B = B^c = \mathbf{X} \setminus B$, $\Gamma_x = \{x\}^c$ our definition of predictability coincides with the definition of visibility (or previsibility) given by van der Hoeven (1982) and Kallenberg (1983).

Below we denote by $x_i, y_i, \ldots$ the ith component of vectors $x, y, \ldots \in \mathbb{R}^2$. Also we write $x \leqq y$ $(x < y)$ if $x_i \leqq y_i$ $(x_i < y_i)$ for $i = 1, 2$. The next two examples will be further discussed in Sect. 5.

Example 2.3 Let $\mathbf{X} = (0, \infty)^2$ and take $\mathcal{S}$ as the system of all rectangles $(a, b] := (a_1, b_1] \times (a_2, b_2]$ where $a < b$. Put $\Gamma_x = (0, x) := \{y\colon y < x\}$ and $\Gamma_{(a,b]} = (0, a]$, where $0 := (0, 0)$ if there is no risk of ambiguity. The predictable σ-field is then given by the classical definition of Cairoli and Walsh (1975) in terms of the filtration $\mathcal{F}_x^\Phi := \sigma\{\Phi((0, y])\colon y \leqq x\}$.

Example 2.4 Consider the framework and the notation of Example 2.3. For $s > 0$ let $\mathcal{F}_{s,\infty}^\Phi$ be the σ-field generated by all the $\mathcal{F}_{(s,t)}^\Phi$, $t > 0$ and define $\mathcal{F}_{\infty,s}^\Phi$ similarly. Based on the filtration $\mathcal{F}_{x_1,\infty}^\Phi \vee \mathcal{F}_{\infty,x_2}^\Phi$, $x \in \mathbf{X}$, one can define (see Dozzi, 1981) another predictable σ-field which is in our framework given by the choice $\Gamma_x = \{y\colon y_1 < x_1 \text{ or } y_2 < x_2\}$ and $\Gamma_{(a,b]} = \{y\colon y_1 \leqq a_1 \text{ or } y_2 \leqq b_2\}$.

There is a general method for constructing systems Γ needed in our definition of predictability.

Example 2.5 Let $f\colon \mathbf{X}\times\mathbf{X}\to\mathbb{R}$ be measurable and continuous from below in its first argument. Then

$$\Gamma_x=\{y: f(x,y)>0\}, \tag{2.8}$$

$$\Gamma_B=\{y: f(x,y)>0 \text{ for all } x\in B\} \tag{2.9}$$

defines a system which satisfies (2.3) and (2.4). The measurability of Γ_B has to be checked in each case. All the examples above fit into this framework. For example we have to take $f(x,y)=\mathbb{1}\{x \neq y\}$ in order to obtain Example 2.2. Other choices are $f(x,y)=d(x,y)-r$, $f(x,y)=r-d(x,y)$, where $r>0$ and, in case of $\mathbf{X}\subseteq\mathbb{R}^n$, $n\in\mathbb{N}$, $f(x,y)=\mathbb{1}\{f_i(y_i)<x_i,\ i=1,\dots,n\}$, where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ and the f_i are measurable functions on $\mathbb{R}$. A further example in case of $\mathbf{X}=\mathbb{R}_+^2$ is $f(x,y)=g(y_1-x_1,y_2-x_2)$ where g is a measurable function on $\mathbb{R}^2$ which is continuous from below in both arguments. The measurability of Γ_B can then be ensured by claiming certain monotonicity and continuity properties for g. If, e.g., g is monotone non-decreasing in both arguments then $\Gamma_B=\{y: g(y_1-b_1,y_2-b_2)>0\}$.

In this paper we look also at other filtrations generated by point processes which can distinguish its point. To be more precisely we let x_∞ be a point external to $\mathbf{X}$, $\mathbf{X}_\infty:=\mathbf{X}\cup\{x_\infty\}$ and $\mathscr{X}_\infty$ the σ-field generated by $\mathscr{X}$ and $\{x_\infty\}$. Let $\zeta=(\tau_n)_{n\geqq 1}$ be a sequence of random elements of $\mathbf{X}_\infty$ and define a random measure on $\mathbf{X}$ by

$$\Phi=\sum_{n\geqq 1}\mathbb{1}\{\tau_n\neq x_\infty\}\,\delta_{\tau_n}. \tag{2.10}$$

If $\Phi(\mathbf{X})\leqq 1$ (one-point process) then Φ and ζ are the same objects. Otherwise the latter contains more information. There is no need for Φ to be locally bounded. We claim, however, the validity of (2.5). For $B\in\mathscr{X}$ we define another sequence $B\zeta=(\tau_n^B)$ by

$$\tau_n^B=\begin{cases}\tau_n & \text{if } \tau_n\in B,\\ x_\infty & \text{otherwise.}\end{cases}$$

Now we set

$$\mathscr{F}(B):=\sigma(\Gamma_B\zeta), \tag{2.11}$$

$$\mathscr{F}(x):=\sigma(\Gamma_x\zeta), \tag{2.12}$$

to obtain again a filtration in the sense of (2.1) and (2.2). For Example 2.3 the corresponding predictable σ-field has been studied in Buckdahn (1984) and Mazziotto and Merzbach (1988).

The next useful lemma covers both types of filtrations.

Lemma 2.6 (i) *Let $\tilde{\mathscr{S}}\subseteq\mathscr{S}$ be another DC-semiring. Then the predictable σ-field is generated by the sets*

$$H\times B,\quad B\in\tilde{\mathscr{S}},\quad H\in\mathscr{F}(B).$$

(ii) *The predictable σ-field is also generated by the mapping* $(\omega, x) \mapsto (\Gamma_x \Phi(\omega), x)$ *or by the mapping* $(\omega, x) \mapsto (\Gamma_x \zeta(\omega), x)$, *respectively.*

Proof. Since the proof of the second case is similar we deal only with the case (2.6), (2.7). Let the measurable space of all possible realizations of point processes with the phase space $(\mathbf{X}, \mathscr{X})$ be denoted by $(N, \mathscr{N})$ (cf. Kallenberg, 1983). By (2.4) the mapping $(\omega, x) \mapsto (\Gamma_x \Phi(\omega), x)$ is $\mathscr{F} \otimes \mathscr{X} - \mathscr{N} \otimes \mathscr{X}$-measurable. The σ-fields which are defined by the claims (i) and (ii) are denoted by $\mathscr{P}_1$ and $\mathscr{P}_2$.

Firstly, we want to show $\mathscr{P}_1 \subseteq \mathscr{P}_2$ and choose $B \in \tilde{\mathscr{S}}$ and $G \in \mathscr{N}$. For $\varphi \in N$ and $x \in \mathbf{X}$ we set $g(\varphi, x) = \mathbb{1}\{\Gamma_B \varphi \in G, x \in B\}$, where $C\varphi$ is the restriction of φ to a set $C \in \mathscr{X}$. For $x \in B$ we have $\Gamma_x \supseteq \Gamma_B$ (cf. (2.3)) and therefore

$$g(\Gamma_x \Phi, x) = \mathbb{1}\{\Gamma_B(\Gamma_x \Phi) \in G, x \in B\} = \mathbb{1}\{\Gamma_B \Phi \in G, x \in B\}.$$

Hence $\mathscr{P}_1 \subseteq \mathscr{P}_2$. To show the other inclusion we take a continuous function g on $N \times \mathbf{X}$ (the space $(N, \mathscr{N})$ is Polish with respect to the vague convergence on N) and a null-array $(\{B_{n,i}: i \geqq 1\})_{n \geqq 1}$ of $\tilde{\mathscr{S}}$-measurable partitions of $\mathbf{X}$. From (2.2) we obtain the vague convergence $\Gamma_{B_n} \varphi \to \Gamma_x \varphi$ if $B_n \downarrow \{x\}$ and $d(B_n) \downarrow 0$. Hence

$$g(\Gamma_x \Phi(\omega), x) = \lim_{n \to \infty} \sum_{i \geqq 1} \mathbb{1}\{x \in B_{n,i}\}\, g(\Gamma_{B_{n,i}} \Phi(\omega), x)$$

is $\mathscr{P}_1$-measurable. Since the continuous functions on $N \times \mathbf{X}$ generate $\mathscr{N} \otimes \mathscr{X}$, the latter conclusion remains true for arbitrary bounded measurable functions on $N \times \mathbf{X}$. Therefore, we have $\mathscr{P}_2 \subseteq \mathscr{P}_1$.

Finally, we have to show that, for an arbitrary $B \in \mathscr{S}$ and for $H \in \mathscr{F}(B)$, the set $H \times B$ is an element of $\mathscr{P}_1$. To this end we choose a DC-semiring $\mathscr{S}'$ with $\mathscr{S}' \subseteq \mathscr{S}$ and $B \in \mathscr{S}'$. Then we have just proved that $H \times B \in \mathscr{P}_2 = \mathscr{P}_1$ and this completes the proof of the lemma. □

We conclude this section with the definition of predictability of a random measure. Let R be the set of all random measures μ on $(\mathbf{X}, \mathscr{X})$ that are locally integrable, i.e., for which $E\mu(B) < \infty$ if $B \in \mathscr{X}$ is bounded. We call $\mu \in R$ *predictable* if for any null-array $(\{B_{n,i}: i \geqq 1\})_{n \geqq 1}$ of $\mathscr{S}$-measurable nested partitions of $\mathbf{X}$

$$(2.13)\quad \sum_{i \geqq 1} E[\mu(B_{n,i} \cap B) \mid \mathscr{F}(B_{n,i})] \xrightarrow[n \to \infty]{\sigma(L^1, L^\infty)} \mu(B), \qquad B \in \{B_{n,i}: n \geqq 1, i \geqq 1\},$$

where a sequence $\xi, \xi_1, \xi_2, \ldots$ of integrable random variables satisfies $\xi_n \xrightarrow[n \to \infty]{\sigma(L^1, L^\infty)} \xi$ if $E\eta\xi_n$ converges to $E\eta\xi$ for all bounded random variables

η. Note that the predictability of a random measure depends on the probability measure P. If a random measure $\mu \in R$ coincides P-almost surely with a predictable random measure then it is predictable too.

3 Predictable projections of random processes

In this section the filtration is either given by (2.6), (2.7) or by (2.11) and (2.12). In the latter case we define Φ by (2.10). Recall the assumption (2.5). We introduce now a crucial regularity condition $\Sigma(\Gamma)$ on Φ:

$$\Sigma(\Gamma):\ P(\Phi(\Gamma_B^*)=0 \mid \mathscr{F}(B))>0\ \ P\text{-a.s.},\qquad B\in\mathscr{S},$$

where

$$\Gamma_B^* := \bigcup_{x\in B} \Gamma_x \setminus \Gamma_B .$$

The set Γ_B^* is measurable. Indeed, choosing a null-array $(\{B_{n,i}\colon i\geqq 1\})_{n\geqq 1}$ of $\mathscr{S}$-measurable nested partitions of $\mathbf{X}$ with $B\in\{B_{1,1}, B_{1,2}, \ldots\}$ and using the relations (2.3) and (2.4) we obtain

$$\Gamma_B^* = \bigcup_{n,i:B_{n,i}\subseteq B} \Gamma_{B_{n,i}} \setminus \Gamma_B .$$

Example 3.1 In Example 2.2 we have for $B \neq \{x\}$ the equality $\Gamma_B^* = B$ and $\Sigma(\Gamma)$ reduces to the well-known regularity condition (Σ) introduced by Papangelou (1974). In general Γ_B is contained in the complement of Γ_B^*. Therefore it is easy to see that (Σ) implies $\Sigma(\Gamma)$ whenever $P(\Phi(\Gamma_B^*)<\infty)=1$. It is well-known (see Kallenberg, 1983) that Gibbsian point processes satisfy (Σ).

Based on the fact that Γ_B^* and Γ_B are disjoint sets one can prove the following result (see Kallenberg, 1983, for the case of condition (Σ)). We use the notion of a p-thinning ζ of another such sequence $\zeta'=(\tau_n')$ in the obvious sense: τ_n is taken equal to τ_n' with probability p and equal to x_∞, otherwise. Given ζ', these thinnings are conditionally independent for different n.

Example 3.2 Let $0\leqq p<1$ and suppose that Φ (or ζ) is a p-thinning of another point process Ψ (resp. another random sequence $\zeta'=(\tau_n')$) satisfying $P(\Psi(\Gamma_B^*)<\infty)=1$ (resp. $\operatorname{card}\{n\colon \tau_n'\in\Gamma_B^*\}<\infty$ P-a.s.) for all $B\in\mathscr{S}$. Then Φ satisfies $\Sigma(\Gamma)$.

Let F be the set of all measurable and bounded real-valued functions on $\Omega\times\mathbf{X}$. For $f\in F$ and a random measure $\mu\in R$ we set

$$\langle \mu, f\rangle = E\int f(x)\,\mu(dx). \tag{3.1}$$

The measure $C_\mu := \langle \mu, \mathbb{1}\{\cdot\}\rangle$ on $(\Omega\times\mathbf{X}, \mathscr{F}\otimes\mathscr{X})$ is known as the Campbell measure or the Doleans measure of μ. The equality $\langle\mu,\cdot\rangle=\langle\nu,\cdot\rangle$ for μ,

$\nu \in R$ implies $\mu = \nu P$-almost surely. If $\langle \cdot, f \rangle = \langle \cdot, g \rangle$ for $f, g \in F$ then f and g are *indistinguishable* (with respect to P). This means (see Dellacherie and Meyer, 1978) that the universal measurable set $\{\omega$: there is an $x \in \mathbf{X}$ with $f(\omega, x) \neq g(\omega, x)\}$ has the measure zero with respect to the universal completion of P. We abbreviate this as $f \doteq g$ and define the relation $f \dot{\leqq} g$ similarly.

We will suppose the existence of regular conditional probability measures given $\mathscr{F}(B)$, $B \in S$, and that $\mathscr{F}$ is countably generated. This is no severe restriction of generality since the assumption is satisfied for $\mathscr{F} = \sigma(\Phi)$ and other suitable restrictions of $\mathscr{F}$. Conditional expectations are then always defined as integrals with respect to the regular conditional probability measures.

Our main result is as follows.

Theorem 3.3 *Assume $\Sigma(\Gamma)$ and let $f \in F$. Then there exists an up to indistinguishability unique predictable $f^* \in F$ which satisfies $\langle \mu, f^* \rangle = \langle \mu, f \rangle$ for all predictable $\mu \in R_*$.*

We refer to f^* as to the *predictable projection* of $f \in F$. The next theorem lists some properties of the predictable projection.

Theorem 3.4 *Assume $\Sigma(\Gamma)$. Then:*
(i) $(cf+g)^* \doteq cf^* + g^*$ *for all* $c \in \mathbb{R}$ *and* $f, g \in F$.
(ii) *If* $f, f_1, f_2, \ldots \in F$ *satisfy* $f_n \uparrow f$, *then* $f_n^* \dot{\leqq} f_{n+1}^* \dot{\leqq} f^*$ *and* $\lim_{n \to \infty} f_n^* \doteq f^*$.
(iii) $(f g^*)^* \doteq f^* g^*$ *for all* $f, g \in F$.
(iv) *For any* $f \in F$ *we have* $f \doteq f^*$ *iff* f *is indistinguishable from a predictable element of* F. *In particular,* $f \doteq 0$ *implies* $f^* \doteq 0$.
(v) *For any null-array* $(\{B_{n,i}: i \geqq 1\})_{n \geqq 1}$ *of* $\mathscr{S}$*-measurable nested partitions of* $\mathbf{X}$ *and any* $f \in F$ *we have the pointwise convergence*

$$f^*(x) \doteq \lim_{n \to \infty} \sum_{i \geqq 1} \mathbb{1}\{x \in B_{n,i}\} \, E[f(x) \mid \mathscr{F}(B_{n,i})]. \tag{3.2}$$

The idea of the proof of Theorem 3.3 is to show first the convergence of the right-hand side of (3.2) using the martingale convergence theorem on $\Omega \times \mathbf{X}$. In virtue of (2.4) we have

$$\sigma\Big(\bigcup_{n \geqq 1} \mathscr{F}(B_n)\Big) = \mathscr{F}(x) \qquad \text{if } B_n \downarrow \{x\},\ d(B_n) \downarrow 0,$$

and (3.2) implies in particular that

$$f^*(x) = E[f(x) \mid \mathscr{F}(x)] \quad P\text{-a.s.}$$

At this point we note that operators obeying the relations (i)–(iv) of Theorem 3.4, such as the mapping $f \mapsto f^*$, have been studied in a very general framework by Dynkin (1978) and Kerstan and Wakolbinger (1981).

Below we give an alternative expression for the predictable projection which avoids limits and which will also be used in Sect. 5. It is based on the reduced Palm measures $q^!_\psi(\cdot|D\Phi)$, $\psi\in N$, $D\in\mathscr{X}$, conditional on $D\Phi$ as used in Last (1990). Up to technicalities $\{q^!_\psi(\cdot|D\Phi): \psi\in N\}$ can be defined as the reduced Palm measures of Φ as introduced on p. 111 of Kallenberg (1983) with the original probability measure P replaced by the conditional probability measure $P(\cdot|D\Phi)$.

Theorem 3.5 *Consider the filtration as given by* (2.6) *and* (2.7) *and let* $\{B_1, B_2, \ldots\}$ *be a* $\mathscr{S}$*-measurable partition of* $\mathbf{X}$. *Let* $f(x):=g(\Phi, x)$, $x\in\mathbf{X}$, *for some bounded measurable function* g *on* $N\times\mathbf{X}$. *Then*

$$(3.3)\quad f^*(x) \doteq \sum_k \cdot \mathbb{1}\{x\in B_k\} \frac{\int g(\varphi+(\Gamma_x\setminus\Gamma_{B_k})\Phi)\, \mathbb{1}\{\varphi(\Gamma_x\setminus\Gamma_{B_k})=0\}\, q^!_{(\Gamma_x\setminus\Gamma_{B_k})\Phi}(d\varphi|\Gamma_{B_k}\Phi)}{q^!_{(\Gamma_x\setminus\Gamma_{B_k})\Phi}(\varphi(\Gamma_x\setminus\Gamma_{B_k})=0|\Gamma_{B_k}\Phi)},$$

where $0/0:=0$.

Proof. Let $\{B_{n,i}\}$ be as in Theorem 3.4(v) and assume $\{B_1, B_2, \ldots\} = \{B_{1,1}, B_{1,2}, \ldots\}$. Up to indistinguishability the predictable projection of f does not depend on the choice of the conditional probabilities $P(\Phi\in\cdot|\mathscr{F}(B_{n,i}))$. Due to Theorem 4.1 in Last (1990) we can therefore assume for $B_{n,i}\subseteq B_k$

$$(3.4)\quad P(\Phi\in\cdot|\mathscr{F}(B_{n,i})) = \frac{\int \mathbb{1}\{\varphi+(\Gamma_{B_{n,i}}\setminus\Gamma_{B_k})\Phi\in\cdot,\ \varphi(\Gamma_{B_{n,i}}\setminus\Gamma_{B_k})=0\}\, q^!_{(\Gamma_{B_{n,i}}\setminus\Gamma_{B_k})\Phi}(d\varphi|\Gamma_{B_k}\Phi)}{q^!_{(\Gamma_{B_{n,i}}\setminus\Gamma_{B_k})\Phi}(\varphi(\Gamma_{B_{n,i}}\setminus\Gamma_{B_k})=0|\Gamma_{B_k}\Phi)}$$

if the denumerator is positive and finite. The other cases have probability zero.

We will prove in Sect. 6 (see Lemma 6.13 and (6.31)) that

$$\lim_n \sum_i \mathbb{1}\{x\in B_{n,i}\}\, P(\Phi(\Gamma_x\setminus\Gamma_{B_{n,i}})=0|\mathscr{F}(B_{n,i})) \doteq 1.$$

Using (3.4) we hence get

$$\lim_n \sum_k \sum_{B_{n,i}\subseteq B_k} \mathbb{1}\{x\in B_{n,i}\} \frac{q^!_{(\Gamma_{B_{n,i}}\setminus\Gamma_{B_k})\Phi}(\varphi(\Gamma_x\setminus\Gamma_{B_k})=0|\Gamma_{B_k}\Phi)}{q^!_{(\Gamma_{B_{n,i}}\setminus\Gamma_{B_k})\Phi}(\varphi(\Gamma_{B_{n,i}}\setminus\Gamma_{B_k})=0|\Gamma_{B_k}\Phi)} \doteq 1.$$

In view of (2.4) and (2.5) this happens if and only if

$$(3.5)\quad \sum_k \mathbb{1}\{x\in B_k\}\, q^!_{(\Gamma_x\setminus\Gamma_{B_k})\Phi}(\varphi(\Gamma_x\setminus\Gamma_{B_k})=0|\Gamma_{B_k}\Phi)>0$$

up to indistinguishability. Now (3.3) is a quick consequence of (3.2) and (3.5). □

Theorem 3.5 gives a little bit more insight into the predictable projections. If, for example, Φ is almost surely finite on $\mathbf{X}$ and satisfies the so-called condition Σ' (see Kallenberg, 1983, for this and related matters) then we may write (3.3) in terms of the Gibbs kernel instead of the (conditional) reduced Palm measures. If Φ is a Gibbs process (see also van der Hoeven, 1983) then the Gibbs kernel can be calculated in terms of a so-called local energy or, more specifically, in terms of a pair potential. The Eq. (3.3) then reveals the dependence on the filtration in a more explicit form. In Example 3.2 the predictable projection coincides with the original process. In a general situation such an equality is rather exceptional.

4 Dual predictable projections of random measures

We keep the framework of Sect. 3. In the whole section the condition $\Sigma(\Gamma)$ is in force.

Theorem 4.1 *Let $\mu\in R$. Then there exists a P-almost surely unique predictable random measure $\mu^*\in R$ satisfying $\langle\mu,f\rangle=\langle\mu^*,f\rangle$, for all predictable $f\in F$. Given the sets $B_{n,i}$, n, $i\in\mathbb{N}$, as in Theorem* 3.4(v) *and given a bounded set $A\in\mathscr{X}$ the convergence*

$$(4.1)\qquad \sum_{i\geq 1} E[\mu(B_{n,i}\cap B)|\mathscr{F}(B_{n,i})] \xrightarrow[n\to\infty]{L^1(P)} \mu^*(B)$$

takes place uniformly in $B\subseteq A$, $B\in\mathscr{X}$.

We refer to μ^* as to the *dual predictable projection* of $\mu\in R$. We will prove in Sect. 7 (see Theorem 7.3) that μ is predictable iff $\mu=\mu^*$ P-a.s. In fact it suffices to consider (2.13) for one null-array of nested partitions in order to check the predictability of μ.

Remark 4.2 Let $\mu\in R$. Then the signed random measure $\nu:=\mu-\mu^*$ is a *martingalelike* random measure in the sense that

$$(4.2)\qquad E[\nu(B)|\mathscr{F}(B)]=0 \ P\text{-a.s.},\qquad B\in\mathscr{S}.$$

It is easy to see that $\mu=\mu^*+\nu$ is a P-a.s. unique decomposition of μ into a sum of a predictable random measure and a locally integrable martingale-like measure.

As mentioned in the introduction, van der Hoeven (1982, 1983) proved the Theorems 3.3 and 4.1 for Example 2.2 without regularity condition. In that case a random measure is predictable if and only if the process of its atom sizes is predictable. Thanks to this and other special properties of his filtration van der Hoeven derived expressions for the (dual) predictable projections which are more explicit than (3.2), (3.3) and (4.1). We can not present similar formulas in the general case, where usually $\Gamma_x\cup\{x\}$ is a genuine subset of **X**. We give, however, an explicit expression for the dual predictable projection of a zero-one point process.

Example 4.3 Let $\Phi = X\,\delta_Y$, where X and Y are independent random elements of $\{0, 1\}$ or **X**, respectively, and $P(X=1)=p$ for some $p<1$. Define the filtration by (2.6) and (2.7) and assume

$$B \cap \Gamma_B = \emptyset, \qquad B \in \mathcal{S}. \tag{4.3}$$

Then the dual predictable projection of Φ is P-almost surely given by

$$\Phi^*(dx) = \frac{\mathbb{1}\{\Phi(\Gamma_x)=0\}}{1-pF(\Gamma_x)}\, pF(dx), \tag{4.4}$$

where F is the distribution of Y.

Proof. The point process Φ is a p-thinning of δ_Y and obeys $\Sigma(\Gamma)$ according to Example 3.2. Since $\{\Phi(\Gamma_B)=0\}$ is an atom of $\mathcal{F}(B)$ we may assume without loss of generality that

$$P(\cdot\,|\,\mathcal{F}(B_{n,i})) = P(\cdot\,|\,\Phi(\Gamma_{B_{n,i}})=0) \quad \text{on } \{\Phi(\Gamma_{B_{n,i}})=0\}, \tag{4.5}$$

where the $B_{n,i}$ are as in Theorem 3.4(v) with $B \in \{B_{1,1}, B_{1,2}, \dots\}$ for some arbitrary but fixed $B \in \mathcal{S}$. By (4.3) it is obvious that

$$P(\Phi(B_{n,i})=1\,|\,\mathcal{F}(B_{n,i}))=0 \ \ P\text{-a.s. on } \{\Phi(\Gamma_{B_{n,i}})=1\}. \tag{4.6}$$

From Theorem 4.1 we obtain that $\Phi^*(B)$ is the $L^1(P)$-limit of

$$\Phi_n^*(B) := \sum_{i:B_{n,i}\subseteq B} P(\Phi(B_{n,i})=1\,|\,\mathcal{F}(B_{n,i})).$$

The Eqs. (4.5) and (4.6) yield P-almost surely that

$$\begin{aligned}\Phi_n^*(B) &= \sum_{i:B_{n,i}\subseteq B} \mathbb{1}\{\Phi(\Gamma_{B_{n,i}})=0\}\,\frac{P(\Phi(B_{n,i})=1,\, \Phi(\Gamma_{B_{n,i}})=0)}{P(\Phi(\Gamma_{B_{n,i}})=0)}\\ &= \sum_{i:B_{n,i}\subseteq B} \mathbb{1}\{\Phi(\Gamma_{B_{n,i}})=0\}\,\frac{pF(B_{n,i})}{(1-p)+pF((\Gamma_{B_{n,i}})^c)},\end{aligned}$$

where the latter equation follows from a simple calculation. Now we can write

$$\Phi_n^*(B) = \int \mathbb{1}\{x\in B\}\, f_n(x)\, F(dx) \ \ P\text{-a.s.},$$

where

$$f_n(x) := \sum_i \mathbb{1}\{x\in B_{n,i}\}\, \mathbb{1}\{\Phi(\Gamma_{B_{n,i}})=0\}\,\frac{p}{1-pF(\Gamma_{B_{n,i}})}$$

tends to

$$\mathbb{1}\{\Phi(\Gamma_x)=0\}\,\frac{p}{1-pF(\Gamma_x)}$$

as $n \to \infty$ in virtue of (2.4). Since $f_n(x)$ is bounded by $p/(1-p)$ we can conclude the assertion (4.4). □

The expression (4.4) reveals the dependence of a dual predictable projection on the filtration. In particular we note:

Remark 4.4 Consider the framework of Example 4.3 and set $\tau = Y$ if $X = 1$ and $\tau = x_\infty$ otherwise. Then $\Phi = \mathbb{1}\{\tau \neq x_\infty\}\,\delta_\tau$ and $pF + (1-p)\,\delta_{x_\infty}$ is the distribution of τ. For Example 2.1 Eq. (4.4) reduces to the well-known formula for the compensator of zero-one point process. In the setting of Example 2.2 and if F is a diffuse measure, then Φ^* simplifies to $p/(1-p)\,\mathbb{1}\{\Phi(\mathbf{X})=0\}\,F$ (cf. van der Hoeven, 1983). The above calculation goes through for $p=1$ as long as $F(\Gamma_x)$ is bounded away from 1. If not, then one could take the limit of the $\Phi_n^*(B)$ directly. The point process δ_Y, however, does in general not satisfy our regularity condition, as will easily be seen in Example 7.6.

In the next section we will specialize our model (2.3), (2.4) in case of $\mathbf{X} = \mathbb{R}_+^2$ and discuss some further properties of predictable projections. Also then it seems to be hard to do some calculation being more specific than (3.2), (4.1) or (3.3). At least in Example 2.3 one can find some expressions for the dual predictable projection of Φ which are similar to Jacod's formula for the compensator of point process on $\mathbb{R}_+$ (see Buckdahn, 1984, Mazziotto and Merzbach, 1988).

5 The two-dimensional case

We consider a point process Φ on $\mathbf{X} := (0, \infty)^2$ and let $\mathcal{S}$ be the system of all rectangles $(a, b]$. The results of this section can be generalized to $(0, \infty)^n$. In this section we study filtrations of the type (2.6) and (2.7). Let Γ be a system satisfying (2.3)–(2.5). We introduce some further assumption on Γ and begin with the claim

$$\Gamma_{(a,b]} = \bar{\Gamma}_a, \qquad a < b, \tag{5.1}$$

where

$$\bar{\Gamma}_a := \bigcap_{x > a} \Gamma_x. \tag{5.2}$$

We assume further that

$$\Gamma_x \subseteq \Gamma_y \qquad \text{if } x \leqq y \tag{5.3}$$

and

$$\bigcup_{x < y} \Gamma_x = \Gamma_y. \tag{5.4}$$

Then we have in particular

$$(5.5)\qquad \Gamma_x \subseteq \bar{\Gamma}_x \subseteq \Gamma_y \quad \text{if } x<y,$$

$$(5.6)\qquad \Gamma_B \subseteq \Gamma_x \quad \text{if } x \in B \in \mathscr{S}.$$

Let us strengthen (2.5) to

$$(5.7)\qquad \Phi(\bar{\Gamma}_x \setminus \Gamma_B) < \infty \quad \text{if } x \in B \in \mathscr{S}.$$

Writing (recall (2.7))

$$(5.8)\qquad \mathscr{F}_x := \sigma(\bar{\Gamma}_x \Phi), \qquad \mathscr{F}_{x-} := \sigma\Big(\bigcup_{y<x} \mathscr{F}_x\Big),$$

we obtain

$$(5.9)\qquad \mathscr{F}_x = \bigcap_{y>x} \mathscr{F}(y), \qquad \mathscr{F}_{x-} = \mathscr{F}(x)$$

and

$$(5.10)\qquad \mathscr{F}_x \subseteq \mathscr{F}_y \quad \text{if } x<y.$$

We also have

$$(5.11)\qquad \bar{\Gamma}_x = \bigcap_{y>x} \bar{\Gamma}_y$$

and therefore

$$(5.12)\qquad \mathscr{F}_x = \bigcap_{y>x} \mathscr{F}_y.$$

Obviously (5.1) and (5.8) entail that the predictable σ-field is given by the classical definition

$$\mathscr{P} = \sigma(H \times (a, b]: H \in \mathscr{F}_a, a<b).$$

Example 5.1 Let $h: \mathbb{R}^2 \to \mathbb{R}$ be monotone non-increasing in both arguments. Assume the right-continuity $h(x) = \lim_{y \to x, x \leq y} h(y)$ and define $h^-(x) := \lim_{y \to x, y<x} h(y)$. Then the definitions $\Gamma_x := \{y: h(y-x)>0\}$, $\bar{\Gamma}_x := \{y: h^-(y-x) >0\}$ and (5.1) satisfy the assumptions of this section, i.e. (2.3), (2.4), (5.2)–(5.4). The Examples 2.3 and 2.4 are special cases of this model. In the first example we have to take $h(x) = \mathbb{1}\{x_1<0, x_2<0\}$ and in the second $h(x) = \mathbb{1}\{x_1<0$ or $x_2<0\}$.

The next result shows the existence of nice versions of bounded $\{\mathscr{F}_x\}$- and $\{\mathscr{F}(x)\}$-martingales. It reveals the connection to the predictable projections known from the literature.

Theorem 5.2 *Assume $\Sigma(\Gamma)$ and let Y be a bounded random variable.*
(i) *There exist versions $M(x)$ of $E[Y|\mathscr{F}_x]$ and $M^-(x)$ of $E[Y|\mathscr{F}(x)]$ such that*

$$\lim_{y \to x, y < x} M(y) = M^-(x), \tag{5.13}$$

$$\lim_{y \to x, y < x} M^-(y) = M^-(x), \tag{5.14}$$

$$\lim_{y \to x, x \leqq y} M(y) = M(x). \tag{5.15}$$

The process $\{M^-(x)\}$ is predictable.
(ii) *Let $f(x) = \mathbb{1}\{x \in B\}\, Y$ for some $B \in \mathscr{X}$. Then the predictable projection of f satisfies*

$$f^*(x) \doteq \mathbb{1}\{x \in B\}\, M^-(x). \tag{5.16}$$

Proof. Let Z be a bounded version of $E[Y|\Phi]$ and write $Z = h(\Phi)$ for a suitable bounded and measurable function h on $(N, \mathscr{N})$ (the space of all point process realizations). Let $\{B_1, B_2, \ldots\}$ be a $\mathscr{S}$-measurable partition of $\mathscr{X}$ and put

$$B_{k,x} = \Gamma_x \setminus \Gamma_{B_k}, \qquad \bar{B}_{k,x} = \bar{\Gamma}_x \setminus \Gamma_{B_k}.$$

We define

$$M(x) = \sum_k \mathbb{1}\{x \in B_k\} \frac{\int h(\varphi + \bar{B}_{k,x}\, \Phi)\, \mathbb{1}\{\varphi(\bar{B}_{k,x}) = 0\}\, q^!_{\bar{B}_{k,x}\Phi}(d\varphi \,|\, \Gamma_{B_k}\, \Phi)}{q^!_{\bar{B}_{k,x}\Phi}(\varphi(\bar{B}_{k,x}) = 0 \,|\, \Gamma_{B_k}\, \Phi)}, \tag{5.17}$$

$$M^-(x) = \sum_k \mathbb{1}\{x \in B_k\} \frac{\int h(\varphi + B_{k,x}\, \Phi)\, \mathbb{1}\{\varphi(B_{k,x}) = 0\}\, q^!_{B_{k,x}\Phi}(d\varphi \,|\, \Gamma_{B_k}\, \Phi)}{q^!_{B_{k,x}\Phi}(\varphi(B_{k,x}) = 0 \,|\, \Gamma_{B_k}\, \Phi)}, \tag{5.18}$$

where $0/0 := 0$. According to Theorem 4.1 of Last (1990) we have for all $x \in \mathbf{X}$ the desired equations $M(x) = E[Z|\mathscr{F}_x] = E[Y|\mathscr{F}_x]$ and $M^-(x) = E[Y|\mathscr{F}(x)]$ P-almost surely. Now we take $x \in \mathbf{X}$ and a sequence $x_n \in \mathbf{X}$, $n \in \mathbb{N}$, with $x_n < x$ and $\lim x_n = x$. We may assume that $x_n \in B_k$ for all n for some k. In virtue of (5.4) and (5.5) we have $\Gamma_{x_n} \uparrow \Gamma_x$ and $\bar{\Gamma}_{x_n} \uparrow \Gamma_x$ and therefore the monotone non-increasing convergence of $\{\varphi(B_{k,x_n} = 0\}$ as well as of $\{\varphi(\bar{B}_{k,x_n} = 0\}$ towards $\{\varphi(B_{k,x} = 0\}$. On account of assumption (5.7) we find an $n_0 \in \mathbb{N}$ and a $\psi \in N$ such that $B_{k,x_n}\Phi = \bar{B}_{k,x_n}\Phi = \psi$ for $n \geqq n_0$. A look at (5.17) and (5.18) shows that $\lim M(x_n) = M^-(x)$ if the corresponding denumerator in (5.18) is positive. Hence (3.5) implies (5.13) and the same arguments demonstrate (5.14). The proof of (5.15) has to be based on (5.11). We do not need (3.5) such that (5.15) holds also without $\Sigma(\Gamma)$.

(ii) Take a system $\{B_{n,i}\}$ as in Theorem 3.4(v) and assume $\{B_1, B_2, \ldots\} = \{B_{1,1}, B_{1,2}, \ldots\}$. Let $a_{n,i}$ be the left end-point of $B_{n,i}$. In view of (3.2), (5.1) and (3.4) we obtain

$$f^*(x) \doteq \lim_{n\to\infty} \sum_i \mathbb{1}\{x \in B_{n,i} \cap B\}\, M(a_{n,i})$$

and (5.13) implies the assertion (5.16). □

In the examples known from the literature the predictable projection of a process f as in Theorem 5.2 is a left-continuous version of the martingale $E[Y|\mathscr{F}(x)]$ restricted to B. Continuity arguments show that our projections are indeed consistent with the classical notions. We need the regularity condition $\Sigma(\Gamma)$. Although the latter is not compatible with the condition (F4) of Cairoli and Walsh (1975), it is much less restrictive as is evident from the Examples 3.1 and 3.2. (For a point process with exactly one point, Al-Hussaini and Elliott, 1981, showed that (F4) is equivalent to the independence of the components of that point.) Once condition $\Sigma(\Gamma)$ is accepted, our results cover a much more greater variety of models than those presented in Example 2.3 (or in Example 2.4). From a probabilistic point of view there is no reason to focus on these examples only. The relations (5.10) and (5.12) are standard assumptions for stochastic calculus for processes with multidimensional parameter. Satisfying these assumptions and being generated by a point process constitute the main properties of the filtration studied in this section.

In Example 2.3 the predictability of a random measure μ means, by the usual definition, that the process $x \mapsto \mu(x) := \mu((0, x])$ is predictable. In the general case we can give here no similar characterization. However, our notion of predictability generalizes Meyer's definition of a natural increasing process. We refer here to Dellacherie and Meyer (1982) and, for the two-dimensional case, to Cairoli (1971) and Dozzi (1981).

Remark 5.3 Assume $\Sigma(\Gamma)$. A $\mu \in R$ is predictable iff it is *natural* in the following sense. For all bounded random variables Y and all $B \in \mathscr{X}$ we have

$$E \int \mathbb{1}\{x \in B\}\, M^-(x)\, \mu(dx) = EY\mu(B), \tag{5.19}$$

where $M^-(x)$ is a left-continuous version of the martingale $E[Y|\mathscr{F}(x)]$. In that case $\mu(z)$ is P-almost surely $\mathscr{F}(z)$-measurable for all $z \in \mathbf{X}$.

Proof. We will prove in Sect. 7 that μ is predictable if $\mu = \mu^*$ P-a.s. Hence the first assertion is an immediate consequence of (5.16) and the fact that the processes f as in Theorem 5.2 generate the σ-field $\mathscr{F} \otimes \mathscr{X}$.

Now we assume μ to be predictable and fix a $z \in \mathbf{X}$. We have to show that

$$EY\mu(z) = EE[Y|\mathscr{F}(z)]\, \mu(z)$$

for all bounded random variables Y. Let $\tilde{Y}=E[Y|\mathscr{F}(z)]$ and define $M^-(\tilde{M}^-)$ in terms of $Y(\tilde{Y})$ as in Theorem 5.2. Since $E[Y|\mathscr{F}(x)]=E[\tilde{Y}|\mathscr{F}(x)]$ P-a.s. for $x\leqq z$ we have $\mathbb{1}\{x\leqq z\}\,\tilde{M}^-(x)\doteq\mathbb{1}\{x\leqq z\}\,M^-(x)$. Therefore we obtain from (5.19)

$$\begin{aligned} EY\mu(z)&=E\int M^-(x)\,\mathbb{1}\{x\leqq z\}\,\mu(dx)\\ &=E\int \tilde{M}^-(x)\,\mathbb{1}\{x\leqq z\}\,\mu(dx)=E\tilde{Y}\mu(z), \end{aligned}$$

as desired. □

For the next remark we also refer to Remark 4.2.

Remark 5.4 Assume $\Sigma(\Gamma)$. Let $\mu\in R$ and assume $\mu(Z)$ to be $\mathscr{F}_z$-measurable for all $z\in\mathbf{X}$. Then $\{\mu(z)-\mu^*(z)\}$ is a weak martingale in the sense of Cairoli and Walsh (1975). This property uniquely determines μ^* among all predictable random measures.

6 Proofs of Theorem 3.4 and Example 3.2

In the main part of this section we will prove the convergence (3.2) and show that the limit satisfies the other assertions of Theorem 3.4. The proof is divided into five steps. We start with some preliminary definitions and remarks. In the first two steps we need only the properties (2.1) and (2.2) of the filtration.

Step 1 We consider sets $M\subseteq R$ and write $f=g \bmod M$ for $f, g\in F$ or, analogously

$$f(\omega, x)=g(\omega, x)\ M\text{-a.e.}(\omega, x)$$

if $f=g$ C_μ-a.e. for all $\mu\in M$. For example $f\doteq g$ iff $f=g \bmod R$.

For a non-negative $h\in F$ and $\mu\in R$ we define a random measure $h\mu$ by $h\mu(dx)=h(x)\,\mu(dx)$ which satisfies $\langle h\mu, f\rangle=\langle\mu, hf\rangle$ for all $f\in F$.

Lemma 6.1 *Let $M\subseteq R$ and suppose that for all non-negative $h\in F$ and all $\mu\in R$ the random measure $h\mu$ is again in M. Then $f=g \bmod M$ iff $\langle\mu,f\rangle=\langle\mu, g\rangle$ for all $\mu\in M$ with $C_\mu(\Omega\times\mathbf{X})=1$.*

Note that $f=g \bmod M$ implies $f=g$ C_μ-a.e. for all $\mu\in R$ with the property $C_\mu\ll\{C_\nu\colon \nu\in M\}$, where a measure α on a measurable space $(A,\mathscr{A})$ satisfies $\alpha\ll B$ for a set B of measures on $(A,\mathscr{A})$ if $\{C\in\mathscr{A}\colon \beta(C)=0 \text{ for all } \beta\in B\}\subseteq\{C\in\mathscr{A}\colon \alpha(C)=0\}$.

For the next definition we also refer to Dynkin (1978) and Kerstan and Wakolbinger (1981). We call a mapping $T\colon F\to F$ an *M-projection* if the following properties are satisfied:

(6.1) $$T(cf+g)=cTf+Tg \bmod M,\qquad c\in\mathbb{R},\ f, g\in F,$$

(6.2) $$Tf_n\uparrow Tf \bmod M\quad \text{if } \{f_n\}\subseteq F\quad \text{with}\quad f_n\uparrow f \bmod M,$$

(where $f_n \leqq g \bmod M$ if $\mathbb{1}\{f_n > g\} = 0 \bmod M$, and $f_n \uparrow f \bmod M$ if $f_n \leqq f_{n+1} \leqq f \bmod M$ and $\mathbb{1}\{\lim f_n \neq f\} = 0 \bmod M$),

(6.3) $$Tf = 1 \bmod M \qquad \text{if } f = 1 \bmod M,$$

(6.4) $$T(f \cdot Tg) = (Tf)(Tg) \bmod M, \qquad f, g \in F.$$

Let us fix an M-projection T for a moment. Then (6.1) and (6.3) imply

(6.5) $$Tf = 0 \bmod M \qquad \text{if } f = 0 \bmod M,$$

(6.6) $$Tf = Tg \bmod M \qquad \text{if } f = g \bmod M, \quad f, g \in F.$$

The notion of a M-projection is closely related to conditional expectations. To make this more precise we define

$$\mathscr{P}(T) := \{A \in \mathscr{F} \otimes \mathscr{X} \colon T\mathbb{1}_A = \mathbb{1}_A \bmod M\}$$

and may derive from (6.1) and (6.4) that $\mathscr{P}(T)$ is a σ-field. A function $f \in F$ is $\mathscr{P}(T)$-measurable iff $Tf = f \bmod M$. If $\mu \in R$ satisfies $C_\mu \ll \{C_\nu \colon \nu \in M\}$, then $A \mapsto C_\mu T(A) := \langle \mu, T\mathbb{1}_A \rangle$, $A \in \mathscr{F} \otimes \mathscr{X}$, defines a measure. Moreover, if in addition $C_\mu(\Omega \times \mathbf{X}) = C_\mu T(\Omega \times \mathbf{X}) = 1$, then

(6.7) $$E_{C_\mu T}[f \mid \mathscr{P}(T)] = Tf \quad C_\mu T\text{-a.s.},$$

where $E_{C_\mu T}$ refers to a (conditional) expectation with respect to $C_\mu T$.

Step 2 We fix a null-array $\mathscr{B} := (\{B_{n,i} \colon i \geqq 1\})_{n \geqq 1}$ of nested $\mathscr{S}$-measurable partitions of $\mathbf{X}$ as in Theorem 3.4(v). For any $f \in F$, $\mu \in R$ and $n \in \mathbb{N}$ we define $T_n f \in F$ by

(6.8) $$T_n f(x) = \sum_i \mathbb{1}\{x \in B_{n,i}\}\, E[f(x) \mid \mathscr{F}(B_{n,i})]$$

and $\mu T_n \in R$ by

(6.9) $$\mu T_n(dx) = \sum_i \mathbb{1}\{x \in B_{n,i}\}\, E[\mu(dx) \mid \mathscr{F}(B_{n,i})].$$

It can easily be proved that $T_n \mu$ is the P-almost surely unique random measure satisfying

(6.10) $$\langle \mu T_n, f \rangle = \langle \mu, T_n f \rangle, \qquad f \in F.$$

Also it should be clear that T_n is an R-projection and

(6.11) $$T_n T_m f = T_{\min\{m,n\}} f \bmod R, \qquad f \in F.$$

Define

$$(6.12)\qquad \mathcal{P}_n := \sigma(A \times B_{m,j}\colon A \in \mathcal{F}(B_{n,i}),\ B_{m,j} \subseteq B_{n,i},\ m \geqq n,\ i, j \geqq 1),$$

$$(6.13)\qquad \mathcal{M} := \{A \in \mathcal{F} \otimes \mathcal{X} : \mathbb{1}_A = 0 \bmod R\}.$$

Lemma 6.2 *We have*

$$(6.14)\qquad \mathcal{P}(T_n) = \sigma(\mathcal{P}_n \cup \mathcal{M}),$$

$$(6.15)\qquad \mathcal{P}(T_n) \subseteq \mathcal{P}(T_{n+1}), \qquad \mathcal{P}_n \subseteq \mathcal{P}_{n+1},$$

$$(6.16)\qquad \mathcal{P} = \sigma\Big(\bigcup_{n \geqq 1} \mathcal{P}_n\Big).$$

Proof. Clearly $T_n f$ is $\mathcal{P}_n$-measurable for all $f \in F$. Since $A \in \mathcal{P}(T_n)$ implies a representation $\mathbb{1}_A = T_n \mathbb{1}_A + g$, where $g = 0 \bmod R$, we obtain $\mathcal{P}(T_n) \subseteq \sigma(\mathcal{P}_n \cup \mathcal{M})$. Conversely it is obvious that $\mathcal{P}_n \subseteq \mathcal{P}(T_n)$ and the inclusion $\sigma(\mathcal{P}_n \cup \mathcal{M}) \subseteq \mathcal{P}(T_n)$ follows easily from the property (6.5) of a projection operator. In order to prove the first relation of (6.15) we take an $f \in F$ with $f = T_n f \bmod R$ and obtain from (6.11)

$$T_{n+1} f = T_{n+1} T_n f = T_n f = f \bmod R.$$

The other relations are obvious from the definitions. □

For any $f \in F$ we denote

$$(6.17)\qquad B(f) := \{\liminf_{n\to\infty} T_n f = \limsup_{n\to\infty} T_n f\},$$

$$(6.18)\qquad Tf := \mathbb{1}_{B(f)} \liminf_{n\to\infty} T_n f$$

and note that $B(f)$ and Tf are $\mathcal{P}$-measurable. Furthermore we denote

$$(6.19)\qquad R_* := \{\mu \in R\colon C_\mu|\mathcal{P} \ll \{C_{\nu T_n}|\mathcal{P}\colon \nu \in R,\ n \in \mathbb{N}\}\},$$

$$(6.20)\qquad \mathcal{M}_* := \{A \in \mathcal{F} \otimes \mathcal{X} : \mathbb{1}_A = 0 \bmod R_*\},$$

where $C_\mu|\mathcal{P}$ is the restriction of C_μ onto $\mathcal{P}$.

The following result is the first main step towards the proof of Theorem 3.4.

Theorem 6.3 *The mapping T is an R_*-projection and we have*

$$(6.21)\qquad \mathcal{P}(T) = \sigma(\mathcal{P} \cup \mathcal{M}_*),$$

$$(6.22)\qquad T_n T f = T_n f \bmod R, \qquad f \in F,\ n \in \mathbb{N}.$$

The proof of Theorem 6.3 is based on the following lemma.

Lemma 6.4 (i) *Let $\mu \in R$. Then $\mu \in R^*$ iff*

$$C_\mu|\mathscr{P} \ll \{C_{\nu T_n}|\mathscr{P}\colon \nu \in R,\ n \in \mathbb{N},\ C_\nu(\Omega \times \mathbf{X}) = 1\} \tag{6.23}$$

(ii) *Let $m \in \mathbb{N}$, $\mu \in R$, $f \in F$ and suppose that $C_\mu(\Omega \times \mathbf{X}) = 1$. Then*

$$E_{C_\mu T_m}[f|\mathscr{P}_n] = T_n f \quad C_{\mu T_m}\text{-a.s.}, \qquad n \geqq m, \tag{6.24}$$

$$E_{C_\mu T_m}[f|\mathscr{P}] = \lim_{n \to \infty} T_n f \quad C_{\mu T_m}\text{-a.s.} \tag{6.25}$$

Proof of Lemma 6.4 (i) This follows easily from Lemma 6.1.
(ii) In view of (6.14) it suffices firstly to show that $E_{C_\mu T_m}[f|\mathscr{P}(T_n)] = T_n f\ C_\mu$-a.s. for $n \geqq m$. Take a $g \in F$ with $T_n g = g$. Then (6.4) and (6.11) provide

$$T_m(g\, T_n f) = T_m\, T_n(g \cdot f) = T_m(g f) \bmod R$$

as desired. The second relation is now a consequence of (6.15), (6.16) and the martingale convergence theorem. □

Proof of Theorem 6.3 It is illustrative enough to prove the properties (6.3) and (6.4) of a projection. First we note that on account of Lemma 6.4

$$\mathbb{1}_{B(f)} = 1 \bmod R_*, \tag{6.26}$$

$$Tf = E_{C_\mu T_m}[f|\mathscr{P}]\ C_{\mu T_m}\text{-a.e.} \quad \text{if } C_\mu(\Omega \times \mathbf{X}) = 1. \tag{6.27}$$

Let $f \in F$ satisfies $f = 1 \bmod R_*$. In particular, for $\mu \in R$ and $m \in \mathbb{N}$ with $C_\mu(\Omega \times \mathbf{X}) = 1$ we have $f = 1\ C_{\mu T_m}$-a.e. and we derive from (6.27)

$$Tf = E_{C_\mu T_m}[1|\mathscr{P}] = 1\ C_{\mu T_m}\text{-a.e.}$$

Since $\{Tf \neq 1\}$ is predictable we obtain from (6.23) the desired relation $\mathbb{1}\{Tf \neq 1\} = 0 \bmod R^*$.

For $f, g \in F$ and μ, m as above we conclude in virtue of (6.27) $C_{\mu T_m}$-a.e.

$$\begin{aligned} T(f\, Tg) &= E_{C_\mu T_m}[f\, E_{C_\mu T_m}[g|\mathscr{P}]|\mathscr{P}] \\ &= E_{C_\mu T_m}[f|\mathscr{P}]\, E_{C_\mu T_m}[g|\mathscr{P}] = (Tf)(Tg). \end{aligned}$$

The set $A := \{T(f\, Tg) \neq (Tf)(Tg)\}$ is predictable and (6.23) implies $\mathbb{1}_A = 0 \bmod R^*$, which was to be shown.
(ii) Let us assume that f is predictable. Then (6.27) implies $Tf = f \bmod R_*$, i.e., Tf is $\mathscr{P}(T)$-measurable. Since $\mathscr{M}_* \subseteq \mathscr{P}(T)$ by definition, we have $\sigma(\mathscr{P} \cup \mathscr{M}_*) \subseteq \mathscr{P}(T)$. The converse inclusion follows as in the proof of (6.14).
(iii) Let $f \in F$ and $\mu \in R$ with $C_\mu(\Omega \times \mathbf{X}) = 1$. Equation (6.27) implies $\langle \mu T_n, Tf \rangle = \langle \mu T_n, f \rangle$. In view of (6.10) we therefore have $\langle \mu, T_n\, Tf \rangle = \langle \mu, T_n f \rangle$ and Lemma 6.1 implies the assertion. □

Later we will need the following characterization of R^*.

Lemma 6.5 *Let $\mu \in R$. Then $\mu \in R_*$ iff* $\lim_{n\to\infty} T_n f = f\, C_\mu$-a.e. *for all predictable $f \in F$.*

Proof. The necessity of the stated condition is proved in Theorem 6.3. The converse direction follows from (6.23) and bounded convergence. □

Step 3 In this and the following steps we will take advantage from the discrete nature of our filtrations. In order to handle both cases simultaneously we set in case of (2.6) and (2.7)

$$\xi_x := \Gamma_x \Phi, \qquad \xi_B := \Gamma_B \Phi, \qquad x \in \mathbf{X},\ B \in \mathscr{S},$$

and in case of (2.11) and (2.12)

$$\xi_x := \Gamma_x \zeta, \qquad \xi_B := \Gamma_B \zeta, \qquad x \in \mathbf{X},\ B \in \mathscr{S}.$$

For $m \in \mathbb{N}$ we define the predictable sets

$$Y_m := \{(\omega, x) : \xi_x(\omega) = \xi_{B_m(x)}(\omega)\}, \tag{6.28}$$

where the set $B_m(x)$ is the unique element of $\{B_{m,j} : j = 1, 2, \ldots\}$ containing $x \in \mathbf{X}$. Consider a predictable $f \in F$. In view of Lemma 2.6(ii) it has to be of the form $f(\omega, x) = g(\xi_x(\omega), x)$ for a suitable measurable function g. By definition of Y_m we have

$$(\mathbb{1}_{Y_m} f)(x) = \sum_j \mathbb{1}\{x \in B_{m,j}\}\, \mathbb{1}\{\xi_x = \xi_{B_{m,j}}\}\, g(\xi_{B_{m,j}}, x),$$

which is a product of $\mathbb{1}_{Y_m}$ with a $\mathscr{P}_m$-measurable function. Therefore we can deduce from (6.4) and (6.14):

Lemma 6.6 *For all $m \in \mathbb{N}$ and predictable $f \in F$*

$$\mathbb{1}_{Y_m} T_m \mathbb{1}_{Y_m} f = \mathbb{1}_{Y_m} f\, T_m \mathbb{1}_{Y_m} \bmod R. \tag{6.29}$$

In view of (2.4) and (2.5) we have

$$Y_m \uparrow \Omega \times \mathbf{X} \quad \text{as} \quad m \to \infty. \tag{6.30}$$

The Lemmata 6.5 and 6.6 motivate the definition

$$A_* := \{\lim T_m \mathbb{1}_{Y_m} = 1\}. \tag{6.31}$$

This set characterizes R_* as can be seen from Lemma 6.5, Lemma 6.6 and Eq. (6.38) below.

Theorem 6.7 *Let $\mu \in R$. Then $\mu \in R_*$ iff $C_\mu(A_*^c) = 0$.*

For $n \geq m$ we define $\mathscr{P}_n$-measurable sets (cf. (6.13))

$$Y_{m,n} := \bigcup_{i,j : B_{n,i} \subseteq B_{m,j}} \{\xi_{B_{n,i}} = \xi_{B_{m,j}}\} \times B_{n,i},$$

and note that $Y_{m,n} \downarrow Y_m$ as $n \to \infty$.

Lemma 6.8 *Let $f \in F$ and $m, n \in \mathbb{N}$ with $n > m$. Then*

$$\mathbb{1}_{Y_m} T_n f = \mathbb{1}_{Y_m} \frac{T_m \mathbb{1}_{Y_{m,n}} f}{T_m \mathbb{1}_{Y_{m,n}}} \bmod R, \tag{6.32}$$

where 0/0 *is defined by* 0.

Proof. Using the separability of $\mathscr{F}$ and the monotone class theorem, the result can easily be derived from a Lemma of Papangelou (1974) which states that for $B, C \in \mathscr{S}$ with $B \subseteq C$ and for an integrable random variable X

$$\begin{aligned} &\mathbb{1}\{\xi_B = \xi_C\} E[\mathbb{1}\{\xi_B = \xi_C\} X \mid \xi_B] \\ &= \mathbb{1}\{\xi_B = \xi_C\} \frac{E[\mathbb{1}\{\xi_B = \xi_C\} X \mid \xi_C]}{P(\xi_B = \xi_C \mid \xi_C)} \quad P\text{-a.s.} \end{aligned} \tag{6.33}$$

We note here that

$$\{\xi_B = \xi_C\} \subseteq \{P(\xi_B = \xi_C \mid \xi_C) > 0\} \; P\text{-a.s.} \quad \square \tag{6.34}$$

Inserting into (6.32) $f = \mathbb{1}_{Y_n}$ or $f = \mathbb{1}_{Y_m}$ and taking into account $Y_{m,n} \cap Y_n = Y_m$ for $m \leqq n$ we obtain in particular

$$\mathbb{1}_{Y_m} T_n \mathbb{1}_{Y_n} = \mathbb{1}_{Y_m} \frac{T_m \mathbb{1}_{Y_m}}{T_m \mathbb{1}_{Y_{m,n}}}, \bmod R, \qquad n > m \tag{6.35}$$

$$\mathbb{1}_{Y_n} T_n \mathbb{1}_{Y_m} = \mathbb{1}_{Y_m} \frac{T_m \mathbb{1}_{Y_m}}{T_m \mathbb{1}_{Y_{m,n}}}, \bmod R, \qquad n > m. \tag{6.36}$$

For $A, B \in \mathscr{F} \otimes \mathscr{X}$ we write $A \subseteq B \bmod R$ if $\mathbb{1}_A \leqq \mathbb{1}_B \bmod R$ and we write $A = B \bmod R$ if $A \subseteq B \bmod R$ and $B \subseteq A \bmod R$. With the aid of (6.35) and (6.36) it can be shown in a few lines that

$$A_* = \bigcup_m \bigcap_{n > m} \{T_n \mathbb{1}_{Y_n} > 0\} \bmod R, \tag{6.37}$$

$$A_* = \{\lim_{m \to \infty} \lim_{n \to \infty} T_n \mathbb{1}_{Y_m} = 1\} \bmod R. \tag{6.38}$$

Step 4 We denote the class of all null-arrays of nested $\mathscr{S}$-measurable partitions of $\mathbf{X}$ by $\mathscr{Z}$ and want to show that the results of the preceding steps are essentially independent of $\mathscr{B} \in \mathscr{Z}$, which was fixed in Step 2 and Step 3. To denote the dependence on $\mathscr{B}$ we write $T_n^{\mathscr{B}}$, $T^{\mathscr{B}}$, $R_*^{\mathscr{B}}$, $A_*^{\mathscr{B}}$ ... instead of T_n, T, R_*, A_* ... For $\mathscr{B} = (\{B_{n,i}: i \geqq 1\})_{n \geqq 1} \in \mathscr{Z}$ and $\mathscr{C} = (\{C_{n,i}: i \geqq 1\})_{n \geqq 1} \in \mathscr{Z}$ we denote by $\mathscr{B} \wedge \mathscr{C}$ the unique element $\mathscr{D} = (\{D_{n,i}: i \geqq 1\})_{n \geqq 1} \in \mathscr{Z}$ with

$\{D_{n,i}: i \geqq 1\} = \{B_{n,i} \cap C_{n,j}: i, j \geqq 1\}$. Then it is obvious from the definitions that

$$(6.39) \qquad T_m^{\mathscr{B}} T_n^{\mathscr{D}} f = T_m^{\mathscr{B}} f \mod R, \quad n \geqq m.$$

Lemma 6.9 *Let $\mathscr{B}, \mathscr{C} \in \mathscr{X}$. Then*

$$(6.40) \qquad A_*^{\mathscr{B}} = A_*^{\mathscr{C}} \mod R,$$

$$(6.41) \qquad R_*^{\mathscr{B}} = R_*^{\mathscr{C}}.$$

Proof. Let $\mathscr{D} = (\{D_{n,i}: i \geqq 1\})_{n \geqq 1} := \mathscr{B} \wedge \mathscr{C}$. Similarly as in Lemma 6.8 we may derive from (6.33) and (6.34)

$$(6.42) \qquad f_{m,n} T_n^{\mathscr{D}} \mathbb{1}_{Y_n} = f_{m,n} \frac{T_m \mathbb{1}_{Y_m^{\mathscr{B}}}}{T_m f_{m,n}} \mod R, \quad n > m,$$

and

$$(6.43) \qquad f_{m,n} \leqq \mathbb{1}\{T_m^{\mathscr{B}} f_{m,n} > 0\} \mod R,$$

where

$$f_{m,n} := \mathbb{1}\{(\omega, x): \xi_{D_n(x)}(\omega) = \xi_{B_m(x)}(\omega)\}, \quad n \geqq m.$$

With the help of (6.37) we can now derive $A_*^{\mathscr{B}} = A_*^{\mathscr{D}} \mod R$. Then we have also $A_*^{\mathscr{C}} = A_*^{\mathscr{D}} \mod R$ and (6.40) follows. The relation (6.41) follows from (6.40) and Theorem 6.7. □

We may now write R_* instead of $R_*^{\mathscr{B}}$, $\mathscr{B} \in \mathscr{X}$.

Theorem 6.10 *Let $\mathscr{B}, \mathscr{C} \in \mathscr{X}$ and $f \in F$. Then*

$$(6.44) \qquad T^{\mathscr{B}} f = T^{\mathscr{C}} f \mod R_*.$$

Proof. Again we define $\mathscr{D}$ as in the proof of Lemma 6.9. Let $f, g \in F$ and suppose g to be predictable. Since $T^{\mathscr{D}}$ is a R_*-projection according to Theorem 6.3 and (6.41), we conclude from $T^{\mathscr{D}} g = g \mod R_*$

$$(6.45) \qquad g\, T^{\mathscr{D}} f = T^{\mathscr{D}} g f \mod R.$$

Let $\mu \in R$ with $C_\mu(\Omega \times \mathbf{X}) = 1$ and take $m \in \mathbb{N}$. Since by definition $\mu T_m^{\mathscr{B}} \in R_* = R_*^{\mathscr{B}}$, we obtain from (6.39) and (6.45)

$$\langle \mu T_m^{\mathscr{B}}, g\, T^{\mathscr{D}} f \rangle = \langle \mu, T_m^{\mathscr{B}} T^{\mathscr{D}} g f \rangle = \lim_{n \to \infty} \langle \mu, T_m^{\mathscr{B}} T_n^{\mathscr{D}} g f \rangle$$
$$= \langle \mu, T_m^{\mathscr{B}} g f \rangle = \langle \mu T_m^{\mathscr{B}}, g f \rangle.$$

Therefore we have

$$E_{C_\mu T_m^{\mathscr{B}}}[f \mid \mathscr{P}] = T^{\mathscr{D}} f \ \ C_\mu T_m^{\mathscr{B}}\text{-a.s.}$$

On the other hand, by (6.27)

$$E_{C_\mu T_m^{\mathscr{B}}}[f\,|\,\mathscr{P}] = T^{\mathscr{B}} f \quad C_\mu T_m^{\mathscr{B}}\text{-a.s.}$$

and, consequently,

$$T^{\mathscr{B}} f = T^{\mathscr{D}} f \quad C_\mu T_m^{\mathscr{B}}\text{-a.s.}$$

Now (6.23) finally implies

$$T^{\mathscr{B}} f = T^{\mathscr{D}} f \bmod R_* .$$

By symmetry this remains true with $\mathscr{B}$ replaced by $\mathscr{C}$ and (6.44) follows. □

In order to summarize our results in Theorem 6.12 below we take some $\mathscr{C} = (\{C_{n,i}: i \geqq 1\})_{n \geqq 1} \in \mathscr{Z}$ and define for $f \in F$

$$f^* := T^{\mathscr{C}} f. \tag{6.46}$$

We restate the definition of R_* in the equivalent form

$$R^* = \{\mu \in R: \lim_{n\to\infty} T_n^{\mathscr{C}} \mathbb{1}_{Y_n} = 1 \; C_\mu\text{-a.e.}\}$$

and note that Theorem 6.7 implies:

Lemma 6.11 (i) *Let* $A \in \mathscr{F} \otimes \mathscr{X}$. *Then* $A \in \mathscr{M}_*$, (*i.e.* $A = \emptyset \bmod R_*$) *iff* $A \subseteq A_*^c \bmod R$.
(ii) *We have* $f = g \bmod R_*$ *for* $f, g \in F$ *iff* $\mathbb{1}_{A_*} f = \mathbb{1}_{A_*} g \bmod R$.

Theorem 6.12 (i) *The mapping* $f \mapsto f^*$ *is an* R_**-projection.*
(ii) $f = f^* \bmod R^*$ *iff there is a predictable* $g \in F$ *with* $f = g \bmod R_*$, *i.e.* $\mathbb{1}_{A_*} f = \mathbb{1}_{A_*} g \bmod R$.
(iii) *For any* $\mathscr{B} = (\{B_{n,i} := i = 1, 2, \ldots\})_{n \geqq 1} \in \mathscr{Z}$

$$f^*(\omega, x) = \lim_{n\to\infty} \sum_i \mathbb{1}\{x \in B_{n,i}\} \, E[f(x)\,|\,\mathscr{F}(B_{n,i})](\omega) \; R_*\text{-a.e.}(\omega, x).$$

Step 6 All the properties in Theorem 3.4 are implied by Theorem 6.12 and the following observation.

Lemma 6.13 *Assume* $\Sigma(\Gamma)$. *Then* $A_* = \Omega \times \mathbf{X} \bmod R$ *and, consequently,* $R = R_*$.

Proof. We define

$$Y(B) := \bigcap_{x \in B} \{\xi_x = \xi_B\} = \{\Phi(\bigcup_{x \in B} \Gamma_x \setminus \Gamma_B) = 0\}, \qquad B \in \mathscr{S},$$

$$Y := \bigcap_{i,n} \{P(Y(B_{n,i})\,|\,\mathscr{F}(B_{n,i})) > 0\},$$

where $\mathscr{B}=(\{B_{n,i}: i=1,2,\ldots\})_{n\geqq 1}\in\mathscr{Z}$. By $\Sigma(\Gamma)$ we have $P(Y)=1$. Furthermore

$$T_n^{\mathscr{B}}\mathbb{1}_{Y_n^{\mathscr{B}}}(x)=\sum_i \mathbb{1}\{x\in B_{n,i}\}\, P(\xi_x=\xi_{B_{n,i}}|\mathscr{F}(B_{n,i}))$$
$$\geqq \sum_i \mathbb{1}\{x\in B_{n,i}\}\, P(Y(B_{n,i})|\mathscr{F}(B_{n,i})$$

and therefore

$$Y\times \mathbf{X}\subseteq\{T_n^{\mathscr{B}}\mathbb{1}_{Y_n^{\mathscr{B}}}>0\}.$$

Since $Y\times\mathbf{X}=\Omega\times\mathbf{X} \bmod R$ we obtain the first assertion from (6.37). Theorem 6.7 yields the second assertion of the lemma. □

Proof of Example 3.2 We do the case (2.6), (2.7). The other case can be done with obvious changes.

Let $B\in\mathscr{S}$ and $Y\in\mathscr{F}(B)$ with $P(Y)>0$. We have to show that $P(Y\cap\{\Phi(\Gamma_B^*)=0\})>0$. Since $P(\Psi(\Gamma_B^*)<\infty)=1$, we find an integer k with $P(Y\cap\{\Psi(\Gamma_B^*)=k\})>0$. Using the fact that $\Gamma_B^*\Phi$ and $\Gamma_B\Phi$ are conditionally independent given Ψ and the definition of a p-thinning we obtain

$$\begin{aligned}P(Y\cap\{\Phi(\Gamma_B^*)=0\})&\geqq E\,\mathbb{1}\{\Psi(\Gamma_B^*)=k\}\; P(Y\cap\{\Phi(\Gamma_B^*)=0\}\,|\,\Psi)\\ &=E\,\mathbb{1}\{\Psi(\Gamma_B^*)=k\}\; P(Y|\Psi)\, P(\Psi(\Gamma_B^*)=0\,|\,\Psi)\\ &=E\,\mathbb{1}\{\Psi(\Gamma_B^*)=k\}\; P(Y|\Psi)(1-p)^k\\ &=(1-p)^k\, P(Y\cap\{\Psi(\Gamma_B^*)=k\})>0. \quad \square\end{aligned}$$

7 Proof of the Theorems 3.3 and 4.1

Let $\mathscr{B}=(\{B_{n,i}: i\geqq 1\})_{n\geqq 1}\in\mathscr{Z}$ and recall by a look at (6.46) that f^*, $f\in F$, was defined with the aid of a fixed $\mathscr{C}=(\{C_{n,i}: i=1,2,\ldots\})_{n\geqq 1}\in\mathscr{Z}$. We use the notations of the Steps 2 and 3 of Sect. 6, where $\mathscr{B}$ was fixed.

Lemma 7.1 *Let $F^1(\mathscr{P})$ be the set of all predictable $f\in F$ with values in* [0, 1]. *Let $\mu\in R_*$ and assume $C_\mu(\Omega\times\mathbf{X})=E_\mu(\mathbf{X})<\infty$. Then*

$$\langle\mu, T_n f\rangle\to\langle\mu, f\rangle \quad \textit{as} \quad n\to\infty, \textit{ uniformly in } f\in F^1(\mathscr{P}).$$

Proof. For $f\in F^1(\mathscr{P})$ and $m\in\mathbb{N}$ we have

$$\begin{aligned}|\mathbb{1}_{Y_m}T_m\mathbb{1}_{Y_m}f-T_m f|&\leqq|\mathbb{1}_{Y_m}T_m\mathbb{1}_{Y_m}f-\mathbb{1}_{Y_m}T_m f|\\ &\quad+\mathbb{1}\,Y_m^c\,T_m f\leqq|\mathbb{1}_{Y_m}T_m\mathbb{1}_{Y_m}f-\mathbb{1}_{Y_m}T_m\mathbb{1}_{Y_m}f|\\ &\quad+\mathbb{1}_{Y_m}T_m\mathbb{1}_{Y_m^c}f+\mathbb{1}_{Y_m^c}T_m f\leqq T_m\mathbb{1}_{Y_m^c}+\mathbb{1}_{Y_m^c}\end{aligned}$$

and, analogously,

$$|\mathbb{1}_{Y_m}f\,T_m\mathbb{1}_{Y_m}-f|\leqq T_m\mathbb{1}_{Y_m^c}+\mathbb{1}_{Y_m^c}.$$

Hence (6.29) yields

$$\begin{aligned}\langle\mu, |f-T_m f|\rangle &\leqq \langle\mu, |f-\mathbb{1}_{Y_m} f\, T_m \mathbb{1}_{Y_m}|\rangle \\ &\quad+\langle\mu, |\mathbb{1}_{Y_m} f\, T_m \mathbb{1}_{Y_m}-\mathbb{1}_{Y_m} T_m f\, \mathbb{1}_{Y_m}|\rangle \\ &\quad+\langle\mu, |\mathbb{1}_{Y_m} T_m f\, \mathbb{1}_{Y_m}-T_m f|\rangle \\ &\leqq 2\langle\mu, \mathbb{1}_{Y_m^c}+T_m \mathbb{1}_{Y_m^c}\rangle .\end{aligned}$$

The latter term tends to zero in view of (6.30), Theorem 6.7 and (6.31). □

Theorem 7.2 *Let $\mu\in R_*$. Then there exists a P-almost surely unique random measure $\mu^*\in R_*$ which satisfies*

(7.1) $$\langle\mu^*, f\rangle=\langle\mu, f^*\rangle$$

as well as the convergence (4.1).

Proof. By $\alpha(A):=\langle\mu, (\mathbb{1}_A)^*\rangle$ we define a measure α on $\Omega\times\mathbf{X}$. Since $\Omega\times B$ is predictable for $B\in\mathscr{X}$, we have in particular

$$\alpha(\Omega\times B)=\langle\mu, \mathbb{1}_{\Omega\times B}\rangle=E\,\mu(B),$$

which is finite for bounded B. We can now conclude the existence and uniqueness of μ^* from $\alpha(\cdot\times B)\ll P$ and a well-known result on disintegration (cf. e.g. Theorem 16.3.3 in Kallenberg, 1983).

The relation $\mu^*\in R_*$ follows easily from the property (6.5) of a projection in connection with Theorem 6.12 and the definition (6.19) of R_*. To prove the second assertion of the theorem we choose a bounded set $A\in\mathscr{X}$ and may assume without loss of generality that $\mu(\cdot)=\mu(\cdot\cap A)$ and, consequently, $C_\mu(\Omega\times\mathbf{X})=E\,\mu(A)<\infty$. We want to apply Lemma 7.1 and note that $\{T\mathbb{1}_{Y\times B}: Y\in\mathscr{F}, B\in\mathscr{X}\}\subseteq F^1(\mathscr{P})$. Taking into account (6.10), $T^{\mathscr{B}}f=f^* \bmod R_*$ for $f\in F$ (cf. Theorem 6.10) and (6.22) we obtain

$$\begin{aligned}E\,\mathbb{1}_Y\,\mu\, T_n(B) &=\langle\mu\, T_n, \mathbb{1}_{Y\times B}\rangle=\langle\mu, T_n \mathbb{1}_{Y\times B}\rangle \\ &=\langle\mu, T_n\, T\mathbb{1}_{Y\times B}\rangle \xrightarrow[n\to\infty]{} \langle\mu, T\mathbb{1}_{Y\times B}\rangle \\ &=\langle\mu, (\mathbb{1}_{Y\times B})^*\rangle=\langle\mu^*, \mathbb{1}_{Y\times B}\rangle=E\,\mathbb{1}_Y\,\mu^*(B),\end{aligned}$$

where the convergence is uniformly in $Y\in\mathscr{F}$ and $B\in\mathscr{X}$. This easily implies

$$\mu\, T_n(B) \xrightarrow[n\to\infty]{L^1(P)} \mu^*(B)$$

uniformly in $B\in\mathscr{X}$ and (4.3) follows from the representation (6.9). □

We call $\mu\in R_*$ *predictable* if $\mu=\mu^*$ P-almost surely. This is in accordance with the definition in Sect. 2 as shown by the next theorem.

Theorem 7.3 *Let $\mu\in R_*$. Then μ is predictable iff it satisfies* (2.13).

Proof. Let us assume (2.13) and choose $Y \in \mathscr{F}$ and $B \in \{B_{n,i}: n, i \geqq 1\}$. Using (7.1) and Theorem 6.12(iii) we obtain by dominated convergence

$$\begin{aligned} C_{\mu^*}(Y \times B) &= \langle \mu, (\mathbb{1}_{Y \times B})^* \rangle \\ &= \langle \mu, \lim_{n \to \infty} \sum_{i: B_{n,i} \subseteq B} P(Y \mid \mathscr{F}(B_{n,i})) \mathbb{1}_{B_{n,i}} \rangle \\ &= \lim_{n \to \infty} \sum_{i: B_{n,i} \subseteq B} EP(Y \mid \mathscr{F}(B_{n,i})) \, \mu(B_{n,i}) \\ &= \lim_{n \to \infty} \sum_{i: B_{n,i} \subseteq B} E \mathbb{1}_Y \, E[\mu(B_{n,i}) \mid \mathscr{F}(B_{n,i})] \\ &= E \mathbb{1}_Y \, \mu(B) = C_\mu(Y \times B). \end{aligned}$$

The monotone class theorem implies $C_\mu = C_{\mu^*}$ and hence $\mu = \mu^*$ P-almost surely. The other part of the theorem follows from Theorem 7.2. □

In view of Lemma 6.13 the Theorems 3.3 and 4.1 are special cases of the next more general result:

Theorem 7.4 (i) *Let $f \in F$. Then $f^* \in F$ is the* mod R_* *unique predictable element of F satisfying $\langle \mu, f^* \rangle = \langle \mu, f \rangle$ for all predictable $\mu \in R_*$.*
(ii) *Let $\mu \in R_*$. Then μ^* is the P-almost surely unique predictable element of R_* satisfying $\langle \mu^*, f \rangle = \langle \mu, f \rangle$ for all predictable $f \in F$.*

Proof. (i) Let $f \in F$. By (7.1) we have $\langle \mu, f^* \rangle = \langle \mu, f \rangle$ for all predictable $\mu \in R_*$. Assume that an other predictable $g \in F$ also satisfies $\langle \mu, f \rangle = \langle \mu, g \rangle$ for all predictable $\mu \in R_*$. Then we obtain for an arbitrary $\nu \in R_*$ by Theorem 7.2 and Theorem 6.12(ii)

$$\langle \nu, g \rangle = \langle \nu^*, g \rangle = \langle \nu^*, f \rangle = \langle \nu, f^* \rangle.$$

This yields $g = f^*$ mod R_* as desired.
(ii) Let $\mu \in R^*$. Then $\langle \mu, f \rangle = \langle \mu^*, f \rangle$ for all predictable $f \in F$ as noted above. In particular we have $E \mathbb{1}_H \, \mu(B) = E \mathbb{1}_H \, \mu^*(B)$ for $B \in \mathscr{S}$ and $H \in \mathscr{F}(B)$ and hence (see also Remark 4.2)

$$E[\mu(B) \mid \mathscr{F}(B)] = E[\mu^*(B) \mid \mathscr{F}(B)] \ P\text{-a.s.}$$

Therefore Theorem 7.2 entails the P-almost sure equality $(\mu^*)^* = \mu^*$, i.e., the predictability of μ^*.

Let ν be another predictable element of R_* which also satisfies $\langle \mu, f \rangle = \langle \nu, f \rangle$ for all predictable f. Then we obtain for an arbitrary $g \in F$:

$$\langle \nu, f \rangle = \langle \nu^*, f \rangle = \langle \nu, f^* \rangle = \langle \mu, f^* \rangle = \langle \mu^*, f \rangle.$$

Hence $C_\nu = C_{\mu^*}$ and $\nu = \mu^*$ P-a.s. follows. □

Remark 7.5 Consider the assumptions and notations of Theorem 3.5. It can be seen from the proof of that theorem that

$$A_* = \{\sum_k \mathbb{1}\{x \in B_k\}\, q^{!}_{(\Gamma_x \setminus \Gamma_{B_k})\Phi}(\varphi(\Gamma_x \setminus \Gamma_{B_k}) = 0 \,|\, \Gamma_{B_k} \Phi) > 0\} \bmod R.$$

The next simple example shows that $A_* = \Omega \times \mathbf{X} \bmod R_*$ is not satisfied in general. Hence, one has to look for alternative methods in order to define the notion of a predictable projection also outside of A_*.

Example 7.6 Consider the situation of Example 2.2. Let X be a random element of $\mathbf{X}$ and assume that $\Phi(B) = \mathbb{1}\{X \in B\}$, $B \in \mathscr{X}$. Then

$$\{(\omega, x)\colon X(\omega) = x,\ P(X = x) = 0\} \subseteq A_*^c \bmod R.$$

Proof. The event $\{\Phi(B^c) = 0\} = \{X \in B\}$ is an $\mathscr{F}(B)$-atom for all $B \in \mathscr{S}$. Therefore we may assume

$$P(\cdot\,|\,\mathscr{F}(B)) = \frac{P(\{X \in B\} \cap \cdot)}{P(X \in B)} \quad \text{on } \{X \in B\}.$$

It is easy to verify that

$$P(\xi_x = \xi_{B_n(x)} \,|\, \mathscr{F}(B_n(x)))(\omega) = P(X \in (B_n(x) \setminus \{x\}) \,|\, \mathscr{F}(B_n(x)))(\omega)$$

tends to 1 as $n \to \infty$ if $X(\omega) = x$ and $P(X = x) = 0$. This means $(\omega, x) \in A_*^c$. □

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References

Al Hussaini, A., Elliott, R.J.: Martingales, potentials and exponentials associated with a two parameter jump process. Stochastics **6**, 23–42 (1981)

Al Hussaini, A., Elliott, R.J.: Filtrations for the two-parameter jump process. J. Multivariate Anal. **16**, 118–139 (1985)

Buckdahn, R.: Projektionen von Zufallsfeldern über Punktprozessen. Dissertation A. Friedrich-Schiller-Universität Jena, Sektion Mathematik, 1984

Cairoli, R.: Décomposition de processus à indices doubles. (Lect. Notes Math., vol. 124, pp. 1–27) Berlin Heidelberg New York: Springer, 1971

Cairoli R., Walsh, J.B.: Stochastic integrals in the plane. Acta Math. **134**, 111–183 (1975)

Dellacherie, C., Meyer, P.A.: Probabilities and Potential. North-Holland, Amsterdam, 1978

Dellacherie, C., Meyer, P.A.: Probabilities and Potential B. North-Holland, Amsterdam, 1982

Doléans, C.: Existence du processus croissant naturel associé à un potentiel de la classe (D). Z. Wahrscheinlichkeitstheor. Verw. Geb. **9**, 309–314 (1968)

Doléans-Dade, C., Meyer, P.A.: Un petit théorème de projection pour processus à deux indices. Sem. de Probab. XIII. (Lect. Notes Math., vol. 721, pp. 204–215) Berlin Heidelberg New York: Springer, 1979

Dozzi, M.: On the decomposition and integration of two-parameter stochastic processes. (Lect. Notes Math., vol. 863, pp. 162–171) Berlin Heidelberg New York: Springer, 1981
Dynkin, E.B.: Sufficient statistics and extreme points. Ann. Probab. **6**, 705–730 (1978)
Hoeven, P.C.T. van der: Une projection de processus ponctuels. Z. Wahrscheinlichkeitstheor. Verw. Geb. **61**, 483–499 (1982)
Hoeven, P.C.T. van der: On point processes. Math. Centre Tracts 165. Amsterdam: Mathematisch Centrum, 1983
Ivanoff, B.G., Merzbach, E.: Characterization of compensators for point processes on the plane. Stochastics Stochastic Rep. **29**, 395–405 (1990)
Kallenberg, O.: On conditional intensities of point processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **41**, 205–220 (1978)
Kallenberg, O.: Random measures. 3rd ed. Berlin: Akademie, 1983; London: Academic Press, 1983
Kerstan, J., Wakolbinger, A.: Ergodic decomposition of probability laws. Z. Wahrscheinlichkeitstheor. Verw. Geb. **56**, 399–414 (1981)
Last, G.: Some remarks on conditional distributions for point processes. Stochastic Processes Appl. **34**, 121–135 (1990)
Mazziotto, G., Merzbach, E.: Point processes indexed by directed sets. Stochastic Processes Appl. **30**, 105–119 (1988)
Merzbach, E., Nualart, D.: A martingale approach to point processes in the plane. Ann. Probab. **16**, 265–274 (1988)
Merzbach, E., Zakai, M.: Predictable and dual predictable projections of two-parameter stochastic processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **53**, 263–269 (1980)
Papangelou, F.: The conditional intensity of general point processes and an application to line processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **28**, 207–226 (1974)
Rao, K.M.: On decomposition theorems of Meyer. Math. Scand. **24**, 66–78 (1969)