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# Gauge theorems for resolvents with application to Markov processes

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**Summary.** Let $\mathbb{V} = (V_\alpha)_{\alpha \geqq 0}$ be a (not necessarily sub-Markovian) resolvent such that the kernel $V_\alpha$ for some $\alpha \geqq 0$ is compact and irreducible. We prove the following general gauge theorem: If there exists at least one $\mathbb{V}$-excessive function which is not $\mathbb{V}$-invariant, then $V_0$ is bounded.

This result will be applied to resolvents $\mathbb{U}^M$ arising from perturbation of sub-Markovian right resolvents $\mathbb{U}$ by multiplicative functionals $M$ (not necessarily supermartingale), for instance, by Feynman-Kac functionals. Among others, this leads to an extension of the gauge theorem of Chung/Rao and even of one direction of the conditional gauge theorem of Falkner and Zhao.

## 0. Introduction

In the last decade there has been a wave of results on gauge theorems and conditional gauge theorems ([C1], [C2], [CR1], [CR2], [CrFZ], [Fa], [Z1], [Z2], . . .). Closely related are results on perturbation of harmonic spaces ([BoHH], [HH], [HM], . . .). We prove several gauge theorems in the general context of strong resolvents $\mathbb{V} = (V_\alpha)_{\alpha \geqq 0}$. For instance, suppose that the kernel $V_\alpha$ for some $\alpha \geqq 0$ is compact and irreducible. (This assumption is fulfilled in any of the above-mentioned articles.) If in that case there exists at least one $\mathbb{V}$-excessive function which is not $\mathbb{V}$-invariant, then the potential kernel $V_0$ is bounded. In particular, if $V_0$ is non-recurrent, then it is already bounded.

The notion of a strong resolvent is slightly more restrictive than the usual notion of a resolvent. Obviously, some restriction is necessary to avoid pathologies (because we do not subject the kernels $V_\alpha$ to any boundedness or finiteness condition whatsoever). However, this restriction does not matter, since every resolvent derived from a measurable semigroup of kernels is a strong resolvent. This in particular applies to every resolvent $\mathbb{U}^M = (U_\alpha^M)_{\alpha \geqq 0}$ arising from perturbation of a sub-Markovian right resolvent $\mathbb{U}$ by a multiplicative functional $M$ (not necessarily supermartingale), i.e.

$$U_\alpha^M f(x) := \mathbf{E}^x \int_0^\zeta e^{-\alpha \cdot t} \cdot M_t \cdot f(X_t)\, dt \,.$$

Without any further assumption, $\mathbb{U}^M$ is always a strong resolvent.

In order to discuss the remaining assumptions on $\mathbb{U}^M$, we introduce a certain compatibility condition for multiplicative functionals $M$ (being considerably less restrictive than the usual Kato condition). For compatible multiplicative functionals $M$, the compactness and irreducibility assumptions on $\mathbb{U}^M$ reduce to corresponding assumptions on $\mathbb{U}^{M'}$ where $M'$ is a suitable decreasing (!) multiplicative functional. For such sub-Markovian resolvents $\mathbb{U}^{M'}$, things are easy and well-known. We mention several sufficient criteria for compactness respectively irreducibility.

According to the above general gauge theorem, the main task is to check the existence of $\mathbb{U}^M$-excessive functions which are not $\mathbb{U}^M$-invariant. Typical $\mathbb{U}^M$-excessive functions are of course the potentials

$$U_0^M f : x \mapsto \mathbf{E}^x \int_0^\zeta M_t \cdot f(X_t)\,dt \tag{0.1}$$

of functions $f \geqq 0$. But also the gauge function

$$\Gamma^M : x \mapsto \mathbf{E}^x [M_\zeta \cdot 1_{\{\zeta < \infty\}}] \tag{0.2}$$

turns out to be $\mathbb{U}^M$-excessive. Moreover, even the conditional gauge function

$$\gamma^M(x, s) := \mathbf{E}^{x/s} [M_\zeta \cdot 1_{\{\zeta < \infty\}}]$$

gives rise to $\mathbb{U}^M$-excessive functions, namely, for any $\mathbb{U}$-excessive function $s$, the function

$$x \mapsto s(x) \cdot \gamma^M(x, s) \tag{0.3}$$

is (up to a regularization) $\mathbb{U}^M$-excessive. Here $\mathbf{E}^{x/s}$ denotes expectation with respect to Doob's $s$-transformation (i.e. transformation by means of the excessive function $s$) of the original process starting in $x$.

Suppose now that $U_\alpha^M$ is irreducible (for some $\alpha \geqq 0$). Then all these $\mathbb{U}^M$-excessive functions (0.1)–(0.3) are $\mathbb{U}^M$-invariant if and only if they are degenerate (i.e. $\equiv 0$ or $\equiv \infty$). Thus each of them is a good indicator (or "gauge") for $U_0^M$ to be bounded.

If the multiplicative function $M$ satisfies the already mentioned compatibility condition, then the boundedness of the potential kernel $U_0^M$ implies the boundedness of the gauge function $\Gamma^M$. Hence our general gauge theorem for resolvents leads to an extension of the gauge theorem of Chung/Rao. Furthermore, it contains in a very general framework one direction of the conditional gauge theorem of Falkner and Zhao.

## 1. Gauge theorems for resolvents

Throughout this paper we assume that $E$ is a Radon space and $\mathscr{E}$ its $\sigma$-field of universally measurable subsets. (We recall that a Radon space is a topological space which is homeomorphic to an universally measurable subspace of a compact metric space, cf. [DM], [Sh].)

**(1.1) Definition.** A family $\mathbb{V} = (V_\alpha)_{\alpha \geqq 0}$ of kernels

$$V_\alpha : (E, \mathscr{E}) \to (E, \mathscr{E})$$

on the measurable space $(E, \mathscr{E})$ will be called a *strong resolvent* iff the following strong resolvent equation holds for all $0 \leqq \beta < \alpha < \infty$:

$$\text{(R1)} \qquad V_\beta = \sum_{n=1}^{\infty} (\alpha - \beta)^{n-1} \cdot (V_\alpha)^n .$$

(1.2) *Remarks.* a) It should be mentioned that this strong resolvent equation is always fulfilled if $\mathbb{V}$ is defined by a *measurable semigroup* $\mathbb{Q} = (Q_t)_{t \geqq 0}$ of kernels on $(E, \mathscr{E})$ (which have not to be sub-Markovian or bounded):

$$V_\alpha = \int_0^\infty Q_t \cdot e^{-\alpha \cdot t} dt \quad \text{for } \alpha \geqq 0 .$$

This follows immediately from the fact that in this case the $n$-th iteration of the kernel $V_\alpha$ is given by $(V_\alpha)^n = \frac{1}{(n-1)!} \int_0^\infty Q_t \cdot e^{-\alpha \cdot t} \cdot t^{n-1} dt$.

b) On the other hand, every strong resolvent $\mathbb{V}$ is of course a *resolvent in the usual sense*, i.e. $\mathbb{V}$ satisfies the resolvent equation

$$V_\beta = V_\alpha + (\alpha - \beta) \cdot V_\alpha \circ V_\beta, \quad V_\alpha \circ V_\beta = V_\beta \circ V_\alpha \quad \text{for all } 0 \leqq \beta < \alpha < \infty .$$

Under suitable continuity assumptions on $\alpha \mapsto V_\alpha$ the converse is also true. For example, every *sub-Markovian resolvent* $\mathbb{V}$ with $V_0 = \lim_{\alpha \to 0} V_\alpha$ is a strong resolvent (cf. [DM], XII.8.3).

c) In order to illustrate the difference between the resolvent equation and the strong resolvent equation, let $\mathbb{U}$ be the (sub-Markovian) resolvent derived from the Brownian semigroup on $E = \mathbb{R}^d$ and define (for $f \in \mathscr{E}_+$ and $\alpha \geqq 0$)

$$V_\alpha f := \begin{cases} U_\alpha f & \text{if } \alpha \geqq 1 \\ 0 & \text{if } \alpha < 1 \text{ and } U_\alpha f \equiv 0 \\ \infty & \text{else .} \end{cases}$$

It is easy to see that the family of kernels $(V_\alpha)_{\alpha \geqq 0}$ satisfies the resolvent equation but not the strong resolvent equation.

As usual, the kernel $V_\alpha$ (for $\alpha \geqq 0$) can be considered as a map from $\mathscr{E}_+$ to $\mathscr{E}_+$ (where $\mathscr{E}_+$ denotes the set of nonnegative universally measurable functions on $E$). If the function $V_\alpha 1$ is bounded, the kernel $V_\alpha$ defines also a bounded operator on $\mathscr{E}_b$, the Banach space of bounded universally measurable functions on $E$. For example, if $\mathbb{V}$ is sub-Markovian, then $\| V_\alpha \| = \| V_\alpha 1 \| \leqq 1/\alpha$ for every $\alpha > 0$. Here and henceforth $\| . \|$ denotes the supremum norm on $E$ and also the corresponding operator norm on $\mathscr{E}_b$.

However, in this article we are mainly interested in resolvents which are not sub-Markovian. In general, the functions $V_\alpha 1$ for $\alpha > 0$ are not even assumed to be finite. Our essential assumption will be that the operator

$$\text{(R2)} \qquad V_\alpha : \mathscr{E}_b \to \mathscr{E}_b \quad \text{is compact for some } \alpha \geqq 0.$$

Several times we will additionally assume that the kernel $V_\alpha$ for some (hence every) $\alpha \geqq 0$ is non-degenerate and irreducible in the following sense:

(R3) $\quad V_\alpha 1 \not\equiv 0$ on $E$ and $\forall f \in \mathscr{E}_+$: either $V_\alpha f \equiv 0$ or $V_\alpha f > 0$ on $E$.

Typical examples of strong resolvents (satisfying (R2) and/or (R3)) will be discussed in the third part of this paper.

Our aim is to obtain conditions which imply that the potential kernel $V := V_0$ is bounded, i.e. $\|V\| := \sup_{x \in E} V1(x) < \infty$. Every result in this direction will be called "gauge theorem" (for reasons which become clear in the third part). The first one is the following

**(1.3) Theorem.** *Let* $\mathbb{V}$ *satisfy* (*R*1) *and* (*R*2). *Then the following statements are equivalent*:

(1.3.1) $\quad \|V\| < \infty$;

(1.3.2) $\quad Vf < \infty$ *a.e. for some* $f \in \mathscr{E}_+$ *with* $f > 0$ *a.e.*

(*Here* "*a.e.*" *means as usual* "*on a set* $F \in \mathscr{E}$ *with* $V1_{E \setminus F} \equiv 0$".)
*If* $\mathbb{V}$ *moreover satisfies* (*R*3), *then the above statements are also equivalent to*:

(1.3.3) $\quad 0 \not\equiv Vf \not\equiv \infty$ *for some* $f \in \mathscr{E}_+$.

The *proof* of this Theorem (as well as that of Theorem (1.5)) will be deferred to the end of this section

(1.4) *Remark.* In general, a kernel $V$ (or a resolvent $\mathbb{V}$) satisfying (1.3.2) is called transient. (Among other possible definitions for transience, (1.3.2) is one of the weakest, cf. [Ge].) The kernel $V$ is called non-recurrent iff the property (1.3.3) is fulfilled. So the Theorem states that under (R1)–(R3) boundedness, transience and non-recurrence of $V$ are equivalent. This however will be sharpened in the sequel. The importance of Theorem (1.3) lies in the equivalence (1.3.1)$\Leftrightarrow$(1.3.2) (i.e. of boundedness and transience) which requires only (R1) + (R2). All "gauge theorems" up to now use essentially the irreducibility (R3) of $\mathbb{V}$.

In order to obtain sharper results it turns out to be reasonable not only to distinguish between $\|V\| < \infty$ and $\|V\| = \infty$, but to consider the following three cases:

- $\|V\| < \infty$,
- $\|V\| = \infty$ and $\|V_\alpha\| < \infty$ for all $\alpha > 0$,
- $\|V_\alpha\| = \infty$ for some $\alpha > 0$.

These three possible cases will be compared with properties of the set

$$^*\mathscr{H} = \left\{ u \in \mathscr{E}_+ : u = \sup_{\alpha > 0} \alpha \cdot V_\alpha u \right\}$$

of excessive functions for $\mathbb{V}$ and with properties of the set

$$_*\mathscr{H} = \left\{ u \in \mathscr{E}_+ : u = \inf_{\alpha > 0} \alpha \cdot V_\alpha u \right\}$$

of defective functions for $\mathbb{V}$. Let $\mathscr{H} = {}^*\mathscr{H} \cap {}_*\mathscr{H}$ be the set of invariant functions for $\mathbb{V}$. Then obviously for every non-degenerate strong resolvent

$$^*\mathscr{H} \supset \mathscr{H} \supset \{0, \infty\}, \quad ^*\mathscr{H}_b \supset \mathscr{H}_b \supset \{0\} \quad \text{and} \quad {}_*\mathscr{H}_b \supset \mathscr{H}_b \supset \{0\}\,.$$

(Here and henceforth the index "$_b$" denotes the subspace of bounded functions in a given function space.)

**(1.5) Theorem.** *Let* $\mathbb{V}$ *satisfy* (*R*1)–(*R*3).

*a) The following statements are equivalent:*

(1.5.1) $\quad \|V\| < \infty;$

(1.5.2) $\quad {}_*\mathscr{H}_b = \{0\};$

(1.5.3) $\quad {}^*\mathscr{H} \neq \mathscr{H}.$

*b) On the other hand, equivalent are:*

(1.5.4) $\quad \|V\| = \infty,\ \|V_\alpha\| < \infty$ *for all* $\alpha > 0;$

(1.5.5) $\quad \mathscr{H}_b \neq \{0\};$

(1.5.6) $\quad {}^*\mathscr{H}\backslash\{\infty\} = {}^*\mathscr{H}_b = {}_*\mathscr{H}_b = \mathscr{H}_b = \mathbb{R}_+ \cdot h$ *with some* $h > 0.$

*c) Finally, the statements below are equivalent:*

(1.5.7) $\quad \|V_\alpha\| = \infty$ *for some* $\alpha > 0;$

(1.5.8) $\quad {}^*\mathscr{H} = \{0, \infty\};$

(1.5.9) $\quad {}_*\mathscr{H}_b \neq \mathscr{H}_b.$

(1.6) *Remarks.* a) The equivalence (1.5.1)$\Leftrightarrow$(1.5.2) is true, yet under (R1) + (R2) (the implication (1.5.1)$\Rightarrow$(1.5.2) even under (R1)). Moreover, in (1.5.2) the set ${}_*\mathscr{H}_b$ of bounded defective functions can be replaced by ${}_*\tilde{\mathscr{H}}_b := \{u \in \mathscr{E}_{b,+} : u \leqq \alpha \cdot V_\alpha u$ for all $\alpha > 0\}$, the set of bounded submedian functions of $E$:

(1.6.1) $$_*\tilde{\mathscr{H}}_b = \{0\}\,.$$

The statement (1.6.1) (or (1.5.2)) can be considered as some kind of minimum principle. Indeed, let us for the moment call a $\mathscr{E}$-measurable function $u$ nearly hyperharmonic if for all $\alpha > 0$ the function $V_\alpha u$ is well defined and $\alpha \cdot V_\alpha u \leqq u$. (If $\|V_\alpha\| < \infty$, obviously $V_\alpha u$ is well defined for all lower bounded, $\mathscr{E}$-measurable functions.) Then for every nearly hyperharmonic function $u$ the negative part $u^-$ is a submedian function. So (1.6.1) implies that every lower bounded, nearly hyperharmonic function is nonnegative.

b) The assertion of the Theorem remains valid if all the sets ${}^*\mathscr{H}$ and $\mathscr{H}$ are replaced by ${}^*\mathscr{H}_b$ and $\mathscr{H}_b$, respectively. This yields a complete symmetry between ${}^*\mathscr{H}_b$ and ${}_*\mathscr{H}_b$. However, our main interest lies in the implication

(1.6.2) $$^*\mathscr{H} \neq \mathscr{H} \Rightarrow \|V\| < \infty$$

which is sharper than ${}^*\mathscr{H}_b \neq \mathscr{H}_b \Rightarrow \|V\| < \infty$. (From Theorem (1.5) one can indeed deduce an even sharper version of (1.6.2): ${}^*\mathscr{H} \neq \mathscr{H}_b \cup \{\infty\} \Rightarrow \|V\| < \infty$.) It is easy to see that the essential implication (1.3.3) $\Rightarrow$ (1.3.1) in Theorem (1.3) is also a direct consequence of (1.6.2). The statement (1.6.2) (or, more generally, Theorem (1.5)) can be considered as a general gauge theorem. We shall see in the third part of this paper that (1.6.2) includes the gauge theorem of K. L. Chung and M. Rao ([CR1], cf. also [CR2]) which is meanwhile a well known theorem in probabilistic potential theory. A more general gauge theorem than in [CR1] was obtained in analytic potential theory by W. Hansen and H. Hueber ([HH], cf. also [BoHH], [HM]). It is a remarkable fact that all these gauge theorems can be deduced from a general result on resolvents (Theorem (1.5) or statement (1.6.2)). Moreover, we shall obtain from (1.6.2) one part of the so-called conditional gauge theorem of N. Falkner and Z. Zhao ([Fa], [Z1], cf. also [Z2], [CrFZ]).

The above-mentioned results depend essentially on the following two Lemmas.

**(1.7) Lemma.** *Let* $\mathbb{V}$ *satisfy* $(R1)+(R2)$. *Let* $u \in {}^*\mathscr{H}$ *and* $v \in {}_*\mathscr{H}_b$ *with* $\{u<\infty\} \cap \{v>0\} \neq \emptyset$. *Then there exist* $\gamma \in \mathbb{R}_+$ *and* $x \in \{u<\infty\} \cap \{v>0\}$ *such that*

$$u \geqq \gamma \cdot v \text{ on } E \quad \text{and} \quad u(x) = \gamma \cdot v(x) .$$

*If* $\mathbb{V}$ *also satisfies* $(R3)$ *we have* $u \equiv \gamma \cdot v$.

*Proof.* Let $g := u/v$ on $\{u<\infty\} \cap \{v>0\}$ and $g := \infty$ elsewhere. Then obviously $g \in \mathscr{E}_+$ and $g \not\equiv \infty$. Furthermore, take some $\alpha > 0$ such that $V_\alpha : \mathscr{E}_b \to \mathscr{E}_b$ is a compact operator. This implies that the kernel $V_\alpha$ is basic ([DM], IX.15), i.e. that there exists a finite measure $m$ on $(E, \mathscr{E})$ (called reference measure) such that for all $F \in \mathscr{E}$

$$m(F) = 0 \Leftrightarrow V_\alpha 1_F \equiv 0 .$$

Now $E$ is a Radon space. So we can apply Lusin's theorem to the finite measure $m$ and to the universally measurable function $g$ (cf. [Sh], A2.3). This yields the existence of an increasing sequence $(K_n)_n$ of compact sets $K_n \subset E$ such that

$$m(E \backslash K_n) \to 0 \quad (n \to \infty)$$

and such that the restriction of $g$ to $K_n$, i.e. the map

$$g : K_n \to \bar{\mathbb{R}}_+ ,$$

is continuous in the extended sense (that is, with respect to the topology of $\bar{\mathbb{R}}$). Since $V_\alpha(x, .)$ is (for every $x \in E$) a finite measure, which is absolutely continuous with respect to $m$, we obtain

$$V_\alpha 1_{E \backslash K_n}(x) \to 0 \quad (n \to \infty)$$

for every $x \in E$. Now $V_\alpha$ is a compact operator, so this convergence is uniform:

$$\| V_\alpha 1_{E \backslash K_n} \| \to 0 \quad (n \to \infty) .$$

In particular, there exists a compact set $K \subset E$ such that $\| V_\alpha 1_{E \backslash K} \| \leqq \frac{1}{2\alpha}$ and such that the map $v : K \to \bar{\mathbb{R}}_+$ is continuous.

Hence there exists some $x \in K$ with $g(x) = \inf g(K) =: \gamma$. We may assume that $\gamma < \infty$ (otherwise we add to $K$ some point $y \in E$ with $g(y) < \infty$). It remains to show that $\gamma = \inf g(E)$.

For this purpose let $w := u - \gamma \cdot v$. Then we have $w \geqq 0$ on $K$ and $\alpha \cdot V_\alpha w \leqq w$ on $E$. This implies

$$w^- \leqq \alpha \cdot V_\alpha w^- = \alpha \cdot V_\alpha(w^- \cdot 1_{E\setminus K}) \leqq \alpha \cdot \| V_\alpha 1_{E\setminus K} \| \cdot \|w^-\| \leqq \frac{1}{2} \cdot \|w^-\|$$

and therefore $\|w^-\| = 0$, i.e. $w \geqq 0$.

If additionally (R3) holds this obviously yields $w \equiv 0$ since $w$ was recognized as a nonnegative, hence excessive function and since $w(x) = 0$ for some $x \in E$. □

**(1.8) Lemma.** *Let* $\mathbb{V}$ *satisfy* (R1) + (R2). *Let* $\|V\| = \infty$ *and* $\|V_\alpha\| < \infty$ *for all* $\alpha > 0$. *Then* $\mathscr{H}_b \neq \{0\}$, *i.e. there exists an invariant function* $h \neq 0$. *If* $\mathbb{V}$ *also satisfies* (R3), *we have* $h > 0$ *on* $E$.

*Proof.* Fix some $\alpha > 0$ for which $V_\alpha$ is compact. By the strong resolvent equation (R1) we have for all $\beta \in [0, \alpha[$:

$$V_\beta = \frac{1}{\alpha - \beta} \cdot \sum_{n=1}^{\infty} [(\alpha - \beta) \cdot V_\alpha]^n .$$

Assuming that $\|V\| = \infty$ and $\|V_\beta\| < \infty$ for all $\beta > 0$, this implies that the spectral radius of the operator $V_\alpha$ on $\mathscr{E}_b$ is $1/\alpha$. But $V_\alpha$ is a compact and positivity preserving operator on $\mathscr{E}_b$. Applying the Krein–Rutman theorem we obtain that $1/\alpha$ is an eigenvalue of $V_\alpha$ for which there exists a nonnegative eigenfunction $h \not\equiv 0$. That is, there exists a function $h \in \mathscr{E}_{b,+} \setminus \{0\}$ such that

$$h = \alpha \cdot V_\alpha h .$$

By the resolvent equation we get for any $\beta > 0$:

$$h = \alpha \cdot V_\alpha h = \alpha \cdot [V_\beta h + (\beta - \alpha) \cdot V_\beta \circ V_\alpha h] =$$
$$= \alpha \cdot V_\beta h + (\beta - \alpha) \cdot V_\beta h = \beta \cdot V_\beta h .$$

So we have $h \in \mathscr{H}_b \setminus \{0\}$. □

We now turn to the proof of the Theorems.

*Proof of Theorem (1.3).*

(1.3.1) ⇒ (1.3.2) ⇒ (1.3.3): Trivial.

(1.3.2) ⇒ (1.3.1): Let us assume that $\|V\| = \infty$ and that $V_\gamma$ is compact for some $\gamma > 0$. By (R1) there is a number $\beta \in [0, \gamma[$ such that $\|V_\beta\| = \infty$ and $\|V_\alpha\| < \infty$ for all $\alpha > \beta$. Applying lemma 2 to the resolvent $\mathbb{V}^\beta := (V_{\beta+\alpha})_{\alpha \geqq 0}$ we obtain a function $v \in \mathscr{E}_b \setminus \{0\}$ invariant for $\mathbb{V}^\beta$, i.e. a bounded, nonnegative function $v \neq 0$ with

$$v = \alpha \cdot V_{\beta+\alpha} v \quad \text{for all } \alpha > 0 .$$

By the resolvent equation we get $\alpha \cdot V_\alpha v - v = \beta \cdot V_\alpha v$, which is $\geqq 0$ for all $\alpha > 0$ and equals $\frac{\beta}{\alpha - \beta} \cdot v$ for $\alpha > \beta$, hence tends to 0 for $\alpha \to \infty$. In particular, $v \in {}_*\mathscr{H}_b$.

Furthermore, let $u := Vf - V_\infty f$ with $V_\infty f := \lim_{\alpha \to \infty} V_\alpha f$. An easy calculation shows that $u - \alpha \cdot V_\alpha u = V_\alpha f - V_\infty f$, which is $\geqq 0$ for all $\alpha > 0$ and tends to 0 for $\alpha \to \infty$. Thus $u \in {}^*\mathscr{H}$.

Obviously, $\{u < \infty\} \cap \{v > 0\} \neq \emptyset$ (since $v = 0$ a.e. would imply $v \equiv 0$). Hence applying Lemma 1 we obtain $x_0 \in \{u < \infty\} \cap \{v > 0\}$ and $\gamma \in \mathbb{R}_+$ such that $u \geqq \gamma \cdot v$ in $E$ and $u(x_0) = \gamma \cdot v(x_0)$. Now

$$u - \gamma \cdot v = \alpha \cdot V_\alpha(u - \gamma \cdot v) + V_\alpha f - V_\infty f + \gamma \cdot \beta \cdot V_\alpha v \geqq V_\alpha f - V_\infty f \geqq 0 .$$

Thus $u(x_0) = \gamma \cdot v(x_0)$ implies $V_\alpha f(x_0) = V_\infty f(x_0)$ for all $\alpha > 0$. Therefore by (R1)

$$0 = V_\beta f(x_0) - V_{\alpha+\beta} f(x_0) = \alpha \cdot V_\beta \circ V_{\alpha+\beta} f(x_0)$$

for all $\alpha > 0$, which implies (due to $f > 0$ a.e.)

$$V_\beta v(x_0) - V_{\alpha+\beta} v(x_0) = \alpha \cdot V_\beta \circ V_{\alpha+\beta} v(x_0) = 0$$

and thus $v(x_0) = \alpha \cdot V_\beta v(x_0)$ (since $v = \alpha \cdot V_{\alpha+\beta} v$) for all $\alpha > 0$. So $v(x_0) = 0$, in contradiction to $x_0 \in \{v > 0\}$.

(1.3.3) $\Rightarrow$ (1.3.1): If $\mathbb{V}$ satisfies (R3), we obviously have $v > 0$ on $E$ for the function $v$ in the above proof. Therefore we have again $\{u < \infty\} \cap \{v > 0\} \neq \emptyset$ for $u := Vf - V_\infty f$. So with the same argument as above we obtain $V_\alpha f(x_0) = V_\infty f(x_0)$ for some $x_0 \in E$. But by (R3) $Vf \not\equiv 0$ implies $V_\beta f > 0$ and thus $V_\alpha f - V_\infty f \geqq V_\alpha f - V_\beta f = (\beta - \alpha) \cdot V_\alpha \circ V_\beta f > 0$ for all $\alpha > 0$ and all sufficiently large $\beta$.

*Proof of Theorem* (1.5).

(1.5.1) $\Rightarrow$ (1.5.2): Since $\| V \| < \infty$ we can take some $\alpha > 0$ such that $\| V \| \leqq \frac{1}{2\alpha}$. Then for every $u \in {}_*\mathscr{H}_b$ we obtain

$$\| u \| \leqq \| \alpha \cdot V_\alpha u \| \leqq \alpha \cdot \| V \| \cdot \| u \| \leqq \frac{1}{2} \| u \| ,$$

hence $\| u \| = 0$. (Indeed, this argument works for all $u \in {}_*\tilde{\mathscr{H}}_b$.)

(1.5.1) $\Rightarrow$ (1.5.3): Obviously, the function $u := V1 - V_\infty 1$ is excessive, in particular $u = \alpha \cdot V_\alpha u + (V_\alpha 1 - V_\infty 1) \geqq \alpha \cdot V_\alpha u$ for all $\alpha > 0$. Now (R3) implies $V_\beta 1 > 0$ and thus $V_\alpha 1 - V_\infty 1 \geqq V_\alpha 1 - V_\beta 1 = (\beta - \alpha) \cdot V_\alpha \circ V_\beta 1 > 0$ for all $\alpha > 0$ and all sufficiently large $\beta$. So $u \in \mathscr{H}$ if and only if $u \equiv \infty$.

(1.5.4) $\Rightarrow$ (1.5.5): Lemma (1.8).

(1.5.7) $\Rightarrow$ (1.5.9): Since $\| V_\alpha \| = \infty$ for some $\alpha > 0$ and since on the other hand $V_\alpha$ is compact for some $\alpha > 0$, there is a $\beta > 0$ such that $\| V_\beta \| = \infty$ and $\| V_{\beta+\alpha} \| < \infty$ for all $\alpha > 0$. Applying the implication (1.5.4) $\Rightarrow$ (1.5.5) to the resolvent $\mathbb{V}^\beta := (V_{\beta+\alpha})_{\alpha \geqq 0}$ we obtain a function $h \in \mathscr{E}_{b,+} \backslash \{0\}$ such that

$$h = \alpha \cdot V_{\beta+\alpha} h \quad \text{for all } \alpha > 0 .$$

By the resolvent equation we obtain $\alpha \cdot V_\alpha h - h = \beta \cdot V_\alpha h$ for all $\alpha > 0$, which is $\geqq 0$ and $\not\equiv 0$, and thus for $\alpha > \beta$ we get $\alpha \cdot V_\alpha h - h = \frac{\beta}{\alpha - \beta} \cdot h$, which tends to 0 for $\alpha \to \infty$. Hence $h \in {}_*\mathscr{H}_b \backslash \mathscr{H}_b$.

(1.5.2) $\Rightarrow$ (1.5.1): This follows immediately from (1.5.7) $\Rightarrow$ (1.5.9) and (1.5.4) $\Rightarrow$ (1.5.5).

(1.5.5) ⇒ (1.5.6): Let $h$ be a function in $\mathscr{H}_b \setminus \{0\}$. By (R3) we have, of course, $h > 0$. Therefore, the assertion follows directly from the above Lemma 1.

(1.5.7) ⇒ (1.5.8): The argument is similar to the proof of (1.5.7) ⇒ (1.5.9). Choose $\beta > 0$ such that $\| V_\beta \| = \infty$ and $\| V_{\beta+\alpha} \| < \infty$ for all $\alpha > 0$. Since every function $u$ which is excessive for $\mathbb{V}$ is obviously also excessive for $\mathbb{V}^\beta$, we obtain from (1.5.4) ⇒ (1.5.6) that $u$ is indeed invariant for $\mathbb{V}^\beta$, i.e.

$$u = \alpha \cdot V_{\beta+\alpha} u \quad \text{for all } \alpha > 0 \,.$$

Therefore, $\alpha \cdot V_\alpha u = u + \beta \cdot V_\alpha u$. Hence $u \in {}^*\mathscr{H}$ if and only if $u \equiv 0$ or $u \equiv \infty$.

All the remaining implications are immediate conseqences of the implications proved up to now. □

## 2. Application to Markov processes

Let again $E$ be a Radon space with $\mathscr{E}$ its $\sigma$-field of universally measurable subsets. Furthermore, let $\mathbb{P} = (P_t)_{t \geqq 0}$ be a right semigroup of sub-Markovian kernels on $(E, \mathscr{E})$, i.e. a semigroup which satisfies the so-called "right hypotheses" (cf. [DM], [Sh]). For example, the transition semigroup of an Hunt or standard process is a right semigroup. From the semigroup $(P_t)_{t \geqq 0}$ we obtain (by Laplace transformation) the sub-Markovian resolvent $\mathbb{U} = (U_\alpha)_{\alpha \geqq 0}$ and the associated set of excessive functions ${}^*\mathscr{H}$. For every function $s \in {}^*\mathscr{H}$ we define a semigroup $\mathbb{P}^{/s} = (P_t^{/s})_{t \geqq 0}$ of kernels on the Radon space $E^{/s} := \{x \in E : 0 < s(x) < \infty\}$ by

$$P_t^{/s} f(x) = \frac{1}{s(x)} \cdot P_t(s \cdot f)(x) \,.$$

$\mathbb{P}^{/s}$ is again a right semigroup of sub-Markovian kernels (called the Doob–Meyer transformation of $\mathbb{P}$ by $s$).

Now let $\bar{E}$ be a Radon space, equipped with its $\sigma$-field $\bar{\mathscr{E}}$ of universally measurable subsets and containing $E$ as an universally measurable subset, and let $\bar{\mathbb{P}}^{/s}$ be a right semigroup of Markovian kernels on $(\bar{E}, \bar{\mathscr{E}})$ such that

$$\bar{P}_t^{/s}(x, F) = P_t^{/s}(x, F) \quad \text{for } x \in E^{/s}, F \subset E^{/s} \quad \text{and}$$

$$\bar{P}_t^{/s}(x, .) = \varepsilon_x \qquad \text{for } x \in \bar{E} \setminus E^{/s} \,.$$

(There are various ways to extend the sub-Markovian semigroup $\mathbb{P}^{/s}$ on $E^{/s}$ to such a Markovian semigroup $\bar{\mathbb{P}}^{/s}$ on $\bar{E}$. The most common way is to choose $\bar{E} = E \cup \{\delta\}$ with some point $\delta$ used as cemetery and to define $\bar{P}_t^{/s}(x, \{\delta\}) = 1 - P_t^{/s}(x, E^{/s})$ for $x \in E^{/s}$.)

Let $\Omega$ be the set of right continuous maps from $\mathbb{R}_+$ to $\bar{E}$ and let $\zeta$ be the lifetime in $E$, i.e. the debut of $\bar{E} \setminus E$. On $\Omega$ we have the filtration $(\mathscr{F}_t^o)_{t \geqq 0}$ generated by the projection maps $(X_t)_{t \geqq 0}$ from $\Omega$ to $\bar{E}$. Since $\bar{\mathbb{P}}^{/s}$ is a right semigroup we obtain for every probability measure $\mu$ on $\bar{E}$ a probability measure $\mathbf{P}^{\mu/s}$ on $\Omega$ such that $(X_t)_{t \geqq 0}$ is a Markov process with initial law $\mu$ and transition semigroup $\bar{\mathbb{P}}^{/s}$. Let $\mathscr{F}_t^{\mu/s}$ be the augmentation of $\mathscr{F}_t^o$ with respect to $(\Omega, \mathscr{F}_\infty^o, \mathbf{P}^{\mu/s})$ and

$$\mathscr{F}_t^* := \bigcap_{\mu, s} \mathscr{F}_t^{\mu/s} \,,$$

where the intersection is taken over all probability measures on $\bar{E}$ and all functions $s \in {}^*\mathscr{H}$. We shall say that a statement holds $\mathbf{P}^*$-a.s. iff it holds $\mathbf{P}^{\mu/s}$-a.s. for every probability measure $\mu$ on $\bar{E}$ and every $s \in {}^*\mathscr{H}$.

**(2.1) Definition.** A map $M: \mathbb{R}_+ \times \Omega \to \bar{\mathbb{R}}_+$ will be called a *multiplicative functional* if for all $t \geqq 0$

(2.1.1) $\quad M_t$ is $\mathscr{F}_t^*$-measurable

and the following holds $\mathbf{P}^*$-a.s.

(2.1.2) $\quad M_{r+t} = M_t \cdot (M_r \circ \theta_t)$ for all $r \geqq 0$;

(2.1.3) $\quad r \to M_r$ is right continuous on $[0, \infty[$.

For (2.1.2) to be well defined it is necessary to give a meaning to the expression $0 \cdot \infty$. We make the convention $0 \cdot \infty = 0$ which is common in measure theory.

(2.2) *Examples.* Typical multiplicative functionals $M$ are:
 a) the Feynman–Kac functional

$$M_t := (I^q)_t := \exp\left( - \int_0^{t+0} q(X_s)\, \mathrm{d}s \right) \tag{2.2.1}$$

for a lower or upper bounded, universally measurable function $q$ on $\bar{E}$ (cf. [St3], Lemma 1.3), or more generally, $M_t = (I^{q^+})_t \cdot (I^{-q^-})_t$ for an arbitrary, universally measurable function $q$ on $\bar{E}$, here again using the convention $0 \cdot \infty = 0$;
 b) the killing functional

$$M_t := I(F)(t) := \begin{cases} 1, & \text{for } t < D(F) \\ 0, & \text{for } t \geqq D(F) \end{cases} \tag{2.2.2}$$

for a Borel set $F \subset \bar{E}$ with $D(F)$ being the debut of $\bar{E} \backslash F$;
 c) the Doob functional

$$M_t := (/h)_t := \frac{h(X_t)}{h(X_0)} \tag{2.2.3}$$

for a finely continuous function $h$ on $\bar{E}$ with $0 < h < \infty$ (where fine continuity is defined with respect to the semigroup $\bar{\mathbb{P}}$).

For a multiplicative functional $M$ we define families $\mathbb{P}^M = (P_t^M)_{t \geqq 0}$ and $\mathbb{U}^M = (U_\alpha^M)_{\alpha \geqq 0}$ of kernels on $(E, \mathscr{E})$ by

$$P_t^M f(x) := \mathbf{E}^x[M_t \cdot f(X_t) \cdot 1_{\{t < \zeta\}}]\,,$$

$$U_\alpha^M f(x) := \int_0^\infty \mathrm{e}^{-\alpha \cdot t} \cdot P_t^M f(x)\, \mathrm{d}t = \mathbf{E}^x \int_0^\zeta \mathrm{e}^{-\alpha \cdot t} \cdot M_t \cdot f(X_t)\, \mathrm{d}t\,.$$

It is easy to see that $\mathbb{P}^M$ is a measurable semigroup of kernels and that therefore $\mathbb{U}^M$ is a resolvent of kernels satisfying the strong resolvent equation (R1) of the first part of this paper. The resolvent $\mathbb{U}^M$ can be considered as a perturbation of the resolvent $\mathbb{U}$ by the multiplicative functional $M$. If $M$ is decreasing (i.e. if $M \leqq 1$, that is, if $M_t \leqq 1$ for all $t \geqq 0$), this perturbation reduces to the usual *subordination* of resolvents (semigroups, processes, . . .).

(2.3) *Remarks.* a) Every decreasing multiplicative functional $M$ can slightly be modified (or "regularized") in order to obtain a decreasing multiplicative functional $\tilde{M}$ (called perfect exact regularization of $M$) which is (perfect and) exact (cf. [Sh], 55.19). The resolvent $\mathbb{U}^{\tilde{M}}$ (obtained as perturbation of $\mathbb{U}$ by $\tilde{M}$ or as regularization of $\mathbb{U}^M$) is *exactly subordinated* to $\mathbb{U}$.

On the other hand, for every resolvent $\mathbb{V}$ which is exactly subordinated to $\mathbb{U}$ there exists an exact multiplicative functional $M$ (unique up to indistinguishability) such that $\mathbb{V} = \mathbb{U}^M$ (cf. [Sh], 56.14).

b) In the case of Brownian motion on $\mathbb{R}^d$, there is a one-to-one correspondence between (equivalence classes of indistinguishable) exact decreasing multiplicative functionals and (suitable equivalence classes of) *measures on* $\mathbb{R}^d$ *charging no polar sets* ([St4], 7.4). The multiplicative functional $M$ corresponds to the measure $\mu$ iff $\mathbb{U}^M$ is the resolvent for the quadratic form

$$u \mapsto \frac{1}{2} \int |\nabla u(x)|^2 \, dx + \int |u(x)|^2 \mu(dx)$$

(resp. for the associated "Schrödinger operator" $-\frac{1}{2}\Delta + \mu$).

The main feature here is that we do not assume that $M$ is decreasing and even not that it is a supermartingale, so in general the perturbed resolvent $\mathbb{U}^M$ is not sub-Markovian.

We are interested in boundedness criteria for the potential kernel $U^M$ and also in such criteria for the *gauge function* $\Gamma^M$ on $E$:

$$\Gamma^M : x \mapsto \mathbf{E}^x[M_\zeta \cdot 1_{\{\zeta < \infty\}}] .$$

Its supremum norm $\|\Gamma^M\| := \sup_{x \in E} \Gamma^M(x)$ is the so-called gauge.

(2.4) *Remark.* Let $E$ be a Borel set in $\mathbb{R}^d$ and $q$ be an universally measurable function on $\mathbb{R}^d$. To get the "gauge function for $(E, q)$" in the sense of K.L. Chung (cf. [C1], [C2]), we have to choose $\bar{\mathbb{P}}$ (resp. $\mathbb{P}$) to be the Brownian semigroup on $\bar{E} = \mathbb{R}^d$, stopped (resp. killed) at the debut of $\mathbb{R}^d \backslash E$ and we have to choose $M$ to be the Feynman–Kac functional $I^q$. Then indeed

$$\Gamma^M(x) = \mathbf{E}^x\left[ e^{-\int_0^D q(X_s)\,ds} \cdot 1_{\{D < \infty\}} \right]$$

where $\mathbf{E}^x$ denotes expectation with respect to Brownian motion, starting in $x \in \mathbb{R}^d$, or—which comes to the same thing—with respect to Brownian motion, starting in $x \in \mathbb{R}^d$ and stopped at $D := D(\mathbb{R}^d \backslash E)$, the debut of $\mathbb{R}^d \backslash E$. (Note that in our context, it is more convenient to use the debut of the set $\mathbb{R}^d \backslash E$ instead of its hitting time.)

In order to obtain weak conditions, which imply the finiteness of the gauge or the boundedness of the potential kernel, we generalize the gauge function in two directions. First of all we go from the gauge function $\Gamma^M$ on $E$ to the *conditional gauge function* $\gamma^M$ on $E \times {}^*\mathscr{H}$:

$$\gamma^M : (x, s) \mapsto \mathbf{E}^{x/s}[M_\zeta \cdot 1_{\{\zeta < \infty\}}] .$$

Of course, we get back $\Gamma^M$ as $\gamma^M(., 1)$.

In order to obtain a second generalization of the gauge function we define the following *balayage kernel* $H^M$ as a kernel from $(E, \mathscr{E})$ to $(\bar{E}, \bar{\mathscr{E}})$ by

$$H^M\varphi(x) := \mathbf{E}^x[M_\zeta \cdot \varphi(X_\zeta) \cdot 1_{\{\zeta<\infty\}}] .$$

Obviously, the gauge function $\Gamma^M$ is nothing else than (the "equilibrium potential") $H^M 1$, hence the gauge $\|\Gamma^M\|$ is also given by the operator norm $\|H^M\|$ of the balayage kernel, regarded as map $H^M : \bar{\mathscr{E}}_b \to \mathscr{E}_b$.

(2.5) *Remark.* To give an idea of this balayage kernel $H^M$ and of the above-mentioned conditional gauge function $\gamma^M$, let us consider the special situation of [CrFZ] (or [Z1], [Fa]). That is, $E$ is a bounded domain in $\mathbb{R}^d$ and $\mathbb{P}$ is the semigroup for a diffusion killed at the debut of $\mathbb{R}^d \backslash E$. In this case one can take $\bar{E}$ to be the closure of $E$ in $\mathbb{R}^d$ (or to be $\mathbb{R}^d$ itself) and $\bar{\mathbb{P}}$ to be the semigroup for the diffusion stopped at the debut of $\bar{E}\backslash E$. Then the measures $H^M(x, .)$ for $x \in E$ are carried by the boundary $\partial E$ of $E$. (On the other hand, if one chooses $\bar{E} = E \cup \{\delta\}$, then all the measures $H^M(x, .)$ are carried by $\{\delta\}$, in particular on $E$ we have $H^M\varphi \equiv \varphi(\delta) \cdot \Gamma^M$ for all $\varphi \in \bar{\mathscr{E}}_+$.) Now let $u(.,.)$ be the Green function and $h(.,.)$ be the Poisson (or Martin) kernel function for the diffusion on $E$, provided these functions are well defined (which certainly is the case if $E$ has Lipschitz boundary and if the diffusion is generated by a second order, uniformly elliptic operator in divergence form like in [CrFZ]).

Typical excessive functions $s \in {}^*\mathscr{H}$ are then given by $s = u(., y)$ for $y \in E$ and by $s = h(., z)$ for $z \in \partial E$. By means of the conditional gauge function $\gamma^M$ on $E \times {}^*\mathscr{H}$ we can define functions

$$(x, y) \mapsto \mathbf{E}^{x,y}[M_\zeta \cdot 1_{\{\zeta<\infty\}}] := \gamma^M(x, u(., y))$$

on $E \times E$ and

$$(x, z) \mapsto \mathbf{E}^{x,z}[M_\zeta \cdot 1_{\{\zeta<\infty\}}] := \gamma^M(x, h(., z))$$

on $E \times \partial E$. These functions are precisely the "conditional gauge functions" considered in [Z1], [Fa], [Z2] and [CrFZ].

**(2.6) Theorem.** *Let $M$ be a multiplicative functional and let $\mathbb{U}^M$ satisfy (R2) and (R3). Then each of the following conditions implies $\|U^M\| < \infty$:*

(2.6.1) $\quad 0 \not\equiv U^M f \not\equiv \infty$ *on $E$ for some $f \in \mathscr{E}_+$;*

(2.6.2) $\quad 0 \not\equiv H^M\varphi \not\equiv \infty$ *on $E$ for some $\varphi \in \bar{\mathscr{E}}_+$;*

(2.6.3) $\quad 0 \not\equiv \gamma^M(., s) \not\equiv \infty$ *on $E^{/s}$ for some $s \in {}^*\mathscr{H}$.*

(2.7) *Remarks.* a) In order to obtain the above conditional gauge condition (2.6.3) we had to require the $\mathscr{F}_t^*$-measurability of $M_t$ ($t \geqq 0$) in (2.1.1) and we could only allow $\mathbf{P}^*$-exceptional sets in (2.1.2) and (2.1.3). Since all of our examples (2.2.1)–(2.2.3) satisfy these conditions we don't consider them as serious restrictions (though it should be clear that the other statements hold for all multiplicative functionals in the usual sense).

b) The condition (2.6.2) reduces to the gauge condition

$$0 \not\equiv \Gamma^M \not\equiv \infty \quad \text{on } E$$

if $\bar{E} = E \cup \{\delta\}$. In general, however, (2.6.2) is much weaker. So, for our purpose, it turns out reasonable not to require $\bar{E} = E \cup \{\delta\}$, but to allow $\bar{E}$ to be any larger set.

*Proof of Theorem (2.6).* Let $^*\mathscr{H}^M$ be the set of excessive functions for the resolvent $\mathbb{U}^M$, and let $\mathscr{H}^M$ be the set of invariant functions for $\mathbb{U}^M$. It should be remarked that all of the above assertions can be deduced from the statement (1.6.2) which in our situation reads as follows

$$^*\mathscr{H}^M \neq \mathscr{H}^M \Rightarrow \|U^M\| < \infty \ .$$

What we have to do in all the three cases is to prove that the given function is excessive for $\mathbb{U}^M$, but not invariant. In the case of (2.6.1) we omit this, since the assertion is yet proved in Theorem (1.3).

(2.6.2): For a function $u \in \mathscr{E}_+$ to be in $^*\mathscr{H}^M$ it is obviously sufficient to satisfy $P_t^M u \leqq u$ for $t > 0$ and $\lim_{t\to 0} P_t^M u = u$. But for $x \in E$ we have $\zeta > 0$ $\mathbf{P}^x$-a.s. and therefore

$$P_t^M \circ H^M \varphi(x) = \mathbf{E}^x[M_t \cdot 1_{\{t<\zeta\}} \cdot \mathbf{E}^{X_t}[M_\zeta \cdot \varphi(X_\zeta)]]$$

$$= \mathbf{E}^x[M_\zeta \cdot \varphi(X_\zeta) \cdot 1_{\{t<\zeta\}}] \nearrow H^M \varphi(x) \quad (t \searrow 0) \ .$$

Thus $H^M \varphi \in {}^*\mathscr{H}^M$. Now take some $x \in E$ with $H^M \varphi(x) < \infty$ and assume $H^M \varphi \in \mathscr{H}^M$. Then for Lebesgue almost all $t > 0$

$$0 = H^M \varphi(x) - P_t^M \circ H^M \varphi(x) = \mathbf{E}^x[M_\zeta \cdot \varphi(X_\zeta) \cdot 1_{\{\zeta \leqq t\}}]$$

and therefore $H^M \varphi(x) = 0$. Under (R3), however, this implies $H^M \varphi \equiv 0$.

(2.6.3): First of all we will prove that the following function $\tilde{u}$ on $E$ is supermedian for the semigroup $\mathbb{P}^M$:

$$\tilde{u}(x) = \begin{cases} s(x) \cdot \gamma^M(x, s), & \text{if } s(x) < \infty \ , \\ \infty \ , & \text{if } s(x) = \infty \ . \end{cases}$$

Then obviously $u := \lim_{t\to 0} P_t^M \tilde{u}$ is an excessive function for the semigroup $\mathbb{P}^M$ (hence $u \in {}^*\mathscr{H}^M$) which in a second step will be shown to coincide with $\tilde{u}$ on $\{s < \infty\}$. Finally, we prove that $u \in \mathscr{H}^M$ implies $u \equiv 0$ or $u \equiv \infty$ which contradicts the assumption (2.6.3).

a) For $x \in \{s = \infty\}$ there is nothing to prove. Let $F := \{x \in E : s(x) < \infty\}$. Then $F$ is an absorbing set, i.e. for all $x \in F$ we have $\mathbf{P}^x$-a.s.

$$X_t \in F \quad \text{for all } t < \zeta \ .$$

Thus for $x \in F$

$$P_t^M \tilde{u}(x) = \mathbf{E}^x[M_t \cdot \tilde{u}(X_t) \cdot 1_{\{t<\zeta\}}] =$$

$$= \mathbf{E}^x[M_t \cdot s(X_t) \cdot \gamma^M(X_t, s) \cdot 1_{\{t<\zeta\}}] =$$

$$= s(x) \cdot \mathbf{E}^{x/s}[M_t \cdot \gamma^M(X_t, s) \cdot 1_{\{t<\zeta\}}] \ ,$$

here essentially using the $\mathscr{F}_t^*$-measurability of $M_t$ ([DM], XVI. 28–34). From the Markov property of the $s$-conditioned process and from the multiplicativity of $M$ we obtain

$$s(x)\cdot \mathbf{E}^{x/s}[M_t\cdot \mathbf{E}^{X_t/s}[M_\zeta\cdot 1_{\{\zeta<\infty\}}]\cdot 1_{\{t<\zeta\}}] = s(x)\cdot \mathbf{E}^{x/s}[M_\zeta\cdot 1_{\{t<\zeta<\infty\}}] \leqq \tilde{u}(x)\,.$$

b) For $x\in\{s<\infty\}$ we obtain from a) that

$$u(x) := \lim_{t\to 0} P_t^M \tilde{u}(x) = s(x)\cdot \mathbf{E}^{x/s}[M_\zeta\cdot 1_{\{0<\zeta<\infty\}}]\,.$$

Thus $u(x)=\tilde{u}(x)$ since $\zeta>0$ $\mathbf{P}^{x/s}$-a.s. for $x\in E$.

c) Finally, assume $u\not\equiv\infty$ (which implies $\tilde{u}\not\equiv\infty$) and $u\in\mathscr{H}^M$, i.e.

$$u=\alpha\cdot U_\alpha^M u \quad \text{for all } \alpha>0\,.$$

Since $\tilde{u}$ coincides with $u$ on the absorbing set $\{s<\infty\}$ we obtain

$$\tilde{u}=\alpha\cdot U_\alpha^M \tilde{u} \quad \text{on } \{s<\infty\} \quad \text{for all } \alpha>0\,.$$

Thus for every $x\in\{\tilde{u}<\infty\}\subset\{s<\infty\}$ and Lebesgue almost all $t>0$ we get

$$0=\tilde{u}(x)-P_t^M\tilde{u}(x)=\mathbf{E}^x[M_\zeta\cdot 1_{\{\zeta<t\}}]$$

and therefore $\tilde{u}(x)=0$. This, however, implies $u(x)=0$ and (by (R3)) $u\equiv 0$. □

## 3. Compatible multiplicative functionals

We recall that up to now we have imposed no explicit finiteness or boundedness assumptions on the multiplicative functionals. In particular, we emphasize that Theorem (2.6) holds true for arbitrary multiplicative functionals $M$.

The point, however, is that we have to check the assumptions on the resolvents $\mathbb{U}^M$. In order to obtain sufficient criteria for compactness (and also for irreducibility) we introduce the notion of compatible multiplicative functionals.

**(3.1) Definition.** The multiplicative functional $M$ will be called *compatible* if there exist two multiplicative functionals $K$ and $L$ such that

$$M_t=K_t\cdot L_t,\quad K_t\leqq 1 \quad\text{and}\quad L_t\geqq L_0=1 \quad \text{for all } t\geqq 0$$

and such that the following compatibility condition is fulfilled

$$\left\|\mathbf{E}^{\cdot}\int_0^\zeta e^{-\alpha\cdot t}\cdot K_t\,\mathrm{d}L_t\right\|<\infty \quad \text{for some } \alpha\in\mathbb{R} \tag{3.1.1}$$

(where $\|\,.\,\|$ denotes the supremum norm on $E$).

(3.2) *Remarks.* a) Given an arbitrary multiplicative functional $K\leqq 1$, the compatibility condition for $K\cdot L$ is obviously fulfilled if the multiplicative functional $L\geqq 1$ satisfies

$$\left\|\mathbf{E}^{\cdot}\int_0^\zeta e^{-\alpha\cdot t}\,\mathrm{d}L_t\right\|<\infty \quad \text{for some } \alpha\in\mathbb{R}\,, \tag{3.2.1}$$

in particular, if $L$ satisfies the so-called (generalized) Kato condition

(3.2.2) $$\lim_{\alpha\to\infty} \left\| \mathbf{E}^{\cdot} \int_0^{\zeta} e^{-\alpha\cdot t}\, dL_t \right\| = 0\,.$$

If $L$ is of the special form

$$L_t = (I^{-q})_t = \exp\left( \int_0^{t+} q(X_s)\, ds \right)$$

with a function $q \in \mathscr{E}_+$, then (3.2.2) is equivalent to the usual Kato condition

(3.2.3) $$\lim_{t\to 0} \left\| \mathbf{E}^{\cdot} \int_0^{t\wedge\zeta} q(X_s)\, ds \right\| = 0$$

(cf. [St3], Sect. 4.A).

b) To illustrate the difference between the above conditions (3.1.1) and (3.2.2), let $\mathbb{P}$ be the Brownian semigroup on $E = \mathbb{R}^d$ ($d \geqq 2$) and let $M$ be given by

$$M = I^q \quad \text{with} \quad q(x) = \|x\|^{-k} \cdot \left( \frac{1}{2} - \sin(\|x\|^{-k}) \right).$$

Then $M$ is a compatible multiplicative functional for all $k > 0$ ([St3], Beispiel 4.10; see also [St5] for further examples). But $L := I^{-q^-}$ satisfies (3.2.2) if and only if $0 < k < 2$. This can easily be seen because in the case of Brownian motion a multiplicative functional $L \geqq 1$ satisfies the Kato condition (3.2.2) iff the associated additive functional $\log L$ generates a bounded and uniformly continuous $\alpha$-potential (for some—hence all—$\alpha > 0$), i.e. iff the function

$$x \mapsto u_L^\alpha(x) := \mathbf{E}^x \int_0^{\zeta} e^{-\alpha\cdot t}\, d(\log L_t)$$

on $\mathbb{R}^d$ is bounded and uniformly continuous ([St3], Korollar 4.7).

c) There are two reasons to consider multiplicative functionals which are compatible. First of all, for compatible multiplicative functionals $M = K \cdot L$ the conditions (R2) and (R3) on the resolvent $\mathbb{U}^{K\cdot L}$ (which is in general not sub-Markovian) reduce (according to Proposition (3.3) below) to corresponding conditions on the sub-Markovian resolvent $\mathbb{U}^K$. This is a remarkable fact! The conditions on $\mathbb{U}^K$ are easy to check, see Remark (3.5).

Secondly, for compatible multiplicative functionals $M$ the boundedness of the potential kernel $U^M$ implies the boundedness of the balayage kernel $H^M$ (due to 3.3.1)). Thus in this case every result which states that $U^M$ is bounded also states that the gauge $\|\Gamma^M\|$ is finite. This is why we call every theorem which yields sufficient conditions for $\|U^M\| < \infty$ a gauge theorem.

**(3.3) Proposition.** *For a compatible multiplicative functional $M = K \cdot L$ the following holds:*

(3.3.1) $\|U^M\| < \infty \Rightarrow \|H^M\| < \infty$.

(3.3.2) $\mathbb{U}^{K\cdot L}$ *satisfies* (R2) $\Leftrightarrow$ $\mathbb{U}^K$ *satisfies* (R2).

(3.3.3) $\mathbb{U}^{K\cdot L}$ *satisfies* (R3) $\Leftrightarrow$ $\mathbb{U}^K$ *satisfies* (R3).

*Proof.*

(3.3.1): For $x \in E$ we have

$$H^M 1(x) = \mathbf{E}^x[K_\zeta \cdot L_\zeta \cdot 1_{\{\zeta < \infty\}}] = \mathbf{E}^x\left[K_\zeta \cdot \left(1 + \int_0^\zeta \mathrm{d}L_s\right) \cdot 1_{\{\zeta<\infty\}}\right] \leqq$$

$$\leqq 1 + \mathbf{E}^x \int_0^\zeta K_s \,\mathrm{d}L_s \,.$$

Now choose some $\alpha > 0$ such $C := \| \mathbf{E}^{\cdot} \int_0^\zeta \mathrm{e}^{-\alpha \cdot s} \cdot K_s \mathrm{d}L_s \| < \infty$. Then

$$\mathbf{E}^x \int_0^\zeta K_s \mathrm{d}L_s \leqq C + \mathbf{E}^x \int_0^\zeta (1 - \mathrm{e}^{-\alpha \cdot s}) \cdot K_s \mathrm{d}L_s =$$

$$= C + \alpha \cdot \mathbf{E}^x \int_0^\zeta \int_0^s \mathrm{e}^{\alpha \cdot t} \,\mathrm{d}t \cdot \mathrm{e}^{-\alpha \cdot s} \cdot K_s \mathrm{d}L_s =$$

$$= C + \alpha \int_0^\infty \mathrm{e}^{\alpha \cdot t} \cdot \mathbf{E}^x \int_t^\zeta \mathrm{e}^{-\alpha \cdot s} \cdot K_s \mathrm{d}L_s \,\mathrm{d}t =$$

$$= C + \alpha \cdot \mathbf{E}^x \int_0^\zeta K_t \cdot L_t \cdot \mathbf{E}^{X_t} \int_0^\zeta \mathrm{e}^{-\alpha \cdot s} \cdot K_s \mathrm{d}L_s \,\mathrm{d}t \leqq$$

$$\leqq C \cdot (1 + \alpha \cdot \| U^M \|) \,.$$

(3.3.2): We prove the implication "$\Leftarrow$". The converse is less important and the proof is quite similar. First of all we remark that if $U_\alpha^K$ is compact for some $\alpha \geqq 0$ then (by the resolvent equation (R1)) $U_\beta^K$ is compact for all $\beta \geqq \alpha$. So let us choose an $\alpha > 0$ such that the operator $U_\alpha^K$ on $\mathscr{E}_b$ is compact and that the operator

$$W_\alpha : f \mapsto \mathbf{E}^{\cdot} \int_0^\zeta \mathrm{e}^{-\alpha \cdot t} \cdot K_t \cdot f(X_t) \,\mathrm{d}L_t$$

on $\mathscr{E}_b$ is bounded. To simplify notation let $\tilde{K}_t := \exp(-\alpha \cdot t) \cdot K_t \cdot 1_{\{t<\zeta\}}$. Then

$$U_\alpha^{K \cdot L} f(x) - U_\alpha^K f(x) = \mathbf{E}^x \int_0^\infty (L_t - 1) \cdot \tilde{K}_t \cdot f(X_t) \,\mathrm{d}t =$$

$$= \mathbf{E}^x \int_0^\infty \int_s^\infty \tilde{K}_t \cdot f(X_t) \,\mathrm{d}t \,\mathrm{d}L_s =$$

$$= \mathbf{E}^x \int_0^\infty \tilde{K}_s \cdot \mathbf{E}^{X_s} \int_0^\infty \tilde{K}_t \cdot f(X_t) \,\mathrm{d}t \,\mathrm{d}L_s =$$

$$= W_\alpha \circ U_\alpha^K f(x) \,.$$

Thus $U_\alpha^{K \cdot L} = (I + W_\alpha) \circ U_\alpha^K$ is the composition of a bounded and a compact operator and is thereby itself compact.

(3.3.3): Since $K_t \cdot L_t > 0 \Leftrightarrow K_t > 0$ (due to our convention $0 \cdot \infty = 0$) we have for all $\alpha \geqq 0$, all $x \in E$ and all $f \in \mathscr{E}_+$:

$$U_\alpha^{K \cdot L} f(x) > 0 \Leftrightarrow U_\alpha^K f(x) > 0 \,. \qquad \square$$

**(3.4) Theorem.** *Let $M = K \cdot L$ be a compatible multiplicative functional such that $\mathbb{U}^K$ satisfies (R2) and (R3) and assume $H^M \not\equiv 0$. Then the following statements are equivalent:*

(3.4.1) $\quad {}^*\mathscr{H}^M \neq \mathscr{H}^M$;

(3.4.2) $\quad 0 \not\equiv \gamma^M(.,s) \not\equiv \infty$ *on $E^{/s}$ for some $s \in {}^*\mathscr{H}$;*

(3.4.3) $\quad 0 \not\equiv H^M \varphi \not\equiv \infty$ *on $E$ for some $\varphi \in \bar{\mathscr{E}}_+$;*

(3.4.4) $\quad 0 \not\equiv U^M f \not\equiv \infty$ *on $E$ for some $f \in \mathscr{E}_+$;*

(3.4.5) $\quad \| H^M \| < \infty$;

(3.4.6) $\quad \| U^M \| < \infty$;

(3.4.7) $\quad {}_*\mathscr{H}_b^M = \{0\}$.

*Proof.* Since $M = K \cdot L$ is assumed to be compatible, the compactness and irreducibility of $\mathbb{U}^K$ imply that also $\mathbb{U}^M$ is compact and irreducible. Hence, Theorem (1.5) (which yields the equivalence of (3.4.1), (3.4.6) and (3.4.7)) and Theorem (2.6) (which states that each of (3.4.2), (3.4.3), (3.4.4) implies (3.4.6)) are applicable. According to the above Proposition (3.3), property (3.4.6) implies (3.4.5) and the latter implies (3.4.2) and (3.4.3) (since we assumed $H^M 1 \not\equiv 0$). But an easy calculation shows that under $H^M 1 \not\equiv 0$ we also have $U^M 1 \not\equiv 0$, hence, (3.4.6) implies (3.4.4), too. This finishes the proof. □

(3.5) *Remark.* It remains to check the assumptions (R2) and (R3) on $\mathbb{U}^K$ for a decreasing (!) multiplicative functional $K \leqq 1$. (The assumption $H^M 1 \not\equiv 0$ in Theorem (3.4) is only to avoid pathologies.) In [St3] we have deduced various (sufficient and/or necessary) criteria for $\mathbb{U}^K$ to satisfy (R2) or (R3). However, in this article we can only mention some special results.

a) For a multiplicative functional $K \leqq 1$ satisfying $U^K 1 > 0$ (that is, $K_0 > 0$ $\mathbf{P}^x$-a.s. for all $x \in E$) the operator $U_\alpha^K$ (for $\alpha \geqq 0$) is irreducible under each of the following conditions (cf. [St3], 3.13, 3.18):

- $E$ is a fine domain;
- $E$ is a domain and $(E, {}^*\mathscr{H})$ defines an elliptic harmonic space (in the sense of Brelot).

However, none of these conditions is necessary as it can be seen by the symmetric stable processes of index $2r$ for $0 < r < 1$ on a non-empty open subset $E$ of $\mathbb{R}^d$. In this case the resolvent $\mathbb{U}$ satisfies (R3) without any further assumption on $E$ (cf. [St3], Beispiel 3.16).

b) The operator $U_\alpha^K$ (for some—hence all—$\alpha > 0$) is compact provided for some multiplicative functional $\tilde{K}$ with $K \leqq \tilde{K} \leqq 1$ (eg. for $\tilde{K} \equiv 1$) one of the following conditions is fulfilled:

- $U_\alpha^{\tilde{K}}$ is a compact operator;
- $E$ is a compact set and $\mathbb{U}^{\tilde{K}}$ is a strong Feller resolvent on $E$, i.e.

$$U_\alpha^{\tilde{K}}: \mathscr{E}_b \to \mathscr{C} \quad (\alpha > 0)$$

where $\mathscr{C}$ denotes the set of continuous functions on $E$, or $\bar{E}$ is a compact set and $\bar{\mathbb{U}}^{\tilde{K}}$ is a strong Feller resolvent on $\bar{E}$ (cf. [DM], IX.18).

- $E$ is a locally compact set, there exists an increasing sequence of compact sets $(F_n)_n \uparrow E$ such that the operators $U_\alpha^{\tilde{K}\cdot I(F_n)}$ (for $n \in \mathbb{N}$) are compact and

$$\lim_{x\to\infty} U_\alpha^K 1(x) = 0 ,$$

cf. [St3], 6.4.
(Let us note parenthetically that $U_\alpha^{\tilde{K}\cdot I(F_n)}$ is compact (for $n \in \mathbb{N}$) if $\mathbb{U}^{\tilde{K}}$ is a strong Feller resolvent on $E$ vanishing at infinity. If moreover $K \leqq I^q$ for some $q \in \mathscr{E}_+$ with $\lim_{x\to\infty} q(x) = \infty$, then $\lim_{x\to\infty} U_\alpha^K 1(x) = 0$, cf. [St3], 6.2 resp. 6.17.1).

(3.6) *Final remarks.* a) Let us recall that the quantity $\|H^M\|$ in (3.4.5) is nothing else than the gauge $\|\Gamma^M\|$. Furthermore, note that the finiteness of the gauge function $\Gamma^M$ in one point $x \in E$ obviously implies (3.4.2) as well as (3.4.3). Hence, Theorem (3.4) indeed contains *the gauge theorem* in the sense of K.L. Chung. Namely, under the assumptions of Theorem (3.4)

$$\Gamma^M \not\equiv \infty \Rightarrow \|\Gamma^M\| < \infty , \tag{3.6.1}$$

that is, *if the gauge function $\Gamma^M$ is finite at one point $x \in E$ then it is already bounded in $\mathbb{R}^d$.*

Moreover, in the (rather general) situation of Theorem (3.4) we even got one implication of the famous *conditional gauge theorem* (cf. [Z1], [Fa], [Z2], [CrFZ])

$$0 \not\equiv \gamma^M(.,s) \not\equiv \infty \quad \text{on } E^{/s} \quad \text{for some } s \in {}^*\mathscr{H} \Rightarrow \|\Gamma^M\| < \infty. \tag{3.6.2}$$

(The other implication of the conditional gauge theorem

$$\|\Gamma^M\| < \infty \Rightarrow \sup_{\substack{x\in E^{/s}\\ s\in {}^*\mathscr{H}}} \gamma^M(x,s) < \infty$$

seems to depend essentially on very restrictive assumptions on $E$, $\mathbb{P}$ and $M$.)

b) Now let us have a look at the assumptions in Theorem (3.4) and let us compare them with the assumptions in the gauge theorems of [CR1], [C1], [C2] resp. in the conditional gauge theorems of [Fa], [Z1], [Z2]. In all these cases, we have the particular situation developed in Remark (2.4). That is, $E$ is a Borel set in $\mathbb{R}^d$, $\mathbb{P}$ is the semigroup for the Brownian motion on $E$, killed at the debut of $\mathbb{R}^d \backslash E$, and $M$ is the Feynman–Kac functional $I^q$ associated with an universally measurable function $q$ on $E$. Moreover, one imposes the following additional conditions:

- $E$ is open and connected;
- $E$ is bounded (or at least of finite Lebesgue measure);
- $q^+$ and $q^-$ satisfy the Kato condition (3.2.3).

By means of the criteria (mentioned above in (3.5)) for irreducibility and/or compactness, it should be easy to see that

- the fact that the set $E$ is open and connected implies that the resolvent $\mathbb{U}^{q^+} := \mathbb{U}^{I^{q^+}}$ is irreducible (since $q^+$ is assumed to satisfy the Kato condition which obviously is much stronger than the condition $U^{q^+}1 > 0$ on $E$);
- the boundedness of the set $E$ (or the finiteness of its Lebesgue measure) implies that the resolvent $\mathbb{U}$ is compact; the compactness of $\mathbb{U}$ implies the compactness of $\mathbb{U}^{q^+}$ (for arbitrary $q^+ \geqq 0$);

- the Kato condition for $q^-$ implies the compatibility of $M = I^{q^+} \cdot I^{-q^-}$ (for arbitrary $q^+ \geqq 0$).

Note that, of course, none of the conditions on $E$ and $q^\pm$ mentioned above is necessary in order to imply our essential assumptions in Theorem (3.4), namely, irreducibility and compactness of $\mathbb{U}^{q^+}$ and compatibility of $M = I^q$. For instance,

- irreducibility of $\mathbb{U}^{q^+}$ still holds if we merely assume that $E$ is a *fine* domain (instead of a domain);
- compactness of $\mathbb{U}^{q^+}$ still holds (for arbitrary $q^+ \geqq 0$) if we merely assume that $E$ is Green-compact, i.e. $\lim_{x\to\infty} \mathrm{E}^x[D(\mathbb{R}^d \backslash E)] = 0$ (instead of having finite Lebesgue measure); on the other hand, compactness of $\mathbb{U}^{q^+}$ holds for arbitrary Borel sets $E$ if we assume that $\lim_{x\to\infty} q^+(x) = \infty$;
- compatibility of $I^q$ also holds for highly singular, oscillating potentials like in (3.2.b).

We emphasize that (due to Theorem (3.4) and the above remarks) the gauge theorem (for Brownian motion on $E \subset \mathbb{R}^d$) holds in many situations where *Harnack's inequality* fails to be true!

Finally, it should be clear that all of the above results remain true if we replace the Brownian motion on $E \subset \mathbb{R}^d$ by the diffusion generated by a second order, uniformly elliptic differential operator $A$ on $\mathbb{R}^d$ of the form $A = \sum a_{ij} \cdot \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i \cdot \frac{\partial}{\partial x_i}$ with bounded continuous coefficients or of the form $A = \sum \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} + \sum b_i \cdot \frac{\partial}{\partial x_i}$ with bounded measurable coefficients (since such a diffusion defines a strong Feller resolvent and also an elliptic harmonic space, cf. [Kr], Th. 2′).

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