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Probability Theory and Related Fields

Quantum and non-causal stochastic calculus

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Summary. The quantum stochastic calculus initiated by Hudson and Parthasarathy, and the non-causal stochastic calculus originating with the papers of Hitsuda and Skorohod, are two potent extensions of the Itô calculus, currently enjoying intensive development. The former provides a quantum probabilistic extension of Schrödinger's equation, enabling the construction of a Markov process for a quantum dynamical semigroup. The latter allows the treatment of stochastic differential equations which involve terms which anticipate the future. In this paper the close relationship between these theories is displayed, and a non-causal quantum stochastic calculus, already in demand from physics, is described.

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0 Introduction

Close scrutiny of the quantum stochastic integrals of Hudson and Parthasarathy [HuP, Par] reveals that each may be obtained from a combination of *classical* operations – the Hitsuda–Skorohod integral ([Hit, Sko] see e.g. [NuZ, NuP]) and the gradient operator on Wiener space (see e.g. [Zak]). In their action on Fock space – more precisely, Guichardet space (see below) – both these operations take a particularly simple form; and, due to a combinatorial property of Fock space (the $\not{f}$-Lemma, see below), are easy to work with. This is illustrated by an elementary (Hilbert space) proof of the mutual adjointness of gradient and integral, first established by Gaveau and Trauber [GaT]. Many other results in the non-causal calculus are made simple by exploiting combinatorial properties of Guichardet-Fock space. The Quantum Itô Lemma [HuP] is seen as a corollary of the so-called Skorohod isometry for non-causal integrals. Moreover, exploiting these operations, one is led to a natural formulation of non-adapted integrals – and, with equal ease, of multiple integrals – in the quantum context. Demands for a quantum calculus able to deal with anticipating integrands have come from quantum optics. By employing a form of non-adapted calculus Maassen and Robinson are able to account for the spectral shape of an atom made fluorescent by a laser beam tuned to a transition frequency of the atom [RoM]. Barchielli's work on input-output channels, and electron shelving also involves anticipating processes [Bar]. Generalised quantum stochastic integrals have also been constructed by Belavkin [Bel].

In the Hudson–Parthasarathy calculus there are three fundamental processes: creation (A_t^*), preservation (Λ_t) and annihilation (A_t). The annihilation integral is the simplest of the corresponding stochastic integrals. All operators act on a domain of exponential vectors ε_φ (defined in (1.1)) in Fock space. Exponential vectors correspond, under the natural isomorphism between Fock space and *Wiener space* (expressed in (0.3)), with stochastic exponentials: $\exp\{\int \varphi\, dB - \frac{1}{2}\int \varphi^2\}$. The formal eigen-relation $dA\varepsilon_\varphi = \varphi(t)\varepsilon_\varphi\, dt$ led these authors to the formula

$$\int_0^t F\, dA\varepsilon_\varphi = \int_0^t \varphi(s)\, F(s)\varepsilon_\varphi\, ds \tag{0.1}$$

for an operator-valued process $\{F(s): s \geqq 0\}$ which satisfies the condition of *local square-integrability* of the (Hilbert space-valued) map $s \mapsto F(s)\varepsilon_\varphi$. In order to define the *creation integral* $\int_0^t F\, dA^*$, beyond the case of simple integrands, estimates were sought. These were obtained by imposing two conditions. Firstly the integrand F should be *adapted* in an operator sense: for each t, $F(t)$ acts non-trivially only in *Fock space up to time t*. Secondly the increments of the integrator should be in the Itô sense – namely *future pointing*. Under these conditions, the commutation relations for creation and annihilation operators, together with Gronwall's Lemma, give the estimate

$$\left\| \int_0^t F\, dA^* \varepsilon_\varphi \right\|^2 \leqq 2e^{\|\varphi\|^2} \int_0^t \|F(s)\varepsilon_\varphi\|^2\, ds\,.$$

This permits an extension by continuity of the creation integral to locally square integrable F. Furthermore the preservation integral yields to the same treatment, once the test functions φ are restricted to be locally essentially bounded.

The adjoint relations

$$\left(\int F\, dA^*\right)^* \supset \int F^*\, dA; \quad \left(\int F\, d\Lambda\right)^* \supset \int F^*\, d\Lambda; \quad \left(\int F\, dA\right)^* \supset \int F^*\, dA^*$$

– valid for all reasonably well-behaved integrands – may be *explained* by the commutativity of each of $dA^*(s)$, $d\Lambda(s)$ and $dA(s)$ with $F(s)$, at each instant s (and the self-adjointness of the preservation process). Adaptedness and the Itô convention, together obviate the need for separate consideration of integrals of the form $\int dAF$ etc (cf. Clifford and Fermi theories [BSW 1.2, ApH, L1] and the free stochastic calculus [KuS], Example 4.2 below).

The key observation in the present work is as follows: the creation integral of F acting on an exponential vector ε_φ is nothing but the Hitsuda–Skorohod (H–S) integral of the classical process obtained by letting $F(\cdot)$ act on ε_φ:

$$\int F\, dA^* \varepsilon_\varphi = \mathscr{S}(F(\cdot)\varepsilon_\varphi)\,. \tag{0.2}$$

It is worth remarking that the operator-adaptedness assumption on F does not help in the least in giving sense to the right-hand side. $F(\cdot)\varepsilon_\varphi$ itself will not be adapted (in the classical sense) unless $\varphi = 0$. The H–S integral is an extension of the Itô integral to non-adapted integrands. The cost of dropping operator-adaptedness is the imposition of a certain *smoothness assumption*. In Fock space (as opposed to Wiener space) this amounts to a growth restriction as one moves up through the *particle levels*.

The H–S formulation of the creation integral (0.2) suggests that the natural class of integrands is those operator processes F for which the classical process $F(\cdot)k$ is Skorohod-integrable (for a reasonable (dense) family of Fock vectors/Wiener functionals k):

$$A^*(F)k = \mathscr{S}(F(\cdot)k) .$$

Fortunately the H–S integral, when formulated in Guichardet–Fock space, is beautifully simple. We have gained ground on two counts: an extension of the theory is effected by a simplification; the creation integral now being defined directly, without recourse to a limiting procedure.

What about the other integrals? What is the appropriate extension preserving the desirable adjoint relations

$$A(F)^* \supset A^*(F^*); \quad \Lambda(F)^* \supset \Lambda(F^*); \quad A^*(F)^* \supset A(F^*)?$$

(0.1) is already non-adapted but, in the new context, exponential vectors no longer play such a central role. The key here is to invoke the gradient operator on Wiener functionals. This also takes a strikingly simple form in Fock space. If the formal relation $dA = a(t)dt$ is taken too seriously one is confronted with the problematic domain of the unsmeared annihilation operators $\{a(t)\colon t \geqq 0\}$. However the gradient ∇ is very respectable as an operator $\mathscr{H} \mapsto \mathscr{H} \otimes L^2(\mathbb{R}_+)$ (where $\mathscr{H}$ is Wiener–Fock space). The relation (0.1) then reads

$$\int_0^t F\, dA\, \varepsilon_\varphi = \int_0^t F(s) \nabla_s\, \varepsilon_\varphi\, ds ,$$

where $\nabla_s k = \nabla k(\cdot, s)$ is defined for a.a. s when $k \in \mathrm{Dom}(\nabla) = \mathrm{Dom}(\sqrt{N})$ – N being the number operator (see below). In this way one naturally exceeds the exponential domain, and arrives at the definition

$$A(F)k = \int F(s) \nabla_s k\, ds ,$$

for processes F for which $\nabla_s k \in \mathrm{Dom}(F(s))$ for a.a. s and $F(\cdot)\nabla . k$ is (Bochner) integrable for a reasonable (dense) family of vectors k. The form of the extended preservation integral is now evident:

$$\Lambda(F)k = \mathscr{S}(F(\cdot)\nabla . k) .$$

Notice the pattern of order in which operations occur – the gradient, if it is involved, is applied first, followed by the operator integrand, with (H–S) integration being applied last. This pattern persists in multidimensions (see Sect. 3). It is an echo of the extended Wick ordering rule of thumb suggested in [HuS].

Any square integrable Wiener functional F may be represented as a sum of multiple Wiener–Itô integrals:

$$f_0 + \sum_{n \geqq 1} \int \ldots \int_{t_1 < \ldots < t_n} f_n(t_1, \ldots, t_n) dB_{t_1} \ldots dB_{t_n} .$$

This provides the isomorphic identification of Wiener space and Fock space alluded to above. Using the finite sets language of Guichardet [Gui] this may be neatly expressed by the formula

$$F = \int f(\sigma) dB_\sigma , \tag{0.3}$$

the integral being over finite subsets σ of $\mathbb{R}_+$. In other words, multiple integrals of all orders are treated at once. The algebraic character of the resulting identification of Wiener space with Guichardet space is discussed in [LM1] and [LP]. In this spirit, operators have been defined (successively by Maassen [Maa], Meyer [Me1] and Lindsay [L2]) which have the formal expression

$$\iiiint x(\alpha, \beta, \gamma, \delta)\, dA_\alpha^*\, d\Lambda_\beta\, dA_\gamma\, d\delta\,. \tag{0.4}$$

x is an $\not{\mathcal{I}}$-kernel for the operator (see [L2]). In the present context it is no more difficult to take multiple integrals of operator-valued integrands. These may be constructed from obvious generalisations of the gradient and integral operations in Fock space.

Section 1 serves to fix notation, and ends with the Fock space proof of Gaveau and Trauber's result. In Sect. 2 the non-adapted integrals are defined; and a quantum Skorohod isometry is proved, from which both the classical Skorohod isometry and the quantum Itô Lemma follow. In Sect. 3 the case of multidimensional noise is made explicit; and the reader is taken through a series of amalgamations, heading for the analogue of (0.4) for operator-valued kernels. The resulting integral coincides with a procedure developed independently by Belavkin [Bel]. In Sect. 4 it is shown how several previous ad hoc extensions of the Hudson–Parthasarathy theory, together with the Fichtner–Freudenberg class of Fock-space operators [FiF], are subsumed by the non-causal calculus presented here; and also how Speicher's free integrals [Spe] may be viewed as non-adapted quantum stochastic integrals.

1 Gradient–Skorohod adjoint relation

Fixing a σ-finite, non-atomic, separable measure space $M = (S, \mathcal{F}, m)$ in which each singleton set $\{s\}$ belongs to $\mathcal{F}$, let Γ denote the collection of subsets of S having finite cardinality: $\{\sigma \subset S: \#\sigma < \infty\}$. Then Γ has the countable partition $\bigcup_{n \geqq 0} \Gamma_n$ where $\Gamma_n = \{\sigma \subset S: \#\sigma = n\}$ and S will frequently be identified with Γ_1. The measurable structure on Γ is defined as follows: $U \subset \Gamma$ is measurable if, for each n, $\Phi^{-1}(U \cap \Gamma_n) \in \mathcal{F}^n \cap S^{(n)}$, where $S^{(n)}$ is the collection of points lying *in general position*: $\{s \in S^n: s_i \neq s_j \text{ for } i \neq j\}$ and $\Phi: \bigcup_{n \geqq 0} S^{(n)} \to \Gamma$ is the map taking each point $s \in S^{(n)}$ to the set of its coordinates $\{s_1, \ldots, s_n\}$ with $S^{(0)} := \{0\}$ being mapped to $\{\emptyset\}$. The union maps $\sigma \in \Gamma^d \mapsto |\sigma| := \sigma_1 \cup \ldots \cup \sigma_d$ $(d \geqq 2)$ are then measurable. The sets $S^{(n)}$ may not themselves be measurable, in the product algebra $\mathcal{F}^n$, however each differs from S^n by a null set so that the following defines a measure μ on Γ:

$$U \mapsto \iota_\emptyset(U) + \sum_{n \geqq 1} (n!)^{-1}\, \overline{m^n}(\Phi^{-1}(U \cap \Gamma_n))\,,$$

where $\iota_\emptyset(U) = 1$ if $\emptyset \in U$, and 0 otherwise, and $\overline{m^n}$ is the completion of the product measure m^n. (Γ, μ) is the symmetric measure space of M [Gui], and $L^2(\Gamma)$ is naturally isomorphic (through the map Φ) to the symmetric Fock space over $L^2(M)$. The abbreviation $d\sigma$ for $d\mu(\sigma)$ will be adopted throughout and $L^2(\Gamma)$ will be referred to as *Guichardet space*. Exponential vectors take the following form in

Guichardet space:

$$\varepsilon_\varphi(\sigma) = \prod_{s \in \sigma} \varphi(s), \quad \varphi \in L^2(M) . \tag{1.1}$$

The crucial property of these vectors is that they are linearly independent and total in $L^2(\Gamma)$, moreover the correspondence $\varphi \mapsto \varepsilon_\varphi$ is continuous. The following identity is frequently useful [LM 2, LP].

$\int$-Lemma. For $d \geqq 2$ let $g : \Gamma^d \to \mathbb{C}$ be integrable, or measurable and non-negative, then

$$\int \ldots \int g(\sigma_1, \ldots, \sigma_d) d\sigma_1 \ldots d\sigma_d = \int \sum g(\alpha_1, \ldots, \alpha_d) d\sigma$$

the sum being over partitions of σ into d parts: $(\alpha_1, \ldots, \alpha_d)$.

This result clearly extends to separably-valued, integrable maps into a Banach space.

Definition 1.1 For $f : \Gamma \to \mathbb{C}$, $\mathscr{U} f : \Gamma \times \Gamma \to \mathbb{C}$ is defined by $\mathscr{U} f(\alpha, \beta) = f(\alpha \cup \beta)$, and for $x : \Gamma \times \Gamma \to \mathbb{C}$, $\mathscr{P} x : \Gamma \to \mathbb{C}$ is defined by $\mathscr{P} x(\sigma) = \sum_{\alpha \subset \sigma} x(\alpha, \bar{\alpha})$, where $\bar{\alpha}$ denotes the complement $\sigma \backslash \alpha$ of α in σ.

These will be referred to as *union* and *partition* operators. Notice that

$$\mathscr{U} \varepsilon_\varphi = \varepsilon_\varphi \otimes \varepsilon_\varphi \quad \text{and} \quad \mathscr{P}(\varepsilon_\varphi \otimes \varepsilon_\psi) = \varepsilon_{\varphi + \psi} ,$$

a consequence of which is that $\mathscr{P}\mathscr{U} = 2^N$ where N is the *number operator*:

$$Nk(\sigma) = \# \sigma k(\sigma) .$$

Proposition 1.2 *Considering $\mathscr{U}$ and $\mathscr{P}$ as unbounded Hilbert space operators between $L^2(\Gamma)$ and $L^2(\Gamma \times \Gamma)$, with maximal domains,*

(i) $\mathscr{U}^* = \mathscr{P}$ *and* $\mathscr{P}^* = \mathscr{U}$ – *in particular each operator is closed;*
(ii) $\operatorname{Dom} \mathscr{U} = \operatorname{Dom}(\sqrt{2}^N)$ *and* $\operatorname{Dom} \mathscr{P} = \operatorname{Dom}(\sqrt{2}^{(N_1 + N_2)} P_{\mathrm{Sym.}})$

where $P_{\mathrm{Sym.}}$ is the orthogonal projection given by $P_{\mathrm{Sym.}} x(\sigma, \tau) = 2^{-(\# \sigma \cup \tau)} \sum_{\alpha \subset \sigma \cup \tau} x(\alpha, \bar{\alpha})$ *and* $N_i x(\sigma_1, \sigma_2) := (\# \sigma_i) x(\sigma_1, \sigma_2)$.

Proof. This is given in [L2, Proposition 2.5]. □

Notice that for $k \in \operatorname{Dom} \mathscr{U}$ and a.a. ω, $\mathscr{U}_\omega k = k(\cdot \cup \omega)$ defines an element of $L^2(\Gamma)$. Now consider the operators $\nabla : L^2(\Gamma) \to L^2(\Gamma \times S)$ and $\mathscr{S} : L^2(\Gamma \times S) \to L^2(\Gamma)$ given by

$$\nabla f(\alpha, s) = f(\alpha \cup \{s\}); \quad \mathscr{S} x(\sigma) = \sum_{s \in \sigma} x(\sigma \backslash \{s\}, s)$$

with their maximal domains:

$$\{f \in L^2(\Gamma) : \iint |f(\alpha \cup \{s\})|^2 \, d\alpha \, ds < \infty\} \quad \text{and}$$
$$\{x \in L^2(\Gamma \times S) : \int |\mathscr{S} x(\sigma)|^2 \, d\sigma < \infty\} \tag{1.2}$$

respectively. ∇ and $\mathscr{S}$ are restrictions of $\mathscr{U}$ and $\mathscr{P}$ in the sense that $\mathscr{U} f|_{\Gamma \times S} = \nabla f$ and $\mathscr{P} x = \mathscr{S} x$ whenever the support of x lies in $\Gamma \times S$. Thus $\nabla \varepsilon_\varphi = \varepsilon_\varphi \otimes \varphi$ and, in the notation of [LP], $\mathscr{S}(\varepsilon_\varphi \otimes \psi) = \varepsilon_\varphi \circ \psi$, so that $\mathscr{S} \nabla = N$.

Theorem 1.3 (cf [GaT]) *When these unbounded operators are given their maximal domains* (1.2),

(i) $\nabla^* = \mathscr{S}$; $\mathscr{S}^* = \nabla$

(ii) Dom ∇ = Dom $\sqrt{N}$; Dom $\mathscr{S}$ = Dom$(\sqrt{N_1}\, P_{\text{sym.}})$

where $P_{\text{sym.}}$ *is the orthogonal projection given by*

$$P_{\text{sym.}} x(\sigma, t) = (1 + \#\sigma)^{-1} \sum_{s \in \sigma \cup \{t\}} x(\sigma \cup \{t\} \setminus \{s\}, s),$$

and $N_1 x(\sigma, t) = \#\sigma x(\sigma, t)$.

Proof. Let $V: L^2(\Gamma) \to L^2(\Gamma \times S)$ be given by $Vk(\sigma, s) = (1 + \#\sigma)^{-\frac{1}{2}} k(\sigma \cup \{s\})$. Then, by the $\int$-Lemma,

$$\|Vk\|^2 = \iint (1 + \#\sigma)^{-1} |k(\sigma \cup s)|^2 \, dr \, ds$$

$$= \int_{\Gamma_{\geqq 1}} \sum_{s \in \tau} (\#\tau)^{-1} |k(\tau)|^2 \, d\tau = \|P_{\geqq 1} k\|^2$$

so that V is a partial isometry with initial space $L^2(\Gamma_{\geqq 1})$. Since $\nabla = \sqrt{I + N_1}\, V$ and $\sqrt{I + N_1}$ is self-adjoint, ∇ is a closed operator. From the above calculation it is clear that Dom ∇ = Dom $\sqrt{N}$, so that $V\sqrt{N}$ is *the* polar decomposition of ∇. By another application of the $\int$-Lemma,

$$\iint (1 + \#\sigma)^{-\frac{1}{2}} k(\sigma \cup \{s\}) x(\sigma, s) d\sigma \, ds$$

$$= \int_{\Gamma_{\geqq 1}} \sum_{s \in \tau} k(\tau) (\#\tau)^{-\frac{1}{2}} x(\tau \setminus \{s\}, s) d\tau$$

so that V^* is given by

$$V^* x(\tau) = \chi_{\Gamma_{\geqq 1}}(\tau) (\#\tau)^{-\frac{1}{2}} \sum_{s \in \tau} x(\tau \setminus \{s\}, s).$$

In particular, $\mathscr{S} = \sqrt{N}\, V^*$. Thus $\mathscr{S} = \nabla^*$ and, since ∇ is closed, $\mathscr{S}^* = \nabla$. Finally

$$\int \left| \sum_{s \in \sigma} x(\sigma \setminus \{s\}, s) \right|^2 d\sigma = \int \#\sigma \sum_{s \in \sigma} |x_{\text{sym.}}(\sigma \setminus \{s\}, s)|^2 \, d\sigma$$

$$= \iint (\#\tau + 1) |x_{\text{sym.}}(\tau, s)|^2 \, d\tau \, ds$$

so that Dom $\mathscr{S}$ = Dom$(\sqrt{1 + N_1}\, P_{\text{sym.}})$. □

Theorem 1.3 remains valid if the target space $\mathbb{C}$ is replaced by a separable hilbert space $\mathfrak{h}$ since the $\int$-Lemma does. When $S = \mathbb{R}_+$ we have the identification of Fock space and Wiener space via chaos decomposition expressed through multiple Wiener–Itô integrals (0.3). Under this identification $\mathscr{S}$ becomes the Hitsuda–Skorohod integral and ∇ becomes the gradient operator, moreover the multidimensional case may be treated with equal ease by choosing $S = \mathbb{R}_+ \times \{1, \ldots, d\}$. Functions $x: \Gamma \to \mathscr{H} := L^2(\Gamma; \mathfrak{h})$ which are square-integrable and for which the corresponding element of $L^2(\Gamma \times \Gamma; \mathfrak{h})$ lies in the domain of $\mathscr{S}$ will be called *Skorohod integrable.* Interesting results on this duality and the iterated Hitsuda–Skorohod and gradient operations, together with further references, may be found in Meyer's Quantum Probability Notes [Me2].

2 Quantum Skorohod isometry

We are now ready for the formal definitions first given tentatively in [LP]. Let $\{F(\omega): \omega \in \Gamma\}$ be a family of operators on $\mathscr{H} = L^2(\Gamma; \mathfrak{h})$, let k be a vector in $\mathscr{H}$ and consider the conditions

(a) $k \in \operatorname{Dom} F(\omega)$ for a.a. ω;
(b) $k \in \operatorname{Dom} \mathscr{U} = \operatorname{Dom} \sqrt{2^N}$;
(c) $\mathscr{U}_\omega k \in \operatorname{Dom} F(\omega)$ for a.a.ω

Definition 2.1 $\operatorname{Dom}(T(F)) = \{k \in \mathscr{H}: ,k$ satisfies (a) and $F(\cdot)k$ is Bochner integrable$\}$

$$T(F)k = \int F(\omega)k d\omega$$

$\operatorname{Dom}(A^*(F)) = \{k \in \mathscr{H}: k$ satisfies (a) and $F(\cdot)k$ is Skorohod integrable$\}$

$$A^*(F)k = \mathscr{P}(F(\cdot)k)$$

$\operatorname{Dom}(A(F)) = \{k \in \mathscr{H}: k$ satisfies (b), (c) and $F(\cdot)\mathscr{U}.k$ is Bochner integrable$\}$

$$A(F)k = \int F(\omega)\mathscr{U}_\omega k\, d\omega \ (= T(F\mathscr{U}.)k)$$

$\operatorname{Dom}(\Lambda(F)) = \{k \in \mathscr{H}: k$ satisfies (b), (c) and $F(\cdot)\mathscr{U}.k$ is Skorhod integrable$\}$

$$\Lambda(F)k = \mathscr{P}(F(\cdot)\mathscr{U}.k) \quad (= A^*(F\mathscr{U}.)k)\,.$$

When F is supported by $S = \Gamma_1$, then (b) should be replaced by

$$\text{(b}'\text{)} \ k \in \operatorname{Dom} \nabla = \operatorname{Dom} \sqrt{N}$$

and the definitions then read

$$A^*(F)k = \mathscr{S}(F(\cdot)k); \quad A(F)k = \int F(s)\nabla_s k\, ds; \quad \Lambda(F)k = \mathscr{S}(F(\cdot)\nabla.k)\,.$$

First note that if $v\varepsilon_\varphi \in \operatorname{Dom} F(\omega)$, for almost all ω, and $\varepsilon_\varphi(\cdot)F(\cdot)v\varepsilon_\varphi \in L^1(\Gamma; \mathscr{H})$, then $v\varepsilon_\varphi \in \operatorname{Dom} A(F)$ and

$$A(F)v\varepsilon_\varphi = \int \varepsilon_\varphi(\omega)F(\omega)v\varepsilon_\varphi\, d\omega\,. \tag{2.1}$$

Moreover $v\varepsilon_\varphi \in \operatorname{Dom} \Lambda(F)$ if and only if $v\varepsilon_\varphi \in \operatorname{Dom} A^*(\varepsilon_\varphi F)$, in which case

$$\Lambda(F)v\varepsilon_\varphi = A^*(\varepsilon_\varphi F)v\varepsilon_\varphi\,. \tag{2.2}$$

The adjoint relations

$$A^*(F^*) \subset A(F)^*; \quad A(F^*) \subset A^*(F)^*; \quad \Lambda(F^*) \subset \Lambda(F)^*$$

follow immediately from Proposition 1.2, and there is a *quantum Skorohod isometry.*

Theorem 2.2 *Let* $\{F_i(s): s \in S\}$, $i = 1, 2$, *be two families of operators on* $\mathscr{H}$ *and let* $k_i \in \operatorname{Dom} A^*(F_i)$. *If*

$$\iint (1 + \#\sigma)\, \|[F_i(t)k]\ (\sigma)\|_{\mathfrak{h}}^2\, d\sigma\, dt < \infty \tag{2.3}$$

then

$$\langle A^*(F_1)k_1, A^*(F_2)k_2\rangle = \int \langle F_1(s)k_1, F_2(s)k_2\rangle ds + \iint \langle \nabla_s[F_1(t)k_1], \nabla_t[F_2(s)k_2]\rangle\, ds\, dt\,. \tag{2.4}$$

Proof. The condition (2.3) is sufficient for each of the three terms in (2.4) to be well-defined. Let $x_i:(\sigma, t)\mapsto [F_i(t)k_i](\sigma)$. then by the $\not{I}$-Lemma each x_i satisfies,

$$\iint \|\nabla_s x_t\|^2\, ds\, dt = \int \|\sqrt{N} x_t\|^2\, dt < \infty \ ,$$

justifying two further applications of the $\not{I}$-Lemma below:

$$\begin{aligned}
&\int \langle F_1(t)k_1, F_2(t)k_2\rangle dt + \iint \langle \nabla_s[F_1(t)k_1], \nabla_t[F_2(s)k]\rangle\, ds\, dt \\
&\quad = \iint \langle x_1(\sigma, t), x_2(\sigma, t)\rangle_{\mathfrak{h}}\, d\sigma\, dt + \iiint \langle x_1(\omega\cup s, t), x_2(\omega\cup t, s)\rangle_{\mathfrak{h}}\, d\omega\, ds\, dt \\
&\quad = \iint \langle x_1(\sigma, t), x_2(\sigma, t)\rangle_{\mathfrak{h}}\, d\sigma\, dt + \iint \sum_{s\in\sigma} \langle x_1(\sigma, t), x_2(\sigma\backslash s\cup t, s)\rangle_{\mathfrak{h}}\, d\sigma\, dt \\
&\quad = \iint \sum_{s\in\sigma\cup t} \langle x_1(\sigma, t), x_2(\sigma\cup t\backslash s, s)\rangle_{\mathfrak{h}}\, d\sigma\, dt \\
&\quad = \int \sum_{t\in\omega} \sum_{s\in\omega} \langle x_1(\omega\backslash t, t), x_2(\omega\backslash s, s)\rangle_{\mathfrak{h}}\, d\omega \\
&\quad = \langle A^*(F_1)k_1, A^*(F_2)k_2\rangle\ .
\end{aligned}$$

□

This contains the classical Skorohod isometry.

Corollary 2.3 [Sko] *Let x be an L^2-process on a standard Wiener (probability) space for which $x_t \in \operatorname{Dom}\nabla$ for a.a. t, and $\int \mathbb{E}|\nabla_s x_t|^2\, ds\, dt < \infty$. Then x is Skorohod integrable and*

$$\mathbb{E}|\mathscr{S}x|^2 = \int \mathbb{E}|x_t|^2\, dt + \iint \mathbb{E}(\overline{\nabla_t x_s}\nabla_s x_t)\, ds\, dt\ .$$

Proof. Apply the theorem with $S = \mathbb{R}_+$: let $k_1 = k_2 = 1$ and, for each t, put $F_1(t)$ and $F_2(t)$ equal to the rank one operator $|x_t\rangle\langle 1|$. The result then follows by the natural identification of Wiener space and Fock space. □

This is not an isometry at all – there are non-zero processes whose H–S integral is zero. However it is an extension of the Itô isometry since when x is adapted the second term vanishes: $\nabla_s x_t = 0$ for $s > t$, and $\nabla_t x_s = 0$ for $s < t$. It may be seen from the proof of the theorem that the classical result actually implies the quantum *isometry.* Moreover, one may view the proof as commutation relations of the form "$\nabla_s^*\nabla_t = \nabla_t\nabla_s^* - \delta(s-t)$" at work. Such arguments are made rigorous in white noise analysis (see e.g. [Kuo, KuR]). Clearly these differing points of view are complementary – see [Oba] for a Hida-type distribution theory for quantum stochastic integrals.

When $S = \mathbb{R}_+$ the integrals defined above are multiple-integral, non-adapted extensions of the Hudson–Parthasarathy integrals and, as we see next, the (quantum) Skorohod isometry yields a new proof of the quantum Itô Lemma which is both simple and direct. Let $\mathscr{D}$ be a dense subspace of a Hilbert space $\mathfrak{h}$, and let D be a dense subspace of $L^2(\mathbb{R}_+)$ with the invariance property: $f\in D \Rightarrow f\chi_{[0,t]}\in D\ \forall t > 0$.

Proposition 2.4 *Let $\{F(s): s \geqq 0\}$ be an operator-adapted, square-integrable $(\mathscr{D}, D)$-process (see* [HuP, Definitions 3.1, 3.3, L 3]). *Then $\int F\, dA^*$, $\int F\, dA$ and $\int F\, d\Lambda$ are the restrictions of the respective operators $A^*(F)$, $A(F)$ and $\Lambda(F)$ to the exponential domain $\mathscr{D}\otimes\mathscr{E}(D)$. (For $\int F\, d\Lambda v\varepsilon_\varphi$ to be well-defined, $\int |\varphi(s)|^2\, \|F(s)v\varepsilon_\varphi\|^2\, ds$ must be finite.)*

Proof. We have already observed (before (2.1)) that $\mathscr{D} \otimes \mathscr{E}(D) \subset \operatorname{Dom} A(F)$. Since F has support in Γ_1, (2.1) implies that

$$A(F)v\varepsilon_\varphi = \int \varphi(s)\, F(s) v\varepsilon_\varphi\, ds = \int F dA\, v\varepsilon_\psi\,,$$

for $v \in \mathscr{D}$, $\varphi \in D$. Thus $A(F)$ extends $\int F dA$. Let $x : \sigma \mapsto \sum_{s\in\sigma} [F(s)v\varepsilon_\varphi](\sigma\setminus s)$ and $y = \int F dA^*\, v\varepsilon_\psi$. Now $(\tau, s) \mapsto \langle \psi(s)\varepsilon_\psi(\tau), [F(s)v\varepsilon_\varphi](\tau)\rangle_{\mathfrak{h}}$ is integrable so, by the $\oint$-Lemma, $\sigma \mapsto \langle u\varepsilon_\psi(\sigma), x(\sigma)\rangle_{\mathfrak{h}}$ is integrable with integral

$$\int \overline{\psi(s)}\, \langle u\varepsilon_\psi, F(s)v\varepsilon_\varphi\rangle\, ds = \langle u\varepsilon_\psi, y\rangle\,.$$

Let $a > 1$, then, by another application of the $\oint$-Lemma,

$$\begin{aligned}\int \|a^{-\#\sigma} x(\sigma)\|_{\mathfrak{h}}^2\, d\sigma &\leqq \int (\#\sigma/a^{2\#\sigma}) \sum_{s\in\sigma} \|[F(s)v\varepsilon_\varphi](\sigma\setminus s)\|_{\mathfrak{h}}^2 \\ &= \iint\ [(1 + \#\tau)/(a^2)^{1+\#\tau}]\ \|F(s)v\varepsilon_\varphi(\tau)\|_{\mathfrak{h}}^2\, d\tau\, ds \\ &\leqq \gamma(a) \int \|F(s)v\varepsilon_\varphi\|^2\, ds\end{aligned}$$

for some $\gamma(a) < \infty$. Combining these we have

$$\int \langle u\varepsilon_\psi(\sigma), a^{-\#\sigma}\{y(\sigma) - x(\sigma)\}\rangle_{\mathfrak{h}}\, d\sigma = \int \langle u\varepsilon_{a^{-1}\psi}(\sigma), y(\sigma) - x(\sigma)\rangle_{\mathfrak{h}}\, d\sigma = 0\,.$$

By the totality of $\mathscr{D} \otimes \mathscr{E}(D)$ in $\mathscr{H} = L^2(\Gamma; \mathfrak{h}) = \mathfrak{h} \otimes L^2(\Gamma)$, $x = y$ a.e. In particular $x \in \mathscr{H}$ and $F(\cdot)v\varepsilon_\psi$ is Skorohod integrable. Thus $A^*(F)$ extends $\int F dA^*$. That $A(F)$ extends $\int F dA$ now follows from (2.2) and the corresponding relation for Hudson–Parthasarathy integrals:

$$\int F dA\, v\varepsilon_\varphi = \int \varphi F\, dA^*\, v\varepsilon_\varphi\,. \qquad \square$$

Corollary 2.5 [HuP] *Let $M_i = \int_0^\cdot F_i dA^*$ where F_i is an operator-adapted, locally square integrable $(\mathscr{D}_i, D_i)$-process, and let $k_i = v_i \varepsilon_{\varphi_i}$ where $v_i \in \mathscr{D}_i$ and $\varphi_i \in D_i$ $(i = 1, 2)$. If*

$$\int_0^T \|\varphi_i(s) M_i(s) k_i\|^2\, ds < \infty\,, \quad \textit{for each } T > 0\,, \tag{2.5}$$

then

$$\begin{aligned}\langle M_1(T)k_1, M_2(T)k_2\rangle = &\int_0^T \overline{\varphi_1(s)}\, \langle M_1(s)k_1, F_2(s)k_2\rangle\, ds \\ &+ \int_0^T \varphi_2(s)\, \langle F_1(s)k_1, M_2(s)k_2\rangle\, ds \\ &+ \int_0^T \langle F_1(s)k_1, F_2(s)k_2\rangle\, ds\,.\end{aligned}$$

Proof. Let $a = k/(1 + k)$ and $T > 0$ then $F_i' := \sqrt{a^N}\, F_i \chi_{[0,T]}$ satisfies the conditions of Theorem 2.2. Let $x_i : (\sigma, t) \mapsto [F_i(t)k_i](\sigma, t)$. By operator-adaptedness, $x_1(\sigma \cup t, s) = \varphi_1(t) x_1(\sigma, s)$ for $s < t$ and similarly $x_2(\sigma \cup s, t) = \varphi_2(s) x_2(\sigma, t)$ for

$s > t$, so that

$$\int d\sigma\, a^{\#\sigma-1} \langle [A^*(F_1')k_1](\sigma), [A^*(F_2')k_2](\sigma)\rangle_{\mathfrak{h}} - \int_0^T dt \int d\sigma\, a^{\#\sigma} \langle x_1(\sigma, t), x_2(\sigma, t)\rangle_{\mathfrak{h}}$$

$$= \int_0^T dt \int_0^T ds \int d\sigma\, a^{\#\sigma+1} \langle x_1(\sigma \cup t, s), x_2(\sigma \cup s, t)\rangle_{\mathfrak{h}}$$

$$= \int_0^T dt\, \overline{\varphi_1(t)} \int_0^t ds \int d\sigma\, a^{\#\sigma+1} \langle x_1(\sigma, s), [\nabla x_2(\cdot, t)](\sigma, s)\rangle_{\mathfrak{h}}$$

$$+ \int_0^T ds\, \varphi_2(s) \int_0^s dt \int d\sigma\, a^{\#\sigma+1} \langle [\nabla x_1(\cdot, s)](\sigma, t), x_2(\sigma, t)\rangle_{\mathfrak{h}}$$

$$= \int_0^T dt \overline{\varphi_1(t)} \int d\sigma a^{\#\sigma} \langle [M_1(t)k_1](\sigma), [F_2(t)k_2](\sigma)\rangle_{\mathfrak{h}}$$

$$+ \int_0^T ds\, \varphi_2(s) \int d\sigma\, a^{\#\sigma} \langle [F_1(s)k_1](\sigma), [M_2(s)k_2](\sigma)\rangle_{\mathfrak{h}}$$

where Theorem 1.3 and Proposition 2.4 are used in the last step. Letting $k \to \infty$ the result follows by dominated convergence. □

Corollary 2.6 *Let F, φ, k and M be as in Corollary 2.5, then*

$$\|M(t)k\|^2 \leqq 2 \int_0^t \exp\left\{ \int_s^t |\varphi|^2 \right\} \|F(s)k\|^2\, ds\,. \tag{2.6}$$

Proof. By Corollary 2.5 $M(\cdot)k$ satisfies

$$\|M(t)k\|^2 = \int_0^t \{2\|F(s)k\|^2 + |\varphi(s)|^2\, \|M(s)k\|^2 - \alpha(s)\}\, ds$$

where $\alpha := \|\varphi(\cdot)M(\cdot)k\|^2 + \|F(\cdot)k\|^2 - 2\,\mathrm{Re}\langle \varphi(\cdot)M(\cdot)k, F(\cdot)k\rangle$ is non-negative. $\|M(\cdot)k\|^2$ is thus (a.e.) differentiable, and a simple integrating factor argument gives (2.6). □

Remark. In fact the condition (2.5) is redundant since $\int_0^{\cdot} F_i\, dA^* k_i$ is continuous, and so locally bounded.

3 Multidimensions

For this section let $S = S_\bullet \times \{1, \ldots, d\}$ and write $\Gamma_\bullet$ for $\Gamma(S_\bullet)$, so that $\Gamma = \Gamma(S)$ is naturally indentifiable with $\Gamma_\bullet^d := \Gamma_\bullet \times \ldots \times \Gamma_\bullet$. We shall use the grad/Skorohod notation for the union/partition operations. Thus, for $i, j = 1, \ldots, d$,

$$\nabla^i : L^2(\Gamma; \mathfrak{h}) \to L^2(\Gamma \times \Gamma_\bullet; \mathfrak{h}) \quad \mathscr{S}_j : L^2(\Gamma \times \Gamma_\bullet; \mathfrak{h}) \to L^2(\Gamma; \mathfrak{h})$$

$$\nabla^i f(\sigma, \tau) = f(\sigma \cup \tau^i) \quad \mathscr{S}_j x : \sigma \mapsto \sum_{\alpha \subset \sigma_j} x(\sigma \backslash \alpha^j, \alpha)$$

where, for $\tau \in \Gamma_\bullet$, $\tau^i = \{(t,i): t \in \tau\}$ and, for $\sigma \in \Gamma$, $\sigma_j = \{s: (s,j) \in \sigma\}$. Generalised creation, exchange, annihilation and time integrals may then be defined as follows

$$A_j^*(F)k = \mathscr{S}_j(F(\cdot)k); \qquad \Lambda_j^i(F)k = \mathscr{S}_j(F(\cdot)\nabla^i_\cdot k);$$

$$A^i(F)k = \int_{\Gamma_\bullet} F(\sigma)\nabla^i_\sigma k \, d\sigma; \quad T(F) = \int_{\Gamma_\bullet} F(\sigma)k \, d\sigma \tag{3.1}$$

for suitable operator-valued maps $F: \Gamma_\bullet \to \mathscr{L}(\mathscr{H})$. If we define $\mathscr{S}_0$ to be the operation of (Lebesgue) integration over $\Gamma_\bullet$, and ∇^0 to be the injection given by $\nabla^0 k(\omega, \sigma) = k(\omega)$, then the operations (3.1) may be (individually) written

$$\Lambda_\beta^\alpha(F)k = \mathscr{S}_\beta(F(\cdot)\nabla^\alpha_\cdot k) .$$

Our aim is to amalgamate these $(d+1)^2$-operations. The creation and annihilation integrals simply give us back those of Definition 1.4:

$$A^*(F)k: \sigma \mapsto \sum_{\alpha \subset \sigma} [F(\alpha)k](\sigma \backslash \alpha); \quad A(F)k = \int_\Gamma F(\gamma)\nabla_\gamma k \, d\gamma \tag{3.2}$$

with the difference that we may think of an element of Γ as a d-tuple of elements of $\Gamma_\bullet$, so that F in (3.2) is a d-argument function. For preservation/exchange d^2 arguments are required, and a little thought leads to the definition.

$$\Lambda(F)k: \sigma \mapsto \sum_{\omega^\cdot \subset \sigma} [F(\underset{\sim}{\omega})\nabla_{\omega_\cdot} k](\sigma \backslash \omega^\cdot)$$

where the sum is over $d \times d$ arrays $\underset{\sim}{\omega} = (\omega_j^i)$ of elements of $\Gamma_\bullet$ for which the d-tuple $\omega^\cdot = (\bigcup_i \omega_1^i, \ldots, \bigcup_i \omega_d^i)$ is a *subset* of σ in the sense that $(\omega^\cdot)_j \subset \sigma_j$ for $j = 1, \ldots, d$, and $\omega_\cdot$ is defined similarly in terms of the rows of $\underset{\sim}{\omega}$, instead of its columns. The final amalgamation of these three operations, and $T(\cdot)$, makes contact with 2-, 3-, and 4-argument integral-sum kernel operators [Maa, Me1, L2]. In one dimensional form, a four argument kernel x determines an operator X by

$$Xk: \sigma \mapsto \sum \iint x(\alpha_1, \alpha_2, \omega_1, \omega_2)\, k(\omega_1 \cup \alpha_2 \cup \alpha_3)\, d\omega_1\, d\omega_2 \tag{3.3}$$

where the sum is over partitions of σ into three: $(\alpha_1, \alpha_2, \alpha_3)$. Under suitable regularity conditions the product of two such operators X and Y is a third integral-sum kernel operator. Remarkably the value of the kernel of XY is given simply by a (finite) sum over partitions of products of values of x and values of y [L2].

A creation-annihilation amalgamation is quite straight forward:

$$A^*A(F)k: \sigma \mapsto \sum_{\alpha \subset \sigma} \int [F(\alpha, \gamma)\nabla_\gamma k](\bar{\alpha})\, d\gamma$$

as is preservation-annihilation:

$$\Lambda A(F)k: \sigma \mapsto \sum_{\omega^\cdot \subset \sigma} \int [F(\underset{\sim}{\omega}, \gamma)\nabla_{\gamma \cup \omega_\cdot} k](\sigma \backslash \omega^\cdot)\, d\gamma .$$

If we combine all three, and T, we obain

$$\Xi(F)k: \sigma \mapsto \sum_{\alpha \cup \beta^\cdot \subset \sigma} \iint [F(\alpha, \underset{\sim}{\beta}, \gamma, \delta)\nabla_{\gamma \cup \beta_\cdot} k](\xi)\, d\gamma\, d\delta$$

where σ, α, γ, and ξ are d-tuples from $\Gamma_\bullet$, β is a $d \times d$ array from $\Gamma_\bullet$ and ξ is the complement of $\alpha \cup \beta^\cdot$ in σ. Ξ is a quantum stochastic operation on a $(d+1)^2$-argument operator-valued function F, generalising the scalar-valued case of integral-sum operators (3.3). It is equivalent to the generalised quantum stochastic integral discussed in [Bel]. Belavkin gives an estimate for the integration operator in terms of a scale of Fock space norms and $L^2 \times L^\infty \times L^2 \times L^1$ norm in the argument of the integrand. He also obtains a formula for the product of non-adapted integrals generalising the product formula for 4-argument $\not{\mathcal{I}}$-kernel operators [L2].

Notice that our multiple integral does not coincide with *iterated* Wick-ordered integrals. In other words if $F = F_1 \otimes F_2 \otimes F_3 \otimes F_4$ then, in general, $\Xi(F) \neq A^*(F_1)\Lambda(F_2)A(F_3)T(F_4)$. For example, although $A^*(F_1)A(F_3) = A^*A(F_1 \otimes F_3)$, it is easily seen that $\Lambda(F_2)A(F_3) \neq \Lambda A(F_2 \otimes F_3)$.

4 Examples

In this section we view various constructions as non-adapted quantum stochastic integrals.

Example 4.1 In their construction of locally normal states of infinite Bose systems Fichtner and Freudenberg are lead to a class of Fock space operators which incorporate a position measurement with a local observable [FiF]. These fit into the present scheme as preservation integrals of the form $\Lambda(\chi_U(\cdot)X)$ where U is a measurable subset of configuration space Γ_S and X is a (local) observable. The action of these operators is given by

$$\Lambda(\chi_U(\cdot)X)k : \sigma \mapsto \sum_{\alpha \subset \sigma} \chi_U(\alpha)\,(X\nabla_\alpha k)\,(\sigma \setminus \alpha)\,. \tag{4.1}$$

This has a nice interpretation: pick out each configuration lying in U and, holding it fixed, measure observable X. Let $\delta_\varnothing : \Gamma \to \mathbb{C}$ denote the function which maps σ to 1 if $\sigma = \varnothing$, and to 0 otherwise. If X is the rank one operator $|f\rangle\langle\delta_\varnothing|$ respectively $|\delta_\varnothing\rangle\langle g|$) with $f \in L^2(\Gamma_n)$, $g \in L^2(\Gamma_m)$ then one gets (generalised) creation (respectively annihilation) operators. More generally if X is an integral operator, with kernel x, (4.1) gives the corresponding 2-argument integral-sum operator [Maa, L2]. If X is a self-adjoint, one-particle operator then (4.1) gives its differential second quantisation, whose action in the notation of [LP] is $\varepsilon_\varphi \mapsto X\varphi \circ \varepsilon_\varphi$.

Example 4.2 Kümmerer and Speicher have developed an Itô-type stochastic calculus based on a non-commutative notion of independence due to Voiculescu [KüS, Spe]. Voiculescu's independence is related to a continuous free product (as opposed to tenor product) of algebras [Voi]. Parthasarathy and Sinha showed that the basic processes of this free stochastic calculus may be represented in Guichardet space as slightly extended Hudson–Parthasarathy-type integrals [PS2]. By exploiting the basic tools of non-causal calculus we may take this much further and represent stochastic integrals with respect to these processes. The key ingredients are the *shift process* $(S^t)_{t \geq 0}$ and a projection-valued process $(P^t)_{t \geq 0}$ defined as follows

$$S^t k(\sigma) = \begin{cases} k(\sigma - t) \\ 0 \end{cases} \quad \text{and} \quad P^t k(\sigma) = \begin{cases} k(\sigma) & \text{if } \sigma \subset [t, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Psi : \bigoplus_{n \geqq 0} L^2(T^n) \to L^2(\Gamma_T)$ be given by

$$(\Psi f)(\sigma) = f_n(s_1, s_2 - s_1, \ldots, s_n - s_{n-1})$$

for $f = (f_n)$ and $\sigma = \{s_1, \ldots, s_n\}$ where $0 \leqq s_1 < \ldots < s_n$ and $T = [0, \infty)$. Then Ψ is an isomorphism of full (unsymmetrised) Fock space with Guichardet space which intertwines

$$\int dA_f^* \, G^\Psi := A^*(S^\cdot G^\Psi) \quad \text{and} \quad \int G^\Psi \, dA_f := A(G^\Psi (S^\cdot)^*) \tag{4.2}$$

respectively with the creation and annihilation integrals

$$\int dl^* G \quad \text{and} \quad \int G dl \tag{4.3}$$

of the free calculus, where $G^\Psi(s) = \Psi G(s) \Psi^{-1}$. In (4.3) G should be adapted to the Cuntz algebra filtration and should belong to $L^2(T; \mathscr{B})$ where $\mathscr{B}$ is a certain Banach algebra consisting of bounded operators [KuS]. The natural constraints in (4.2) are rather different. In the creation integral only one of the terms in the Skorohod sum survives, due to the shift:

$$[A^*(S^\cdot F)k](\sigma) = \mathscr{S}(S^\cdot F(\cdot)k)(\sigma) = [F(\min \sigma)k]((\sigma \setminus \{\min \sigma\}) - \min \sigma)$$

for $\sigma \neq \emptyset$, and the $\not{\int}$-Lemma yields the *isometric relation*:

$$\| \int dA_f^* \, Fk \|_{\mathscr{H}} = \| F(\cdot)k \|_{L^2(T; \mathscr{H})} \tag{4.4}$$

and natural domain: $\{k \in \mathscr{H} : F(\cdot)k \in L^2(T; \mathscr{H})\}$. Moreover, if F is bounded-operator-valued, as in [KuS], then

$$\| \int dA_f^* \, F \|_{\mathscr{B}(\mathscr{H})} \leqq \| F(\cdot) \|_{L^2(T; \mathscr{B}(\mathscr{H}))} .$$

Notice that the Cuntz algebra relations

$$l(\varphi) l^*(\psi) = \langle \varphi, \psi \rangle I$$

follow from the isometry (4.4) by letting $G = \varphi(\cdot)I$ and then polarising. When F belongs to $L^2(T; \mathscr{B}(\mathscr{H}))$ its free annihilation integral is bounded, being $(\int dA_f^* \, F^*)^*$, but more generally

$$\int F \, dA_f k = \int F(t) \, (S^t)^* \, \nabla_t k \, \mathrm{d}t$$

with corresponding domain.

Free preservation integrals [Spe] may also be constructed on Guichardet space from the Skorohod integral and gradient operator:

$$\int d\Lambda_f \, Fk = \mathscr{S}(\nabla . P^\cdot F(\cdot)k) . \tag{4.5}$$

Again only one term in the Skorohod sum survives, so

$$[\int d\Lambda_f \, Fk](\sigma) = \sum_{s \in \sigma} \chi_{\Gamma[s,\infty)}(\sigma \setminus \{s\}) \, [F(s)k](\sigma) = [F(\min \sigma)k](\sigma) .$$

This is *not* a non-causal preservation integral in our sense, since the gradient operator is not acting first. For this reason the domain of the free preservation integral (4.5) is more delicate: an assumption of the form – *for each* $s \geqq 0$ $F(s)k$ *is continuous on* $\Gamma[s, \infty)$ – is appropriate. The free preservation process $\Lambda_f(t) = \int_0^t d\Lambda_f$ is projection valued: $\Lambda_f(t) = (P^t)^\perp$. In order to establish algebraic relations in the free stochastic calculus (Itô-type product formula), and to construct stochastic dilations of quantum dynamical semigroups using the calculus, it is necessary to consider two-sided integrals of the form $\int F dA_f G$ etc. Taking care of continuity for the domain, the 2-sided annihilation integral may be defined directly:

$$\int F\, dA_f\, Gk = \int G(t)\,(S^t)^*\, \nabla_t F(t)k\, \mathrm{d}t\,.$$

The other 2-sided integrals may be defined first as forms:

$$\int F^*\, dA_f^* G : (h, k) \mapsto \int \langle \nabla_t F(t)h,\, S^t G(t)k \rangle\, \mathrm{d}t$$

$$\int F^*\, dA_f G : (h, k) \mapsto \int \langle P^t \nabla_t F(t)h,\, P^t \nabla_t G(t)k \rangle\, \mathrm{d}t$$

with domain constraints beginning with $P^{\cdot} \nabla_{\cdot} F(\cdot)h \in L^2(T; \mathscr{H})$. This again connects with the work of Obata [Oba].

Example 4.3 The quantum Itô calculus is extended by Vincent–Smith to cover certain processes of the form $X(s) = F_s \otimes R^s$ where F_s acts on $\mathscr{H}_s := \{k \in \mathscr{H} : \operatorname{supp} k \subset \Gamma[0, s]\}$ and R^s is a second quantised multiplication operator acting on $\mathscr{H}^s := \{k \in \mathscr{H} : \operatorname{supp} k \subset \Gamma[s, \infty]\}$ [Vin]. The corresponding integrals $\int X\, dA$ and $\int dA^* X$ coincide with our non-adapted integrals $A(X)$ and $A^*(X)$ respectively, and have the following action on the exponential domain:

$$\int X\, dA\, v\varepsilon_\varphi = \int \psi(s) F_s v \varepsilon_{\varphi_{s]}} \otimes R^s \varepsilon_{\varphi_{[s}}\, ds$$

$$\int dA^* X v\varepsilon_\varphi : \sigma \mapsto \sum_{s \in \sigma} F_s v \varepsilon_{\varphi_{s]}}(\sigma \cap [0, s)) R^s \varepsilon_{\varphi_{[s}}(\sigma \cap (s, \infty))\,.$$

In Vincent-Smith's applications R^s is the projection onto the zero particle subspace of $\mathscr{H}^s$ so that X is *Ω-adapted*: $X(s) = F(s)P_s$ where $F(s) = F_s \otimes I^s$ and P_s is the orthogonal projection onto $\mathscr{H}_s$. The solution of the non-adapted quantum stochastic differential equation

$$W(t) = I - A(\chi_{[0,t]}\, V^* P.) + A^*(\chi_{[0,t]} V P. W) + T(\chi_{[0,t]} K P. W)$$

is then the dilation of the semigroup on $\mathscr{B}(\mathfrak{h})$ with generator $X \mapsto XK + K^*X + V^*XV (K, V \in \mathscr{B}(\mathfrak{h}))$ obtained in [AlF]. Integrals in which integrators occur on the 'wrong side', namely $\int X\, dA^*$ and $\int dAX$, are also discussed in [Vin], but these are not used in his applications. Extended preservation integrals are not defined – our analysis shows that these should be neither left nor right integrals: first (stochastically) differentiate, then act with an operator integrand and finally stochastically integrate in the Hitsuda–Skorohod sense.

Example 4.4 Let T be a finite quantum stop time in the following sense: $\{T(t) : t \geqq 0\}$ is a spectral resolution adapted to the Fock space filtration $(\mathscr{B}(\mathscr{H}_t) \otimes I^t)$. The following associated operators may be defined rigorously:

$$S^T = \int \mathrm{d}T(t) S^t; \quad P_{T,\varphi} := \int \mathrm{d}T(t)\, \varphi(t) P_t, \quad \varphi \in L^\infty(\mathbb{R}_+)$$

where, as before, S^t is the shift through t and P_t the orthogonal projection on $\mathscr{H}_t$ [PS1]. The time increments here are *backward pointing*. Parthasarathy and Sinha showed that if $\mathscr{H}^T$ is the range of the isometric operator S^T and $\mathscr{H}_T$ is the closed linear span of the ranges of $(P_{T,\phi} : \phi \in L^\infty(\mathbb{R}_+))$ then a natural isomorphism exists between $\mathscr{H}_T \otimes \mathscr{H}^T$ and $\mathscr{H}$, factorising $\mathscr{H}$ into before and after time T spaces. This is not really an example of the *present* theory, but due to its obvious resemblance to the free integral and Ω-adapted integral respectively, one might hope that a broader picture may eventually emerge, in which time may be random, and such integrals incorporated into the non-causal quantum stochastic calculus.

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Note added in proof. References [Me 1, 2] have now been subsumed by P-A Meyer's recent book: Quantum probability for probabilists. (Lect. Notes Math., vol. 1538) Berlin Heidelberg: Springer 1993

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