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Probability Theory and Related Fields

A phase transition for a stochastic PDE related to the contact process *

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Summary: We consider the one-dimensional heat equation, with a semilinear term and with a nonlinear white noise term. R. Durrett conjectured that this equation arises as a weak limit of the contact process with long-range interactions. We show that our equation possesses a phase transition. To be more precise, we assume that the initial function is nonnegative with bounded total mass. If a certain parameter in the equation is small enough, then the solution dies out to 0 in finite time, with probability 1. If this parameter is large enough, then the solution has a positive probability of never dying out to 0. This result answers a question of Durett.

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1 Introduction

Consider the equation

$$(1.1)\qquad \begin{aligned} u_t &= \tfrac{1}{6} u_{xx} + \theta u - u^2 + u^{\frac{1}{2}} \dot{W}, \qquad t>0,\ x\in\mathbb{R},\ \theta>0 \\ u(0,x) &= u_0(x) \geqq 0 \end{aligned}$$

where $\dot{W}=\dot{W}(t,x)$ is spacetime white noise. Durrett suggested that (1.1) should arise as a limit of the long-range contact process studied in Bramson, Durrett, and Swindle [1]. In addition, Durrett guessed that (1.1) exhibits a phase transition as θ varies. For small values of $\theta, u(t,x)$ should die out to 0 in finite time. For large values of $\theta, u(t,x)$ should survive with nonzero probability.

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The purpose of this paper is to prove the existence of a phase transition. In a companion paper, we show that our solution arises as the limit of a long-range contact process.

For our main result, we assume that the initial function $u_0(x)$ in (1.1) is continuous with compact support, nonnegative, and not identically 0. We write $u_0 \in C_c^+$. We say that $u(t, x)$ survives if for all $t>0$, $u(t, 0)$ is not identically 0.

Theorem 1 *Let $u(t, x)$ be a solution to* (1.1). *There exists a constant $\theta_c > 0$, not depending on u_0, such that*

(i) *If $\theta < \theta_c$, then $P(u(t, x)$ survives$) = 0$*
(ii) *If $\theta > \theta_c$, then $P(u(t, x)$ survives$) > 0$.*

For future use, we let T be the first time such that $u(t, x)$ is identically 0. Let $T = \infty$ if there is no such time.

Next we discuss the proof of Theorem 1. First consider case (i), in which $\theta < \theta_c$, and we must show $P(u(t, x)$ survives$) = 0$). We compare $u(t, x)$ to a continuous time branching process with expected offspring size $\mu < 1$. Such a branching process dies out with Probability 1. To implement this comparison, we write $u_0(x) = \sum_{i=1}^{N} u_0^{(i)}(x)$, where $u_0^{(i)}$ is supported on interval $[z_i, z_i + 1]$, $\int_{z_i}^{z_i+1} u_0^{(i)}(x)\,dx \leqq 1$, and $u_0^{(i)}(x) \geqq 0$. We show that one can think of the $u^{(i)}(t, x)$ as almost evolving independently, starting from $u_0^{(i)}(x)$, and such that $u(t, x) \geqq \sum_{i=1}^{N} u^{(i)}(t, x)$. The $u^{(i)}$ satisfy an equation similar to (1.1). We call the $u^{(i)}$ "bricks" that make up u. We choose a stopping time τ, and again majorize $u^i(\tau, x)$ by a sum of $N(i, \tau)$ bricks. These are the offspring of the original brick. If θ is small, we show that the expected offspring size $EN(i, \tau) < 1$, completing the proof. This final step involves scaling (1.1), thereby transforming it into

$$v_t = \frac{\theta^3}{6} v_{xx} + v - v^2 + v^{\frac{1}{2}} \dot{W}. \tag{1.2}$$

Here, θ is small, so the solution does not spread quickly. Furthermore, the term $-v^2$ keeps the solution from becoming too large. Assume that we start with a brick: $v(0, x) = 1(n \leqq x \leqq n+1)$. If θ is small, then with high probability, the noise will drive $v(t, x)$ to 0 before it spreads out too much.

Secondly, consider Theorem 1, case (ii). Here $\theta > \theta_c$, and we must prove that $P(u(t, x)$ survives$) > 0$. We compare $u(t, x)$ to oriented percolation. This idea has already been used in particle systems, as explained in [4]. We begin with another scaling of $u(t, x)$, obtaining

$$v_t = \tfrac{1}{6} v_{xx} + v - v^2 + \theta^{-\frac{3}{4}} v^{\frac{1}{2}} \dot{W}. \tag{1.3}$$

If θ is large, the noise in (1.3) is small, and we almost have

$$w_t = \frac{1}{6} w_{xx} + w - w^2. \tag{1.4}$$

But (1.4) is the classical Kolmogorov-Petrovskii-Piscuinov equation for which many solutions converge to travelling waves. In particular, if $w(0, x) > \frac{1}{2} 1\ (n \leqq x \leqq n+1)$ then for some future time t_0, $w(t_0, x) \geqq \frac{1}{2} 1(n-1 \leqq x \leqq n+2)$. The same holds true for $v(t, x)$ satisfying (1.2), with high probability. Again, we can show approximate independence of the bricks. By this method, we compare $v(t, x)$ to an oriented site percolation process. This process will be described in Sect. 2.2. It is similar to ordinary percolation, except that only paths having certain directions (the forward time direction) are allowed. There is some mild dependence in our percolation process, but this causes no trouble. We show that if $\theta > \theta_c$, then there is a positive probability that, in the percolation model, the origin is part of an infinite connected cluster. This in turn implies that $u(t, x)$ survives with positive probability.

Finally, we note that there is a further connection between (1.1) and the Kolmogorov-Petrovskii-Piscuinov equation. In a future paper, Tribe will show that (1.1) possesses random travelling wave solutions. The proof will use techniques of Durrett [3].

2 Phase transition

2.1 Preliminary results

First we specify the martingale problem mentioned in the introduction. Let $C([0, \infty), M_F)$ be the space of continuous M_F valued paths and $D([0, \infty), M_F)$ the corresponding space of cadlag paths. On either space X_t are the coordinate variables, $\mathscr{H}_t$ the canonical right continuous filtration and $\mathscr{H} = \sigma(\mathscr{H}_t : t \geqq 0)$. We write $X_t(f)$ for $\int f(x) X_t(dx)$ whenever this is defined. For $m \in M_F$ define a candidate density at $x \in \mathbb{R}$ by

$$U(m, x) = \begin{cases} \lim_{n \to \infty} (n/2)\, m([x-(1/n), x+(1/n)]) & \text{if the limit exists} \\ 0 & \text{otherwise.} \end{cases}$$

Consider the martingale problem for probabilities on $(C[0, \infty), M_F)$, $\mathscr{H}_t, \mathscr{H})$: for all $\phi \in C_c^\infty$ the space of infinitely differentiable functions of compact support, $Z_t(\phi)$ is an almost surely continuous local martingale where

$$Z_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s((1/6)\phi_{xx} + \theta\phi - U(X_s, \cdot)\phi)\, ds, \tag{2.1}$$

$$\langle Z(\phi)\rangle_t = \int_0^t X_s(\phi^2)\, ds.$$

If $u_0 \in L^1$ and $u(t, x)$ is a solution to (1.1) then letting $X_t(dx) = u(t, x)\,dx$, the law of X solves (2.1). We also give a martingale problem for a related process which will prove useful: for all $\phi \in C_c^\infty$, $\bar{Z}_t(\phi)$ is an almost surely continuous local martingale where

$$\bar{Z}_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s((1/6)\,\phi_{xx} + \theta\phi)\,ds, \tag{2.2}$$

$$[\bar{Z}(\phi)]_t = \int_0^t X_s(\phi^2)\,ds.$$

If $u_0 \in L^1$ and $\bar{u}(t, x)$ solves

$$\bar{u}_t = \tfrac{1}{6}\bar{u}_{xx} + \theta\bar{u} + \bar{u}^{1/2}\dot{W}, \qquad \bar{u}(0, x) = u_0 \tag{2.3}$$

then the law of $\bar{X}_t(dx) = \bar{u}(t, x)\,dx$ solves (2.2). Note that uniqueness in law for the martingale problems implies uniqueness in law for the stochastic P.D.E.'s. (2.2) is the martingale problem for the measure-valued branching process studied by Dawson, Perkins, and others. It is known that (2.2) has a unique solution if $\bar{u}(0, x)$ is nonnegative and integrable. For uniqueness of (2.1) we have the following lemma, given in Evans and Perkins [8], Theorem 3.9. Let M_F be the set of finite measures on $\mathbb{R}$, and define

$$\mathscr{F} = (m \in M_F \colon \iint \log_+(1/|x-y|)\,m(dx)\,m(dy) < \infty).$$

Lemma 2.1.1 *Let $m \in \mathscr{F}$. Then there exists a unique solution P to* (2.1) *satisfying $P(X_0 = m) = 1$.*

Next, we give a scaling result which will prove useful later. A similar idea appeared in Mueller and Sowers [13], Lemma 2.4.

Lemma 2.1.2 *Let $u(t, x)$ satisfy* (1.1), *and let*

$$v(t, x) = c\,u(at, bx).$$

Then $v(t, x)$ satisfies

$$v_t = \frac{a}{6b^2} v_{xx} + \theta a v - \frac{a}{c} v^2 + \frac{c^{\frac{1}{2}} a^{\frac{1}{2}}}{b^{\frac{1}{2}}} v^{\frac{1}{2}} \dot{W}, \tag{2.4}$$

where the $\dot{W}$ in (2.4) *may be a different white noise than the $\dot{W}$ in* (1.1).

Proof. By Theorem 3.2 of Walsh [18], (1.1) is equivalent to the following weak form: for all $\phi(x) \in C_c^\infty$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} u(t, x)\,\phi(x)\,dx = & \int_{-\infty}^{\infty} u(0, x)\,\phi(x)\,dx \\ & + \int_0^t \int_{-\infty}^{\infty} u(s, x)[\phi_{xx} + \theta\phi(x) - u(s, x)\,\phi(x)]\,dx \\ & + \int_0^t \int_{-\infty}^{\infty} u(s, x)^{\frac{1}{2}}\,W(dx\,ds).\end{aligned} \tag{2.5}$$

Lemma 2.1.2 follows by scaling (2.5), and observing that if $W(t, x)$ is a Brownian sheet, then $a^{-\frac{1}{2}} b^{-\frac{1}{2}} W(at, bx)$ is also a Brownian sheet.

We also need comparison theorems. Similar theorems appear in Kotelenez [11], Mueller [12], Pardoux [14], and Shiga [16]. In passing, we note that the proof of the existence of a critical parameter θ_c is buried in the middle of this sequence of lemmas, in 2.1.6. We nead a slightly different equation than (1.1). Suppose that $a>0$, $b>0$. Consider

(2.6)
$$u_t = \frac{a}{6} u_{xx} + u - u^2 + b u^{\frac{1}{2}} \dot{W},$$
$$u(0, x) = u_0(x).$$

Of course, (2.6) can be given rigorous meaning via an integral equation similar to (2.5). We also consider the following equations

(2.7)
$$u_t = \frac{a}{6} u_{xx} + u + b u^{\frac{1}{2}} \dot{W},$$
$$u(0, x) = u_0(x),$$

(2.8)
$$u_t = \frac{a}{6} u_{xx} + u - u^2 + b u^{\frac{1}{2}} \dot{W}, \qquad L < x < R,$$
$$u(t, L) = u(t, R) = 0,$$
$$u(0, x) = u_0(x).$$

Lemma 2.1.3 *Let $u_0^{(i)} \in C_c^+$ for $i=1,2$; and assume that $u_0^{(1)}(x) \leqq u_0^{(2)}(x)$ for all $x \in \mathbb{R}$. Then, on a common probability space, we can find solutions $u^{(i)}(t, x)$ $i=1,2$ to (2.6), such that $u^{(i)}(0, x) = u_0^{(i)}(x)$, $i=1,2$; and such that $u^{(1)}(t, x) \leqq u^{(2)}(t, x)$ for all $t \geqq 0$, $x \in \mathbb{R}$, a.s.*

Lemma 2.1.4 *Let $u_0^{(i)} \in C_c^+$ for $i=1,2$; and assume that $u_0^{(1)}(x) \leqq u_0^{(2)}(x)$ for all $x \in R$. Then, on a common probability space, we can construct solutions $u^{(1)}(t, x)$ to (2.6) and $u^{(2)}(t, x)$ to (2.7), such that $u^{(i)}(0, x) = u_0^{(i)}(x)$; $i=1,2$; and such that $u^{(1)}(t, x) \leqq u^{(2)}(t, x)$ for all $t \geqq 0$, $x \in R$, a.s.*

Lemma 2.1.5 *Let $u_0^{(i)} \in C_c^+$ for $i \geqq 0$; and assume that $u_0^{(0)}(x) \geqq u_0^{(i)}(x)$ for all $x \in \mathbb{R}$, $i \geqq 1$. Then, on a common probability space, we can find a solution $u^{(0)}(t, x)$ to (2.6) and $u^{(i)}(t, x)$; $i \geqq 1$ to (2.8), such that $u^{(i)}(0, x) = u_0^{(i)}(x)$, $i \geqq 0$; and such that $u^{(0)}(t, x) \geqq u^{(i)}(t, x)$ for all $i \geqq 1$, $t \geqq 0$, $x \in \mathbb{R}$, a.s. Here, we are assuming that $u^{(i)}(t, x)$ are given by (2.8) for $i \geqq 1$ and $L_i \leqq x \leqq R_i$. Outside of $[L_i, R_i]$, we assume that $u^{(i)}(t, x) = 0$.*

The following lemma proves the existence of a critical parameter $\theta_c \in [0, \infty]$ for a given initial condition. Using absolute continuity results of Evans and Perkins [7], it follows that for $t>0$, solutions $u(t, \cdot)$ with different initial conditions induce absolutely continuous measures on function space. Thus, the critical parameter must be the same for all initial conditions.

We include the following lemma here because its proof is similar to that of the preceding lemmas.

Lemma 2.1.6 *Suppose that $u_1(t,x)$ and $u_2(t,x)$ are two solutions of* (1.1) *with the same initial conditions $u_0(x)$, but with different parameters $\theta=\theta_1$ and $\theta=\theta_2$, respectively. Suppose further that $\theta_1 \leqq \theta_2$. Then, with Probability 1, $u_1(t,x) \leqq u_2(t,x)$ for all $t>0$, $x \in R$.*

Lemmas 2.1.3–2.1.6 have similar proofs, so we omit some details. But first, note that these lemmas are easy to see on the heuristic level. We think of u as a limit of contact processes. Lemma 2.1.6 says that for two contact processes with the same initial conditions, the contact process with more growth will be larger. The other lemmas say that for two contact processes, if one has a larger set of occupied sites at time 0, it will still have a larger set of occupied sites at some later time. Standard coupling methods can be used to make these statements precise.

Now we give rigorous proofs. We first consider Lemmas 2.1.3 and 2.1.4. Let $f_n(u)$ be a sequence of Lipschitz functions converging uniformly to $u^{\frac{1}{2}}$ for $u \in [0, \infty]$. Also assume that $f_n(0)=0$.

Consider the equations

(2.9)
$$u_t = \frac{a}{6} u_{xx} + u - u^2 + b f_n(u)\, \dot{W},$$
$$u(0,x) = u_0^{(1)}(x),$$

(2.10)
$$u_t = \frac{a}{6} u_{xx} + u + b f_n(u)\, \dot{W},$$
$$u(0,x) = u_0^{(2)}(x).$$

By Theorem 3.3 of Kotelenez [11], if $u^{(1,n)}(t,x)$ and $u^{(2,n)}(t,x)$ are solutions to (2.9) and (2.10) respectively, and if $u_0^{(1)}(x) \leqq u_0^{(2)}(x)$ for all $x \in \mathbb{R}$, then for all $t \geqq 0$, $x \in \mathbb{R}$, we have $u^{(1,n)}(t,x) \leqq u^{(2,n)}(t,x)$ a.s. Now we use the same argument as in the proof of Theorem 2.5 of Shiga [16]. First, this argument gives the tightness of $u^{(2,n)}(t,x)$. Then since $u^{(1,n)}(t,x) \leqq u^{(2,n)}(t,x)$, and since $-u^2$ is locally Lipschitz, Shiga's argument also shows the tightness of $(u^{(1,n)}(t,x), u^{(2,n)}(t,x))$. Choosing a subsequence which converges in distribution, we obtain solutions $u^{(1)}(t,x)$ and $u^{(2)}(t,x)$ to (2.6) and (2.7), respectively, such that for all $t>0$, $x \in \mathbb{R}$, we have $u^{(1)}(t,x) \leqq u^{(2)}(t,x)$. This proves Lemma 2.1.4. We can also prove Lemma 2.1.3, by considering solutions $u^{(1,n)}$ and $u^{(2,n)}$ to (2.9) which are both majorized by a solution $u^{(3,n)}$ to (2.10). Using Kotelenez's Theorem and extracting a convergent subsequence proves Lemma 2.1.3.

Let us outline the proof of Lemma 2.1.5. Again, we replace $u^{\frac{1}{2}}$ by $f_n(u)$ in (2.6) and (2.8), where f_n was given earlier in the proof of Lemmas 2.1.3 and 2.1.4. To analyze such equations, Kotelenez [11] considers Picard iteration for the corresponding integral equations, where $\dot{W}$ is approximated by a smoother noise. We have a countable number of processes while Kotelenez has just 2, but in the approximation, for each $i \geqq 1$ we would have that $u^{(0)}(t, x) \geqq u^{(i)}(t, x)$ for all $t>0$, $x \in R$. The union of the null sets would still be a null set. We would have to show tightness for an infinite set of processes instead of just 2, but again this would cause no difficulties. Lemma 2.1.5 would fit into his scheme, except that (2.6) involves the Laplacian Δ on $\mathbb{R}$, and (2.8) involves the Dirichlet Laplacian on $[L, R]$. Let $S(t, x, f)$ be the semigroup generated by Δ on $\mathbb{R}$, and let $\underline{S}(t, x, f)$ be the semigroup generated by Δ on $[L, R]$. To prove Lemma 2.1.5 by Kotelenez's methods, we would merely use the fact that for $x \in [L, R]$, $t \geqq 0$, and $f \geqq 0$, we have $\underline{S}(t, x, f) \leqq S(t, x, f)$.

Finally, 2.1.6 also follows from Kotelenez's comparison theorem, after replacing $u^{\frac{1}{2}}$ by a sequence of Lipschitz approximations $f_n(x)$ and taking the weak limit.

Lemma 2.1.7 *Suppose that $u_0^{(k)} \in C_c^+$ for $k=0, \ldots, n$; and that $\{\dot{W}_k(t, x)\}_{k=1}^n$ are independent white noises. Suppose that $u_0^{(0)}(x) \leqq \sum_{k=1}^{n} u_0^{(k)}(x)$.*

Let $\alpha, \beta > 0$. Then for some white noise $\dot{W}_0(t, x)$ there exist solutions $\{u^{(k)}(t, x)\}_{k=0}^n$ to

$$u^{(k)} = \alpha u_{xx}^{(k)} + u^{(k)} - (u^{(k)})^2 + \beta (u^{(k)})^{\frac{1}{2}} \dot{W}_k \tag{2.11}$$

$$u^{(k)}(0, x) = u_0^{(k)}(x); \qquad k=0, \ldots, n$$

such that for all $t \geqq 0$, $x \in \mathbb{R}$, we have

$$u^{(0)}(t, x) \leqq \sum_{k=1}^{n} u^{(k)}(t, x).$$

Proof. We reformulate (2.11) as a martingale problem. For a function $\phi \in C_c^\infty(\mathbb{R})$, let $x_k(t, \phi) = \int_{-\infty}^{\infty} u(t, x)\, \phi(x)\, dx$. Fix k in (2.11). As is well known from Walsh [18] or Shiga [16], (2.11) holds for some white noise $\dot{W}_k$ iff $Z_k(t, \phi)$ is a continuous local martingale for all $\phi \in C_c^\infty(\mathbb{R})$, where

$$Z_k(t, \phi) = x_k(t, \phi) - x_k(0, \phi) + \int_0^t x_k(s, \alpha \phi_{xx} + \phi - \phi u^{(k)}(s, \cdot))\, ds, \tag{2.12}$$

$$\langle Z_k(\cdot, \phi) \rangle_t = \beta^2 \int_0^t x_k(s, \phi^2)\, ds.$$

Furthermore, if W_k and W_l are independent white noises, we have

$$\langle Z_k(\cdot, \phi), Z_l(\cdot, \phi) \rangle_t = 0.$$

Let $\underline{u}(t,x)=\sum_{k=1}^{n} u^{(k)}(t,x)$ and let $X(t,\phi)=\int_{-\infty}^{\infty} \underline{u}(t,x)\,\phi(x)\,dx$, for $g\in C_c^\infty(\mathbb{R})$. Adding (2.12) for $k=1,\ldots,n$; and taking into account (2.1), we find that

$$\underline{u}_t=\alpha\underline{u}_{xx}+\underline{u}-\underline{u}^2+2\sum_{1\leqq i<j\leqq n} u^{(i)}u^{(j)}+\beta\underline{u}^{\frac{1}{2}}\dot{W}_0, \tag{2.13}$$

$$\underline{u}(0,x)\geqq u_0^{(0)}(x),$$

for some white noise $\dot{W}$. Applying arguments similar to those used to prove Lemmas 2.1.4–2.1.7, we find that for $t\geqq 0$, $x\in\mathbb{R}$, we have

$$u^{(0)}(t,x)\leqq\underline{u}(t,x)\quad\text{a.s.}$$

Here, the W_0 in (2.11) is the same as in (2.13). This proves Lemma 2.1.7.

We wish to show that if $u_0\in C_c^+$, then solutions $u(t,x)$ of (2.6) have compact support in x, and to estimate the size of the support. By Lemma 2.1.4, we need only study solutions $\bar{u}(t,x)$ of (2.7). Note that $\bar{u}(t,x)$ is the density of a super-Brownian motion, with different parameters than usual. Consider the equation

$$v_t^{(\gamma)}=\frac{a}{6}v_{xx}^{(\gamma)}+\gamma v^{(\gamma)}+b(v^{(\gamma)})^{\frac{1}{2}}\dot{W}, \tag{2.14}$$

$$v(0,x)=u_0(x). \tag{2.15}$$

Iscoe [10] has shown that $v^{(0)}(t,x)$ has compact support in x, if $u_0\in C_c^+$. However $v^{(1)}$, not $v^{(0)}$, satisfies (2.6). To compare $v^{(1)}$ to $v^{(0)}$, we need a result of Dawson [2]. Dawson proved the following lemma for (2.15) with $a=6$, $b=1$; but his proof easily generalizes to our case.

Lemma 2.1.8 *Let $P_{\gamma,t}$ denote the measure induced on path space by $v^{(\gamma)}(s,x)$; $s\leqq t$, $x\in\mathbb{R}$. Assume that $v^{(\gamma)}(0,x)=u_0(x)$. Then $P_{\gamma,t}$ is absolutely continuous with respect to $P_{0,t}$, and*

$$\frac{dP_{\gamma,t}}{dP_{0,t}}=\exp\left(-\frac{\gamma}{b}\int_0^t\int_{-\infty}^{\infty} v^{\frac{1}{2}}(s,x)\,W(dx\,ds)-\frac{\gamma^2}{2b^2}\int_0^t\int_{-\infty}^{\infty} v(s,x)\,dx\,ds\right). \tag{2.16}$$

Let $A=A(t,y)$ be the event that for $0\leqq s\leqq t$ and $|x|>y$, we have $v^{(\gamma)}(s,x)=0$. Suppose that $v^{(\gamma)}(0,x)=0$ for $|x|>R$, and that the initial mass $\int_{-\infty}^{\infty} v^{(\gamma)}(0,x)\,dx=I$.

Lemma 2.1.9 *If $y\geqq R$ and $t\leqq\frac{1}{4\gamma}$, then*

$$P_{\gamma,t}(A^c(t,R+y))\leqq cI\frac{\sqrt{a}}{b(R+y)}$$

Proof. By Lemma 2.1.8 and Cauchy's inequality,

$$(2.17)\qquad P_{\gamma,t}(A^c)=E_{0,t}\Big(I(A^c)\exp\Big(-\frac{\gamma}{b}\int_0^t\int_{-\infty}^{\infty} v^{\frac{1}{2}}(s,x)\,W(dx\,ds)$$

$$-\frac{\gamma^2}{2b^2}\int_0^t\int_{-\infty}^{\infty} v(s,x)\,dx\,ds\Big)$$

$$\leqq P_{0,t}(A^c)^{\frac{1}{2}}\Big(E_{0,t}\Big(\exp\Big(-\frac{2\gamma}{b}\int_0^t\int_{-\infty}^{\infty} v^{\frac{1}{2}}(s,x)\,W(dx\,ds)\Big)\Big)\Big)^{1/2}.$$

However, results of Iscoe [10] allow us to estimate $P_{0,t}(A)$. He only dealt with the case $a=6$, $b=1$, but some simple modifications give us the following result. This is a modification of Theorem 1 of Iscoe [10]. Here, $v(t,x)=v^{(0)}(t,x)$.

Lemma 2.1.10 *Suppose $I=1$. Then*

$$P_{0,\infty}(A)=1-\exp\Big[-(R+y)^{-2}\int_{-\infty}^{\infty} v(0,x)\,h\Big(\frac{x}{R-y}\Big)\,dx\Big]$$

where $h(x)$ is the solution of:

$$(2.18)\qquad \frac{a}{6}h_{xx}=b^2h^2$$

$$\lim_{|x|\to 1} h(x)=\infty.$$

Let $h_0(x)$ be the solution of (2.18) with $a=6$, $b=1$. By scaling, we find that

$$h(x)=\frac{a}{6b^2}h_0(x).$$

Iscoe [10] notes that $h_0(0)\approx 8.38$. Therefore

$$P_{0,t}(A^c)\leqq P_{0,\infty}(A^c)$$

$$\leqq 1-\exp\Big[-(R+y)^{-2}\frac{a}{6b^2}I h_0\Big(\frac{R}{R+y}\Big)\Big]$$

since $h_0(x)$ is nondecreasing in $[0,1]$. Recall that $I=\int_{-\infty}^{\infty} v(0,x)\,dx$. Using the inequality $1-e^{-x}\leqq x$ for $x>0$, we have

$$P_{0,t}(A^c)\leqq (R+y)^{-2}\frac{a}{6b^2}\,I h_0\Big(\frac{R}{R+y}\Big)$$

$$\leqq c\frac{a}{b^2(R+y)^2}$$

for $y\geqq R$, where $c=6h_0(\frac{1}{2})$.

Now consider $X=\int_0^t \int_{-\infty}^{\infty} v^{\frac{1}{2}}(s,x)\, W(dx\,ds)$, where $v=v^{(0)}$. Since X is a white noise integral in the sense of Walsh [18], X is equal in distribution to a time-changed Brownian motion. That is, for some Brownian motion B

$$X=B(T), \qquad T=\int_0^t \int_{-\infty}^{\infty} v(s,x)\, dx\, ds.$$

It is well known that the total mass $M(t)=\int_{-\infty}^{\infty} v(t,x)\, dx$ satisfies the Feller equation $dM=bM^{\frac{1}{2}}\, d\bar{B}$, for some Brownian motion $\bar{B}$. We have:

$$\begin{aligned} &P(|X|>\lambda) \leqq P(\sup_{s\leqq r} |B(s)|>\lambda)+P\left(\sup_{s\leqq t} M(s)>\frac{r}{t}\right) \\ &\quad =(I)+(II). \end{aligned} \tag{2.19}$$

By standard estimates,

$$(I) \leqq \exp\left[-\frac{\lambda^2}{r}\right] \tag{2.20}$$

for λ^2/r sufficiently large. To estimate (II), we let $Y(t)=\frac{2}{b} M(t)^{\frac{1}{2}}$ and use Ito's lemma:

$$dY=d\bar{B}-\frac{b}{4} M^{-\frac{1}{2}}\, dt \leqq d\bar{B}.$$

Therefore, since $Y(0)=(2/b)\, M(0)^{\frac{1}{2}}=(2/b)\, I^{\frac{1}{2}}$,

$$\begin{aligned} (II) &= P\left(\sup_{s\leqq t} \frac{2}{b} M(s)^{\frac{1}{2}} > \frac{2}{b}\sqrt{\frac{r}{t}}\right) \\ &\leqq P\left(\sup_{s\leqq t} \left[\bar{B}(s)+\frac{2}{b} I^{\frac{1}{2}}\right] > \frac{2}{b}\sqrt{\frac{r}{t}}\right) \\ &\leqq P\left(\sup_{s\leqq t} \bar{B}(s) > \frac{2}{b}\left[\sqrt{\frac{r}{t}}-I^{\frac{1}{2}}\right]\right) \\ &\leqq \exp\left[-\frac{\frac{4}{b^2}\left(\sqrt{\frac{r}{t}}-I^{\frac{1}{2}}\right)^2}{2t}\right] \\ &\leqq \exp\left[-\frac{r}{b^2 t^2}\right] \end{aligned} \tag{2.21}$$

if $I < 0.08\,r/t$ and if $r/(b^2 t^2)$ is sufficiently large. Putting together (2.19), (2.20), and (2.21), with $r = \lambda b t$, we have

$$P(|X| > \lambda) \leqq \exp\left[-\frac{\lambda^2}{r}\right] + \exp\left[-\frac{r}{b^2 t^2}\right] \tag{2.22}$$

$$\leqq 2\exp\left[-\frac{\lambda}{bt}\right] \tag{2.23}$$

for λ sufficiently large, and such that $I < 0.08\,\lambda b$.

Now, using (2.22), for $t < \dfrac{1}{4\gamma}$, we have

$$\begin{aligned}
&E_{0,t}\exp\left[\frac{-2\gamma}{b}\int_0^t\int_{-\infty}^{\infty} v^{\frac{1}{2}}(s,x)\, W(dx\,ds)\right]\\
&= E_{0,t}\exp\left[\frac{-2\gamma}{b}X\right]\\
&\leqq \int_{-\infty}^{0}\exp\left[\frac{-2\gamma y}{b}\right] P(X\in dy) + 1\\
&\leqq \int_{-\infty}^{0}\left[\int_0^y\left(-\frac{2\gamma}{b}\right)\exp\left[\frac{-2\gamma z}{b}\right] + 1\right] dz\, P\{X\in dy\} + 1\\
&\leqq 2 + \int_0^{\infty}\int_z^{\infty} P(|X|\in dy)\frac{2\gamma}{b}\exp\left[\frac{-2\gamma z}{b}\right] dz\\
&\leqq 2 + \int_0^{\infty} P\{|X| > z\}\frac{2\gamma}{b}\exp\left[\frac{-2\gamma z}{b}\right] dz\\
&\leqq 2 + \int_0^{\infty}\frac{2\gamma}{b}\exp\left[\frac{-2\gamma z}{b}\right]\exp\left[-\frac{z}{bt}\right] dz\\
&< c < \infty
\end{aligned}$$

where c does not depend on a, b, or γ, but only on the fact that $t > \dfrac{1}{4\gamma}$. Putting together (2.17), (2.19), and (2.24), we obtain Lemma 2.1.9.

Next, we prove a large deviations lemma. As stated earlier, several authors have given similar theorems. For $t \geqq 0$, $-M \leqq x \leqq M$, let $v(t, x)$ and $w(t, x)$ satisfy the following equations.

$$\begin{aligned}
&v_t = \tfrac{1}{6} v_{xx} + v - v^2 + b v^{\frac{1}{2}}\dot{W}, &(2.24)\\
&v(t, -M) = v(t, M) = 0,\\
&v(0, x) = u_0(x),\\
&w_t = \tfrac{1}{6} w_{xx} + w - w^2, &(2.25)
\end{aligned}$$

$$w(t,-M)=w(t,M)=0,$$
$$w(0,x)=u_0(x).$$

Our goal is to show the following lemma. Its proof will take up the rest of the subsection.

Lemma 2.1.11 *Let $L, M, T>1$, and suppose that $\sup_{x\in\mathbb{R}} u_0(x)<\frac{L}{2}e^{-T}$. There exists a constant $C(T, M)$ such that if $\delta^2/(Lb^2)>C(T, M)$ then*

$$P(\sup_{\substack{0\le t\le T\\ -M\le x\le M}} |v(t,x)-w(t,x)|>\delta)<\exp\left[-\frac{c\delta^2}{Lb^2}\right]+\exp\left[-\frac{cL}{b^2}\right].$$

It is convenient to cut off v when it becomes large. Let $v^{(L)}(t,x)$ satisfy

$$v_t^{(L)}=\tfrac{1}{6}v_{xx}^{(L)}+v^{(L)}-(v^{(L)}\wedge L)^2+b(v^{(L)}\wedge L)^{\frac{1}{2}}\dot{W}, \tag{2.26}$$
$$v^{(L)}(t,-M)=v^{(L)}(t,M)=0,$$
$$v^{(L)}(0,x)=u_0(x).$$

Let $\tau=\tau(L)$ be the first time $t\ge 0$ that $\sup_{x\in\mathbb{R}} v^{(L)}(t,x)=L$. Clearly, we can construct $u(t,x)$ and $v^{(L)}(t,x)$ satisfying (2.25) and (2.26), respectively, such that $v(t,x)=v^{(L)}(t,x)$ for $t\le\tau$, $-M\le x\le M$.

Let $H(t,x,y)$ be the fundamental solution of the equation

$$u_t=u_{xx}+u; \qquad t\ge 0,\ -M\le x\le M,$$
$$u(t,-M)=u(t,M)=0,$$
$$u(0,x)=\delta(x-y).$$

Let

$$N(t,x)=b\int_0^t\int_{-\infty}^{\infty} H(t-s,x,y)[v^{(L)}(s,y)\wedge L]^{\frac{1}{2}}\,W(dy\,ds)).$$

Of course, $N(t,x)$ is the noise term in the integral equation equivalent to (2.26). The following lemma is very similar to proposition (A.2) of Sowers [17].

Lemma 2.1.12 *Fix $T, M>0$. There is a constant $c>0$ such that if $\frac{\delta^2}{Lb^2}$ is sufficiently large, then*

$$P(\sup_{\substack{0\le s\le T\\ -M\le x\le M}} |N(s,x)|>\delta)\le\exp\left[-\frac{c\delta^2}{Lb^2}\right].$$

The proof of Lemma 2.1.12 rests on the following result.

Lemma 2.1.13 *Let $0<h<M$. If $0<\sigma<\frac{1}{2}$, then*

(A) $\int_0^T \int_{-M}^M |H(t, x+h, y)-H(t, x, y)|^2 \, dy\, dt \leqq c h^{2\sigma}$

(B) $\int_0^T \int_{-M}^M |H(t+h, x, y)-H(t, x, y)|^2 \, dy\, ds \leqq c h^2$

(C) $\int_0^h \int_{-M}^M |H(t+s, x, y)|^2 \, dy\, ds \leqq c h^{\frac{1}{2}}$

The constants c depend on T and M.

Proof of Lemma 2.1.13. Sowers [17] Lemma A.1, gives a similar proof, for periodic boundary conditions. Note that, for $t>0$,

$$H(t, x, y)=e^t \sum_{k=1}^{\infty} e^{-\theta_k t} \phi_k(x)\, \phi_k(y) \tag{2.27}$$

where

$$\phi_k(x)=\sqrt{\frac{2}{M}} \sin\left(\frac{\pi k x}{2M}-\frac{\pi k}{2}\right)$$

$$\theta_k=\frac{\pi^2 k^2}{24 M^2}.$$

To prove part A of Lemma 2.1.13, by (2.27),

$$\begin{aligned}
&\int_0^T \int_{-M}^M |H(t, x+h, y)-H(t, x, y)|^2 \, dy\, dt \\
&\leqq e^{2T} \sum_{k=1}^{\infty} \frac{1-e^{-2\theta_k T}}{2\theta_k} |\phi_k(x+h)-\phi_k(x)|^2 \\
&\leqq \sum_{k=1}^{\infty} \frac{e^{2T}}{2\theta_k} |\phi_k(x+h)-\phi_k(x)|^{2-2\sigma} \, |\phi_k(x+h)-\phi, (x)|^{2\sigma}.
\end{aligned}$$

Clearly, $|\phi_k(x+h)-\phi_k(x)| \leqq \frac{2\sqrt{2}}{\sqrt{M}}$, and by the mean value theorem, $|\phi_k(x+h)-\phi_k(x)| \leqq \frac{\pi k}{2^{\frac{1}{2}} M^{\frac{3}{2}}} h$. Therefore

$$\begin{aligned}
\int_0^T \int_{-M}^M |H(t, x+h, y)-H(t, x, y)|^2 \, dy &\leqq c h^{2\sigma} \sum_{k=1}^{\infty} \frac{k^{2\sigma}}{\theta_k} \\
&\leqq c h^{2\sigma}.
\end{aligned}$$

To prove part B, by (2.27),

$$\int_0^T \int_{-M}^M |H(t+h, x, y) - H(t, x, y)|^2 \, dy$$

$$\leqq e^{2T} \sum_{k=1}^{\infty} \phi_k^2(x) \int_0^T [e^{-\theta_k(t+h)} - e^{-\theta_k t}]^2 \, dt$$

$$\leqq \frac{2e^{2T}}{M} [e^h - 1]^2 \sum_{k=1}^{\infty} \frac{1 - e^{-2\theta_k T}}{2\theta_k}$$

$$< ch^2$$

if $h < M$.

To prove part C, by the maximum principle, $H(t, x, y) \leqq e^{2T} G(t, x, y)$ for $t \leqq T$.

$$\int_0^h \int_{-M}^M |H(t, +s, x, y)|^2 \, dy \, ds \leqq e^{2T} \int_0^h \int_{-M}^M G(t+s, y)^2 \, dy \, ds$$

$$= c \int_0^h \frac{1}{\sqrt{t+s}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{t+s}} e^{-c\frac{y^2}{t+s}} \, dy \, ds$$

$$\leqq c \int_0^h \frac{1}{\sqrt{5}} \, ds$$

$$= c h^{\frac{1}{2}}.$$

This completes the proof of Lemma 2.1.13. Now we proceed with the proof of Lemma 2.1.12. This part is similar to the proof of Lemma 2.1 of Mueller [12]. We know by Walsh [18], Chap. 2, that $N(t, x)$ is a.s. continuous in (t, x). Therefore, the following estimate will prove Lemma 2.1.12.

Let $\mathscr{I} = \mathscr{I}(T, M)$ be the set of integers (i, k, ℓ, n) such that $i = 1, 2$; $n \geqq 1$; $0 \leqq k < 2^n$; $-2^n < \ell < 2^n$. Define events $A(i, k, \ell, n)$ by

$$A(1, k, \ell, n) = \left\{ \left| N\left(\frac{(k+1)T}{2^n}, \frac{\ell M}{2^n}\right) - N\left(\frac{kT}{2^n}, \frac{\ell M}{2^n}\right) \right| \leqq \frac{\delta}{(M+T) 2^{\frac{n}{8}+7}} \right\}$$

$$A(2, k, \ell, n) = \left\{ \left| N\left(\frac{kT}{2^n}, \frac{(\ell+1)M}{2^n}\right) - N\left(\frac{kT}{2^n}, \frac{\ell M}{2^n}\right) \right| \leqq \frac{\delta}{(M+T) 2^{\frac{n}{8}+7}} \right\}.$$

Let

$$A = \bigcap_{(i, k, \ell, n) \in \mathscr{I}} A(i, k, \ell, n).$$

Note that

$$\{ \sup_{\substack{s \leqq T \\ -M \leqq x \leqq M}} |N(t, x)| > \delta \} \subset A^c. \tag{2.28}$$

Indeed, consider the binary expansion of $\left(\frac{s}{t}, \frac{x}{M}\right)$ for $(s, x) \in [0, t] \times [-M, M]$. By adding digits one by one, we can construct a path from $(0, 0)$ to $\left(\frac{s}{t}, \frac{x}{M}\right)$ (with infinitely many steps), such that there is only 1 step each of the type, except for $n=1$. In that case there may be as many as $2(M+T)$ steps

$$\left(\frac{kT}{2^n}, \frac{\ell M}{2^n}\right) \to \left(\frac{(k+1)\,T}{2^n}, \frac{\ell M}{2^n}\right) \quad \text{or}$$

$$\left(\frac{kT}{2^n}, \frac{\ell M}{2^n}\right) \to \left(\frac{kT}{2^n}, \frac{(\ell \pm 1)\,M}{2^n}\right).$$

$N(s, x)$ is an infinite sum of terms $N(s_2, x_2) - N(s_1, x_1)$, where each step $(s_1, x_1) \to (s_2, x_2)$ is one of the above type. If A holds, then $|N(s, x)|$ is bounded by the sum of such terms, and the sum is less than or equal to

$$2(M+T) \sum_{k=1}^{\infty} \frac{\delta}{(M+T)} 2^{\frac{n}{8}+7} < \delta.$$

Now we estimate $P(A^c(i, k, \ell, n))$. Note that

$$\begin{aligned} &N(t, x) - N(t, y) \\ &\quad = b \int_0^t \int_{-M}^{M} [H(t-s, x, z) - H(t-s, y, z)]\,(v^{(L)}(s, z) \wedge L)^{\frac{1}{2}}\, W(dz\,ds) \end{aligned}$$

is a white noise integral. Thus, it is a time changed Brownian motion with time scale S bounded by

$$S \leqq L b^2 \int_0^T \int_{-M}^{M} |H(t, x, z) - H(t, y, z)|^2\, dz\, dt.$$

Therefore, by Lemma 2.1.13, with $\sigma = \frac{1}{4}$,

$$\begin{aligned} (2.29) \qquad P(A^c(2, k, \ell, n)) &\leqq P\left(\sup_{s \leqq cLb^2 2^{-\frac{n}{2}}} |B(s)| \geqq \frac{\delta}{(M+T)\, 2^{\frac{n}{8}+7}}\right) \\ &\leqq \exp\left(-\frac{c\,\delta^2\, 2^{\frac{n}{4}}}{L b^2}\right) \end{aligned}$$

for large δ, where c depends on T and M. Now

$$N(t+h, x)-N(t, x)$$
$$=b \int_0^t \int_{-M}^M [H(t+h-s, x, z)-H(t-s, x, z)](v^{(L)}(s, z) \wedge L)^{\frac{1}{2}} W(dz\,ds)$$
$$+\int_0^h \int_{-M}^M H(t+h-s, x, z)(v^{(L)}(s, z) \wedge L)^{\frac{1}{2}} W(dz\,d(t-s)).$$

Thus, $N(t+h, x)-N(t, x)$ is a time-changed Brownian motion with time scale bounded by

$$S \leqq 2L\left[\int_0^T \int_{-M}^M |H(t+h-s, x, z)-H(t-s, x, z)|^2\,dz\,ds\right.$$
$$\left.+\int_0^h \int_{-\infty}^\infty H(t+h-s, x, z)^2\,dz\,ds\right]^2.$$

Therefore, by Lemma 2.1.13,

$$P(A^c(1, k, \ell, n)) \leqq P\left(\sup_{s \leqq cLb^2 2^{\frac{-n}{2}}} |B(s)| > \frac{\delta}{(M+T)\,2^{\frac{n}{8}+7}}\right) \tag{2.30}$$
$$\leqq \exp\left(-\frac{c\delta^2 2^{\frac{n}{4}}}{Lb^2}\right)$$

for large δ, where c depends on t and M.

Adding together (2.29) and (2.30) for $(i, k, \ell, n) \in \mathscr{I}$, we obtain Lemma 2.1.12.

Now we wish to relate $v^{(L)}$ to v.

Lemma 2.1.14 *Fix* $T, L, M>0$, *and suppose that* $\sup_{-M \leqq x \leqq M} u_0(x) \leqq \frac{L}{2} e^{-T}$. *If* $\frac{L}{b^2}$ *is large enough, then*

$$P\Big(\sup_{\substack{0 \leqq t \leqq T \\ -M \leqq x \leqq M}} v^{(L)}(t, x) > L\Big) \leqq \exp\left[-\frac{cL}{b^2}\right].$$

Proof of Lemma 2.1.14. The following integral equation is equivalent to (2.26).

$$v^{(L)}(t, x)=\int_{-M}^M H(t, x, y)\,u_0(y)\,dy \tag{2.31}$$
$$-\int_0^T \int_{-M}^M H(t-s, x, y)[v^{(L)}_{cs, y}) \wedge L]^2\,dy\,ds$$
$$+N(t, x)$$

If $t<T$, then the maximum principle shows that

$$\int_{-M}^{M} H(t,x,y)\,u_0(y)\,dy \leqq e^T \sup_{-M\leqq x\leqq M} u_0(x) \leqq \frac{L}{2}.$$

Lemma 2.1.14 would follow if we had

$$P\Bigg(\sup_{\substack{0\leqq t\leqq T\\ -M\leqq x\leqq M}} |N(t,x)| > \frac{L}{2}\Bigg) < \exp\left[-\frac{cL}{b^2}\right].$$

This estimate follows from Lemma 2.1.12. Now we can complete the proof of Lemma 2.1.11, using a Gronwall type argument. The key step is:

Lemma 2.1.15 *Under the same assumptions as in Lemma* 2.1.11, *we have*

$$P\Big(\sup_{\substack{0\leqq t\leqq T\\ -M\leqq x\leqq M}} |v^{(L)}(t,x)-w(t,x)|>\delta\Big)<\exp\left[-\frac{c\delta^2}{Lb^2}\right].$$

Clearly, Lemmas 2.1.14 and 2.1.15 imply Lemma 2.1.11.

Proof of Lemma 2.1.15. Note that (2.25) and (2.26) are equivalent to the following integral equations.

$$\begin{aligned} v^{(L)}(t,x) = {} & \int_{-M}^{M} H(t,x,y)\,u_0(y)\,dy \\ & -\int_0^t \int_{-M}^{M} H(t-s,x,y)[v^{(L)}(s,y)\wedge L]^2\,dy\,ds \\ & + N(t,x) \\ w(t,x) = {} & \int_{-M}^{M} H(t,x,y)\,u_0(y)\,dy \\ & -\int_0^t \int_{-M}^{M} H(t-s,x,y)\,w(s,y)^2\,dy\,ds. \end{aligned}$$

Subtracting, we obtain

$$\begin{aligned} (2.32)\quad v^{(L)}(t,x)-w(t,x) = {} & \int_0^t \int_{-M}^{M} H(t-s,x,y)[v^{(L)}(s,y)\wedge L + w(s,y)] \\ & \cdot [v^{(L)}(s,y)\wedge L - w(s,y)]\,dy\,ds \\ & + N(t,x). \end{aligned}$$

Let

$$D(t,x) = |v^{(L)}(t,x)-w(t,x)|.$$

Since $L>1$ and $\sup_{-M\leqq x\leqq M} u_0(x)<\frac{L}{2}e^{-T}$, the maximum principle shows that, for $t\leqq T$, $w(t,x)\leqq L$ for all $x\in[-M,M]$. Therefore, if $s\leqq T$, $y\in[-M,M]$,

$$|v^{(L)}(s,y)\wedge L-w(s,y)|\leqq D(s,y).$$

Thus, (2.32) implies, for $t\leqq T$,

$$\begin{aligned}(2.33)\qquad D(t,x)&\leqq 2L\int_0^t\int_{-M}^M H(t-s,x,y)\,D(s,y)\,dy\,ds+|N(t,x)|\\ &\leqq 2Le^T\int_0^T \sup_{-M\leqq y\leqq M} D(s,y)\,dy\,ds+|N(t,x)|.\end{aligned}$$

Let

$$\bar{D}(t)=\sup_{-M\leqq x\leqq M} D(t,x),$$

$$\bar{N}=\sup_{\substack{0\leqq t\leqq T\\ -M\leqq x\leqq M}} |N(t,x)|.$$

Then, (2.33) implies

$$\bar{D}(t)\leqq 2Le^T\int_0^t \bar{D}(s)\,ds+\bar{N}.$$

Gronwall's lemma then gives

$$\begin{aligned}\bar{D}(t)&\leqq\bar{N}\exp[2Le^T t]\\ &\leqq c\bar{N}\end{aligned}$$

where c depends on L and T. This completes the proof of Lemma 2.1.15.

2.2 *Survival*

In this section we show that if $\theta>\theta_1$, then $u(t,x)$ has a positive probability of survival. Of course, we assume that $u(0,x)\in L^1$, and that $u(0,x)$ is not identically 0.

First, we scale (1.1). Using Lemma 2.1.2 with $a=\theta^{-1}$, $b=\theta^{-\frac{1}{2}}$, $c=\theta^{-1}$, we find that $v(t,x)=cu(at,bx)$ satisfies

$$(2.34)\qquad v_t=\tfrac{1}{6}v_{xx}+v-v^2+\theta^{-\frac{3}{4}}v^{\frac{1}{2}}\dot{W}.$$

It is easy to check that since $v(0,x)$ is nonnegative and not identically 0, there is a constant $\delta>0$ such that with positive probability, $v(1,x)$

$\geqq \delta 1_{[0,1]}(x)$. By the Markov property, we can start afresh at time $t=1$. Thus, we may assume without loss that

$$v(0, x) \geqq \delta 1_{[0,1]}(x).$$

We wish to compare $v(t, x)$ with $\underline{v}(t, x)$ and $w(t, x)$. Here,

$$\underline{v}(t, x)=\underline{v}(t, x, s, y), \qquad w(t, x)=w(t, x, s, y)$$

are defined for $s \in \mathbb{N} \cup \{0\}$, $y \in \mathbb{Z}$, $s \leqq t \leqq s+T$, and $y-M \leqq x \leqq y+M$. Let $\underline{v}$ and w satisfy

$$\begin{aligned} \underline{v}_t &= \tfrac{1}{6} \underline{v}_{xx}+\underline{v}-\underline{v}^2+\theta^{-\frac{3}{4}} \underline{v}^{\frac{1}{2}} \dot{W} \\ \underline{v}(t, y-M) &= \underline{v}(t, y+M)=0 \\ \underline{v}(s, x) &= v(s, x) \wedge 1 \end{aligned} \tag{2.35}$$

$$\begin{aligned} w_t &= \tfrac{1}{6} w_{xx}+w-w^2 \\ w(t, y-M) &= w(t, y+M)=0 \\ w(s, x) &= v(s, x) \wedge 1 . \end{aligned} \tag{2.36}$$

Lemma 2.1.5 implies that we may construct v and $\{\underline{v}(\cdot, \cdot, s, y)\}_{s=1, y \in \mathbb{Z}}^{\infty}$ on a common probability space, such that with Probability 1,

$$v(t, x) \geqq \underline{v}(t, x, s, y) \tag{2.37}$$

for all s, y and all appropriate t, x.

Fix δ, ε, M, $T>0$. Lemma 2.1.11 shows we can choose θ so large that

$$P\Big(\sup_{\substack{s \leqq t \leqq s+T \\ y-M \leqq x \leqq y+M}} |\underline{v}(t, x)-w(t, x)|>\delta\Big)<\varepsilon . \tag{2.38}$$

Following Durrett and Neuhauser [5], our strategy is to compare $v(t, x)$ to N-dependent oriented site percolation with density at least $1-\rho$. Let

$$\mathscr{L}=\{(x, m) \in \mathbb{Z}^2 : x+m \text{ is even, and } m \geqq 0\}.$$

Given random variables $\omega(x, n)$, $(x, n) \in \mathscr{L}$ that indicate whether the sites are open (1) or closed (0), we say that (y, n) can be reached from (x, m) and write $(x, m) \rightarrow (y, n)$ if there is a sequence of points $x=x_m, \ldots, x_n=y$ so that $|x_k-x_{k-1}|=1$ for $m<k \leqq n$ and $\omega(x_k, k)=1$ for $m \leqq k \leqq n$. Up to this point the $\omega(x, n)$ could be arbitrary random variables. The phrase "N dependent with density at least $1-\rho$" means that whenever (x_i, n_i), $1 \leqq i \leqq I$ is a sequence with $|x_i-x_j| \geqq N$ or $|n_i-n_j| \geqq N$ for $i \neq j$ then

$$P(\omega(x_i, n_i)=0 \text{ for } 1 \leqq i \leqq I) \leqq \rho^I .$$

Let ε_0 be the set of occupied sites at time 0: $\varepsilon_0 = \{x \in \mathbb{Z}: \omega(x, 0) = 1\}$. Let $\varepsilon_n = \{x \in \mathbb{Z}: (y, 0) \to (x, n)$ for some $y \in \varepsilon_0\}$. Let $\mathscr{C}$ be the cluster of points (x, n), $n > 0$, such that $(0, 0) \to (x, n)$. Durrett and Neuhauser [5], (Lemma 3.5) prove:

Lemma 2.2.1 *Suppose* $\varepsilon_0 = 2\mathbb{Z}$. *If* $\rho < 6^{-4(2N-1)^2}$ *then* $P(0 \notin \varepsilon_{2n}) \leqq \rho + 162\rho^{\frac{1}{4}(2N-1)^2}$.

Let $\mathscr{C}_n = \{(x, m) \in \mathscr{C}: m = n\}$. Suppose that we reverse time in Lemma 2.2.1, letting $\bar{m}(m) = 2n - m$. We find that

$$P(0 \notin \varepsilon_{2n}) = P(\mathscr{C}_{2n} = \phi).$$

Letting $n \to \infty$, we deduce

Lemma 2.2.2

$$\begin{aligned} P(|\mathscr{C}| < \infty) &= \lim_{n \to \infty} P(\mathscr{C}_{2n} = \phi) \\ &\leqq \rho + 162\rho^{\frac{1}{4}(2N-1)^2}. \end{aligned}$$

Now we describe how to couple v with N-dependent oriented percolation $\omega(y, n)$ where N depends on M and T. For $(y, n) \in \mathscr{L}$, let

$$\eta(y, n) = \begin{cases} 1 & \text{if } \underline{v}(Tn, y) \geqq \delta 1(y \leqq x \leqq y+1) \\ 0 & \text{otherwise.} \end{cases}$$

Secondly, we construct a process $\omega(y, n)$ for $(y, n) \in \mathscr{L}$. If $\eta(y-1, n-1) = \eta(y+1, n-1) = 0$, we choose

$$\omega(y, n) = \begin{cases} 1 & \text{with probability } 1-\rho \\ 0 & \text{with probability } \rho, \end{cases}$$

independently of the other random variables in the construction. Otherwise, let $\omega(y, n) = \eta(y, n)$.

To rigorously show that ω is N-dependent, we focus on $\underline{v}(t, x)$. For $(y, s) \in \mathscr{L}$, let $\mathscr{F}_{y,s}$ be the σ-field generated by $\{v(t, x): 0 \leqq t \leqq s + T$, and either $t \leqq s$ or $|x - y| > M + 1\}$. Clearly, ω is $N = \frac{M+1}{2}$ dependent with density at least $1 - \rho$ if, for all $(y, s) \in \mathscr{L}$,

$$P(\omega(y, s+T) = 0 \mid \mathscr{F}_{y,s}) \leqq \rho. \tag{2.39}$$

Suppose for the moment that (2.39) holds. Then Lemma 2.2.2 implies that $v(t, x)$ survives with positive probability, and we have accomplished our goal for Sect. 3.

Clearly, (2.39) is implied by the following lemma. For simplicity, we take s, $y=0$; the general result follows from translation. Recall that $\underline{v}$ implicitly depends on M.

Lemma 2.2.3 *Suppose $\rho>0$ and that $\underline{v}(0, x)\geqq\delta 1_{[0,1]}(x)$. There exist M, $T>0$ such that for θ large enough, we have*

$$P(\underline{v}(T, x)\geqq\delta 1_{[-1,2]}(x))>1-\rho.$$

Lemma 2.2.3 immediately follows from Lemma 2.1.11 and the following lemma.

Lemma 2.2.4 *There exist T, M, $\delta>0$ such that the following holds. Suppose that $w(0, x)\geqq\delta 1_{[0,1]}(x)$. Then*

$$w(T, x)\geqq 2\delta 1_{[-2,3]}(x).$$

Proof of Lemma 2.2.4. Let $M=6$. We work with subsolutions of (2.36). Recall $w(t, x)$ is a subsolution of (2.36) if $w_t\leqq(1/6)\, w_{xx}+w-w^2$, and w satisfies the initial conditions and boundary conditions given in (2.36). Let κ be a nonnegative constant, and let $h(t, x)$; $t\geqq 0$, $-M\leqq x\leqq M$ satisfy

$$\begin{aligned} h_t &= \tfrac{1}{6} h_{xx}+\kappa h \\ h(t, -M) &= h(t, M)=0 \\ h(0, x) &= \delta 1_{[0,1]}(x). \end{aligned}$$

Then $h(t, x)$ is a subsolution of (2.36) for all t, x such that $\sup_{0\leqq s\leqq t,\, y\in R} h(s, y)\leqq 1/2$. Note that $h(t, x)\, e^{-\kappa t}$ solves the heat equation $h_t=(1/6)\, h_{xx}$, with the same boundary and initial conditions as in (2.40).

Consider the eigenvalue expansion for $h(t, x)$:

$$h(t, x)=\delta \sum_{k=0}^{\infty} c_k\, e^{(\kappa-\lambda_k)t}\, \varphi_k(x)$$

where

$$\begin{aligned} \varphi_k(x) &= \sqrt{\frac{1}{M}} \sin\left[k\pi \frac{x+M}{2M}\right] \\ \lambda_k &= \left(\frac{k\pi}{2M}\right)^2 \\ c_k &= \int_{-M}^{M} \varphi_k(x)\, 1_{[0,1]}(x)\, dx. \end{aligned}$$

Of course,

$$|c_k|\leqq\sqrt{\frac{1}{M}}.$$

See [6], Sects. 1.8 and 1.9, for the theory of such expansions.

Choose $\kappa > 0$ such that $\kappa - \lambda_1 > 0$, but $\kappa - \lambda_i < 0$ for all $i \geqq 2$. Now choose T sufficiently large that

$$\sum_{k=2}^{\infty} c_k e^{(\kappa - \lambda_k) T} \leqq \delta$$

and such that for $-M \leqq x \leqq M$,

$$|c_1| e^{(\kappa - \lambda_k) T} \sin\left[\pi \frac{x+M}{2M}\right] \geqq 3 \cdot 1[-2 \leqq x \leqq 3].$$

Finally, choose $\delta > 0$ such that

$$\sup_{0 \leqq t \leqq T} \sup_{-M \leqq x \leqq M} h(t, x) \leqq \tfrac{1}{2}$$

so that $h(t, x)$ is a subsolution of (2.36). Then,

$$\begin{aligned} w(T, x) &\geqq h(T, x) \\ &\geqq \delta c_1 e^{(\kappa - \lambda_1) T} \varphi_1(x) + \sum_{k=2}^{\infty} \delta e^{(\kappa - \lambda_k) T} \varphi_k(x) \\ &\geqq 2\delta \quad \text{if } -2 \leqq x \leqq 3. \end{aligned}$$

Of course, $w(T, x) \geqq 0$. This proves Lemma 2.2.4, and completes the proof that v survives with positive probability.

2.3 Extinction

In this section, we show that if $\theta < \theta_0$, then with Probability 1, $u(t, x)$ does not survive. Again, we assume that $u_0 \in C_c^+$.

For this section we use Lemma 2.1.2 with $a = \theta^{-1}$, $b = \theta^{-1}$, $c = \theta^{-1}$. We find that $v(t, x) = c u(at, bx)$ satisfies

$$v_t = \frac{\theta}{6} v_{xx} + v - v^2 + \theta^{-\frac{1}{2}} v^{\frac{1}{2}} \dot{W}. \tag{2.40}$$

Let $V(t) = \int_{-\infty}^{\infty} v(t, x)\, dx$, and let $S(t)$ be the support of $v(t, x)$ in the x variable. Suppose that

$$\begin{aligned} V(0) &= K \\ S(0) &\subset [y, y+1]. \end{aligned} \tag{2.41}$$

Finally, let τ be the first time t that either $V(t)=0$, $V(t)=2K$, or $S(t)\not\subset[y-1, y+2]$.

We claim that $\tau<\infty$ a.s. We only outline the argument, leaving it to the reader to fill in the details. Unless $v(t,x)$ is small for most values of x, the noise term will not be small. Then the noise will have an appreciable chance of driving V above $2K$. If $v(t,x)$ is small for most values of x, and if $S(t)\subset[y-1, y+2]$, then the terms $v-v^2$ will be small in comparison to the noise term, and v will evolve like a super-Brownian motion. We know that the super-Brownian motion dies out in finite time.

Lemma 2.3.1 *For θ small enough and K large enough,*

$$P(V(\tau)>0)<\tfrac{1}{6}.$$

Proof. Without loss of generality, let $y=0$. Let $t<\tau$, and integrate (2.40) over x. Since $S(t)\subset[-1,2]$ for $t<\tau$, we may use Jensen's inequality:

$$\begin{aligned} dV &= V\,dt-\int_{-1}^{2} v(t,x)^2\,dx+\theta^{-\frac{1}{2}}V^{\frac{1}{2}}\,dB \\ &\leqq (V-\tfrac{1}{3}V^2)\,dt+\theta^{-\frac{1}{2}}V^{\frac{1}{2}}\,dB. \end{aligned}$$

Here, $B(t)=\int_0^t V(s)^{-\frac{1}{2}}\int_{-1}^{2} v(s,x)^{\frac{1}{2}}\,W(dx\,ds)$ is a Brownian motion.

Let $Y(t)$ satisfy

$$\begin{aligned} dY &= (Y-\tfrac{1}{3}Y^2)\,dt+\theta^{-\frac{1}{2}}Y^{\frac{1}{2}}\,dB \\ Y(0) &= V(0)=K \\ K &\equiv \theta^{-\frac{2}{3}}. \end{aligned}$$

By Theorem 1.1, Chap. VI, of [9], $V(t)\leqq Y(t)$ a.s., for $t<\tau$. Let σ be the first time t that $Y(t)=0$ or $Y(t)=2K$. Of course, $\sigma<\infty$ a.s.

Lemma 2.3.2 *Given $\varepsilon>0$, we can find θ_0 small enough so that $0<\theta<\theta_0$ implies*

$$P(Y(\sigma)=2K)\leqq\varepsilon.$$

Proof. We make a scale change, seeking a function f such that $f(Y)$ is a martingale, and such that $f(0)=0$, $f(2K)=1$. For such an f, we have $f(K)=P(Y(\sigma)=2K)$. Using Ito's lemma, we find that $\frac{1}{2\theta}xf''+(x-\frac{1}{3}x^2)f'=0$, and thus

$$f(x)=\frac{\int_0^K \exp[\theta(-2y+\frac{1}{3}y^2)]\,dy}{\int_0^{2K} \exp[\theta(-2y+\frac{1}{3}y^2)]\,dy}.$$

For large y, the dominant term in the exponent is y^2, and hence when $x=K$ and K is large, the denominator in $f(K)$ is much larger than the numerator. Now Lemma 2.3.2 easily follows.

Lemma 2.3.3 *Fix* $\varepsilon>0$ *and* $t>0$. *If* θ_0 *is small enough, and if* $0<\theta<\theta_0$, *then*

$$P(\sigma>t)<\varepsilon.$$

Proof. Let

$$Z(t)=2K^{-\frac{3}{4}}Y^{\frac{1}{2}}.$$

Using Ito's lemma with (2.42), and recalling that $\theta^{-\frac{1}{2}}=K^{\frac{3}{4}}$, we find that $Z(t)$ satisfies

$$\begin{aligned} dZ&=dB+K^{-\frac{3}{4}}Y^{\frac{1}{2}}dt-K^{-\frac{3}{4}}Y^{\frac{3}{2}}dt-\tfrac{1}{2}K^{\frac{3}{4}}Y^{-\frac{1}{2}}dt\\ &\leqq dB+\tfrac{1}{2}Z\,dt\\ Z(0)&=2K^{-\frac{1}{4}}.\end{aligned}$$

Now, let $\bar{Z}(t)$ satisfy

$$(2.42)\qquad \begin{aligned} d\bar{Z}&=dB+\tfrac{1}{2}\bar{Z}\,dt\\ \bar{Z}(0)&=2K^{-\frac{1}{4}}.\end{aligned}$$

Using Theorem 1.1, Chap. VI of [9], we conclude that

$$0\leqq Z(t)\leqq \bar{Z}(t).$$

Let $\bar{\sigma}$ be the first time $s\geqq 0$ such that $Z(s)=0$. If $Z(s)$ never equals 0, let $\bar{\sigma}=\infty$. Then $\sigma\leqq\bar{\sigma}$, and

$$P(\sigma>t)\leqq P(\bar{\sigma}>t).$$

However, it is easy to check from (2.42) that if θ_0 is small enough, and hence K is large enough, then

$$(2.43)\qquad P(\bar{\sigma}>t)\leqq\varepsilon.$$

To see (2.43), note that for large K, $\bar{Z}(0)$ is close to 0. For small time, the fluctuations of dB will dominate the dt term in (2.42), and will send $Z(t)$ to 0. We leave the details to the reader.

This proves (2.43), and completes the proof of Lemma 2.3.3.

Now we finish the proof of Lemma 2.3.1. First, using Lemma 2.3.2 and Lemma 2.3.3, choose θ_0 so small that $P(Y(\sigma)=2K)<\frac{1}{24}$, and $P(\sigma>t)<\frac{1}{24}$. Then, using Lemma 2.1.9, choose $\theta\leqq\theta_0$ so small that if $t=1/(4\gamma)$, then $P(S(s)\notin[-1,2]$, for some $s\leqq t)<1/24$. Clearly, these three estimates imply that

$$P(S(\tau)\notin[-1,2])<\tfrac{1}{12}.$$

Since $V(t) \leqq Y(t)$ by construction, we have

$$P(V(\tau)>0) \leqq P(S(\tau) \nsubseteq [-1, 2]) + P(Y(\sigma)=2K) + P(\sigma>t) < \tfrac{1}{6}.$$

This proves Lemma 2.3.1.

Now we proceed with the proof of extinction. Since $u(0, x)$ is continuous with compact support, we may choose $v^{(k)}(0, x)$ such that

$$v(0, x) \leqq \sum_{k=1}^{n} v^{(k)}(0, x)$$

with $v^{(k)}(0, x)$ supported on an interval $[x_k, x_k+1]$, and $\int_{x_k}^{x_{k+1}} v^{(k)}(0, x)\, dx = K$. By Lemma 2.1.7, we may extend the $v^{(k)}(0, x)$ to solutions $v^{(k)}(t, x)$ of

$$v_t^{(k)} = \frac{\theta}{6} v_{xx}^{(k)} + v^{(k)} - v^{(k)2} + \theta^{-\frac{1}{2}} v^{(k)\frac{1}{2}} \dot{W}_k,$$

where the $\dot{W}_k$ are independent white noises, and such that

$$v(t, x) \leqq \sum_{k=1}^{n} v^{(k)}(t, x).$$

As mentioned in the introduction, we wish to regard the $v^{(k)}$ as individuals in a branching process. Define τ_k with respect to $v^{(k)}$, as in Lemma 2.3.2. Then $\operatorname{supp}(v^{(k)}(\tau_k, \cdot)) \subset [x_k - 1, x_k + 2]$ and $V^{(k)}(\tau_k) \equiv \int_{x_k-1}^{x_k+2} v^{(k)}(\tau_k, x)\, dx \leqq 2K$. If $V^{(k)}(\tau_k) = 0$, we say that $v^{(k)}$ has no offspring. If $V^{(k)}(\tau_k) > 0$, then there exist 6 nonnegative, continuous functions $\{v^{(k,i)}\}_{i=1}^{6}$ such that each $v^{(k,i)}(\tau_k, \cdot)$ is supported on an interval $[x_{ki}, x_{ki}+1]$, $\int_{x_{ki}}^{x_{ki}+1} v^{(k,i)}(\tau_k, x)\, dx = K$; and $v^{(k)}(\tau_k, x) \leqq \sum_{k=1}^{6} v^{(k,i)}(\tau_k, x)$. The functions $\{v^{(k,i)}\}_{i=1}^{6}$ are regarded as the offspring of $v^{(k)}$. We can define $v^{(k,i)}(\tau_k + t, x)$ as before, and continue the argument. By Lemma 2.3.1, the expected offspring size is less than 1, so the branching process dies out almost surely. This means that eventually, all of the functions have reached 0. This proves extinction for $v(t, x)$.

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