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Some fractal sets determined by stable processes

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Summary. Let Y_i be independent stable subordinators on $(\Omega, \mathscr{F}, P)$ with indices $0<\beta_i<1$ and R_i are the ranges of Y_i, $i=1, 2$. We are able to find the exact Hausdorff measure and packing measure results for the product sets $R_1 \times R_2$, and whenever $\beta_1+\beta_2 \leqq 1/2$, we deduce results for the vector sum $R_1 \oplus R_2=\{x+y: x \in R_1, y \in R_2\}$.

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1 Introduction

It is well known that a stable process of index $\alpha>1$ on $\mathbb{R}$ has a continuous local time [1] (Blumenthal and Getoor), $l(t, x)$, and that the stochastic process inverse to $l(t, 0)$ is a stable subordinator of index $\beta=1-1/\alpha$ [9] (Stone). For a stable subordinator with index β, say Y_β, the range $R_\beta \cap [0, 1]$ has positive finite Hausdorff measure with respect to $\phi(s)=s^\beta(\log\log \frac{1}{s})^{1-\beta}$ — see [17] (Taylor and Wendel) — and, if

$$h(s)=s^\beta f(s) \tag{1.1}$$

with $f(s)$ monotone increasing, Taylor [15] showed that the packing measure

$$h-p(Y_\beta[0,1])=\begin{cases} 0 \\ \infty \end{cases} \text{a.s. according as } \int_{0+} \frac{f^2(s)}{s} ds \begin{cases} <+\infty \\ =+\infty, \end{cases} \tag{1.2}$$

where $Y_\beta[0,1]=\{y=Y_\beta(t): t \in [0,1]\}$. We remark that, the subordinator which arises as the inverse of the local time of a stable process of index α with $1<\alpha \leqq 2$ has index β with $0<\beta \leqq \frac{1}{2}$. However, all of our analysis, apart from some results on projections, requires only that $0<\beta<1$, so we state the main results in the context.

We know that the product set $R_1 \times R_2$ and $R_1 \oplus R_2$ are random sets. The general results of [2] (Besicovitch and Moran) imply that, if $\phi_i(s) = s^{\beta_i}(\log\log\frac{1}{s})^{1-\beta_i}$ and $\phi(s) = \phi_1(s)\phi_2(s)$, then

$$\phi - m(R_1 \times R_2 \cap [0,1]^2) > 0 \text{ a.s.}$$

But the general theory does not provide an upper bound. Our first main result (Theorem 3.8) is that there is a finite positive constant c such that

$$\phi - m(R_1 \times R_2 \cap [0, Y_1(1)] \times [0,, Y_2(1)]) = c \text{ a.s.} \tag{1.3}$$

Our result for packing measure (Theorem 4.2) is more surprising. Using the formulation (1.1), with

$$h(s) = s^{\beta_1+\beta_2} f(s),$$

we obtain

$$h - p(Y_1[0,1] \times Y_2[0,1]) = \begin{cases} 0 \\ +\infty \end{cases} \text{ a.s. according as}$$

$$\int_{0+} \frac{f^2(s)\log\frac{1}{f(s)}}{s}\, ds \begin{cases} < +\infty \\ = +\infty. \end{cases} \tag{1.4}$$

Thus the critical functions for (1.4) are $f(s) = (\log 1/s)^{-\frac{1}{2}}(\log\log 1/s)^{-1-\varepsilon}$, while those for (1.2) are of the form $(\log 1/s)^{-\frac{1}{2}}(\log\log 1/s)^{-\frac{1}{2}-\varepsilon}$, $\varepsilon > 0$. As an immediate corollary of these main theorems we remark that

$$dim(R_1 \times R_2) = Dim(R_1 \times R_2) = \beta_1 + \beta_2 \text{ a.s.},$$

which implies that for almost all ω, this product set is a fractal in the sense of [14] (Taylor). Here $dim(E)$ and $Dim(E)$ denote the Hausdorff and packing dimensions of E respectively.

We also remark that the general results about projecting a planar set on a line (e.g. [4] (Falconer)) relate to the fractal properties in almost all directions. These results do not help us with particular projections. As pointed out in [10] (Taylor), the projections on the lines $y = x$ and $y = -x$ of a product set $A \times B$ are scalar multiples of the vector sum $A \oplus B$ and difference $A \ominus B (= \{x - y : x \in A, y \in B\})$. When $\beta_1 + \beta_2 > 1$, $R_1 \cap [0,1] \oplus R_2 \cap [0,1]$ and $R_1 \cap [0,1] \ominus R_2 \cap [0,1]$ have positive Lebesgue measure, while our result (1.3) implies that, when $\beta_1 + \beta_2 \leqq 1$, both of these sets have zero Lebesgue measure. Hence, for almost all $l \neq 0$,

$$P(l \in R_1 \oplus R_2) = 0.$$

Whenever $\beta_1 + \beta_2 \leqq 1/2$, we can determine the exact Hausdorff measure function for $R_1 \oplus R_2$ and$R_1 \ominus R_2$ (Theorem 5.1), which is our third main result in this paper. However, we have not been able to do the interesting critical case $\beta_1 + \beta_2 = 1$, and packing measure results in all cases seem to be difficult.

As usual, we use $c_1, c_2, \ldots$ to denote finite positive constants whose values may or may not be known. They may be different in different theorems. We start by assembling some preliminary results needed in the sequel.

2 Preliminaries

$\phi(t)$ is said to be a measure function if it is right continuous and monotone increasing with $\phi(0+)=0$. Let Φ denote the class of measure functions $\phi:(0,1)\to[0,1]$

We now consider some special classes of sets for covering and packing. Let Γ stand for the class of the open balls $B(x,r)$ in $\mathbb{R}^d$ and Γ^* stand for the class of dyadic cubes in $\mathbb{R}^d$. $C\in\Gamma^*$ if it has side length 2^{-n}, $n\in\mathbb{N}$, and each of its projections $proj_i C$ on the ith axis is a half-open interval of the form $[k_i 2^{-n}, (k_i+1)2^{-n})$ where $k_i\in\mathbb{Z}$. For $x\in\mathbb{R}^d$, let $u_n(x)$ be the unique dyadic cube of side 2^{-n} containing x.

We also need the class Γ^{**} of semidyadic cubes. $C\in\Gamma^{**}$ if it has side length 2^{-n} and $proj_i C=[\frac{1}{2}k_i 2^{-n}, (\frac{1}{2}k_i+1)2^{-n})$ with $k_i\in\mathbb{Z}$. We denote by $v_n(x)$ the unique semidyadic cube in Γ^{**} of side length 2^{-n} whose complement is at distance 2^{-n-2} from $u_{n+2}(x)$.
Now we define set functions $\phi-m$, $\phi-m_s$, and $\phi-m^*$ on sets in $\mathbb{R}^d$ by

$$\phi-m(E)=\lim_{\delta\downarrow 0}\inf\{\sum\phi(d(E_i)),\ \cup E_i\supseteq E,\ d(E_i)<\delta\}, \tag{2.1}$$

$$\phi-m_s(E)=\lim_{\delta\downarrow 0}\inf\{\sum\phi(d(E_i)),\ \cup E_i\supseteq E,\ d(E_i)<\delta,\ E_i\in\Gamma\}, \tag{2.2}$$

$$\phi-m^*(E)=\lim_{\delta\downarrow 0}\inf\{\sum\phi(d(E_i)),\ \cup E_i\supseteq E,\ d(E_i)<\delta,\ E_i\in\Gamma^*\}, \tag{2.3}$$

where $d(E)$ is the diameter of E. It is easy to see that for any E in $\mathbb{R}^d$,

$$\phi-m(E)\leqq\phi-m_s(E)\leqq\phi-m^*(E)\leqq c_1\phi-m(E). \tag{2.4}$$

c_1 is a positive constant and $\phi-m(E)$ is said to be ϕ-Hausdorff measure of E.

In defining $\phi-m_s$ we used economical coverings by open balls with small radii. Now we consider dense packing by disjoint balls with centers in E and differing radii; this yields packing measure, whose definition and initial properties are given in [16] (Taylor and Tricot). Again assume $\phi\in\Phi$ and define

$$\phi-P(E)=\lim_{\delta\downarrow 0}\sup\{\sum\phi(2r_i),\ B(x_i,r_i)\text{ disjoint},\ x_i\in E,\ r_i<\delta\}. \tag{2.5}$$

It is a premeasure and we obtain a metric outer measure by

$$\phi-p(E)=\inf\{\sum\phi-P(E_i),\ E\subseteq\cup E_i\}. \tag{2.6}$$

$\phi-p$ is called ϕ-packing measure.

If we replace open balls in (2.5) by dyadic cubes containing x_i or semidyadic cubes $v_n(x_i)$ we have $\phi-P^*(E)$ or $\phi-P^{**}(E)$. Correspondingly we obtain $\phi-p^*(E)$ and $\phi-p^{**}(E)$. $\phi-p^*$ may be of different order than $\phi-p$ but it is proved in [16] that

$$\phi-m(E)\leqq\phi-p(E), \tag{2.7}$$

and

$$c_2\phi-p^{**}(E)\leqq\phi-p(E)\leqq c_3\phi-p^{**}(E). \tag{2.8}$$

for some constants c_2, c_3 depending only on ϕ and the dimension of the space.

In studying measure properties, the density theorem is a very effective technique. C. A . Rogers and S. J. Taylor [7] and other people first studied the density theorem for Hausdorff measure. Then a similar result for packing measure is obtained in [16]. We state these as a lemma.

Lemma 2.1 *For a given $\phi \in \Phi$ there are constants $\lambda_1, \lambda_2, \lambda_3$ such that for all $E \subset \mathbb{R}^d$ and every finite Borel measure μ in $\mathbb{R}^d$,*

$$\lambda_1 \mu(E) \inf_{x\in E} \left\{ \liminf_{r\to 0} \frac{\phi(2r)}{\mu B(x,r)} \right\} \leqq \phi - m(E) \leqq \lambda_2 \mu(\mathbb{R}^d) \sup_{x\in E} \left\{ \liminf_{r\to 0} \frac{\phi(2r)}{\mu B(x,r)} \right\}, \quad (2.9)$$

$$\lambda_3 \mu(E) \inf_{x\in E} \left\{ \limsup_{r\to 0} \frac{\phi(2r)}{\mu B(x,r)} \right\} \leqq \phi - p(E) \leqq \mu(\mathbb{R}^d) \sup_{x\in E} \left\{ \limsup_{r\to 0} \frac{\phi(2r)}{\mu B(x,r)} \right\}. \quad (2.10)$$

This lemma gives the ordinary form for the density theorem. We will sometimes use equivalent forms.

In addition to the standard Borel–Cantelli lemma we need a version which does not assume independence.

Lemma 2.2 *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $A_i \in \mathscr{F}$, $i=1,2,\ldots$. If $\sum_i P(A_i) = +\infty$ and*

$$\liminf_{n\to 0} \frac{\sum_{i,j=1}^{n} P(A_i \cap A_j)}{(\sum_{i=1}^{n} P(A_i))^2} \leqq c, \ c \text{ is a constant},$$

then $P(\limsup_n A_n) \geqq c^{-1}$. See [5] (Kochen and Stone).

For any two functions $f(t)$ and $g(t)$, we write $f(t) \sim g(t)$ if there exists a constant $c_1 \neq 0$, such that $\frac{f(t)}{g(t)} \to c_1$ as $t \to 0$ or ∞. We write $f(t) \approx g(t)$ if $c_2 f(t) \leqq g(t) \leqq c_3 f(t)$, c_2 and c_3 are positive constants.

We will need to estimate the small tail of $(T_1+T_2)(T_3+T_4)$, where the T_i's are independent nonnegative random variables. This will be done in two steps. First, we estimate the small tail of X_1+X_2 when they are independent and

$$P\{X_i < \lambda\} \sim c_i \lambda \text{ as } \lambda \downarrow 0, i=1, 2.$$

Lemma 2.3 *Suppose that X_1 and X_2 are independent nonnegative random variables with $P(X_1<x)=F_1(x)$ and $P(X_2<x)=F_2(x)$. If*

$$\frac{F_1(x)}{x} \to c_1 \text{ and } \frac{F_2(x)}{x} \to c_2 \text{ as } x \downarrow 0,$$

then

$$\frac{H(x)}{x^2} \to \frac{1}{2} c_1 c_2 \text{ as } x \downarrow 0,$$

where $H(x) = P(X_1+X_2<x)$.

Proof. We know that $H(x)=\int_0^x F_1(x-y)dF_2(y)$, because X_1, X_2 are non-negative. Then $\forall\varepsilon>0$, if x_0 is small, we have

$$c_1-\varepsilon<\frac{F_1(x)}{x}<c_1+\varepsilon,\ c_2-\varepsilon<\frac{F_2(x)}{x}<c_2+\varepsilon,\ x\leqq x_0\,.$$

So when $x\leqq x_0$,

$$\begin{aligned}\frac{H(x)}{x^2}&=\int_0^x\frac{F_1(x-y)}{x^2}dF_2(y)\\&=\sum_{i=0}^{2^k-1}\int_{\frac{i}{2^k}x}^{\frac{i+1}{2^k}x}\frac{F_1(x-y)}{x^2}dF_2(y)\\&\leqq\sum_{i=0}^{2^k-1}\frac{F_1\left(x-\frac{i}{2^k}x\right)}{x^2}\left[F_2\left(\frac{i+1}{2^k}x\right)-F_2\left(\frac{i}{2^k}x\right)\right]\\&\leqq(c_1+\varepsilon)\sum_{i=0}^{2^k-1}\left(1-\frac{i}{2^k}\right)\left(c_2\frac{1}{2^k}+2\varepsilon\frac{i+1}{2^k}\right)\\&\leqq(c_1+\varepsilon)\,c_2\frac{1}{2}\left(\frac{2^k+1}{2^k}\right)+(c_1+\varepsilon)\sum_{i=0}^{2^k-1}2\varepsilon\frac{i+1}{2^k}\,.\end{aligned}$$

Let $\varepsilon\to 0$, then $k\to+\infty$ we have

$$\limsup_{x\downarrow 0}\frac{H(x)}{x^2}\leqq\frac{1}{2}c_1c_2\,.$$

Similarly we can prove $\liminf_{x\downarrow 0}\frac{H(x)}{x^2}\geqq\frac{1}{2}c_1c_2$. Therefore $\frac{H(x)}{x^2}\to\frac{1}{2}c_1c_2$ as $x\downarrow 0$. #

Lemma 2.4 *Let U and V be two independent nonnegative random variables. $P(U>t)\approx e^{-\alpha t}$ and $P(V>t)\approx e^{-\alpha t}$, for $t\geqq z>0$, $\alpha>0$, then*

$$P(U+V>t)\approx te^{-\alpha t}$$

holds when t is large.

Proof. The proof of this lemma is easy. We only need to bound U and V stochastically from above and below by proper shifted exponential random variables. #

If $U=\log\frac{1}{T_1+T_2}$, and $V=\log\frac{1}{T_3+T_4}$, then the asymptotic form of the small tail for (T_1+T_2) gives the form of the large tail of U. The small tail of $(T_1+T_2)(T_3+T_4)$ now comes from the large tail of $U+V$.

We expect that Lemmas 2.3, 2.4 are known results but include above proofs since we could not find a suitable reference. Our next result relates to the large tail of a sum of a random number of i.i.d. variables, each with large tail which is not small enough to allow a Laplace transform argument to work. We believe this result may be of independent interest so call it Theorem 2.5.

Theorem 2.5 *If* $\{X_i\}$ *is a sequence of nonnegative independent random variables and* $\mathrm{P}(X_i > x) \leqq \exp(-\alpha x^\gamma)$ *for* $x \geqq x_0$, *where* $0 < \gamma < 1$. W *is independent of* $\{X_i\}$ *and takes positive integer values with* $\mathrm{P}(W = k) \leqq p^k$, $0 < p < 1$, $k = 1, 2, \ldots$, *then there exist a point* $x_1 \geqq x_0$ *and a constant* c *such that*

$$\mathrm{P}\left(\sum_{i=1}^{W} X_i > x\right) \leqq c \exp(-\alpha x^\gamma), \text{ for } x \geqq x_1 .$$

Proof. We may assume without loss of generality that W is a suitable shifted geometric random variable, let Y_i be i.i.d. random variables and independent of W with $P(Y_i > x) = \exp(-\alpha x^\gamma)$, $x > 0$, $i = 1, \ldots$. Since Y_i's are subexponential random variables (see [6] (Pitman)), so are random variables $Z_i = Y_i + t$, $t > 0$. One may choose t large enough to make Z_i dominate X_i stochastically. Then the tail of $\sum_{i=1}^{W} Z_i$ dominates that of $\sum_{i=1}^{W} X_i$ stochastically. But by Corollary3 in [3] (Embrechts, et al.), the tail of $\sum_{i=1}^{W} Z_i$ is of the order $\exp(-\alpha x^\gamma)$ as $x \to +\infty$ and this completes our proof. #

In the proofs of next three sections we need both tails of the distribution of $Y_\beta(1)$, where Y_β is a stable subordinator of index β with $0 < \beta < 1$ in $\mathbb{R}$. We take those forms from [8] (Skorokhod).
Let $G(x) = \mathrm{P}(Y_\beta(1) \leqq x)$, then for a constant $0 < c < 1$,

$$G(x) \sim x^{\frac{\beta}{2(1-\beta)}} \exp(-c x^{-\frac{\beta}{1-\beta}}), \text{ as } x \to 0 \tag{2.11}$$

and

$$1 - G(x) \sim x^{-\beta}, \; x \to +\infty. \tag{2.12}$$

Let $T(r)$ be the occupation time of Y_β in $[0, r)$, then

$$T(r) = \int_0^{+\infty} I_{B_r(0)}(Y_\beta(s))\, ds .$$

Using the scaling property and (2.12) above we get

$$\begin{aligned} \mathrm{P}(T(1) < x) &= \mathrm{P}(Y_\beta(x) > 1) \\ &= \mathrm{P}(Y_\beta(1) > x^{-\frac{1}{\beta}}) \sim x, \text{ as } x \to 0 \end{aligned} \tag{2.13}$$

3 Hausdorff measure of $R_1 \times R_2$

We use $|E|$ to denote the Lebesgue measure of E and $[x, y)$ to denote $[x_1, y_1) \times [x_2, y_2)$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We will write $R = R_1 \times R_2$, where R_i are the ranges of the two independent stable subordinators Y_i's of indices β_i, $i = 1,2$. From now on to the end of this paper we denote $\phi(s)= s^{\beta_1+\beta_2}\left(\log\log\frac{1}{s}\right)^{2-\beta_1-\beta_2}$. For convenience, we assume both Y_1 and Y_2 start at 0. Define $\mu_i(E)=|\{s:Y_i(s)\in E\}|$, $i=1,2$ and $\mu=\mu_1\times\mu_2$. We set $Y(t_1,t_2)=(Y_1(t_1), Y_2(t_2))$ and $Y(A)=\{y=(Y_1(t_1), Y_2(t_2)):(t_1,t_2)\in A\}$.

In this section, methods of Taylor and Wendel [17] are used to obtain the upper bound of ϕ-Hausdorff measure of $R\cap[0,1]^2$.

Lemma 3.1 *There exists a constant* $K>0$ *such that*

$$\limsup_{h\downarrow 0}\frac{\mu[0,(h,h))}{\phi(h)}\leqq K \text{ a.s.}$$

Proof. Let $\phi_i(h)=h^{\beta_i}\left(\log\frac{1}{h}\right)^{1-\beta_i}$. By using Theorem 5 in [13] (Taylor) we obtain that for certain constants c_1 and c_2,

$$\limsup_{h\downarrow 0}\frac{\mu_i[0,h)}{\phi_i(h)}=c_i \text{ a.s., } i=1,2\,.$$

Thus

$$\limsup_{h\downarrow 0}\frac{\mu_1\times\mu_2[0,(h,h))}{\phi(h)}\leqq\limsup_{h\downarrow 0}\frac{\mu_1[0,h)}{\phi_1(h)}\cdot\limsup_{h\downarrow 0}\frac{\mu_2[0,h)}{\phi_2(h)}$$

$$=c_1c_2 \text{a.s.}$$

Taking $K=c_1c_2$ we obtain this lemma. #

Lemma 3.2 *Given* $(x,y)\in R$, *let* $S=[(x,y),(x,y)+(h,h))$. *Then for a.s.* ω *we have*

$$\limsup_{h\downarrow 0}\frac{\mu(S)}{\phi(h)}\leqq K, \ K \text{ is the same as in Lemma 3.1.}$$

Proof. By the definition of μ and Lemma 3.1 and using the strong Markov property at points $(x,y)\in R$, we obtain this result immediately. #

Remark. Since we have Lemma 3.2, we can use another form of the density theorem. For convenience we state it here:

Suppose that F is a measure defined on the Borel sets in $\mathbb{R}^2$ and that E is a Borel set such that for each $x\in E$

$$\limsup_{t\downarrow 0}\frac{F[x,x+T)}{h(t)}\leqq c, \ c \text{ is a certain constant.}$$

Then $2c\,h-m(E)\geqq F(E)$, where $h(s)\in\Phi$ and $T=(t,t)$.

This result can be proved by the same argument used in Lemma 4 of [17].

Theorem 3.3 *There is a constant $c>0$ such that for a.s. ω*

$$\phi-m(R\cap[0, Y(1, 1)))\geqq c.$$

Proof. By the definition of μ,

$$\mu((0, Y(t_1, t_2)))=\mu(R\cap[0, Y(t_1, t_2)))$$
$$=t_1 t_2.$$

Thus for every $E\in\mathscr{B}(\mathbb{R}^2)$, $\mu(R\cap Y(E))$ is the Lebesgue measure of E. Now let Γ denote the set of points $(\omega, (t_1, t_2))\in\Omega\times[0, 1]^2$ such that

$$\limsup_{h\downarrow 0}\frac{\mu[Y(t_1, t_2), Y(t_1, t_2)+H)}{\phi(h)}\leqq K$$

where $H=(h, h)$ and K is the same positive constant as in Lemma 3.1. One can verify that Γ is product measurable.

By the strong Markov property and Lemma 3.2 one can see that each (t_1, t_2) section of Γ has probability 1, so that almost every ω section $\Lambda=\Lambda(\omega)$ has Lebesgue measure 1. By using Lemma 3.2 and the version of the density theorem in the remark of Lemma 3.2 we have

$$\phi-m(R\cap[0, Y(1, 1)))\geqq(1/2K)\mu(R\cap(Y(\Lambda)))=(1/2K)>0 \text{ a.s. } \#$$

In order to obtain the upper bound for $\phi-m(R)$ it is required to cover not only the good points (x, y) where

$$\limsup_{h\downarrow 0}\frac{\mu(S)}{\phi(h)}\geqq c>0, \; c \text{ is a constant} \tag{3.1}$$

with $S=[(x, y), (x, y)+(h, h))$ and $(x, y)\in R$, but also the bad points where (3.1) is not satisfied. We therefore proceed to obtain a lemma allowing us to deal with the bad points.

Lemma 3.4 *Let $v_k=(v_k^{(1)}, v_k^{(2)})\in\mathbf{R}^2$, $v_k^{(1)}=v_k^{(2)}=\frac{1}{\sqrt{2}}\exp(-k^{1+\delta})$, $\delta>0$, $w_k=\sqrt{2}v_k^{(1)}$. Define $B_k=\{\mu[0, v_k)<a_1 a_2 K\phi(w_k)\}$, where a_1, a_2 are positive constants, K was defined in Lemma 3.1. Then for suitable positive constants c_3, c_4, m_0, we have*

$$P\left(\bigcap_{k=m}^{2m} B_k\right)\leqq\exp(-c_4 m^{c_3}), \textit{ for all } m\geqq m_0 .$$

Proof. Let $E_k=\{\mu[v_{k+1}, v_k)<a_1 a_2 K\phi(w_k)\}$. It is clear that $B_k\subseteq E_k$, $k\geqq 1$. No consider

$$F_{i,k}=\left\{\frac{\mu_i[v_{k+1}^{(i)}, v_k^{(i)})}{(v_k^{(i)}-v_{k+1}^{(i)})^{\beta_i}}\geqq\frac{a_i d_i\sqrt{2}^{\beta_i}}{(1-e^{-1})^{\beta_i}}\left(\log\log\frac{1}{w_k}\right)^{1-\beta_i}\right\}, d_i=\sqrt{K}, i=1, 2.$$

Note that

$$a_1a_2K\phi(w_k)=\prod_{i=1}^{2} a_id_i\sqrt{2}^{\beta_i}\left[\frac{1}{1-\exp(k^{1+\delta}-(k+1)^{1+\delta})}\right]^{\beta_i}$$
$$\times(v_k^{(i)}-v_{k+1}^{(i)})^{\beta_i}\left(\log\log\frac{1}{w_k}\right)^{1-\beta_i}$$
$$\leqq\prod_{i=1}^{2}\frac{a_id_i(\sqrt{2})^{\beta_i}}{(1-e^{-1})^{\beta_i}}\left(\log\log\frac{1}{w_k}\right)^{1-\beta_i}(v_k^{(i)}-v_{k+1}^{(i)})^{\beta_i}$$
$$\equiv\prod_{i=1}^{2}\lambda_{i,k}\cdot(v_k^{(i)}-v_{k+1}^{(i)})^{\beta_i},$$

so $E_k^c\supseteq F_{1,k}\cap F_{2,k}$. But by the results in [13] (Taylor),

$$P(F_{i,k})\geqq\exp(-b_i\lambda_{i,k}^{\frac{1}{1-\beta_i}})\equiv\exp(-b_ir_i\log(k(1+\delta)))\,,$$

where b_i's are constants bigger than 1 and independent of a_i and K, but r_i's are certain positive constants depending on a_i and K. Choose a_i small enough such that

$$\bar{b}_i=b_ir_i<\frac{1}{2(1+\delta)},$$

then for some constants c_3, c_4 and m_0 large enough,

$$\sum_{k=m}^{2m}P(E_k^c)\geqq(m+1)\exp((-\bar{b}_1-\bar{b}_2)(1+\delta)\log 2m)\geqq c_4m^{c_3},\quad m\geqq m_0\,.$$

The definition of E_k in the beginning of this proof makes these sets independent, thus we have

$$P\left(\bigcap_{k=m}^{2m}E_k\right)=\prod_{k=m}^{2m}P(E_k)=\prod_{k=m}^{2m}(1-P(E_k^c))$$
$$\leqq\exp\left(-\sum_{k=m}^{2m}P(E_k^c)\right)\leqq\exp(-c_4m^{c_3})\,.$$

Since $\bigcap_{k=m}^{2m}B_k\subseteq\bigcap_{k=m}^{2m}E_k$, therefore

$$P\left(\bigcap_{k=m}^{2m}B_k\right)\leqq\exp(-c_4m^{c_3}),\quad m\geqq m_0.\quad \#$$

Remark. By Lemma 3.4 we have that

$$P\left(\bigcap_{k=m}^{M}B_k\right)\leqq\exp(-c_4m^{c_3}),\text{ for } m\geqq m_0,\ M\geqq 2m\,. \tag{3.2}$$

Using Lemmas 3.1 and 3.4 we obtain:

Corollary 3.5 *Let* $K_1=a_1a_2K$, a_1, a_2 *and* K *were defined in Lemma* 3.4, *then*

$$K_1\leqq\limsup_{h\downarrow 0}\frac{\mu[0,(h,h))}{\phi(h)}\leqq K\ a.s..$$

Let Λ_h be the collection of semidyadic cubes

$$\left\{(x, y): \frac{j-1}{2^h} \leqq x < \frac{j}{2^h}, \frac{i-1}{2^h} \leqq y < \frac{i}{2^h}\right\}, \ 2i, 2j = 1, 2 \cdots$$

It is easy to see that any rectangle $\{(x, y): a_1 \leqq x < b_1,\ a_2 \leqq y < b_2\}$ with $\max\{b_1 - a_1, b_2 - a_2\} < 2^{-h_0-2}$, $\min\{b_1 - a_1, b_2 - a_2\} \geqq 2^{-m-1}$ can be contained in a member of $\bigcup_{h=h_0}^{m} \Lambda_h$. Furthermore Λ_h is almost nested in the following sense.

Lemma 3.6 *If $E = \bigcup_{j=1}^{m} I_j$ where each I_j is a member of Λ_h with h between h_0 and n, then it is possible to find a subset $\{j_r\}$ of $\{1, 2, \ldots, m\}$ such that $E = \bigcup I_{j_r}$ and no point of E is in more than four of the cubes I_{j_r}.*

Proof. See [11] (Taylor).

Theorem 3.7 *For a.s. ω, there exists $\tilde{c} = \tilde{c}(\omega) < +\infty$ such that*

$$\phi - m(R \cap [0,1]^2) \leqq \tilde{c} .$$

Proof. In the proof of this theorem m_0, $v_k^{(i)}$, w_k, δ, a_1, a_2, K are the same as in Lemma 3.4.

Given $\varepsilon > 0$, choose h_0 so that $2^{-h_0} < \min(\varepsilon/2, w_{m_0})$. Choose $m = [(h_0 \log 2)^{\frac{1}{1+\delta}}]$, where $[x]$ denotes the largest integer $\leqq x$. Given n, let M_n be the largest integer k such that $\exp(-k^{\delta+1})/\sqrt{2} \geqq 2^{-n}$; n should be taken large enough so that $n > h_0 + 4$ and $M_n \geqq 2m$. For suitable positive constants c_5, c_6 we have $M_n > c_5 n^{c_6}$, when n is sufficiently large. For such fixed n, let us consider the collection of dyadic cubes like

$$I_{j,i,n} = \left\{(x, y): \frac{j-1}{2^n} \leqq x < \frac{j}{2^n}, \frac{i-1}{2^n} \leqq y < \frac{i}{2^n}\right\},$$

we say that $I_{j,i,n}$ is bad for the sample point ω if

1. R meets $I_{j,i,n}$ and
2. there is no semidyadic cube $[a, b)$ of $\bigcup_{h=h_0}^{n} \Lambda_h$ such that $[a, b)$ contains $I_{j,i,n} \cap R$ and

$$\frac{\mu[a, b)}{\phi(d[a, b))} \geqq \frac{a_1 a_2 K}{4^{\beta_1 + \beta_2}}, \ d(E) \text{ is the diameter of } E .$$

All other cubes $I_{j,i,n}$ are said to be good. If $I_{j,i,n}$ is good then either $R \bigcap I_{j,i,n}$ is empty or it can be covered by a semidyadic cube $[a, b)$ of $\bigcup_{h=h_0}^{n} \Lambda_h$ with $\mu[a, b)\phi(d[a, b))^{-1} \geqq (1/4^{\beta_1+\beta_2}) a_1 a_2 K$. We complete the covering of $R \cap [0, 1]^2$ by taking $I_{j,i,n}$ itself to cover the set $R \cap I_{j,i,n}$ when the cube is bad, then all cubes of the covering have diameter less than ε.

Now we show that the contribution to the covering from bad cubes is small.

If $R \cap I_{j,i,n}$ is not void, let $s_{j,i}=(s_{j,i}^{(1)}, s_{j,i}^{(2)})$ be the 'least' point in R in the sense that $s_{j,i}^{(1)}=\inf\{s\in R_1 : s \geqq j-1/2^n\}$ and $s_{j,i}^{(2)}=\inf\{s\in R_2 : s\geqq (i-1)/2^n\}$. Define

$$B_{k,j,i}=\{\mu[s_{j,i}, s_{j,i}+v_k)/\phi(d([s_{j,i}, s_{j,i}+v_k)))<a_1a_2K\} ,$$

where $k=m, m+1, \ldots, M_n$. By the remark of Lemma 3.4 and the strong Markov property we know that $\bigcap_{k=m}^{M_n} B_{k,j,i}$ has a probability at most $\exp(-c_7 n^{c_8})$, c_7 and c_8 are two positive constants.

Now the diameter of the cube $[s_{j,i}, s_{j,i}+v_k)$ is a number in the interval $[2^{-n-1}, 2^{-h_0})$. Hence $[s_{j,i}, s_{j,i}+v_k)$ can be covered by a cube $[a, b)$ of $\bigcup_{h=h_0}^{n} \Lambda_h$ such that $d([a, b))\leqq 4w_k$.

Suppose that ω is in the complement of the set $\bigcap_{k=m}^{M_n} B_{k,j,i}$ then there is at least one k between m and M_n with

$$\frac{\mu[s_{j,i}, s_{j,i}+v_k)}{\phi(w_k)}\geqq a_1a_2K .$$

Covering $[s_{j,i}, s_{j,i}+v_k)$ by $[a, b)\in\bigcup_{h=h_0}^{n} \Lambda_h$ with $d([a, b))\leqq 4w_k$, when t is small enough we have

$$\frac{\mu[a, b)}{\phi(d([a, b))}\geqq\frac{a_1a_2K}{4^{\beta_1+\beta_2}} \text{, note: } \phi(4t)<4^{\beta_1+\beta_2}\phi(t) .$$

Thus the cube $I_{j,i,n}$ is good. Therefore for a constant $c_9>0$

$$P(I_{j,i,n} \text{ is bad})\leqq P(R \text{ meets } I_{j,i,n})\cdot \exp(-c_7n^{c_8})$$

$$\leqq c_9 j^{\beta_1-1} i^{\beta_2-1} \exp(-c_7n^{c_8}), \text{ See Lemma 1 in [17].}$$

Now let T_n denote the number of bad cubes $I_{j,i,n}$ with $1\leqq i, j\leqq 2^n$. It follows that

$$E(T_n)\leqq c_9 \exp(-c_7n^{c_8}) \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} i^{\beta_2-1} j^{\beta_1-1} \leqq c_{10} 2^{n(\beta_1+\beta_2)} \exp(-c_7n^{c_8}) ,$$

where $c_{10}>0$ is a constant.

The covering by bad cubes will make a contribution

$$\Sigma_n'=T_n\{2^{-n(\beta_1+\beta_2)}[\log(n\log 2)]^{2-\beta_1-\beta_2}\} ,$$

whose expectation is majorized by an expression of the form

$$\chi_n=c_{11}(\log n)^{2-\beta_1-\beta_2}\exp(-c_7n^{c_8}), \; c_{11} \text{ is a constant.}$$

For any $\eta>0$, we have $P(\Sigma_n'>\eta)<\chi_n/\eta$. Set $\eta=1/n$ and allow n to vary, by the Borel–Cantelli Lemma we deduce that with probability 1 there exists an integer n_0 such that $\Sigma_n'<1/n$, $n\geqq n_0$. Therefore the contribution to the covering by the bad cubes is negligible.

For each good cube which contains a point of R we choose a cube $[a, b)$ in $\bigcup_{h=h_0}^{n} \Lambda_h$ such that

$$\frac{\mu[a,b)}{\phi(d([a, b)))}\geqq\frac{a_1a_2K}{4^{\beta_1+\beta_2}} .$$

This gives a finite collection of cubes to which Lemma 3.6 can be applied. We obtain a set of form $\bigcup[a_i, b_i)$ which still covers the good cubes $I_{j,i,n}$ but none of them are covered more than four times. For this covering

$$\sum \mu[a_i, b_i)<4\cdot\mu_1[0, l_1)\cdot\mu_2[0, l_2),$$

where $l_1=\sup\{b_i^{(1)}\}$, $l_2=\sup\{b_i^{(2)}\}$ and $b_i=(b_i^{(1)}, b_i^{(2)})$, $i=1, 2$. So l_1 and $l_2<1+2^{-h_0}$. Hence for a proper constant $c_{12}>0$,

$$\sum \phi(d([a, b)))\leqq c_{12}\mu_1[0, 1+2^{-h_0})\cdot\mu_2[0, 1+2^{-h_0}).$$

Thus we obtain a finite covering, say $\bigcup J_i$, of R for each $n\geqq n_0$ such that

$$\sum \phi(d(J_i))\leqq c_{12}\mu_1[0, 1+2^{-h_0})\cdot\mu_2[0, 1+2^{-h_0})+1/n.$$

Let $\varepsilon\to 0$ (so $h_0\to+\infty$), then $n\to+\infty$,

$$\phi-m(R\cap[0, 1]^2)\leqq c_{13}\mu[0, (1, 1)) \text{ a.s., } c_{13} \text{ is a positive constant.}$$

Since $\mu[0, (1, 1))$ is finite, the proof is now complete. #

Let $f(t_1, t_2)=\phi-m(R\cap[0, Y(t_1, t_2)))$. Repeat the argument in [17] we obtain that for a constant $c>0$,

$$f(t_1, t_2)=ct_1t_2 \text{ a.s.} \tag{3.3}$$

We state this result in the following theorem.

Theorem 3.8 *Let Y_1 and Y_2 be two independent stable subordinators with indices β_1 and β_2 respectively ($0<\beta_1, \beta_2<1$), and R_i be the range of Y_i, $i=1$, 2 then*

$$\phi-m(R_1\times R_2\cap[0, (Y_1(t_1), Y_2(t_2))))=ct_1t_2 \textit{ a.s.},$$

where $\phi(h)=h^{\beta_1+\beta_2}\left(\operatorname{loglog}\frac{1}{h}\right)^{2-\beta_1-\beta_2}$.

Now for any stable process X with index $\alpha>1$, there is a stable subordinator Y_β with index $\beta=1-1/\alpha$ such that Y_β is the inverse to the local time of X at zero. Using (3.3) we have the following corollary.

Corollary 3.9 *Let X_1 and X_2 be two independent stable processes on the line of indices $\alpha_1, \alpha_2>1$, with zero sets Z_1 and Z_2 and their local times at zero are $A_1(t)$ and $A_2(t)$. Then there is a positive constant c depending only on X_1 and X_2 such that*

$$\varphi-m(Z_1\times Z_2\cap[0, t_1]\times[0, t_2])=cA_1(t_1)\cdot A_2(t_2)$$

for all $t_1, t_2>0$, where $\varphi(h)=h^{2-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}}(\operatorname{loglog}1/h)^{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}$.

4 Packing measure of $R_1\times R_2$

Let $\{\Omega, \mathscr{B}, \mathscr{B}_t, U_t, \zeta_t, Q^x\}$ be a Hunt process (One can find the definitions and properties related to a Hunt process in [1]). The Blumenthal zero one law says that for all $A\in\cap_{h>0}\mathscr{B}_h$, $Q^x(A)$ is either zero or one. Now we need a corresponding law for a pair of independent Hunt processes.

We write $\bigvee_{t\in T}\mathscr{B}_t$ to denote the σ-field generated by the union of all $\mathscr{B}_t$ and $\sigma\{U_s: s\in A\}$ to denote the σ-field generated by U_s, $s\in A$.

Let $X_1=\{\Omega,\mathscr{F},\mathscr{F}_t,X_1(t),\theta_t,\mathrm{P}_1^{x_1}\}$ and $X_2=\{\Omega,\mathscr{G},\mathscr{G}_t,X_2(t),\eta_t,P_2^{x_2}\}$ be two independent Hunt processes on $\mathbb{R}$, where $\mathscr{F}=\bigvee_{t>0}\mathscr{F}_t$, $\mathscr{F}_t=\sigma\{X_1(s): s\leqq t\}$, $\mathscr{G}=\bigvee_{t>0}\mathscr{G}_t$, $\mathscr{G}_t=\sigma\{X_2: s\leqq t\}$, $X_1(s)\circ\theta_t=X_1(s+t)$, $X_2(s)\circ\eta_t=X_2(s+t)$.
For any $x=(x_1,x_2)\in\mathbb{R}^2$, P^x is defined by extension from

$$\mathrm{P}^x\left(\left[\bigcap_{i=1}^n X_1^{-1}(s_i)(A_i)\right]\cap\left[\bigcap_{j=1}^m X_2^{-1}(t_j)(B_j)\right]\right)$$
$$=\mathrm{P}_1^{x_1}\left(\bigcap_{i=1}^n X_1^{-1}(s_i)(A_i)\right)\cdot\mathrm{P}_2^{x_2}\left(\bigcap_{j=1}^m X_2^{-1}(t_j)(B_j)\right),$$

where n and m are any integers and all $A_i, B_j\in\mathscr{B}(\mathbb{R})$ ($\mathscr{B}(\mathbb{R}^d)$ = the Borel σ-field on $\mathbb{R}^d$). Let $X(t)=(X_1(t),X_2(t))$ and $\mathscr{H}_t=\sigma\{X(s): s\leqq t\}$. One can verify that $X=\{\Omega,\ \mathscr{H},\ \mathscr{H}_t,\ X(t),\ \xi_t,\ \mathrm{P}^x\}$ is a Hunt process, where $\mathscr{H}=\bigvee_{t>0}\mathscr{H}_t$, $X(s)\circ\xi_t=X(t+s)$.

Lemma 4.1 *X, X_1 and X_2 are defined as above. Given a set $A\in\cap_{h>0}(\mathscr{F}_h\vee\mathscr{G}_h)$ and $x=(x_1,x_2)$, then $\mathrm{P}^x(A)$ is either zero or one.*

Proof. By the Blumenthal zero one law in [1], $\forall A\in\cap_{t>0}\mathscr{H}_t$, we have

$$\mathrm{P}^x(A)=0 \text{ or } 1 \text{ for all } x \text{ in } \mathbb{R}^2\,.$$

We know that

$$\mathscr{H}_t=\sigma\left(\bigcap_{i=1}^n X_{s_i}^{-1}(A_i),\ s_i\leqq t,\ A_i\in\mathscr{B}(\mathbb{R}^2),\ i=1,\dots,n,\ n\in\mathbb{N}\right)$$
$$=\sigma\left(\bigcap_{i=1}^n X_{s_i}^{-1}(C_{1,i}\times C_{2,i}): s_i\leqq t,\ C_{1,i}\, C_{2,i}\in\mathscr{B}(\mathbb{R}),\ i=1,\dots,n,\ n\in\mathbb{N}\right)$$

and

$$\mathscr{F}_t\vee\mathscr{G}_t=\sigma(\mathscr{F}_t\cup\mathscr{G}_t)$$
$$=\sigma\left\{\left[\bigcap_{i=1}^n X_1^{-1}(s_i)(A_i)\right]\cap\left[\bigcap_{j=1}^m X_2^{-1}(t_j)(B_j)\right],\ A_i,\ B_j\in\mathscr{B}(\mathbb{R}),\ s_i,\ t_j\leqq t,\ n,\ m\in\mathbb{N}\right\},$$

but for any i, j,

$$X_{s_i}^{-1}(C_{1,i}\times C_{2,i})=X_1^{-1}(s_i)(C_{1,i})\cap X_2^{-1}(s_i)(C_{2,i})\in\mathscr{F}_t\vee\mathscr{G}_t$$

and

$$X_1^{-1}(s_i)(A_i)\cap X_2^{-1}(t_j)(B_j)=X_{s_j}^{-1}(A_i\times\mathbb{R})\cap X_{t_j}^{-1}(\mathbb{R}\times B_j)\in\mathscr{H}_t\,,$$

so $\mathscr{H}_t=\mathscr{F}_t\vee\mathscr{G}_t$. Therefore $\forall A\in\cap_{t>0}\mathscr{F}_t\vee\mathscr{G}_t$, $\mathrm{P}^x(A)$ is either one or zero, for all $x\in\mathbb{R}^2$. #

Remark. If X is a Lévy process on $(\Omega, \mathscr{F}, \mathrm{P})$, define

$$\mathrm{P}^0(X(t)\in A)=\mathrm{P}(X(t)\in A),\ \mathscr{F}_t=\sigma\{X(s), s\leqq t\},\ t>0\ ,$$

$$\mathrm{P}^x(X(t)\in A)=\mathrm{P}(X(t)\in A\,|\,X(0)=x),\ x\neq 0\ ,$$

then $\{\Omega, X(t), \mathscr{F}, \mathscr{F}_t, \theta_t, \mathrm{P}^x\}$ is a Hunt process, $p(t, x, A)=\mathrm{P}^x(X(t)\in A)$ are its transition functions.

For Hausdorff measure, as one can see from our results in Sect. 3, the exact measure function for $R_1\times R_2$ is the product of those for R_1 and R_2. For packing measure the situation is more complicated.

The proof of following theorem is relatively long. The first step is to estimate two probabilities needed in the sequel. The second step is to use the density theorem to find the exact packing measure of $Y_1[0, 1]\times Y_2[0, 1]$.

Theorem 4.2 *Let* $h(s)=s^{\beta_1+\beta_2}\psi(s)$, $\psi(s)\in\Phi$, $\psi(2s)/\psi(s)\leqq N<+\infty$, $0<s<\frac{1}{2}$, *then*

$$h-p(Y_1[0, 1]\times Y_2[0, 1])=\begin{cases}0\\+\infty\end{cases}\text{a.s. according as}$$

$$\int_{0+}\frac{[\psi(s)]^2\log[1/\psi(s)]}{s}\,ds\begin{cases}<+\infty\\=+\infty\ .\end{cases}$$

Proof. Let

$$Y_{i,n}=(1/2^n)^{-\beta_i}\mu_i B_{1/2^n}(Y_i(t_i)),\ t_i\in(0, 1)\ ,$$

$$Z_{i,n}=(1/2^n)^{-\beta_i}\mu_i^+(B_{1/2^n}(Y_i(t_i))),\ t_i\in(0, 1),\quad (t_i'\text{s are fixed})\ ,$$

where $\mu_i B_r(x)=|\{t:|Y_i(t)-x|<r\}|$ and $\mu_i^+ B_r(Y_i(t_i))=|\{s>t_i:|Y_i(s)-Y_i(t_i)|<r\}|$, $i=1, 2$.

Given any fixed $\lambda>0$, we now estimate $\mathrm{P}(Y_{1,n}Y_{2,n}<\lambda\psi(2^{-n+1}))$ and $\mathrm{P}(Z_{1,n}Z_{2,n}<\lambda\psi(2^{-n+1}))$.

Let $T_1(r)=\int_0^\infty I_{B_r(0)}(Y_1(s))ds$, $T_3(r)=\int_0^\infty I_{B_r(0)}(Y_2(s))ds$, and let $T_2(r)$, $T_4(r)$ be the corresponding sojourn times for independent copies of the duals $Y_1^{(1)}$, $Y_2^{(1)}$ of Y_1, Y_2 obtained by time reversal, see [1]. We denote $T_i(1)$ as T_i, $i=1,\dots,4$. Hence

$$\mathrm{P}(Y_{1,n}Y_{2,n}<\lambda\psi(2^{-n+1}))=\mathrm{P}((T_1+T_2)(T_3+T_4)<\lambda\psi(2^{-n+1}))\ .$$

According to (2.13) we know that

$$\mathrm{P}(T_i<x)\sim x \text{ as } x\to 0,\ i=1, 2\ .$$

Thus by Lemma 2.3 for any fixed $\lambda>0$,

$$\mathrm{P}(T_1+T_2<\lambda\psi^{\frac{1}{2}}(2^{-n+1}))\sim\frac{1}{2}\lambda^2\psi(2^{-n+1})\ ,$$

$$\mathrm{P}(T_3+T_4<\lambda\psi^{\frac{1}{2}}(2^{-n+1}))\sim\frac{1}{2}\lambda^2\psi(2^{-n+1})\ .$$

Let $U=\log \frac{1}{T_1+T_2}$, $V=\log \frac{1}{T_3+T_4}$, then

$$P(Y_{1,n}Y_{2,n}<\lambda\psi(2^{-n+1}))=P\left(U+V>\log \frac{1}{\lambda\psi(2^{-n+1})}\right).$$

By using Lemma 2.4 we know

$$\psi^2(2^{-n+1})\log\frac{1}{\psi(2^{-n+1})}\approx P(Y_{1,n}Y_{2,n}<\lambda\psi(2^{-n+1}))\,. \tag{4.1}$$

Similarly we can prove that

$$\psi^2(2^{-n+1})\log\frac{1}{\psi(2^{-n+1})}\approx P(Z_{1,n}Z_{2,n}<\lambda\psi(2^{-n+1}))\,. \tag{4.2}$$

(4.1) and (4.2) will be used later in the proof of this theorem.

We now consider the case that $\int_{0+}\frac{[\psi(s)]^2\log[1/\psi(s)]}{s}ds<+\infty$. Set

$$G=\left\{(t_1,t_2)\in[0,1]^2:\liminf_{r\downarrow 0}\frac{\mu_1 B_r(Y_1(t_1))\mu_2 B_r(Y_2(t_2))}{h(2r)}=\infty\right\},$$

then $|G|=1$, using the same Fubini argument as in the proof of Theorem 3.3. By Lemma 2.1 we have $h-p(Y(G))=0$ a.s., where $Y(G)=\{y:y=(Y_1(t_1), Y_2(t_2)), (t_1,t_2)\in G\}$.

As for the bad points, let

$$Q_n=\left\{(t_1,t_2)\in[0,1]^2:\liminf_{r\downarrow 0}\frac{\mu_1 B_r(Y_1(t_1))\mu_2 B_r(Y_2(t_2))}{h(2r)}\leqq n\right\}.$$

We can get a contribution to $h-p^{**}(Y(Q_n))$ from semi-dyadic cubes of side 2^{-k} such that $Y=(Y_1,Y_2)$ hits the inside dyadic cube of side 2^{-k-2} and then leaves the ball of radius 2^{-k-2} in time (t_1,t_2) where $t_1\cdot t_2<nh(2^{-k})$ (note that t_i is the time spent by Y_i, $i=1,2$). The expected number of dyadic cubes of side 2^{-k-2} hit in $[0,1]^2$ is $O(2^{k(\beta_1+\beta_2)})$ and the probability of being bad (being hit but the process gets out quickly) is $O(\psi(1/2^k)\log 1/\psi(1/2^k))$ (by using (4.2)). Denote N_k as the total number of bad cubes, we have

$$\begin{aligned}
Eh-P^{**}Y(Q_n)&\leqq c_{17}\sum_{k=k_0}^{+\infty}E(N_k)h(2^{-k})\\
&\leqq c_{18}\sum_{k=k_0}^{\infty}h(2^{-k})2^{k(\beta_1+\beta_2)}\psi(2^{-k})\log(1/\psi(2^{-k}))\\
&\leqq c_{19}\left(\sum_{k=k_0}^{\infty}\psi^2(2^{-k})\log(1/\psi(2^{-k}))\right),\quad k_0 \text{ large},
\end{aligned}$$

where c_{17}, c_{18} and c_{19} are proper positive constants.

$\int_{0+} \frac{[\psi(s)]^2 \log[1/\psi(s)]}{s} ds < \infty$ implies that this series converges, we can let $k_0 \to \infty$ to deduce that $\mathrm{E}h - p^{**}Y(Q_n) = 0$, which gives $h - p^{**}Y(Q_n) = 0$ a.s.
But $G \cup \cup_n Q_n = [0, 1]^2$, therefore by (2.8)

$$h - p(Y_1[0, 1] \times Y_2[0, 1]) = 0 \text{ a.s.}$$

Now let us consider the other case that $\int_{0+} \frac{\psi^2(s)\log(1/\psi(s))}{s} ds = +\infty$. First we define random variables $a_i^{(n)}$, $b_i^{(n)}$ and events A_n by

$$A_n = \{(a_1^{(n)} + b_1^{(n)})(a_2^{(n)} + b_2^{(n)}) < \varepsilon^2 \lambda^2 h(2^{-n})\} ,$$

$$a_i^{(n)} = |\{-\lambda h_i(2^{-n+1}) + t_i < t \leqq -\lambda h_i(2^{-n}) + t_i, |Y_i(t) - Y_i(t_i)| < 2^{-n}\}| ,$$

$$b_i^{(n)} = |\{\lambda h_i(2^{-n}) + t_i < t \leqq \lambda h_i(2^{-n+1}) + t_i, |Y_i(t) - Y_i(t_i)| < 2^{-n}\}| ,$$

where $h_i(s) = s^{\beta_i} \psi^{1/2}(s)$, $i = 1, 2$, $\lambda > 0$, ε is small enough to make

$$(1 + \varepsilon) h_i(2^{-n}) < h_i(2^{-n+1}),\ i = 1, 2 .$$

By the independence of $a_1^{(n)}$ and $b_1^{(n)}$,

$$\begin{aligned} \mathrm{P}\left(a_1^{(n)} < \frac{\varepsilon}{2} \lambda h_1(2^{-n})\right) \cdot \mathrm{P}\left(b_1^{(n)} < \frac{\varepsilon}{2} \lambda h_2(2^{-n})\right) & \\ \leqq \mathrm{P}(a_1^{(n)} + b_1^{(n)} < \varepsilon \lambda h_1(2^{-n})) & \\ \leqq \mathrm{P}(a_1^{(n)} < \varepsilon \lambda h_1(2^{-n})) \cdot \mathrm{P}(b_1^{(n)} < \varepsilon \lambda h_1(2^{-n})) . & \end{aligned}$$

Using (2.13) we obtain

$$\mathrm{P}(a_1^{(n)} + b_1^{(n)} < \varepsilon \lambda h_1(2^{-n})) \approx \psi(2^{-n}) .$$

Similarly, $\mathrm{P}(a_2^{(n)} + b_2^{(n)} < \varepsilon \lambda h_2(2^{-n})) \approx \psi(2^{-n})$.
By using Lemma 2.4 we can find a constant $c_{20} > 0$ such that

$$\mathrm{P}(A_n) \leqq c_{20} \psi^2(2^{-n+1}) \log(1/\psi(2^{-n+1})) .$$

Let $B_n = \{\mu_1 B_{2^{-n}}(Y_1(t_1)) \mu_2 B_{2^{-n}}(Y_2(t_2)) < \varepsilon^2 \lambda^2 h(2^{-n})\}$, then $B_n \subset A_n$, by (4.1) we know $\mathrm{P}(B_n) \approx \psi^2(2^{-n+1}) \log \frac{1}{\psi(2^{-n+1})}$. So for a constant $c_{21} > 0$, $\mathrm{P}(B_n) \geqq c_{21} \mathrm{P}(A_n)$. But $\{A_n\}$ are independent, so $\mathrm{P}(B_n \cap B_m) \leqq \left(\frac{1}{c_{21}}\right)^2 \mathrm{P}(B_m) \mathrm{P}(B_n)$, if $n \neq m$. By using Lemma 2.2, we obtain

$$\mathrm{P}(\limsup_{n \to +\infty} B_n) \geqq c_{21}^2 > 0 .$$

Thus letting $\lambda \to 0$, for any pair (t_1, t_2), $0 < t_1, t_2 < 1$, we have

$$\mathrm{P}\left(\liminf_{r \downarrow 0} \frac{\mu_1 B_r(Y_1(t_1)) \mu_2 B_r(Y_2(t_2))}{h(2r)} = 0\right) \geqq c_{21}^2 > 0 .$$

Since this event has the same probability as $\left\{\liminf_{r\downarrow 0} \frac{(T_1(r)+T_3(r))(T_2(r)+T_4(r))}{h(2r)}=0\right\}$ and the later one is in the initial σ-field generated by Y_1, Y_2, applying Lemma 4.1 gives

$$\liminf_{r\downarrow 0} \frac{\mu_1 B_r(Y_1(t_1))\mu_2 B_r(Y_2(t_2))}{h(2r)}=0 \text{ a.s.}$$

The standard Fubini argument and Lemma 2.1 now imply that

$$h-p(Y_1[0,1]\times Y_2[0,1])=+\infty \text{ a.s.. } \#$$

5 The measure properties of the projection

Now we consider the measure properties of the projection of R on the line $y=x$.

Let Y_i be independent stable subordinators with indices β_i such that $\beta_1+\beta_2\leqq 1/2$ and $W_i=\{Y_i(t): t\in[0,+\infty)\}\cap[0,1]$ and $W=\frac{1}{\sqrt{2}}(W_1\oplus W_2)=\left\{\frac{u+v}{\sqrt{2}}: u\in W_1,\ v\in W_2\right\}$.

Because $\phi-m(A)\geqq\phi-m(proj_\theta A)$ for all θ ($proj_\theta A$ is the projection of A in the direction θ, using Theorem 3.8 we know $\phi-m(W)<\infty$ a.s., where $\phi(s)=s^{\beta_1+\beta_2}\left(\operatorname{loglog}\frac{1}{s}\right)^{2-\beta_1-\beta_2}$. The proof of $\phi-m(W)>0$ a.s. is much more difficult.

In unit cube $[0,1]^2$, for each $n\geqq 2$ we have 2^n-2 nondegenerate and nonoverlapping strips which are perpendicular to the diagonal $y=x$ such that for the ith strip, $S_{i,n}$, the coordinates of the two interception points with the diagonal are $\left(\frac{i}{2^n},\frac{i}{2^n}\right)$ and $\left(\frac{i+1}{2^n},\frac{i+1}{2^n}\right)$. In fact, $S_{i,n}=[0,1]^2\cap\left\{(x,y): \frac{i}{2^{n-1}}\leqq x+y<\frac{i+1}{2^{n-1}}\right\}$. We call the segment between the above two interception points as $w_{i,n}$. Actually $w_{i,n}=\left\{(x,y): x=y,\ \frac{i}{2^n}\leqq x,\ y<\frac{i+1}{2^n}\right\}$. Define measure $\nu=\nu_\omega$ on W such that

$\nu_\omega(w_{i,n})=\mu(S_{i,n})$, for all i, n, μ was defined in the beginning of Sect. 3.

Fix n, for each $x\in\frac{1}{\sqrt{2}}[0,1]\oplus[0,1]$, there a unique segment $w_{i,n}$ containing x, we denote it as $w_n(x)$. We now consider the upper bound of $\limsup_{n\to 0}\frac{\nu_\omega(w_n(x))}{\phi(2^{-n})}$, $x\in W$.

Let $T_{i,n}$ denote the number of dyadic cubes in the strip $S_{i,n}$ hit by $Y=(Y_1,Y_2)$. Before we estimate the probability of $\{T_{i,n}\geqq k\}$ we should note the following facts: (1) For any stable subordinator X with index $0<\beta<1$, if $I_{i,n}=\left[\frac{i}{2^n},\frac{i+1}{2^n}\right)$, $0\leqq i\leqq 2^n-1$, then

$$P(X \text{ hits } I_{i,n})\leqq\frac{\sin\pi\beta}{\pi(1-\beta)}\cdot\frac{1}{i^{1-\beta}} \text{ (see [17])} . \tag{5.1}$$

(2) For any stable subordinator X and any two integers $j<k$, using the technique of stopping time and the strong Markov property one can prove

$$P(\{X \text{ hits } I_{j,n}\}\cap\{X \text{ hits } I_{k,n}\})\leqq P(\{X \text{ hits } I_{j,n}\})P(\{X \text{ hits } I_{k-j,n}\})$$

and if $l_1<l_2<\cdots<l_k$, then

$$\begin{aligned}&P(X \text{ hits } I_{l_1,n},\dots,I_{l_k,n})\\&=P(X \text{ hits } I_{l_1,n})P(X \text{ hits } I_{l_2,n}\mid X \text{ hits } I_{l_1,n})\cdots P(X \text{ hits } I_{l_k,n}\mid X \text{ hits } I_{l_{k-1},n}) .\end{aligned}$$

(3) For a dyadic cube, say $I_{l,k,n}=\left[\left(\frac{l}{2^{-n}},\frac{k}{2^{-n}}\right),\left(\frac{l+1}{2^{-n}},\frac{k+1}{2^{-n}}\right)\right)$, contained in the ith strip $S_{i,n}$, we have $k=2i-l$. If we call $(l/2^{-n}, k/2^{-n})$ as the least point of $I_{l,k,n}$, then we can number the dyadic cubes in the strip $S_{i,n}$ by the numerators of the first coordinates of their least points.

Now we are ready to estimate $P(T_{i,n}\geqq r)$. In fact,

$$\begin{aligned}&P(T_{i,n}\geqq r)\\&\leqq P\Big(\text{There are at least } \left[\frac{r}{2}\right] \text{ dyadic cubes lying in}\\&\qquad\text{the upper half of the strip } S_{i,n} \text{ hit by } Y\Big)\\&\quad+P\Big(\text{There are at least } \left[\frac{r}{2}\right] \text{ dyadic cubes lying in}\\&\qquad\text{the lower half of the strip } S_{i,n} \text{ hit by } Y\Big)\\&\leqq M_1\sum_{1\leqq l_1<\cdots<l_{[r/2]}\leqq i} P(Y \text{ hits } D_{l_1},\cdots \text{ and } D_{l_{[r/2]}}),\end{aligned}$$

where $[x]$ denotes the largest integer less than x and $D_1,\dots,D_{2i}$ are dyadic cubes in $S_{i,n}$, $M_1>0$ is a constant. Let $k=[r/2]$, by those facts we know the term in the right hand side of the inequality is dominated by

$$\begin{aligned}&M_1c_{22}^k\left[\sum_{l_1=1}^{i-k+1}\cdots\sum_{l_k\geqq l_{k-1}+1}^{i}\frac{1}{l_1^{1-\beta_1}}\frac{1}{(l_2-l_1)^{1-\beta_1}}\cdots\right.\\&\left.\frac{1}{(l_k-l_{k-1})^{1-\beta_1}}\frac{1}{(l_2-l_1)^{1-\beta_2}}\cdots\frac{1}{(l_k-l_{k-1})^{1-\beta_2}}\frac{1}{(2i-l_k)^{1-\beta_2}}\right]\end{aligned}$$

$$\leqq \frac{M_1 c_{22}^k}{i^{1-\beta_2}} \left[\sum_{l_1=1}^{i-k+1} \cdots \sum_{l_{k-1}>l_{k-2}}^{i-1} \frac{1}{l_1^{1-\beta_1}} \frac{1}{(l_2-l_1)^{2-\beta_1-\beta_2}} \cdots \frac{1}{(l_{k-2}-l_{k-1})^{2-\beta_1-\beta_2}} \right.$$

$$\left. \left(1 - \frac{\frac{1}{1-\beta_1-\beta_2}}{1+\frac{1}{1-\beta_1-\beta_2}} \left(\frac{1}{i-k+1} \right)^{1-\beta_1-\beta_2} \right) \left(1 + \frac{1}{1-\beta_1-\beta_2} \right) \right]$$

$$\leqq M_1 c_{22}^k i^{\beta_1+\beta_2-1} \left(1+\frac{1}{\beta_1} \right) \left(1 + \frac{1}{1-\beta_1-\beta_2} \right)^k ,$$

where $c_{22} = \prod_{i=1}^{2} \frac{\sin \pi\beta_i}{\pi(1-\beta_i)}$, and $c_{22}\left(1+\frac{1}{1-\beta_1-\beta_2}\right)<1$. Thus there exists $0<\rho<1$ such that $P(T_{i,n} \geqq r) \leqq c_{23}\rho^r$, c_{23} is a constant.
Therefore when $r \geqq r_0$, we can find a number, say ρ_2, between 0 and 1, such that

$$P(T_{i,n} \geqq r) \leqq \rho_2^r . \tag{5.2}$$

Let $B_i(t) = \inf\{u : Y_i(u) > t\}$, $i=1, 2$. They have the same distributions as $t^\beta B_i(1)$ and $\{B_i(1) \geqq x\} = \{Y_i(x) \leqq 1\}$, $i=1$, 2. It follows from (2.11) that for some constants $c_{28}^{(i)} > 0$ and $0 < c_{29}^{(i)} < 1$, $i=1$, 2, we have

$$\begin{aligned} P(B_i(\alpha) \geqq w) &= P(Y_i(1) \leqq (w^{-\frac{1}{\beta_i}}\alpha)) \\ &\sim c_{28}^{(i)} (w^{-\frac{1}{2(1-\beta_i)}} \alpha^{\frac{\beta_i}{2(1-\beta_i)}}) \exp(-c_{29}^{(i)} w^{\frac{1}{1-\beta_i}} \alpha^{-\frac{\beta_i}{1-\beta_i}}) , \\ &\text{if } w^{-\frac{1}{\beta_i}} \alpha \to 0 . \end{aligned}$$

By the above estimation we have

$$\begin{aligned} P(B_1(1)B_2(1) > \lambda) &\leqq P(B_1(1) > \lambda^\gamma) + P(B_2(1) > \lambda^{1-\gamma}) \\ &\leqq \tfrac{1}{2} \exp(-c_{30} \lambda^{\frac{\gamma}{1-\beta_1}}), \text{ if } \lambda \geqq \lambda_0 , \end{aligned}$$

where $\gamma = \frac{1-\beta_1}{2-\beta_1-\beta_2}$, c_{30} is a positive constant less than 1.
So given any dyadic cube I, there is a constant $c_{31}>0$ such that for any $M>0$,

$$\begin{aligned} P\left(\frac{\mu(I)}{\phi(2^{-n})} > M \right) &= P(B_1(1)B_2(1) > M(\log\log 2^n)^{2-\beta_1-\beta_2}) \\ &\leqq \tfrac{1}{2} \exp(-c_{30} M^{\frac{\gamma}{1-\beta_1}} \log\log 2^n) \\ &\leqq c_{31} n^{-c_{30} M^{\frac{\gamma}{1-\beta_1}}} . \end{aligned} \tag{5.3}$$

Given $x \in W_1 \oplus W_2$, if $S_{i,n} = S_{i,n}(x)$ is the unique strip containing $w_n(x)$, then using (5.2), (5.3) and Theorem 2.5 we have

$$\mathrm{P}\left(\frac{\nu_\omega(w_n(x))}{\phi(2^{-n})} \geqq M\right) \leqq \mathrm{P}\left(\frac{\mu(S_{i,n}(x))}{\phi(2^{-n})} \geqq M\right)$$

$$\leqq \mathrm{P}\left(\sum_{j=1}^{T_{i,n}} \frac{\mu(I^{(j)})}{\phi(2^{-n})} \geqq M\right), \ (I^{(j)}\text{'s are dyadic cubes in } S_{i,n})$$

$$\leqq c_{32} \frac{1}{n^2}, \ (\text{by } (5.3)),$$

if we choose M large enough to make $c_{30} M^{\frac{\gamma}{1-\beta_1}} > 2$, c_{32} is a constant. Hence by the Borel–Cantelli Lemma we obtain

$$\limsup_{n \to +\infty} \frac{\nu_\omega(w_n(x))}{\phi(2^{-n})} \leqq 3M \text{ a.s., for any } x \in W.$$

Using Lemma 2.1 we have $\phi - m(W) > 0$ a.s. So we have proved the following theorem:

Theorem 5.1 *If Y_1 and Y_2 are two independent stable subordinators with indices β_1 and β_2 such that $\beta_1 + \beta_2 \leqq 1/2$, then*

$$0 < \phi - m(W) < +\infty \text{ a.s.,}$$

where $\phi(s) = s^{\beta_1 + \beta_2} \left(\log\log \frac{1}{s}\right)^{2 - \beta_1 - \beta_2}$.

Corollary 5.2 *If X_1, X_2 are two independent stable processes with indices $1 < \alpha_1, \alpha_2 \leqq 2$ and $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \geqq \frac{3}{2}$, Z_1 and Z_2 are zero sets of X_1 and X_2, then*

$$0 < \varphi - m(Z_1 \cap [0, 1] \oplus Z_2 \cap [0, 1]) < +\infty \text{ a.s.,}$$

where $\varphi(s) = s^{2 - \frac{1}{\alpha_1} - \frac{1}{\alpha_2}} \left(\log\log \frac{1}{s}\right)^{\frac{1}{\alpha_1} + \frac{1}{\alpha_2}}$.

Remark. The conditions in Corollary 5.2 exclude the most interesting case of X_1, X_2 both being Wiener processes. In this case the number $T_{i,n}$ of cubes hit in one strip $S_{i,n}$ does not have an exponential tail so we are unable to find the exact measure function for $Z_1 \oplus Z_2$.

As for the packing measure of $Z_1 \cap [0, 1] \oplus Z_2 \cap [0, 1]$, if $g(s) = s^{\beta_1 + \beta_2} f(s)$, $f(s)$ is a measure function, then when $\int_{0+} \frac{f^2(s)|\log f(s)|}{s} ds < +\infty$, $g - p(Z_1 \cap [0, 1] \oplus Z_2 \cap [0, 1]) = 0$. But we can say little about the other direction. The problem is that we can not estimate $\limsup_{r \to 0} \frac{g(2r)}{\mu B(x, r)}$, where μ is the projection of $\mu_1 \times \mu_2$ on the diagonal, because of a lack of independence.

The final result relates to the case $\beta_1 + \beta_2 > 1$.

Theorem 5.3 *If* Y_1 *and* Y_2 *are two independent stable subordinators with indices* β_1 *and* β_2 *such that* $\beta_1+\beta_2>1$, *then*

$$|W_1 \oplus W_2|>0 \ a.s..$$

Proof. In order to prove this theorem we only need to modify the proof of Theorem 1 in [12] (Taylor) and adapt the technique of Lemma 9 in [12] to show that, for each $x \in (0, 1)$, $P(x \in W_1 \oplus W_2) \geqq c > 0$. #

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