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Probability Theory and Related Fields

An extension of a result of Burdzy and Lawler

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Summary. It is shown that for all mean zero, finite variance random walks, the critical non-intersection exponents are equal to those for Brownian motion. The method uses the local time of intersection.

Brownian nonintersections and random walks

We consider two independent Brownian motions $\{X(t): t \geqq 0\}$ and $\{Y(t): t \geqq 0\}$ run until they leave a large disc (or sphere). We recover a Theorem of Burdzy and Lawler relating non-intersection probabilities for the Brownian motions killed on leaving a large sphere with the probabilities that the paths of two independent discrete random walks (again killed on leaving a large sphere) do not meet ever. This paper illustrates a different approach from that taken by Burdzy and Lawler (1989). It also is valid in slightly more generality in that the result is proved for all random walks of mean zero and finite second moments.

Brownian paths may only intersect in dimensions less than four, while the problem is well understood in one dimension, so we only treat the case of two or three dimensions. In writing up our proof we will explicitly assume that the dimension d is two since this case is simpler.

Before stating our main results we require some notation.

Notation

Throughout the paper

X or $\{X(t): t \geqq 0\}$ and Y or $\{Y(t): t \geqq 0\}$ will denote independent Brownian motions in R^d. V or $\{V(n): n \geqq 0\}$ and W or $\{W(n): n \geqq 0\}$ will denote independent random walks in Z^d. C_r will denote the set $\{z: |z| = 2^r\}$, D_r will denote the $\{z: |z| \leqq 2^r\}$. We will drop the subscript for $r=0$, so that for instance C_0 will be written as C.

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If F represents a set then $N \cdot F$ will represent the set $\{N \cdot z: z \in F\}$. So for instance $C_r = 2^r C$. X_r^e or $\{X_r^e(t): t \geqq 0\}$ and Y_r^e or $\{Y_r^e(t): t \geqq 0\}$ will denote independent excursions from C to C_r. For Brownian motion X, the Brownian excursion X^e or the random walk V alike $T(2^N)$ will denote the first hitting time of D_N^c. Likewise $S(2^N)$ will denote this hitting time for Y, Y^e or W. The quantity $P^{z_1, z_2}(\cdot)$ will denote the probability of an event for Brownian motions X and Y starting at z_1 and z_2 respectively. Similarly $Q^{z_1, z_2}(\cdot)$ refers to probabilities pertaining to the random walks V and W and the quantity $P_r^{e, z_1, z_2}(\cdot)$ will denote the probability of an event for Brownian excursions X_r^e and Y_r^e starting at z_1 and z_2 respectively.

The quantity $p_r(z_1, z_2)$ is equal to

$$P^{z_1, z_2}[X(0, T(2^r)) \cap Y(0, S(2^r)) \text{ is empty}].$$

The quantity $q_r(z_1, z_2)$ is analogously defined for the random walks.

F_n will refer to the σ-field generated by $X(0, T(2^n))$ and $Y(0, S(2^n))$. F_n' will refer to the analogous quantity for random walks.

Our major theorem is

Theorem 1 *For z_1 and $z_2 \in R^d$ and x_1 and x_2 in Z^d and a random walk with mean zero and finite second moments*

$$\log_{r \to \infty} \frac{\log(p_r(z_1, z_2))}{r} = \log_{r \to \infty} \frac{\log(q_r(x_1, x_2))}{r} = -k.$$

This result is due to Burdzy and Lawler (1989) when the random walk is simple.

Here and throughout logarithms will be taken to the base 2.

For notational simplicity we will throughout the paper assume that the random walk differences have a covariance matrix equal to the identity.

The strategy of the proof is first to establish the existence of the limit for Brownian motion (Sect. 1). This is a rather easy consequence of scaling and subadditivity. Then (in Sect. 2) the upper bound for the limit for the normalized random walk (with bounded differences) probability is established by using the convergence of the intersection local time for random walks to the like quantity for Brownian motion, then using the fact that Brownian paths meeting corresponds to positive intersection local time. The lower bound for the random walk hitting probabilities is established (in Sect. 3) first by showing the Brownian motion limit is the same if independent excursions are used instead of Brownian paths. The Donsker invariance principle is then used to finish the proof of Theorem 1 for random walks with bounded differences. In Sect. 4 the result is extended to its full generality via dynamic programming arguments.

1. The Brownian case

In this section we prove

Theorem 2 (Burdzy and Lawler) *There exists a constant k in $(0, \infty)$ so that for each z_1 and z_2 on C*

$$\lim_{r \to \infty} \frac{\log(p_r(z_1, z_2))}{r} = -k.$$

Proof. Define the quantity p_r to be the supremum over $z_i \in C$ of $p_r(z_1, z_2)$.

It can be proved that the latter quantity is continuous and that p_r is therefore attained but we will not require this fact.

By considering nested circles it is easily seen that the Theorem will be proved if we can establish it for the sequence p_r. By scaling

$$P[X(T(2^r), T(2^{r+m})) \cap Y(S(2^r), S(2^{r+m})) \quad \text{is empty } |F_r] \leqq p_m.$$

It follows that

$$\log(p_r) + \log(p_m) \geqq \log(p_{r+m}).$$

Therefore by subadditivity the limit $\log(p_r)/r$ must exist. This limit is clearly $\leqq \log(p_1) < 0$. Conversely $p_r \geqq p_r(e_1, -e_1) \geqq P[X^1$ hits 2^r before $0]\, P[Y^1$ hits -2^r before $0] \geqq 2^{-2r}$. So the limit in question must be between zero and -2.

For more information on the bounds for k see Burdzy et al. 1989). □

2. An inequality for random walks

Throughout this section V and W will be independent random walks on Z^d whose differences are of zero mean and bounded range. We will assume that the range is bounded by K. The most important result of this section is

Proposition 2.1 *The quantity $q_n(z_1, z_2)$ satisfies*

$$\limsup_{n\to\infty} \frac{\log(q_n(z_1, z_2))}{n} \leqq -k$$

where k is the constant defined in Theorem 2.

We prove this Proposition by a succession of simple lemmas and propositions. We first further examine the Brownian case. The key tool in this section is the idea of shadows to extend the use of the invariance principle. The approach is detailed in Dynkin (1988) and used in LeGall (1987). In the first paper the idea is credited to Jay Rosen.

First it can easily be seen that for α the intersection local time

$$\{\alpha(t, s) > 0\} \;\Delta\; \{\text{there exists } u < t \;\&\; v < s \text{ s.t. } X(u) = Y(v)\}$$

is a set of Wiener measure zero. The details are worked out as an example in Port and Mountford (1991). (Here Δ denotes the symmetric difference between two sets). We shall make use of the fact established in LeGall (1987) that all moments of α exist.

Secondly, for ε arbitrarily small we can find r so large that $\log(p_r)/r$ is less than $-(k-\varepsilon)$.

Putting these two facts together we obtain

Lemma 2.1 *We can find r, n and $c > 0$, so that*

$$P[\alpha(T(2^r) \wedge n, S(2^r) \wedge n) < c] < 2^{-(k-\varepsilon)r}.$$

Now we fix a sequence ε_m tending to zero so that n/ε_1 and $\varepsilon_m/\varepsilon_{m+1}$ are all integers. Let us fix m_0 for now and define the stopping times

$$T(2^r)' = \min\{\{i\varepsilon_{m_0}: X(i\varepsilon_{m_0}) \text{ not in } D_r\}, n\}$$

and

$$S(2^r)' = \min\{\{i\varepsilon_{m_0}: Y(i\varepsilon_{m_0}) \text{ not in } D_r\}, n\}.$$

Observe that $T(2^r)' > T(2^r) \wedge n$ and that $S(2^r)' > S(2^r) \wedge n$ a.s. unless the common value is n.

It is clear that,

$$P[\alpha(T(2^r)', S(2^r)') < c] < 2^{-(k-\varepsilon)r}.$$

We define the σ-field $G_m = \sigma(X(i\varepsilon_m \wedge n), Y(i\varepsilon_m \wedge n), i=1, 2, \ldots)$. Because of our stipulation on the sequence ε_m we see that $\{G_m\}_{m=1}^{\infty}$ is a filtration. Consequently we readily see that

$$E[\alpha(T(2^r)', S(2^r)') \mid G_m] \xrightarrow{L^2} \alpha(T(2^r)', S(2^r)')$$

as m tends to infinity.

Define the time set

$$\Delta_{m,\delta} = \{(t, s): \text{for all } i: |t - i\varepsilon_m| \geqq \delta\varepsilon_m \text{ and } |s - i\varepsilon_m| \geqq \delta\varepsilon_m\} \cap [0, T(2^r)') \times [0, S(2^r)').$$

Again since all moments of α exist it is readily seen that as δ tends to zero

$$\alpha(\Delta_{m,\delta}) \xrightarrow{L^2} \alpha(T(2^r)', S(2^r)')$$

and it easily follows that by letting m tend to infinity and δ tend to zero we can find m, δ so that

$$P[E[\alpha(\Delta_{m,\delta}) \mid G_m] \leqq c/2] < 2^{-(k-\varepsilon)r} \tag{1}$$

and

$$E[(E[\alpha(\Delta_{m,\delta}) \mid G_m] - \alpha(\Delta_{m,\delta}))^2] \leqq 2^{-(k-\varepsilon)r} c^2/16. \tag{2}$$

The function $E[\alpha(\Delta_{m,\delta}) \mid G_m]$ is equal to

$$\sum_{i=0}^{T(2^r)'/\varepsilon_m - 1} \sum_{j=0}^{S(2^r)'/\varepsilon_m - 1} g(\varepsilon_m, \delta, X(i\varepsilon_m), X((i+1)\varepsilon_m), Y(j\varepsilon_m), Y((j+1)\varepsilon_m))$$

where

$$g(\varepsilon_m, \delta, x_1, x_2, y_1, y_2) = \int_{\varepsilon_m\delta}^{\varepsilon_m(1-\delta)} \int_{\varepsilon_m\delta}^{\varepsilon_m(1-\delta)} \int_{R^d} dz\, f(z, t, x_1, x_2) f(z, s, y_1, y_2)\, ds\, dt$$

and $f(z, t, x_1, x_2)$ is the conditional density of $X(t)$ given that $X(0) = x_1$ and $X(\varepsilon_m) = x_2$.

It is easy to see by dominated convergence that $g(\,)$ is a continuous function of its spatial arguments. It follows that $E[\alpha(\Delta_{m,\delta}) \mid G_m]$ is a continuous path function except for paths where $T(2^r) = T(2^r)' < n$ or $S(2^r) = S(2^r)' < n$. In a like manner it is seen that away from such paths $E[\alpha^2(\Delta_{m,\delta})\mid G_m]$ is a continuous path function. We record these results as

Lemma 2.2 *There exist versions of $E[\alpha^2(\Delta_{m,\delta})\mid G_m]$ which are continuous functions of the Brownian positions at times $i\varepsilon_m$ $(i=0, 1, 2, \ldots, n/(\varepsilon_m))$ except for paths where $X(i\varepsilon_m)$ or $Y(i\varepsilon_m)\in C_r$ for some $i\leqq n/\varepsilon_m$, a set of paths of measure zero.*

We now begin to consider our random walks V and W. Suppose that sequences z_i^N satisfy $z_i^N/N\to z_i\in C$ for $i=1, 2$. We will consider random walks V and W starting at z_1^N and z_2^N respectively; we aim to prove

Proposition 2.2 *Let the stopping times $T_{N,r}$ and $S_{N,r}$ be defined by*

$$T_{N,r}=\inf\{i\varepsilon_{m_0}N^2\colon |V([i\varepsilon_{m_0}N^2])|\geqq N2^r\}\wedge nN^2$$

and

$$S_{N,r}=\inf\{i\varepsilon_{m_0}N^2\colon |W([i\varepsilon_{m_0}N^2])|\geqq N2^r\}\wedge nN^2.$$

Then $Q^{z_1^N, z_2^N}[V(0, T_{N,r})\cap W(0, S_{N,r})$ is empty] is less than $2^{-(k-\varepsilon)r-1}$ for large N.

An obvious analogue of the intersection local time for discrete random walks is $\beta^N(A)=\dfrac{1}{N^2}\,|\{(t, s)\in A\colon V^N(t)=W^N(s)\}|$. Clearly $\beta^N((0, T_{N,r}]\times(0, S_{N,r})>0$ implies that $V^N(0, T_{N,r}]$ does intersect $W^N(0, S_{N,r}]$. (This is for two dimensions, in three the term N^2 is replaced by N.)

Define the subset of the integer lattice $\Delta_{m,\delta}^N$ by

$$\Delta_{m,\delta}^N=[0, T_{N,r}]\times[0, S_{N,r}]\cap\{(t, s)\colon \quad \text{for each } i\colon |t-N^2 i\varepsilon_m|\geqq\delta\varepsilon_m N^2 \text{ and } |s-N^2 i\varepsilon_m|\geqq\delta\varepsilon_m N^2\}.$$

Define the σ-field G_m^N by
$G_m^N=\sigma(V^N(i\varepsilon_m N^2), W^N(i\varepsilon_m N^2), i=1, 2, \ldots)$. In the same way as before we have the function $E[\beta^N(\Delta_{m,\delta}^N)\mid G_m^N]$ is equal to

$$\sum_{i=0}^{(T_{N,r}/\varepsilon_m N^2)-1}\ \sum_{j=0}^{(S_{N,r}/\varepsilon_m N^2)-1} g^N(\varepsilon_m, \delta, V^N(i\varepsilon_m N^2), V^N((i+1)\varepsilon_m N^2), W^N(j\varepsilon_m N^2), W^N((j+1)\varepsilon_m N^2))$$

where

$$g^N(\varepsilon_m, \delta, x_1, x_2, y_1, y_2)=\sum_{i=\varepsilon_m\delta N^2}^{\varepsilon_m(1-\delta)N^2}\ \sum_{j=\varepsilon_m\delta N^2}^{\varepsilon_m(1-\delta)N^2}\frac{1}{N^2}\sum_{Z^d} q(z, i, x_1, x_2)\, q(z, j, y_1, y_2)$$

and $q(z, i, x_1, x_2)$ is the conditional probability that $V^N(i)=z$ given that $V^N(0)=x_1$ and $V^N(\varepsilon_m N^2)=x_2$.

By the local C.L.T. (see e.g. Durrett 1989), it is easy to see that for x_i^N/N and y_i^N/N tending to x_i and y_i we have $g^N(\varepsilon_m, \delta, x_1^N, x_2^N, y_1^N, y_2^N)$ tends to $g(\varepsilon_m, \delta, x_1, x_2, y_1, y_2)$. Note that the question of periodicity is irrelevant here. From this and the invariance principle it follows that

$$E[\beta^N(\Delta_{m,\delta}^N)\mid G_m^N]\xrightarrow{D} E[\alpha(\Delta_{m,\delta})\mid G_m] \tag{4}$$

it follows similarly that

(5)
$$E[(\beta^N(\Delta^N_{m,\delta}))^2 \mid G^N_m] \xrightarrow{D} E[\alpha^2(\Delta_{m,\delta}) \mid G_m].$$

Now for fixed m all the quantities above are bounded. From this we see that

(6)
$$\limsup E[(E^{z_1^N, z_2^N}[\beta^N(\Delta^N_{m,\delta}) \mid G^N_m] - \beta^N(\Delta^N_{m,\delta}))^2] \leqq 2^{-(k-\varepsilon)r} c^2/16.$$

Proposition 2.3 *As N tends to infinity*

$$\limsup Q^{z_1^N, z_2^N}[\beta^N(\Delta^N_{m,\delta}) < c/4] \leqq 2^{-(k-\varepsilon)r}$$

Proof. The quantity $Q[\beta^N(\Delta^N_{m,\delta}) < c/4]$ is majorized by $Q[E[\beta^N(\Delta^N_{m,\delta}) \mid G^N] < c/2] + \frac{16}{c^2} E[(E[\beta^N(\Delta^N_{m,\delta}) \mid G^N_m] - \beta^N(\Delta^N_{m,\delta}))^2]$. Using (1), (4) and (6) the result follows.

Corollary 2.1 *For all N large enough* $Q^{z_1^N, z_2^N}[\beta^N((0, T(N2^r)] \times (0, S(N2^r)]) < c/4]$ *is majorized by* $4.2^{-(k-\varepsilon)r}$ *which in turn is majorized by* $2^{-(k-2\varepsilon)(r+1)}$ *for r large enough.*

Proof. This follows because

$$\begin{aligned} &Q[\beta^N((0, T(N2^r)+1 \wedge N] \times ((0, S(N2^r)+1 \wedge N])) < c/4] \\ &\quad < Q[T_{N,r} > T(N2^{r+1})] + Q[S_{N,r} > S(N2^{r+1})] + Q[\beta^N(\Delta^N_{m,\delta}) < c/4]. \end{aligned}$$

The result now follows by letting m_0 which defines $T_{r,N}$ and $S_{r,N}$, go to infinity.

Proof of Proposition 2.1 Corollary 2 shows that there exists an n_0 such that for each $n \geqq n_0$ and x_i with $|x_i| \leqq 2^n + K$

$$\begin{aligned} &Q^{x_1, x_2}[\text{there does not exist } i < T(2^{n+r+1}),\ j < S(2^{n+r+1}) \text{ s.t. } V(i) = W(j)] \\ &\quad < 2^{-(r+1)(k-2\varepsilon)}. \end{aligned}$$

Now let m equal $n_0 + s(r+1) + v$ where $0 \leqq v \leqq r$. Then for fixed x_i in D_{n_0} we have

$$\begin{aligned} &Q^{x_1, x_2}[\text{there does not exist } i < T(2^m),\ j < S(2^m)\colon V(i) = W(j)] \\ &Q^{x_1, x_2}\Bigg[\bigcap_{j=0}^{s-1} \{V((T(2^{n_0 + j(r+1)}), T(2^{n_0 + (j+1)(r+1)})]) \cap W \\ &((S(2^{n_0 + j(r+1)}), S(2^{n_0 + (j+1)(r+1)})]) \text{ is empty}\}\Bigg] \\ &\leqq 2^{-(r+1)s(k-3\varepsilon)} \leqq 2^{-(m-n_0-r)(k-3\varepsilon)}. \end{aligned}$$

Letting m tend to infinity and recalling that ε is arbitrarily small gives the result.

3.

In this section we attempt to complete the proof of Theorem 1 for the case of random walks with bounded increments by proving

Proposition 3.1 *The quantity $q_r(z_1, z_2)$ satisfies*

$$\limsup_{r\to\infty} \frac{\log(q_r(z_1, z_2))}{r} \geqq -k$$

where k is the constant defined in Theorem 2.

Consider the constant

$$p_r^e = \int_{(z_1, z_2)\in(C)^2} \sigma(\mathrm{d}z_1)\,\sigma(\mathrm{d}z_2)\, p_r^{e, z_1, z_2}$$

where $\sigma(\cdot)$ is normalized surface measure and p_r^{e,z_1,z_2} is the probability that two independent excursions from C to C_r, killed at C_r and beginning at z_1 and z_2 respectively, do not meet.

Lemma 3.1 *As r tends to infinity* $\frac{\log(p_r^e)}{r}$ *tends to k.*

Proof. The last exit distribution from C_1 of Brownian motion killed at C_r has a bounded density with respect to normalized surface measure. Thus if L_2 denotes the last exit time from C_1, then

$$p_r^{x,y} \leqq \int_{C_1} p_{r-1}^{e, z_1/2, z_2/2}\, P^{x,y}[X_{L_2}\in \mathrm{d}z_1,\ Y_{L_2}\in \mathrm{d}z_2] \leqq F p_{r-1}^e$$

for some constant F. In consequence $\underline{\lim}\, \frac{\log p_r^e}{r} \geqq -k$.

On the other hand let $\{X^e(t)\colon 0 \leqq t \leqq T(2^r)\}$ and $\{Y^e(s)\colon 0 \leqq s \leqq S(2^r)\}$ be independent excursions starting from z_1 and z_2 respectively. Then

$$p_r^{e, z_1, z_2} \leqq P[X^e(T(2), T(2^r)) \cap Y^e(S(2), S(2^r)) \text{ is empty}] \leqq r^2 p_{r-1}$$

since $1/r$ is the probability that a Brownian motion starting on C_1 hits C_r before C. This establishes $\overline{\lim}\, \frac{\log p_r^e}{r} \leqq -k$ and hence the lemma. □

Consider a pair of independent excursions from C to C_r, $\{X_r^e(t)\colon 0 \leqq t \leqq T(2^r)\}$ and $\{Y_r^e(s)\colon 0 \leqq s \leqq S(2^r)\}$. Let the random times L_R be defined as $L_R = \sup\{t\colon |X^e(t)| = R\}$. We similarly define the random times L'_R from the Y-excursion. Let G_R be $\sigma\{X_r^e(t), Y_r^e(s)\colon t \leqq L_R,\ s \leqq L'_R\}$. The following facts are well known (see e.g. Williams 1974; Pitman and Yor 1982 or Burdzy (1987)).

1. The process $\{X^e(L_2 + t)\colon t \geqq 0\}$ is an excursion from C_1 to C_r. Similarly for the Y-excursion.
2. The random variables $X_r^e(L_2)$ and $Y_r^e(L'_2)$ have bounded joint density given $G_{1\frac{1}{4}}$.
3. The sigma fields $G_{1\frac{1}{4}}$ and $\sigma\{X_r^e(L_2 + t)\colon t \geqq 0,\ Y_r^e(L'_2 + s)\colon s \geqq 0\}$ are independent given $X_r^e(L_2)$ and $Y_r^e(L'_2)$.

4. The processes $\{X_r^e(t)\colon 0 \leqq t \leqq T(2^n)\}$ and $\{X_n^e(t)\colon 0 \leqq t \leqq T(2^n)\}$ are equal in distribution, $0 \leqq n \leqq r$.

Define $A_{\delta,r}^e$ to be the event that

The distance between $X_r^e[0, T^e(2^r))$ and $Y_r^e(0)\,(\mathrm{d}(X_r^e[0, T^e(2^r)], Y_r^e(0)))$ is less than 4δ

or

$$\mathrm{d}(Y_r^e[0, S^e(2^r)], X_r^e(0)) \text{ is less than } 4\delta$$

or

$$\mathrm{d}(X_r^e(0), Y_r^e(0)) \text{ is less than } 10\delta.$$

and define N_r^e to be the event

$$\{X_r^e[0, T^e(2^r)] \cap Y_r^e[0, S^e(2^r)] \text{ is empty}\}.$$

Define

$$D_{\delta,r} = \int P_r^{e,z_1,z_2}[(A_{\delta,r}^e)^c \cap N_r^e]\,\sigma(\mathrm{d}z_1)\,\sigma(\mathrm{d}z_2).$$

Lemma 3.2 *For δ sufficiently small* $\overline{\lim}\,\dfrac{\log(D_{\delta,r})}{r} \geqq -k.$

Proof. For $\delta < 1/4$ the event $A_{\delta,r}^e$ is $G_{1\frac{1}{4}}$ measurable. The excursion in question is Brownian motion conditioned not to return to the unit circle and killed upon hitting the circle of radius 2^r. That is the h-process with $h(x) = \log(|x|)$. Now $v(x) = (\log(|x|) - \log 2)/\log(|x|)$ is h-harmonic and tends to 1 as $|x|$ tends to infinity. Thus the conditioned Brownian motion is transient and it is easily seen to be the case that

$$\int P_r^{e,z_1,z_2}[A_{\delta,r}^e]\,\sigma(\mathrm{d}z_1)\,\sigma(\mathrm{d}z_2) \to 0 \quad \text{as } \delta \to 0 \quad \text{uniformly in } r.$$

Therefore we can find a δ in $(0, 1/4)$ such that for each r

$$P(A_{\delta,r}^e) = \int P_r^{e,z_1,z_2}[A_{\delta,r}^e]\,\sigma(\mathrm{d}z_1)\,\sigma(\mathrm{d}z_2) < \frac{1}{4\,2^k\,C}$$

where C is the upper bound for the conditional density of $(X_r^e(L_2), Y_r^e(L_2'))$ given $G_{1\frac{1}{4}}$. Now for this δ

$$D_{\delta,r} = p_r^e - \int P_r^{e,z_1,z_2}[A_{\delta,r}^e \cap N_r^e]\,\sigma(\mathrm{d}z_1)\,\sigma(\mathrm{d}z_2) \geqq p_r^e - P[A_{\delta,r}^e]\,C P_{r-1}^e.$$

The last quantity is greater than $\frac{1}{2}p_r^e$ whenever $\dfrac{p_e^r}{p_e^{r-1}} \geqq \dfrac{1}{2}\,\dfrac{1}{2^k}$. But by Lemma 3.1, this must happen infinitely often and the lemma follows.

Define $E_{r,e,\delta}$ to be the intersection of N_r^e, $(A_{\delta,r}^e)^c$ and the events:

The distances $\mathrm{d}(Y_r^e[0, S^e(2^r)], X_r^e(T^e(2^r)))$ and $\mathrm{d}(X_r^e[0, T^e(2^r)], Y_r^e(S^e(2^r)))$ are greater than $2^r\delta$.

Lemma 3.3 *For δ sufficiently small* $\overline{\lim}\,\dfrac{\log(P[E_{r,e,\delta}])}{r} \geqq -k.$

Proof. Consider the event

$$B^r_{n,\delta}=\{d(Y^e_r[0,S^e(2^n)],X^e_r(T^e(2^n)))<2^n\delta\}$$

or

$$\{d(X^e_r[0,T^e(2^n)],Y^e_r(S^e(2^n)))<2^n\delta\},\qquad 0\leqq n\leqq r.$$

Fact 4 mentioned after the proof of Lemma 3.1 ensures that $P[B^r_{n,\delta}]$ does not depend on r. We may choose δ so small that (uniformly on n),

$$P[B^r_{n,\delta}\,|\,\sigma\{X[0,T^e(2^{n-1})],Y[0,S^e(2^{n-1})]\}]<\frac{1}{4.2^{6k}}$$

and the conclusions of Lemma 3.2 are valid.

Fix ε strictly positive but otherwise arbitrarily small. By Lemma 3.2 we can find r arbitrarily large so that $D_{\delta,r}\geqq 2^{-(k+\varepsilon)r}$. We also assume that r is so large that $p_m\leqq 2^{-(k-\varepsilon)m}$ for all $m\geqq r/3$. Let $X^e_r[0,T^e(2^r))$ and $Y^e_r[0,S^e(2^r))$ be independent excursions from C to C_r with $X^e_r(0)$ and $Y^e_r(0)$ independently and uniformly distributed on C.

Define $A^{e,n}_{r,\delta}$ to be the event that

$$\text{The distance between } d(X^e_r[0,T^e(2^n)),Y^e_r(0)) \text{ is less than } 4\delta$$

or

$$d(Y^e_r[0,S^e(2^n)],X^e_r(0)) \text{ is less than } 4\delta$$

or

$$d(X^e_r(0),Y^e_r(0)) \text{ is less than } 10\delta.$$

or

$$\{X^e_r[0,T^e(2^n)]\cap Y^e_r[0,S^e(2^n)] \text{ is empty}\}.$$

Let Z_n be the martingale

$$E[N^e_r\cap(A^e_{\delta,r})^c\,|\,\sigma\{X[0,T^e(2^n)],Y[0,S^e(2^n)]\}]\qquad 0\leqq n\leqq r.$$

Given the definition of the events $A^e_{\delta,r}$ and N^e_r we must have

$$A_n=\{Z_n>0\}\subset A^{e,n}_{r,\delta}.$$

From the definition of conditional expectation

$$\|Z_n\|_1=P[(A^e_{\delta,r})^c\cap N^e_r]=D_{\delta,r}\geqq 2^{-(k+\varepsilon)r}\tag{1}$$

Also we have $Z_n\leqq P[X^e_r[T^e(2^n),T^e(2^r)]\cap Y^e_r[S^e(2^n),S^e(2^r)]$ is empty. The process $X^e_r(t)$, $T^e(2^n)\leqq t\leqq T^e(2^r)$ is simply a Brownian motion conditioned to hit C_r before C. Similarly for Y^e_r. These conditioning events have probability n/r. It follows from this and the above inequality that

$$\|Z_n\|_\infty<\left(\frac{n}{r}\right)^2 p_{r-n}.\tag{2}$$

Putting inequalities (1) and (2) together and using the elementary inequality $\|Z_n\|_1 \leqq P[Z_n>0]\,\|Z_n\|_\infty$, yields

$$\text{for } r/2 \leqq n \leqq 2r/3 \qquad P(Z_n>0) \geqq \frac{D_{\delta,r}}{9p_{n-r}} \geqq \frac{1}{9}\,2^{-(k+5\varepsilon)n}.$$

It also follows (assuming as we may that ε is sufficiently small) that there is an n_0 in $[r/3, 2r/3]$ such that $\frac{P(Z_n>0)}{P(Z_{n-1}>0)} \geqq \frac{1}{2^{6k}}$ since otherwise this would contradict the above lower bound for the probability of $D_{\delta, 2r/3}$. Using our choice of δ we find

$$\begin{aligned} P[\{Z_{n_0}>0\} \cap (B^r_{n_0,\delta})^c] &\geqq P[\{Z_{n_0}>0\}] - P[\{Z_{n_0}>0\} \cap (B^r_{n_0,\delta})] \\ &\geqq P[\{Z_{n_0}>0\}] - P[\{Z_{n_0-1}>0\} \cap (B^r_{n_0,\delta})]. \end{aligned}$$

By our choice of n_0 the last expression is greater than

$$P[Z_{n_0-1}]\,\frac{1}{2.2^{6k}} \geqq \frac{1}{18 2^{6k}}\,2^{-(k+5\varepsilon)n_0} \geqq 2^{-(k+6\varepsilon)n_0}$$

for r large enough. Thus (using Fact 4 again), we have shown that $\frac{\log(P[E_{n_0,e,\delta}])}{n_0} \geqq -(k+6\varepsilon)$. The result follows by the arbitrariness of ε.

We define the set $\operatorname{cup}(z,\delta)$ to be the set $\{w: |w| \leqq 1\} \cap \{w: |w-z| \leqq \delta\}$. The set $r\operatorname{cup}(z,\delta)$ will be the set of points $\{rw: w \in \operatorname{cup}(z,\delta)\}$.

Corollary 3.1 *For independent Brownian motions X and Y define E_{n,δ,z_1,z_2} to be the intersection of the events*

$$X[0, T(2^n)] \cap Y[0, S(2^n)] \text{ is empty.}$$
$$X[0, T(2^n)] \cap D \subset \operatorname{cup}(z_1, 2\delta)$$
$$Y[0, S(2^n)] \cap D \subset \operatorname{cup}(z_2, 2\delta)$$
$$\mathrm{d}(X[0, T(2^n)], Y(S(2^n))) \geqq 2^n\delta$$
$$\mathrm{d}(Y[0, S(2^n)], X(T(2^n))) \geqq 2^n\delta.$$

For each $\varepsilon>0$ and n_0, there exists $n>n_0$, δ and z_1, z_2 on C with $|z_1-z_2|>8\delta$ so that

$$P^{x,y}[E_{n,\delta,z_1,z_2}] \geqq 2^{-(k+5\varepsilon)n}$$

uniformly on $x \in \operatorname{cup}(z_1,\delta)$ and $y \in \operatorname{cup}(z_2,\delta)$.

Proof. Lemma 3.3 states that for δ small and fixed

$$\limsup_{r\to\infty} \frac{\log(P[E_{r,e,\delta}])}{r} \geqq -k.$$

It follows that there exists r arbitrarily large and points z_1 and z_2 on C and more than 8δ apart so that

$$\int_{v_1 \in C \cap \mathrm{cup}(z_1,\delta)} \int_{v_2 \in C \cap \mathrm{cup}(z_2,\delta)} \sigma(\mathrm{d}v_1)\,\sigma(\mathrm{d}v_2)\,P^{e,v_1,v_2}[E_{r,e,\delta}] > K\delta^2\,2^{-(k+\varepsilon)r}.$$

Consider unconditioned Brownian motion X starting at x in $\mathrm{cup}(z_1, \delta)$. Let $D(z_1, \delta)$ be the event that X begins an excursion, denoted X_r^e, from C to C_r before it leaves the ball centred at z_1 of radius 2δ. The following facts follow from Brownian excursion theory (see e.g. Burdzy 1987).

1. On $D(z_1, \delta)$, the density with respect to surface measure σ, of the initial point of the excursion, $X_r^e(0)$, is greater than some strictly positive constant c on $\mathrm{cup}(z_1, \delta)$.
2. Let L be the time of the start of the excursion to C_r. The excursion $X_r^e(t)$: $0 \leqq t \leqq T^e(2^r)$ is conditionally independent of $X[0, L]$, given $X_r^e(0)$.
3. The probability of event $D(z_1, \delta)$ is greater than $K(\delta)/r$ for some $K(\,)$.

Let $D(z_2, \delta)$ and Y_r^e be the analogous quantities for the Brownian motion Y. It follows easily that if $D(z_1, \delta) \cap D(z_2, \delta)$ occurs for X and Y, and $E_{r,e,\delta}$ occurs for X_r^e, Y_r^e then E_{r,δ,z_1,z_2} occurs for X and Y, and $E_{r,e,\delta}$ occurs for X_r^e, Y_r^e then E_{r,δ,z_1,z_2} occurs for X and Y.

Therefore from facts 1, 2 and 3 above it follows that

$$P[E_{n,\delta,z_1,z_2}] > P[D(z_1,\delta) \cap D(z_2,\delta)] \cdot c^2 \int_{v_1 \in C \cap \mathrm{cup}(z_1,\delta)} \int_{v_2 \in C \cap \mathrm{cup}(z_2,\delta)} \cdot\, \sigma(\mathrm{d}v_1)\,\sigma(\mathrm{d}v_2)\,P^{e,v_1,v_2}[E_{r,e,\delta}].$$

By our choice of z_1 and z_2 and fact 3 this latter expression is greater than $k(\delta)\,2^{-(k+5\varepsilon)r}$ which is greater than $2^{-(k+6\varepsilon)r}$ for r large enough. □

For the Brownian motion X (resp. Y) define the stopping time $T(2^n\,\mathrm{cup}(z, \delta))$ (resp. $S(2^n\,\mathrm{cup}(z, \delta))$) to be the first hitting time of $2^n\,\mathrm{cup}(z, \delta)$. These stopping times may be infinite in three dimensions but our interest is only in the event $\{T(2^n) > T(2^n\,\mathrm{cup}(z_1, \delta))$ or the event $\{S(2^n) > S(2^n\,\mathrm{cup}(z_1, \delta))\}$.

Corollary 3.2 *For independent Brownian motions X and Y define F_{n,δ,z_1,z_2} to be the intersection of the events*

$$\mathrm{d}(X[0, T(2^n\,\mathrm{cup}(z_1,\delta))],\; Y[0, S(2^n\,\mathrm{cup}(z_2,\delta))]) > 0$$

$$X[0, T(2^n\,\mathrm{cup}(z_1,\delta))] \cap D \subset \mathrm{cup}(z_1, 2\delta)$$

$$Y[0, S(2^n\,\mathrm{cup}(z_2,\delta))] \cap D \subset \mathrm{cup}(z_2, 2\delta)$$

$$\mathrm{d}(X[0, T(2^n\,\mathrm{cup}(z_2,\delta))], 2^n z_2) \geqq 4.2^n\,\delta$$

$$\mathrm{d}(Y[0, S(2^n\,\mathrm{cup}(z_2,\delta))], 2^n z_1) \geqq 4.2^n\,\delta.$$

$$T(2^n\,\mathrm{cup}(z_1,\delta)) < T(2^n) \quad \textit{and} \quad S(2^n\,\mathrm{cup}(z_2,\delta)) < S(2^n).$$

Define $F_{n,\delta,z_1,z_2,\gamma}$ to be the intersection of the event F_{n,δ,z_1,z_2} with the event $\mathrm{d}(X[0, T(2^n\,\mathrm{cup}(z_1,\delta))], Y[0, S(2^n\,\mathrm{cup}(z_2,\delta))]) \geqq \gamma$.

For each $\varepsilon>0$ and n_0, there exists $n>n_0$, δ and z_1, z_2 on C with $|z_1-z_2|>8\delta$ and $\gamma>0$, so that

$$P^{x,y}[F_{n,\delta,z_1,z_2,\gamma}]\geqq 2^{-(k+7\varepsilon)n}$$

uniformly on $x\in\text{cup}(z_1,\delta)$ and $y\in\text{cup}(z_2,\delta)$

Proof. The preceding corollary states that for δ sufficiently small and a fixed $\varepsilon>0$, we can find n as large as desired with $P[E_{n,\delta,z_1,z_2}]$ larger than $2^{-(k+6\varepsilon)n}$. Now let A_{n,δ,z_1,z_2} be the event:

1. $T(2^{n+1}\,\text{cup}(z_1,\delta))<T(2^{n+1})$
2. $S(2^{n+1}\,\text{cup}(z_2,\delta))<S(2^{n+1})$
3. $\mathrm{d}(X[T(2^n),T(2^{n+1}\,\text{cup}(z_1,\delta))],Y[S(2^n),S(2^{n+1}\,\text{cup}(z_2,\delta))])>0$
4. $\mathrm{d}(X[T(2^n),T(2^{n+1}\,\text{cup}(z_2,\delta))],2^{n+1}z_2)\geqq 4.2^{n+1}\delta$
5. $\mathrm{d}(Y[S(2^n),S(2^{n+1}\,\text{cup}(z_2,\delta))],2^{n+1}z_1)\geqq 4.2^{n+1}\delta.$

Now $E_{n,\delta,z_1,z_2}\cap A_{n,\delta,z_1,z_2}\subset F_{n+1,\delta,z_1,z_2}$. Also it is easy to see that on the event E_{n,δ,z_1,z_2} the conditional probability of A_{n,δ,z_1,z_2} is bounded below by a strictly positive constant depending on δ but not on n. This implies that for n sufficiently large $P[F_{n+1,\delta,z_1,z_2}]>2^{-(k+7\varepsilon)n}$. The statement of the lemma follows by observing that $F_{n+1,\delta,z_1,z_2}=\bigcup_{\gamma>0}F_{n+1,\delta,z_1,z_2,\gamma}$ and the latter events are decreasing in γ.

Recall that V and W are independent random walks on Z^d.

Corollary 3.3 *Let $A(N,n,z_1,z_2,\delta,\gamma)$ be the intersection of the events*

$$\mathrm{d}(V[0,T(N2^n\,\text{cup}(z_1,\delta))]\cap W[0,S(N2^n\,\text{cup}(z_2,\delta))])\geqq\gamma N$$
$$V[0,T(N2^n\text{cup}(z_1,\delta))]\cap N.D\subset N\text{cup}(z_1,2\delta)$$
$$W[0,S(N2^n\,\text{cup}(z_2,\delta))]\cap N.D\subset N\text{cup}(z_2,2\delta)$$
$$\mathrm{d}(V[0,T(N2^n\,\text{cup}(z_1,\delta))],N2^n z_2)\geqq 4.N2^n\delta$$
$$\mathrm{d}(W[0,S(N2^n\,\text{cup}(z_2,\delta))_n],N2^n z_1)\geqq 4.N2^n\delta.$$
$$T(N2^n\,\text{cup}(z_1,\delta))<T(N2^n)$$

and

$$S(N\,\text{cup}(n,z_2,\delta))<S(N2^n).$$

For each $\varepsilon>0$, there exist $n,\delta,z_1,z_2,(|z_1-z_2|>8\delta)$, γ and N_0 such that for each $N\geqq N_0$, $Q^{x_1,x_2}[A(N,n,z_1,z_2,\delta,\gamma)]>2^{-(k+\varepsilon)n}$ uniformly on $x_1\in N\,\text{cup}(z_1,\delta)$ and $x_2\in N\,\text{cup}(z_2,\delta)$.

Proof. Corollary 3.2 enables us to choose an n so that $P^{x,y}[F_{n,\delta,z_1,z_2,\gamma}]>2^{-(k+\varepsilon/2)n}$ uniformly over x in $\text{cup}(z_1,\delta)$ and y in $\text{cup}(z_2,\delta)$. We fix these points and numbers. We now argue by contradiction. If the Corollary were not true then there would exist N_i tending to infinity and $x_i\in\text{cup}(z_1,\delta)$ $y_i\in\text{cup}(z_2,\delta)$ so that

$$Q^{N_i x_i,N_i y_i}[A(N,n,z_1,z_2,\delta,\gamma)]<2^{-(k+\varepsilon)n}.$$

By taking a subsequence if necessary we may assume that $x_i\to x\in\text{cup}(z_1,\delta)$ and $y_i\to y\in\text{cup}(z_2,\delta)$. Then Donsker's Invariance principle implies that

$P^{x,y}[F_{n,\delta,z_1,z_2,\gamma}] \leqq 2^{-(k+\varepsilon)n}$. But this contradicts our choice of n, z_1, z_2, δ and we are done.

Proof of Proposition 3.1 Given $\varepsilon > 0$, let $n, \delta, z_1, z_2, \gamma$ and N_0 be the constants and points furnished by Corollary 3.3. For our random walks V and W let $D(N, n, z_1, z_2, \delta, \gamma)$ be the event that $A(N, n, z_1, z_2, \delta, \gamma)$ (as in Corollary 3.3) occurs for the random walks V^N and W^N where

$$V^N(r) = V(T(2^N \operatorname{cup}(z_1, \delta)) + r) \qquad W^N(r) = W(S(2^N \operatorname{cup}(z_1, \delta)) + r).$$

Thus Corollary 3.3 states that for $N \geqq N_0, P[D(N, n, z_1, z_2, \delta, \gamma)] > 2^{-(k+\varepsilon)n}$. Let D_{N_0} be the intersection of the events

1. $T(2^{N_0} \operatorname{cup}(z_1, \delta)) < T(2^{N_0})$ and $S(2^{N_0} \operatorname{cup}(z_2, \delta)) < S(2^{N_0})$.
2. $V[0, T(2^{N_0} \operatorname{cup}(z_1, \delta))] \cap W[0, S(2^{N_0} \operatorname{cup}(z_2, \delta))$ is empty.
3. $V[0, T(2^{N_0} \operatorname{cup}(z_1, \delta))] \cap 2^{N_0} \operatorname{cup}(z_2, 2\delta))$ is empty.
4. $2^{N_0} \operatorname{cup}(z_1, 2\delta) \cap W[0, S(2^{N_0} \operatorname{cup}(z_2, \delta))$ is empty.

However small it may be $P[D_{N_0}]$ is strictly positive and this is all we require. Now for any m, $D_{N_0} \cap \left(\bigcap_{i=0}^{m} D(N_0 + in, n, z_1, z_2, \delta, \gamma) \right) \subset \{W[0, S(2^{N_0 + (m-1)n})]$ $\cap V[0, T(2^{N_0 + (m-1)n})]$ is empty$\}$. (This was the motivation for our introduction of cups). The Strong Markov Property ensures that

$$P\left[D_{N_0} \cap \left(\bigcap_{i=0}^{m} D(N_0 + in, n, z_1, z_2, \delta, \gamma) \right)\right] \geqq P[D_{N_0}] (2^{-(k+\varepsilon)n})^m$$

which ensures that $\{V[0, T(2^{N_0 + (m-1)n})] \cap W[0, S(2^{N_0 + (m-1)n})]$ is empty$\}$. Has probability at least $P(D_{N_0}) (2^{-(k+\varepsilon)n})^m \geqq C 2^{-(k+\varepsilon)(N_0 + (m-1)n)}$ for some strictly positive C not depending on m. Since ε is arbitarily positive Proposition 3.1 is established. □

4.

In this section we seek to complete the proof of Theorem 1. First note that the boundedness assumption was only used in Sect. 2. To make the arguments of Sect. 3 work it was only required that the random walk could be claimed by Donsker's Invariance principle. Therefore we only need to show that for each $\varepsilon > 0$, x_1, x_2 and all N sufficiently large

$$(*) \qquad Q^{x_1, x_2}[V(0, T(2^N)) \cap W(0, S(2^N)) \text{ is empty}] \leqq 2^{-(k+\varepsilon)N}$$

To do this we require the fact that the constant k of Theorem 1 is less or equal to 2.

We also require the following lemma which is so simple that the proof is omitted.

Lemma 4.1 *Consider a random walk $\{V(n)\colon n \geqq 0\}$ which has zero mean and finite covariance matrix and such that $V(0) \in D(2^N)$. Let $T(2^n)$ be the first time it leaves D_N. Then the quantity*

$$\sup_{i \geqq 1} \frac{P[|V(T(2^N))| > 2^N 2^i]}{2^{2i}} \to 0$$

as N tends to infinity.

We are now ready to prove $_*$. We first fix ε arbitrarily small.
Choose r so that

1. The quantity p_r defined in Section 1 is less than $2^{-(k-\varepsilon/2)r}$.
2. $(k-2\varepsilon)r$ is greater than $(k-3\varepsilon)(r+2)$.
3. $2^{-\varepsilon r} < 1/2$.

Choose M so that

4. For each $m \geqq M$ and x_i of magnitude less than 2^m

$$Q^{x_1, x_2}[V(0, T(2^{r+m})] \cap W(0, S(2^{r+m})] \text{ is empty}] < 2^{-(k-\varepsilon)r}$$

(This is possible by an argument similar to that for Corollary 3.3).
5. For each m greater than or equal to M, the quantity in Lemma 4.1 is less than $2^{-(1+4r)}(1-2^{-\varepsilon r})$.

We now consider $q_N(x, y)$ for $|x|, |y| < M$ and $N > M/\varepsilon$. Write $N = \varepsilon N + s(r+1) + j$ where $j \in [0, r+1]$. In the following we will assume that s is large. We now define the following numbers for $i \leqq s$

$$n_1 = N - r.$$

For $i > 1$, $n_i = n_{i-1} - (r+1)$.

Since all $i \leqq s$, the n_i are all greater than m. For $i > 0$, we define p_i to be the supremum over $x_1, x_2 \in D_{n_i}$ of

$$Q^{x_1, x_2}[V(0, T(2^N)] \cap W(0, S(2^N)] \text{ is empty}]$$

and put $p_0 = 0$. It follows instantly from 3 defining M that $p_1 < 2^{-(k-\varepsilon)r} < 2^{-(k-2\varepsilon)r}$. We consider next p_2. Let x_1 and x_2 be any points in D_{n_2}. Let $T_2 = \inf\{n\colon V(n) \text{ is not in } D_{n_2+r}\}$ and $S_2 = \inf\{n\colon W(n) \text{ is not in } D_{n_2+r}\}$. It follows easily that

$$\begin{aligned}
Q^{x_1, x_2}[V[0, T(2^N)] \\
\cap W[0, S(2^N)] \text{ is empty}] &\leqq Q^{x_1, x_2}[\{V[0, T_2] \cap W[0, S_2] \text{ is empty}\} \\
&\quad \cap \{V[T_2, T(2^N)] \cap W[S_2, S(2^N)] \text{ is empty}] \\
&\leqq Q^{x_1, x_2}[\{V[0, T_2] \cap W[0, S_2] \text{ is empty}\} \\
&\quad \cap \{V[T_2, T(2^N)] \cap W[S_2, S(2^N)] \text{ is empty}\} \\
&\quad \cap \{|V(T_2)|, |W(S_2)| \leqq 2^{n_2+r+1}\}] \\
&\quad + Q^{x_1, x_2}[|V(T_2)| \text{ or } |W(S_2)| > 2^{n_2+r+1}].
\end{aligned}$$

The strong Markov property forces the first term above to be bounded by $2^{-(k-\varepsilon)r}p_1 < 2^{-(2k-3\varepsilon)r}$. While Lemma 4.1 and the choice of m forces the second term to be bounded by $2\times(1-2^{-\varepsilon r})\,2^{-(1+4r)}$. So

$$p_2 \leqq 2^{-(2k-3\varepsilon)r} + 2^{-(4r)}(1-2^{-\varepsilon r}) < 2^{-(k-2\varepsilon)2r}.$$

We now proceed to show that $p_i \leqq 2^{-(k-2\varepsilon)ir}$ for all $i \leqq s$. We use induction. The desired result holds for $i=1, 2$. Suppose that it holds for $j=1, 2, \ldots, i-1$. Let x_1 and x_2 be any two elements of D_{n_i}. Define $T_i = \inf\{n\colon V(n)$ is not in $D_{n_i}\}$ and $S_i = \inf\{n\colon W(n)$ is not in $D_{n_i}\}$. For $1 \leqq j < i$, let $B_{i,j}$ be the event

$$\max\{|V(T_i)|, |W(S_i)|\} \in (2^{n_{j+1}}, 2^{n_j}]$$

and let $B_{i,0}$ be the event $\max|V(T_i)|, |W(S_i)| > 2^{n_1}$. Two observations are important. First, the inductive hypotheses and the strong Markov property imply that on the event $B_{i,j}$ the conditional probability

$$Q^{x_1,x_2}[V[0, T(2^N)] \cap W[0, S(2^N)] \text{ is empty} \mid \sigma\{V[0, T_i], W[0, S_i]\}]$$

is less than or equal to

$$Q^{x_1,x_2}[V[T_i, T(2^N)] \cap W[S_i, S(2^N)] \text{ is empty} \mid \sigma\{V[0, T_i], W[0, S_i]\}] \leqq p_j.$$

Secondly, it follows from Lemma 4.1 and the fact that $n_i \geqq m$, that for $j < i-1$ $P[B_{i,j}] < 2.2^{-(4r+1)}(1-2^{-\varepsilon r})(2^{-2r})^{(i-j-1)}$. For $j = i-1$

$$Q^{x_1,x_2}[\{V[0, T(2^N)] \cap W[0, S(2^N)] \text{ is empty}\} \cap B_{i,i-1}]$$

is less than or equal to

$$\begin{aligned} &Q^{x_1,x_2}[\{V[0, T_i] \cap W[0, S_i] \text{ is empty}\} \\ &\quad \cap \{V[T_2, T(2^N)] \cap W[S_2, S(2^N)] \text{is empty}\} \\ &\quad \cap B_{i,i-1}] \leqq 2^{-(k-\varepsilon)r} p_{i-1}. \end{aligned}$$

Putting these facts together gives

$$p_i \leqq 2^{-(k-\varepsilon)r} p_{i-1} + 2^{-4r}(1-2^{-\varepsilon r})(p_{i-2} + 2^{-2r} p_{i-3} + 2^{-4r} p_{i-4} \ldots$$

and so we may inductively prove that for $i \leqq s$, $p_i \leqq 2^{-(k-2\varepsilon)ir}$.

Consequently for x_i of magnitude less than $2^{\varepsilon N}$

$$Q^{x_1,x_2}[V(0, T(2^N)] \cap W(0, S(2^N)] =] \leqq 2^{-(k-2\varepsilon)sr} \leqq 2^{-(k-3\varepsilon)s(r+2)} \leqq 2^{-(k-3\varepsilon)(1-\varepsilon)N}.$$

This is all that is needed to prove (*) since ε is as small as desired. □

5.

In this section we consider where the maximum of $p_r(z_1, z_2)$ is attained. By rotational symmetry we need only consider when $p_r(e_1, z)$ is maximized by z. It seems intuitively clear that then $p_r(e_1, z)$ will be maximized when $z = -e_1$. Unfortunately we can only prove this in two dimensions. By continuity the maximum p_r is attained and simple scaling arguments show that if $p_r(e_1, z)$ is maximized at z_r, then z_r must tend to $-e_1$ as r tends to infinity.

Theorem 3 *In two dimensions the quantity $p_r(e_1, z)$ is maximized only when $z = -e_1$. Equivalently $p_r(z_1, z_2)$ is maximized only when z_1 and z_2 are antipodal.*

Proof. Consider two independent Brownian motions X and Y beginning respectively at e_1 and z. Let γ be a Mobius transformation fixing C_r and taking e_1 to 0. It is easily checked that γ takes C to a circle G and that $-e_1$ is mapped to the point on G of greatest magnitude. By the invariance of Brownian motion under analytic maps we have

$$p_r(e_1, z) = p_r(0, \gamma(z)).$$

By the rotational symmetry of Brownian motion $p_r(0, |w|\, e_1)$. It only remains to show that $p_r(0, |w|\, e_1)$ is an increasing function of $|w|$. Let $v_1 < v_2 < 2^r$. Let X be a Brownian motion starting at 0 and let Y be an independent Brownian motion starting at $w_1(|w_1| = v_1)$. Let S be the first time that $|Y(t)| = v_2$. From the strong Markov property for the Brownian motion Y,

$$\begin{aligned} p_r(0, v_1 e_1) &= P[X[0, T(2^r)] \cap Y[0, S(2^r)] \text{ is empty}] \\ &< P[X[0, T(2^r)] \cap Y[S, S(2^r)] \text{ is empty}] = p_r(0, v_2 e_1). \quad \square \end{aligned}$$

References

Burdzy, K.: Multidimensional Brownian Excursions and Potential Theory. London: Longman 1987

Burdzy, K., Lawler, G.: Non-intersection exponent for Brownian paths. Part I, existence and an invariance principle. (preprint 1989)

Burdzy, K., Lawler, G., Polaski, T.: On the critical exponent for random walk intersections, preprint 1989

Durett, R.: Stochastic processes: theories and examples. Monterey: Wadsworth 1989

Dynkin, E.: Self intersection gauge. Ann. Probab. **16**, 1–57 (1988)

LeGall, J.F.: Proprietes d'Intersection des Marches alleatoire I, II.

Commun. Math. Phys.**104**, 471–507 (1987)

Pitman, J., Yor, M.: A decomposition of Bessel bridges. Z. Wahrscheinlichkeitstheor. Verw. Geb. **55**, 425–457 (1982)

Port, S., Mountford, T.: Some Brownian path properties. (preprint 1991)

Williams, D.: Path decomposition and continuity of local time for one-dimensional diffusions I. Proc. Lond. Math. Soc., III Ser. **26**, 738–768 (1974)