Probab. Theory Relat. Fields 94, 317–333 (1993)

Probability Theory and Related Fields

Action functional for diffusions in discontinuous media

A.P. Korostelev and S.L. Leonov
Institute for Systems Analysis, Russian Academy of Sciences Prospekt 60-Let Oktyabrya 9, Moscow 117312, Russia

Received April 24, 1991; in revised form April 20, 1992

Summary. The action functional, i.e. the rate function governing the large deviations is obtained for a family of stochastic processes with discontinuous drift and small diffusion. A well-known method of continuous mapping is developed which proves to be efficient in a so called 'stable case'.

Mathematics Subject Classification: 60 F 10

0 Introduction

In this paper we proceed the study started in our previous work (Korostelev and Leonov 1992) on the large deviation principle for small random perturbations of dynamical systems in discontinuous media. We consider two-dimensional diffusion processes with a drift which is discontinuous along the vertical axis $\{(x, y)\colon x = 0\}$.

As far as we are aware, the paper by Borovkov (1967) was the first study of the large deviation principle for the infinite-dimensional case. This principle has been elaborated in detail by Freidlin and Wentzell (1970) for small random perturbations (see also Freidlin and Wentzell 1984; Deuschel and Strook 1989). The term 'action functional' has been proposed by Freidlin (1972) to describe the exponential rate of decay of the large deviation probabilities.

In our study we use a well-known idea of continuous mapping which gives a simple solution of the problem in the stable case, i.e. when the drift may be decomposed into some smooth vector field and a discontinuous summand whose discontinuity is directed towards the vertical axis. We also obtain the large deviation principle for the staying-time in the half-plane $\{(x, y)\colon x > 0\}$.

A general upper large deviation bound which is quite close to our results has been proved by Dupuis et al. (1991). As these authors remark, a lower large deviation bound often requires an analysis on a case by case basis. For discrete-time Markov chains such an analysis has been accomplished by Dupuis and Ellis (1992).

The paper is organized as follows. The main definitions and results are presented in Sect. 1. A comparison of methods and a discussion of possible approaches are given in Sect. 2. Section 3 contains the technical details, while Sect. 4 collects some remarks on the properties of diffusion got by the continuous mapping method. All proofs of the lemmas are postponed to the Appendix.

1 Statement of the main result

Our previous paper (Korostelev and Leonov 1992) has dealt with the large deviation principle for two-dimensional diffusions $(X, Y) = (X^\varepsilon, Y^\varepsilon)$ satisfying Ito stochastic differential equations

$$\begin{cases} dX(t) = b(X(t), Y(t))\,dt + \varepsilon\, dW_1(t) \\ dY(t) = B(X(t), Y(t))\,dt + \varepsilon\, dW_2(t), \quad 0 \leqq t \leqq T, \end{cases} \tag{1.1}$$

with the initial condition

$$X(0) = 0, \quad Y(0) = 0,$$

where ε is a small positive parameter; $W_1(t)$, $W_2(t)$ are two independent standard Wiener processes. It has been assumed that a drift $(b(x, y), B(x, y))$ suffers a jump along the vertical axis $\{(x, y)\colon x = 0\}$.

In the cited paper only the simplest case has been considered when $b(x, y)$ doesn't depend on y: $b(x, y) = b(x)$, and $b(x)$ is piecewise constant:

$$b(x) = \begin{cases} b_+ & \text{if } x > 0 \\ b_- & \text{if } x \leqq 0, \end{cases} \quad B(x, y) = \begin{cases} B_+(y) & \text{if } x > 0 \\ B_-(y) & \text{if } x \leqq 0, \end{cases} \tag{1.2}$$

where the condition $b_- > b_+$ holds; $B_+(y)$ and $B_-(y)$ are smooth bounded functions.

The aim of the present paper is to study the system (1.1) with the drift $(b(x, y), B(x, y))$ which depends on both coordinates. We assume now that there are two pairs of smooth bounded real-valued functions

$$(b_+(x, y), B_+(x, y)), \quad (b_-(x, y), B_-(x, y)),$$

and the drift in (1.1) has the form

$$b(x, y) = \begin{cases} (b_+(x, y) & \text{if } x > 0 \\ (b_-(x, y) & \text{if } x \leqq 0, \end{cases} \quad B(x, y) = \begin{cases} B_+(x, y) & \text{if } x > 0 \\ B_-(x, y) & \text{if } x \leqq 0, \end{cases} \tag{1.3}$$

where $|b_\pm(x, y)|, |B_\pm(x, y)| \leqq C_0$, $C_0 > 0$,

$$|b_\pm(x_1, y_1) - b_\pm(x_2, y_2)| \leqq L(|x_1 - x_2| + |y_1 - y_2|),$$

$$|B_\pm(x_1, y_1) - B_\pm(x_2, y_2)| \leqq L(|x_1 - x_2| + |y_1 - y_2|), \quad L > 0.$$

We will use the concepts and definitions of large deviations developed by Freidlin and Wentzell (1984) to obtain the action functional for the solution of (1.1) (this functional is often called the rate function governing the large deviations, cf. Deuschel and Strook (1989)). We first give a basic definition.

Definition. Let $\|\cdot\|_{0,T}$ be the uniform norm in the space $\mathbb{C}_{[0,T]}$ of continuous functions. Let ψ_1, ψ_2 be elements of $\mathbb{C}_{[0,T]}$ with $\psi_1(0) = \psi_2(0) = 0$. A functional $I(\psi_1, \psi_2) = I_{0,T}(\psi_1, \psi_2)$ is called *action functional* for the solution of (1.1) if

(i) for each $S > 0$ the set

$$\Phi(S) = \{(\psi_1, \psi_2): I(\psi_1, \psi_2) \leqq S\}$$

is a compact subset of $\mathbb{C}_{[0,T]}$;
(ii) for arbitrary small $\delta > 0$, $h > 0$, and for each $S > 0$ the following inequalities hold:

$$\mathbb{P}(\|X - \psi_1\|_{0,T} + \|Y - \psi_2\|_{0,T} < \delta) \geqq \exp\{-\varepsilon^{-2}(I(\psi_1, \psi_2) + h\};$$

$$\mathbb{P}\left(\inf_{(\psi_1, \psi_2) \in \Phi(S)} [\|X - \psi_1\|_{0,T} + \|Y - \psi_2\|_{0,T}] > \delta\right) \leqq \exp\{-\varepsilon^{-2}(S - h)\}$$

provided that $\varepsilon > 0$ is small enough.

It has been shown in Korostelev and Leonov (1992) that the action functional for the solution of Eqs. (1.1) with the simplest discontinuous drift (1.2) has the following form:

$$I(\psi_1, \psi_2) = \inf_{\mu \in M(\psi_1)} (1/2) \int_0^T [(\dot{\psi}_1(t) - \dot{\mu}(t) b_+ - (1 - \dot{\mu}(t)) b_-)^2 + (\dot{\psi}_2(t) - \dot{\mu}(t) B_+(\psi_2(t)) - (1 - \dot{\mu}(t)) B_-(\psi_2(t)))^2] \, dt \tag{1.4}$$

where $M(\psi_1)$ denotes a set of absolutely continuous functions $\mu(t)$ satisfying the following condition:

$$\dot{\mu}(t) = 0 \quad \text{if } \psi_1(t) < 0,$$
$$0 \leqq \dot{\mu}(t) \leqq 1 \quad \text{if } \psi_1(t) = 0,$$
$$\dot{\mu}(t) = 1 \quad \text{if } \psi_1(t) > 0;$$

$I(\psi_1, \psi_2) = +\infty$ if the integral in (1.4) diverges or the functions ψ_1, ψ_2 are not both absolutely continuous.

The physical meaning of $\dot{\mu}$ is obvious: if a diffusion sample path is close to (ψ_1, ψ_2), then the share of time spent by this path in the positive half-plane at time t is close to $\dot{\mu}$, so that μ is approximately the time occupied by (X, Y) in this half-plane.

Example 1.1 If $T = 1$, $\psi_1(t) = 0$, and if the functions $\dot{\psi}_2(t)$, $B_\pm(y)$ are equal to some given constants:

$$\dot{\psi}_2(t) = u_2, \quad B_\pm(y) = B_\pm,$$

then the action functional (1.4) may be written more explicitly:

$$I(0, \psi_2) = I(u_2) = \inf_{0 \leqq \mu_+ \leqq 1} (1/2)[(-\mu_+ b_+ - (1 - \mu_+) b_-)^2 + (u_2 - \mu_+ B_+ - (1 - \mu_+) B_-)^2]. \tag{1.5}$$

The formula (1.4) gives an idea on the form of the action functional in the general case of the drift (1.3):

$$I(\psi_1, \psi_2) = \inf_{\mu \in M(\psi_1)} \frac{1}{2} \int_0^T [(\dot{\psi}_1(t) - \dot{\mu}(t) b_+(\psi_1(t), \psi_2(t)) - (1 - \dot{\mu}(t)) b_-(\psi_1(t), \psi_2(t)))^2 + (\dot{\psi}_2(t) - \dot{\mu}(t) B_+(\psi_1(t), \psi_2(t)) - (1 - \dot{\mu}(t)) B_-(\psi_1(t), \psi_2(t)))^2] \, dt; \tag{1.6}$$

$I(\psi_1, \psi_2) = +\infty$ if the integral in (1.6) diverges or the functions ψ_1, ψ_2 are not both absolutely continuous.

Indeed, we prove that (1.6) holds under the following 'stability' condition.

Assumption 1.1 *Let the jump of $b(x, y)$ on the vertical axis be strictly positive and separated from zero uniformly on y,* i.e.

$$\inf_y \beta(y) \geqq C_1 > 0,$$

where $\beta(y) = b_-(0, y) - b_+(0, y)$.

Theorem 1.1 *If Assumption* 1.1 *holds, then the action functional $I(\psi_1, \psi_2)$ for the solution of stochastic equations* (1.1) *with the drift* (1.3) *is given by* (1.6).

Remark 1.1 Some possible generalizations of Theorem 1.1 are discussed in Sect. 4 (see Remarks 4.1 and 4.2).

Remark 1.2 Let $\dot{\mu}_*(t)$, $0 \leqq \dot{\mu}_*(t) \leqq 1$, be a pointwise minimizer for the quadratic function (in $\dot{\mu}$) on the right-hand side of (1.6). Since $\dot{\mu}_*(t)$ is a bounded measurable function in t, $0 \leqq t \leqq T$, the function $\mu_*(t)$ is absolutely continuous. Hence, the minimization problem on the right-hand side of (1.6) may be solved explicitly with the action functional presented via some function $\hat{L}$,

$$I(\psi_1, \psi_2) = (1/2) \int_0^T \hat{L}(\psi_1, \psi_2, \dot{\psi}_1, \dot{\psi}_2) \, dt.$$

But the truncation condition $\dot{\mu}_* \in [0, 1]$ makes the expression for $\hat{L}$ rather intricate, and we prefer the version (1.6) for the action functional.

2 Remarks on methods of investigation

Continuous mapping method. If the drift in (1.1) has the simplest form (1.2), then the first coordinate $X(t)$ may be studied independently of the second coordinate $Y(t)$. It is worthwhile to note that, in general, the solution of the equation

$$X(t) = \int_0^T b(X(s)) \, ds + \varepsilon W_1(t)$$

does not exist if we substitute εW_1 by an arbitrary continuous function φ. Nevertheless, the solution of this equation does exist for a 'sufficiently large' set of functions which is dense in $\mathbb{C}_{[0,T]}$. It has been shown in Korostelev and Leonov (1992) that if $b_- > b_+$, then the mapping defined by this solution turns out to be continuous and may be extended to the whole space $\mathbb{C}_{[0,T]}$, so that $X = G_1(\varepsilon W_1)$, where G_1 is a continuous mapping:

$$G_1 \colon \mathbb{C}_{[0,T]} \to \mathbb{C}_{[0,T]} \,.$$

Define a functional $\pi(t)$ as a share of time spent by the sample path $X(\cdot)$ in the positive semi-axis:

$$\pi(t) = \int_0^t \chi_+(X(s))\,ds, \qquad 0 \leqq t \leqq T,$$

where $\chi_+(x)$ is the indicator function of the positive semi-axis. For a diffusion in discontinuous media it is useful to study the process $X(t)$ together with the functional $\pi(t)$. The key point of the 'continuous mapping' method is in the following relation between X, π, and εW_1:

$$X(t) = b_+\pi(t) + b_-(t - \pi(t)) + \varepsilon W_1(t), \quad 0 \leqq t \leqq T \,. \tag{2.1}$$

This relation is based on the fact that for any diffusion the staying-time at the origin is equal to zero almost surely. Formula (2.1) makes it possible to consider the functional $\pi = G_2(\varepsilon W_1)$ also as a continuous image of εW_1:

$$\pi(t) = (\varepsilon W_1(t) + b_- t - X(t))/(b_- - b_+) \,, \tag{2.2}$$

the inverse mapping $(X, \pi) \to \varepsilon W_1$ being quite simple:

$$\varepsilon W_1(t) = X(t) - b_+\pi(t) - b_-(1 - \pi(t)) \,.$$

The second coordinate $Y(t)$ of the solution $(X(t), Y(t))$ of (1.1) must satisfy the integral equation

$$Y(t) = \int_0^t B_+(Y(s))\,d\pi(s) + \int_0^t B_-(Y(s))\,d(s - \pi(s)) + \varepsilon W_2(t), \quad 0 \leqq t \leqq T \,. \tag{2.3}$$

Thus, the pair (X, Y) together with functional (2.2) may be considered as an image of the pair $(\varepsilon W_1, \varepsilon W_2)$ under a continuous mapping:

$$(X, \pi, Y) = (G_1(\varepsilon W_1), G_2(\varepsilon W_1), G_3(\varepsilon W_1, \varepsilon W_2)) \,.$$

In this case the inverse mapping is also explicit:

$$\begin{aligned}(\varepsilon W_1(t), \varepsilon W_2(t)) &= (X(t) - b_+\pi(t) - b_-(1 - \pi(t)) \,, \\ & Y(t) - \int_0^t B_+(Y(s))\,d\pi(s) - \int_0^t B_-(Y(s))\,d(s - \pi(s))) \,.\end{aligned} \tag{2.4}$$

The general results imply (Freidlin and Wentzell 1984) that the action functional $I(\psi_1, \psi_2)$ for the pair (X, Y) has the following representation:

$$I(\psi_1, \psi_2) = \inf\left((1/2) \int_0^T (\|\dot\varphi_1(t)\|^2 + \|\dot\varphi_2(t)\|^2)\,dt \right) \tag{2.5}$$

where the infimum is taken over all absolutely continuous functions (φ_1, φ_2) with $\varphi_1(0) = \varphi_2(0) = 0$ such that $\psi_1 = G_1(\varphi_1)$ and $\psi_2 = G_3(\varphi_1, \varphi_2)\,(I(\psi_1, \psi_2) = +\infty$

if there are no such functions φ_1, φ_2). Taking into account (2.4), we see that (2.5) entails representation (1.4) for the action functional.

An additional problem arises when we endeavour to realize this idea for the general case of the drift (1.3). Indeed, the relation analogous to (2.1) must now have the form

$$X(t) = \int_0^t b_+(X(s), Y(s))\,d\pi(s) + \int_0^t b_-(X(s), Y(s))\,d(s - \pi(s)) + \varepsilon W_1(t), \quad 0 \leqq t \leqq T. \tag{2.6}$$

The right-hand side of (2.6) contains the second coordinate $Y(t)$ which is not still defined. To get over this hurdle, we are 'freezing' a continuous path $Y(\cdot)$ and defining two non-anticipating functionals

$$X = X(t;\, Y(\cdot), \varepsilon W_1(\cdot)) \quad \text{and} \quad \pi = \pi(t;\, Y(\cdot), \varepsilon W_1(\cdot)).$$

Since the second coordinate $Y(t)$ is expected to satisfy the integral equation

$$Y(t) = \int_0^t B_+(X(s;\, Y(\cdot), \varepsilon W_1(\cdot)), Y(s))\,d\pi(s;\, Y(\cdot), \varepsilon W_1(\cdot)) + \int_0^t B_-(X(s;\, Y(\cdot), \varepsilon W_1(\cdot)), Y(s))\,d(s - \pi(s;\, Y(\cdot), \varepsilon W_1(\cdot))) + \varepsilon W_2(t), \quad 0 \leqq t \leqq T, \tag{2.7}$$

it is necessary to prove that the solution of equation (2.7) exists, is unique, and continuously depends on εW_1 and εW_2. The inverse mapping in the general case is not much more involved than (2.4).

H. Cramer transform. Dupuis and Ellis (1992) have worked out the direct method to obtain the rate function governing the large deviations for a random motion in discontinuous media. Their result which is remarkable for its generality is based on the Wentzell's generalization of the Cramér transform. They have considered a Markov chain:

$$S(t+1) = S(t) + \xi^+(t+1)\chi(S_1(t) > 0) + \xi^-(t+1)\chi(S_1(t) \leqq 0), \quad t = 0, 1, \ldots, \tag{2.8}$$

where $S(0) = 0$, $S_1(t)$ is the first coordinate of vector $S(t)$, $\xi^+(t)$ and $\xi^-(t)$ are two independent sequences of i.i.d. random vectors with distributions $\mathbb{P}^+$ and $\mathbb{P}^-$ respectively. To formulate the results by Dupuis and Ellis, we restrict ourselves to the two-dimensional case and introduce the following cumulant generating functions

$$H^\pm(z_1, z_2) = \log \int \exp\{z_1 x_1 + z_2 x_2\}\, \mathbb{P}^\pm(dx_1, dx_2)$$

and their Legendre-Fenchel transforms:

$$L^\pm(u_1, u_2) = \sup_{z_1, z_2} [(z_1 u_1 + z_2 u_2) - H^\pm(z_1, z_2)].$$

When studying the large deviation principle for a random walk in discontinuous media, the most crucial point is the description of this motion along the axis of discontinuity. Under some mild conditions the following equality holds:

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(\| \varepsilon^2 S([\varepsilon^{-2}]) - u \| < \delta) = -L(u) \tag{2.9}$$

where $u = (0, u_2)$, $\| \cdot \|$ is the Euclidean norm, and

$$L(u) = \inf(\mu_+ L^+(u^+) + \mu_- L^-(u^-)\colon 0 \leqq \mu_+ \leqq 1, \mu_+ + \mu_- = 1,$$
$$\mu_+ u^+ + \mu_- u^- = u, u_1^+ \leqq 0, u_1^- \geqq 0)\,. \tag{2.10}$$

Here $L(u)$ is the rate function governing the large deviations of the Markov chain (2.8), i.e. the finite-dimensional analogue to the action functional.

Example 2.1 If $\mathbb{P}^{\pm}$ are Gaussian distributions having mean values $(b_{\pm}, B_{\pm})$ and identity variance matrices, then simple calculations of $L(u)$ in (2.10) show that

$$L(u) = \inf_{0 \leqq \mu_+ \leqq 1} (1/2)((-\mu_+ b_+ - (1-\mu_+)b_-)^2 + (u_2 - \mu_+ B_+ - (1-\mu_+)B_-)^2).$$

This result is in a good accordance with the action functional (1.5) in Example 1.1.

3 Definition of continuous mapping

Similar to the simplest case of discontinuous drift (1.2), we start with the first component G_1 of the mapping G. Let $\psi_2(t)$ be an arbitrary fixed continuous function, $\psi_2(t) \in \mathbb{C}_{[0,T]}$. Define a mapping G_1 so that the function $\psi_1 = G_1(\varphi, \psi_2)$ would be a solution of the ordinary differential equation

$$\dot{\psi}_1(t) = b(\psi_1(t), \psi_2(t)) + \dot{\varphi}(t), \quad \psi_1(0) = \varphi(0)\,,$$

provided that this solution exists.

Let $l(t) = l_\delta(t)$ be a broken line, i.e. a continuous piecewise linear function with breaks at points $t_j = j\delta, j = 0, 1, \ldots, T/\delta$, where δ is a given small positive number and T/δ is assumed to be an integer. Define the derivative of the broken line at a break-point as its right-hand derivative, and let $I_j = (t_{j-1}, t_j)$.

Introduce a set $\mathscr{L}(H)$ by

$$\mathscr{L}(H) = \{l(t) \mid l(t) \text{ is a broken line, } \inf_t |\dot{l}(t)| \geqq H > 0\}$$

and denote $\mathscr{L}_0 = \mathscr{L}(2C_0)$ where C_0 is the upper bound for $|b_{\pm}(x, y)|$.

The next technical lemma will be presented without proof since it is a simple generalization of Lemma 1 from Korostelev and Leonov (1992).

Lemma 3.1 *Let $\psi(t) \in \mathbb{C}_{[0,T]}$, $\mu(t) \in M(\psi)$. Then for an arbitrary integer n and for an arbitrary fixed constant $H > 0$ there exist positive $\delta = \delta(n, H)$ and a broken line $l = l_\delta(t)$ such that*

$$l \in \mathscr{L}(H), \; \| l - \psi \|_{0,T} \leqq 1/n,$$

$$\| \mu - \mu_\delta \|_{0,T} \leqq \delta, \textit{ where } \mu_\delta(t) = \operatorname{mes}\{s \mid s \leqq t, l_\delta(s) > 0\},$$

and if $\psi(t) > 0$ *for all* $t \in I_j$, *then* $l(t) > 0$ *for all* $t \in I_j$,

$$\text{if } \psi(t) < 0 \text{ for all } t \in I_j, \text{ then } l(t) < 0 \text{ for all } t \in I_j,$$

$$\text{if } \psi(t) = 0 \text{ for some } \tilde{t} \in I_j, \text{ then either } l(t_{j-1}) = 0 \text{ or } l(t_j) = 0$$

$$\text{and } l(t) \neq 0 \text{ for all } t \in I_j.$$

If $l \in \mathscr{L}_0$, then a function $\psi_1 = G_1(l, \psi_2)$ may be defined as a solution of the differential equation

$$\dot{\psi}_1(t) = b(\psi_1(t), \psi_2(t)) + \dot{l}(t), \quad \psi_1(0) = l(0) .$$

The following integral representation is valid for this solution:

$$\psi_1(t') = \psi_1(t) + \int_t^{t'} b(\psi_1(s), \psi_2(s))\, ds + l(t') - l(t) . \tag{3.1}$$

Furthermore, it is obvious that

$$\text{mes}(s\colon s \leqq T, \psi_1(s) = 0) = 0 \quad \text{if } l \in \mathscr{L}_0 .$$

Lemma 3.2 *If Assumption* 1.1 *holds,* $l, l' \in \mathscr{L}_0$, *and* $\psi_2, \psi_2' \in \mathbb{C}_{[0,T]}$, *then for each* t, $t \leqq T$,

$$\| G_1(l, \psi_2) - G_1(l', \psi_2') \|_{0,T} \leqq K(\| l - l' \|_{0,t} + \int_0^t |\psi_2(s) - \psi_2'(s)|\, ds)$$

where $K = e^{LT} \max(2, L)$, *L is the joint Lipschitz constant for* $b_\pm(x, y)$.

The proofs of this and the subsequent lemmas are given in the Appendix.

Corollary 3.1 *Since the set* $\mathscr{L}_0$ *is dense in* $\mathbb{C}_{[0,T]}$, *the mapping* G_1 *may be extended to the whole space* $\mathbb{C}_{[0,T]}$, *preserving the Lipschitz property: for arbitrary pairs of continuous functions* φ, φ' *and* ψ_2, ψ_2' *the following inequality holds:*

$$\| G_1(\varphi, \psi_2) - G_1(\varphi', \psi_2') \|_{0,t} \leqq (\| \varphi - \varphi' \|_{0,t} + \int_0^t |\psi_2(s) - \psi_2'(s)|\, ds).$$

To introduce the mapping G_2 for a fixed function ψ_2, define auxiliary functions $f(x, y)$ and $v(t)$:

$$f(x, y) = \begin{cases} b_+(x, y) + \beta(y), & x > 0 \\ b_-(x, y), & x \leqq 0; \end{cases}$$

$$v(t) = \hat{G}_2(\varphi, \psi_2)(t) = \int_0^t f(\psi_1(s), \psi_2(s))\, ds + \varphi(t) - \psi_1(t) \tag{3.2}$$

where $\psi_1 = G_1(\varphi, \varphi_2)$. If $\varphi = l \in \mathscr{L}_0$, then it may be easily verified that the function $v(t)$ is absolutely continuous and non-decreasing, as well as the function

$$\mu(t) = G_2(\varphi, \psi_2)(t) = \int_0^t (\dot{v}(s)/\beta(\psi_2(s))\, ds . \tag{3.3}$$

Moreover, for $\varphi = l \in \mathscr{L}_0$ the functions $\mu(t)$ and $\psi_1(t)$ are connected by the following relation:

$$\dot{\mu}(t) = (1/2)(1 + \text{sign}(\psi_1(t))).$$

Now, if $\varphi(t)$ is an arbitrary function from $\mathbb{C}_{[0,T]}$ and $\{l_n\}$ is a sequence of broken lines which uniformly converges to φ, $l_n \in \mathscr{L}_0$, then Lemma 3.2 implies that the functions $v(t)$, $\mu(t)$ defined by (3.2) and (3.3) are non-decreasing absolutely continuous functions of the argument t. Moreover, since the function $f(x, y)$ is Lipschitzian in both arguments, it follows from Lemma 3.2 that $\mu(t) \in M(\psi_1)$ and the mapping $\hat{G}_2 = \hat{G}_2(\varphi, \psi_2)$ is Lipschitzian in φ, ψ_2 in the norm of the space $\mathbb{C}_{[0,T]}$.

The next lemma states the uniqueness of the inverse mapping.

Lemma 3.3 *If ψ_1, ψ_2 are continuous functions and $\mu \in M(\psi_1)$, then there exists a unique function φ, $\varphi \in \mathbb{C}_{[0,T]}$, such that*

$$G\varphi = (G_1(\varphi, \psi_2), G_2(\varphi, \psi_2)) = (\psi_1, \mu),$$

and the following relation holds:

$$\varphi(t) = \int_0^t \dot{\mu}(s)\beta(\psi_2(s))\,ds + \psi_1(t) - \int_0^t f(\psi_1(s), \psi_2(s))\,ds. \tag{3.4}$$

Corollary 3.2 *For any continuous function $\psi_2(t)$, the functions $\psi_1(t)$ and $\mu(t)$ constructed above are two non-anticipating functionals of $\varphi(t)$, the following relation being valid as a mere consequence of* (3.4):

$$\varphi(t) = \psi_1(t) - \int_0^t b_+(\psi_1(s), \psi_2(s))\dot{\mu}(s)\,ds$$
$$- \int_0^t b_-(\psi_1(s), \psi_2(s))(1 - \dot{\mu}(s))\,ds\,. \tag{3.5}$$

When two non-anticipating functionals ψ_1, ψ_1' are given that correspond to different functions ψ_2, ψ_2' and the same function φ, we prove a more precise property of continuity. This property is important for the analysis of the vertical coordinate.

Lemma 3.4 *Let $v = \hat{G}_2(l, \psi_2)$, $v' = \hat{G}_2(l, \psi_2')$ where $l \in \mathscr{L}_0$ and $\psi_2, \psi_2' \in \mathbb{C}_{[0,T]}$. Then the inequality*

$$\int_0^t |\dot{v}(s) - \dot{v}'(s)|\,ds \leqq L_1 \int_0^t \|\psi_2 - \psi_2'\|_{0,s}\,ds$$

holds with some positive constant L_1.

Integral equation for vertical coordinate. Now we proceed to study the vertical coordinate and prove that the integral equation

$$\psi_2(t) = \int_0^t B_+(\psi_1(s; \psi_2(\cdot), \varphi_1(\cdot)), \psi_2(s))\,d\mu(s; \psi_2(\cdot), \varphi_1(\cdot))$$
$$+ \int_0^t B_-(\psi_1(s; \psi_2(\cdot), \varphi_1(\cdot)), \psi_2(s))\,d(s - \mu(s; \psi_2(\cdot), \varphi_1(\cdot))) + \varphi_2(t) \tag{3.6}$$

has a unique solution if ψ_1 and μ are non-anticipating functionals of $\psi_2(\cdot)$ and $\varphi_1(\cdot)$ introduced by (3.1) and (3.3).

Lemma 3.5 *Equation* (3.6) *with the initial condition* $\psi_2(0) = 0$ *has a unique continuous solution in the interval* $0 \leqq t \leqq T$, *this solution being a non-anticipating continuous functional for each pair* $(\varphi_1, \varphi_2) \in \mathbb{C}_{[0,T]} \times \mathbb{C}_{[0,T]}$.

Define the non-anticipating functional from Lemma 3.5 by

$$\psi_2 = G_3(\varphi_1, \varphi_2) .$$

Thus, we have constructed a mapping F which establishes the correspondence between pairs of continuous functions (φ_1, φ_2) and triples (φ_1, μ, ψ_2):

$$F: \mathbb{C}_{[0,T]} \times \mathbb{C}_{[0,T]} \to \mathbb{Q}_{[0,T]} \times \mathbb{C}_{[0,T]} ,$$

where $\mathbb{Q}_{[0,T]} = \{(\psi(t), \mu(t)), 0 \leqq t \leqq T: \psi \in \mathbb{C}_{[0,T]}, \mu \in M(\psi)\}$,

and

$$\psi_1 = F_1(\varphi_1, \varphi_2) = G_1(\varphi_1, G_3(\varphi_1, \varphi_2)) ,$$

$$\mu = F_2(\varphi_1, \varphi_2) = G_2(\varphi_1, G_3(\varphi_1, \varphi_2)) ,$$

$$\psi_2 = F_3(\varphi_1, \varphi_2) = G_3(\varphi_1, \varphi_2) .$$

Define the solution of Eqs. (1.1) by

$$(X, Y) = (F_1(\varepsilon W_1, \varepsilon W_2), F_3(\varepsilon W_1, \varepsilon W_2)) . \tag{3.7}$$

4 Application of continuous mapping to diffusion

Before we pass to the proof of Theorem 1.1, it is necessary to clarify whether definition (3.7) of the processes X, Y is equivalent to the standard definition of diffusion with discontinuous drift. The answer to this question is certainly positive. Introduce the staying-times:

$$\pi(t) = \text{mes}(s: s \leqq t, X(s) > 0) ,$$

$$\pi_0(t) = \text{mes}(s: s \leqq t, X(s) = 0) .$$

The following properties hold almost surely with respect to probability $\mathbb{P}$:

$$\pi \equiv F_2(\varepsilon W_1, \varepsilon W_2) \quad \text{for each } t \in [0, T], \text{ and}$$

$$\pi_0(T) = 0 ,$$

this may be proved by the same arguments as in Korostelev and Leonov (1992, Corollary 3 to Lemma 5). Hence, the second component of the mapping F, applied to the Wiener processes $(\varepsilon W_1, \varepsilon W_2)$, is the staying-time of the path $X(t)$ in the positive semi-axis. Since $\pi_0(T)$ is vanishing, relations (3.5) and (3.6) guarantee that the following identities are true

$$X(t) = \int_0^t b(X(s), Y(s))\, ds + \varepsilon W_1(t),$$

$$Y(t) = \int_0^t B(X(s), Y(s))\, ds + \varepsilon W_2(t)$$

$\mathbb{P}$ – a.s. for each $t \in [0, T]$. This fact means that the pair (X, Y) is the strong solution of Eqs. (1.1) with zero initial conditions. This solution may be regarded as the limit of a sequence of diffusions (X_n, Y_n) with 'smoothed' drifts (see Girsanov 1961).

Proof of Theorem 1.1 According to Lemma 3.3, the mapping F has a unique inverse in $\mathbb{Q}_{[0,T]} \times \mathbb{C}_{[0,T]}$. Moreover, it follows from (3.5) and (3.6) that for each $t \in [0, T]$

$$F^{-1}(\psi_1, \mu, \psi_2)(t)$$
$$= \Bigg(\psi_1(t) - \int_0^t b_+(\psi_1(s), \psi_2(s))\dot{\mu}(s)\,ds - \int_0^t b_-(\psi_1(s), \psi_2(s))(1 - \dot{\mu}(s))\,ds,$$
$$\psi_2(t) - \int_0^t B_+(\psi_1(s), \psi_2(s))\dot{\mu}(s)\,ds - \int_0^t B_-(\psi_1(s), \psi_2(s))(1 - \dot{\mu}(s))\,ds \Bigg).$$

Therefore, the statement of the theorem is a mere consequence of Theorem 3.3.1 in Freidlin and Wentzell (1984). □

Remark 4.1 Independence of the Wiener processes W_1 and W_2 has not been used in our definition of diffusion. Hence, all the results may be extended to the case of the system (1.1) with the correlated white noise having a constant diffusion matrix. In this case the inverse diffusion matrix appears in the corresponding action functional. The study of non-constant diffusion matrices requires more involved arguments.

Remark 4.2 If 'infimum' is omitted on the right-hand side of (1.6), then the functional $I(\psi_1, \mu, \psi_2)$ is the action functional for the triple (X, π, Y). As a consequence, the process-level large deviation principle may be obtained for the staying-time $\pi(t)$ with the action functional

$$I_\pi(\mu) = \inf_{\psi_1, \psi_2} I(\psi_1, \mu, \psi_2)\,.$$

Appendix

Proof of Lemma 3.2 Divide the interval $[0, T]$ into open subintervals (u_j, u_{j+1}), $j = 0, 1, \ldots$, so that on each subinterval the function $g(t) = \psi_1(t) - \psi'_1(t)$ does not change its sign. Note that $g(u_j) = 0$ with the only possible exception at point $u_0 = 0$, and either $g(t)$ is identic zero or $g(t)$ is strictly positive (negative) for $t \in (u_j, u_{j+1})$. Let's assume that $g(t) > 0$ for $t \in (u_j, u_{j+1})$. Using the abbreviation $b(\psi_1, \psi_2)$ instead of $b(\psi_1(s), \psi_2(s))$, we have from (3.1):

$0 < g(t) = J_1 + J_2$, where

$$J_1 = \int_{u_j}^t [b(\psi_1, \psi_2) - b(\psi'_1, \psi'_2)]\,ds = J_{11} + J_{12}\,,$$

$$J_{11} = \int_{u_j}^t [b(\psi_1, \psi_2) - b(\psi'_1, \psi_2)]\,ds,$$

$$J_{12} = \int_{u_j}^t [b(\psi'_1, \psi_2) - b(\psi'_1, \psi'_2)]\,ds\,,$$

$$J_2 = g(u_j) + l(t) - l'(t) - [l(u_j) - l'(u_j)]\,.$$

Since $f(x, y)$ is Lipschitzian in y for fixed x, we have

$$|J_{12}| \leqq L \int_{u_j}^{t} |\psi_2(s) - \psi'_2(s)|\, ds \leqq L \int_0^t |\psi_2(s) - \psi'_2(s)|\, ds .$$

Moreover, it is obvious that $|J_2| \leqq 2\|l - l'\|_{0,t}$. Thus, to prove the lemma, it suffices to show that

$$b(\psi_1, \psi_2) - b(\psi'_1, \psi_2) = R_1 + R_2, \text{ where } R_1 \leqq 0, |R_2| \leqq L|\psi_1 - \psi'_1|$$
$$(R_1 = R_1(\psi_1, \psi'_1, \psi_2), R_2 = R_2(\psi_1, \psi'_1, \psi_2)) . \tag{A1}$$

Indeed, provided that $g(t) > 0$, we get:

$$g(t) = |g(t)| \leqq L \int_{u_j}^{t} |\psi_1(s) - \psi'_1(s)|\, ds + |J_{12}| + |J_2|,$$

and the lemma follows from Gronwall's lemma.

To prove (A1), we first remark that if $\psi_1 > \psi'_1 > 0$ or $0 \geqq \psi_1 > \psi'_1$, then obviously

$$|b(\psi_1, \psi_2) - b(\psi'_1, \psi_2)| \leqq L|\psi_1 - \psi'_1| .$$

Now, if $\psi_1 > 0 \geqslant \psi'_1$, then

$$b(\psi_1, \psi_2) = b_+(\psi_1, \psi_2), b(\psi'_1, \psi_2) = b_-(\psi'_1, \psi_2), \text{ and}$$
$$b(\psi_1, \psi_2) - b(\psi'_1, \psi_2) = [b_+(\psi_1, \psi_2) - b_+(0, \psi_2)]$$
$$+ [b_+(0, \psi_2) - b_-(0, \psi_2)] + [b_-(0, \psi_2) - b_-(\psi'_1, \psi_2)] ,$$

where $[b_+(0, \psi_2) - b_-(0, \psi_2)] < 0$ due to Assumption 1.1, and

$$|[b_+(\psi_1, \psi_2) - b_+(0, \psi_2)] + [b_-(0, \psi_2) - b_-(\psi'_1, \psi_2)]$$
$$\leqq L|\psi_1| + L|\psi'_1| = L|\psi_1 - \psi'_1| .$$

This proves (A1) and completes the proof of the lemma. □

Proof of Lemma 3.3 Approximating the function $\psi(t)$ by broken lines $\psi^n(t) = l_\delta(t)$ from Lemma 3.1 with $\delta = \delta(n, 3C_0)$, we define $\varphi_n(t)$ on intervals I_j recursively:

$$\dot{\varphi}_n(t) = \dot{\psi}^n(t) - b_+(\psi^n(t), \psi_2(t)) \text{ if } \psi^n(t) > 0 \quad \text{for } t \in I_j,$$

and

$$\dot{\varphi}_n(t) = \dot{\psi}^n(t) - b_-(\psi^n(t), \psi_2(t)) \text{ if } \psi^n(t) < 0 \quad \text{for } t \in I_j$$

with the initial condition $\varphi_n(0) = \psi^n(0)$.

Under our assumptions $|\dot{\varphi}_n(t)| \geqq 2C_0$, so relation (3.1) is valid for functions φ_n, ψ^n if we replace l, ψ_1 by φ_n and ψ^n respectively. Now, if we introduce the function

$$\mu_n(t) = \text{mes}(s\colon s \leqq t, \psi^n(s) > 0) ,$$

then $G\varphi_n = (\psi^n, \mu_n)$, and the following relation is valid:

$$\varphi_n(t) = \int_0^t \dot{\mu}_n(s) \beta(\psi_2(s))\, ds + \psi^n(t) - \int_0^t f(\psi^n(s), \psi_2(s))\, ds .$$

Since sequence ψ^n uniformly converges to ψ, it suffices to prove that

$$\sup_{t \leqq T} \left| \int_0^t [\dot{\mu}_n(s) - \dot{\mu}(s)] \beta(\psi_2(s))\, ds \right| \to 0 \quad \text{as} \quad n \to \infty . \tag{A2}$$

To prove (A2), we remark that the function $z(s) = \beta(\psi_2(s))$ is uniformly continuous in the interval $[0, T]$. Therefore, $z(s)$ may be approximated by a broken line $z_\delta(s)$, so that

$$\| z - z_\delta \|_{0,T} < 1/n, \quad \sup_t |\dot{z}_\delta(t)| \leqq (1/n)/\delta, \quad \| z_\delta \|_{0,T} \leqq B. \tag{A3}$$

Further, it is obvious that

$$\left| \int_0^t [\dot{\mu}_n(s) - \dot{\mu}(s)] z(s)\, ds \right|$$
$$\leqq \left| \int_0^t [\dot{\mu}_n(s) - \dot{\mu}(s)] [z(s) - z_\delta(s)]\, ds \right| + \left| \int_0^t [\dot{\mu}_n(s) - \dot{\mu}(s)] z_\delta(s)\, ds \right| .$$

Since $|\dot{\mu}_n(s)| \leqq 1, |\dot{\mu}(s)| \leqq 1$, the first integral term on the right-hand side is not greater than $(2T)/n$. It follows from Lemma 3.1 and (A3) that the second integral term may be estimated as follows:

$$\left| \int_0^t [\dot{\mu}_n(s) - \dot{\mu}(s)] z_\delta(s)\, ds \right| = \Bigg| [\mu_n(t) - \mu(t)] z_\delta(t)$$
$$- \int_0^t [\mu_n(s) - \mu(s)] \dot{z}_\delta(s)\, ds \Bigg| \leqq B\delta + \delta T(1/n)/\delta = B\delta + T/n .$$

This proves (A2) and completes the proof of the lemma.

Proof of Lemma 3.4 We use a construction from the proof of Lemma 3.2. Let the function $g(t) = \psi_1(t) - \psi_1'(t)$ be positive for $t \in (u_j, u_{j+1})$. Using the abbreviation $b(\psi_1, \psi_2)$ instead of $b(\psi_1(s), \psi_2(s))$, we get from (3.1)

$$0 < g(t) = \int_{u_j}^t [b(\psi_1, \psi_2) - b(\psi_1', \psi_2')]\, ds$$
$$= \int_{u_j}^t [b(\psi_1, \psi_2) - b(\psi_1', \psi_2)]\, ds + \int_{u_j}^t [b(\psi_1', \psi_2) - b(\psi_1', \psi_2')]\, ds .$$

The function $g(t)$ is absolutely continuous and its derivative satisfies the equality

$$\dot{g}(s) = [b(\psi_1, \psi_2) - b(\psi_1', \psi_2)] + [b(\psi_1', \psi_2) - b(\psi_1', \psi_2')] . \tag{A4}$$

Denote $\Delta b(s) = b(\psi_1, \psi_2) - b(\psi_1', \psi_2)$. To prove the lemma, it suffices to show that

$$\int_{u_j}^t |\Delta b(s)|\, ds \leqq K_1 \int_{u_j}^t \| \psi_2 - \psi_2' \|_{0,s}\, ds , \tag{A5}$$

with some positive constant K_1. Indeed, integrating (A4) within the interval (u_j, t), $t \in (u_j, u_{j+1})$, we get from (A5)

$$\int_{u_j}^{t} |\dot{g}(s)|\, ds \leqq \int_{u_j}^{t} |\Delta b(s)|\, ds + \int_{u_j}^{t} |b(\psi'_1, \psi_2) - b(\psi'_1, \psi'_2)|\, ds$$

$$\leqq (K_1 + L) \int_{u_j}^{t} \|\psi_2 - \psi'_2\|_{0,s}\, ds ,$$

where L is the joint Lipschitz constant for $b_{\pm}(x.y)$. Thus, Lemma 3.2 and (3.2) imply that

$$\int_0^t |\dot{v}(s) - \dot{v}'(s)|\, ds \leqq \int_0^t |f(\psi_1, \psi_2) - f(\psi'_1, \psi_2)|\, ds$$

$$+ \int_0^t |f(\psi'_1, \psi_2) - f(\psi'_1, \psi'_2)|\, ds + \int_0^t |\dot{g}(s)|\, ds$$

$$\leqq L \int_0^t (|\psi_1(s) - \psi'_1(s)| + |\psi_2(s) - \psi'_2(s)|)\, ds + (K_1 + L) \int_0^t \|\psi_2 - \psi'_2\|_{0,s}\, ds$$

$$\leqq (LKT + 2L + K_1) \int_0^t \|\psi_2 - \psi'_2\|_{0,s}\, ds,$$

where K is the constant from Lemma 3.2.

To prove (A5), we introduce sets

$$S_1 = \{s \in u_j, u_{j+1}) : \psi_1(s) > \psi'_1(s) > 0\} ,$$

$$S_2 = \{s \in (u_j, u_{j+1}) : 0 \geqq \psi_1(s) > \psi'_1(s)\} ,$$

$$S_{31} = \{s \in (u_j, u_{j+1}) : \psi_1(s) > 0 \geqq \psi'_1(s), \Delta b(s) \geqq 0\} ,$$

$$S_{32} = \{s \in (u_j, u_{j+1}) : \psi_1(s) > 0 \geqq \psi'_1(s), \Delta b(s) < 0\} .$$

Then

$$\int_{u_j}^{t} |\Delta b(s)|\, ds = \int_{S_1 \cup S_2 \cup S_{31}} |\Delta b(s)|\, ds - \int_{S_{32}} \Delta b(s)\, ds$$

$$= - \int_{u_j}^{t} \Delta b(s)\, ds + \int_{S_1 \cup S_2 \cup S_{31}} [|\Delta b(s)| + \Delta b(s)]\, ds .$$

It follows from (A1) that for $s \in S_{31}$

$$0 < \Delta b(s) = |\Delta b(s)| \leqq L |\psi_1(s) - \psi'_1(s)| .$$

Further,

$$\left| - \int_{u_j}^{t} \Delta b(s)\, ds \right| \leqq |g(t)| + L \int_{u_j}^{t} |\psi_2(s) - \psi'_2(s)|\, ds ,$$

and it follows from the proof of Lemma 3.2 that

$$|g(t)| \leqq K \int_{u_j}^{t} |\psi_2(s) - \psi'_2(s)|\, ds ,$$

Thus, we finally get

$$\int_{u_j}^{t} |\Delta b(s)|\,ds \leqq (K+L)\int_{u_j}^{t} |\psi_2(s)-\psi_2'(s)|\,ds + 2L\int_{u_j}^{t} |\psi_1(s)-\psi_1'(s)|\,ds$$

$$\leqq (K+L+2LKT)\int_{u_j}^{t} \|\psi_2-\psi_2'\|_{0,s}\,ds\,.$$

Lemma 3.4 is proved. □

Proof of Lemma 3.5 Define functions

$$D_\pm(x,y) = B_\pm(x,y)/\beta(y)$$

where $\beta(y)$ is introduced in Assumption 1.1. Then, according to (3.3), the following identity holds:

$$\int_0^t B_\pm(\psi_1(s;\psi_2(\cdot),\varphi_1(\cdot)),\psi_2(s))\,d\mu(s;\psi_2(\cdot),\varphi_1(\cdot))$$

$$= \int_0^t D_\pm(\psi_1(s;\psi_2(\cdot),\varphi_1(\cdot)),\psi_2(s))\,dv(s;\psi_2(\cdot),\varphi_1(\cdot))\,.$$

So, further on all integrals are considered over dv instead of $d\mu$. Let φ_1 be an arbitrary fixed broken line from $\mathscr{L}_0$ and let φ_2 be an arbitrary fixed continuous function. Define the sequence $\{\psi_2^{(n)}\}$ recursively:

$$\psi_2^{(0)}(t) = 0 \text{ for } 0 \leqq t \leqq T, \text{ and}$$

$$\psi_2^{(n+1)}(t) = \int_0^t D_+(\psi_1(s;\psi_2^{(n)},\varphi_1),\psi_2^{(n)}(s))\,dv(s;\psi_2^{(n)},\varphi_1)$$

$$+ \int_0^t D_-(\psi_1(s;\psi_2^{(n)},\varphi_1),\psi_2^{(n)}(s))\,d(s - v(s;\psi_2^{(n)},\varphi_1)) + \varphi_2(t)$$

for $n = 0, 1, \ldots$. Since the function $\varphi_1(t)$ is the same in the non-anticipating functionals $\psi_1(t;\psi_2^{(n)},\varphi_1)$ and $v(t;\psi_2^{(n)},\varphi_1)$ for all n, Lemma 3.4 leads to the inequality

$$|\psi_2^{(n+1)}(t) - \psi_2^{(n)}(t)| \leqq C\int_0^t \|\psi_2^{(n)}(\cdot) - \psi_2^{(n-1)}(\cdot)\|_{0,s}\,ds\,. \tag{A6}$$

Here and below C denotes some positive constant which is not necessary the same in different expressions. Inequality (A6) guarantees the existence and the uniqueness of the solution ψ_2 of equation (3.6) with $\varphi_1 \in \mathscr{L}_0$ by the routine recursion.

The second step of our proof is to verify the continuity

$$\|\psi_2 - \psi_2'\|_{0,t} \to 0 \quad \text{as} \quad \|\varphi_1 - \varphi_1'\|_{0,t} + \|\varphi_2 - \varphi_2'\|_{0,t} \to 0 \tag{A7}$$

where $\varphi_1, \varphi_1' \in \mathscr{L}_0$, $\varphi_2, \varphi_2' \in \mathbb{C}_{[0,T]}$; ψ_2 and ψ_2' are the solutions of Eq. (3.6) corresponding to the pairs (φ_1, φ_2) and (φ_1', φ_2') respectively. Convergence in (A7) is assumed to be uniform over $t \in [0, T]$. Indeed,

$$|\psi_2(t) - \psi_2'(t)| \leqq \left|\int_0^t D_+(\psi_1,\psi_2)\,dv - \int_0^t D_+(\psi_1',\psi_2')\,dv'\right|$$

$$+ \left|\int_0^t D_-(\psi_1,\psi_2)\,d(s-v) - \int_0^t D_-(\psi_1',\psi_2')\,d(s-v')\right|$$

$$+ |\varphi_2(t) - \varphi_2'(t)| \tag{A8}$$

where

$$\psi_1 = \psi_1(s; \psi_2(\cdot), \varphi_1(\cdot)), \quad \psi'_1 = \psi'_1(s; \psi'_2(\cdot), \varphi'_1(\cdot))$$
$$v = v(s; \psi_2(\cdot), \varphi_1(\cdot)), \quad v' = v'(s; \psi'_2(\cdot), \varphi'_1(\cdot)) .$$

Since the integrals containing functions D_+ and D_- on the right-hand side of (A8) admit the same upper bounds, it suffices to scrutinize only one of them. It is obvious that

$$\left| \int_0^t D_+(\psi_1, \psi_2)\, dv - \int_0^t D_+(\psi'_1, \psi'_2)\, dv' \right|$$
$$\leqq \left| \int_0^t (D_+(\psi_1, \psi_2) - D_+(\psi'_1, \psi'_2))\, dv' \right| + \left| \int_0^t D_+(\psi_1, \psi_2)\, d(v - v') \right|$$
$$\leqq C \int_0^t (\| \psi_1 - \psi'_1 \|_{0,s} + \| \psi_2 - \psi'_2 \|_{0,s})\, ds$$
$$+ \left| \int_0^t D_+(\psi_1, \psi_2)\, d(v(s; \psi_2, \varphi_1) - v(s; \psi'_2, \varphi_1)) \right|$$
$$+ \left| \int_0^t D_+(\psi_1, \psi_2)\, d(v(s; \psi'_2, \varphi_1) - v(s; \psi'_2, \varphi'_1)) \right| .$$

The latter inequality together with (A8) and Lemma 3.4 entails

$$|\psi_2(t) - \psi'_2(t)| \leqq C\left(\int_0^t \| \psi_2 - \psi'_2 \|_{0,s}\, ds + \| \varphi_1 - \varphi'_1 \|_{0,t} + \| \varphi_2 - \varphi'_2 \|_{0,t} \right)$$
$$+ \left| \int_0^t D_+(\psi_1, \psi_2)\, d(v(s; \psi'_2, \varphi_1) - v(s; \psi'_2, \varphi'_1)) \right|$$
$$+ \left| \int_0^t D_-(\psi_1, \psi_2)\, d(v(s; \psi'_2, \varphi_1) - v(s; \psi'_2, \varphi'_1)) \right| . \tag{A.9}$$

Note that the functions $D_+(\psi_1(s), \psi_2(s))$ and $D_-(\psi_1(s), \psi_2(s))$ may be approximated to within an accuracy ε by functions which are piecewise constant in intervals of length $\delta(\varepsilon)$. Hence

$$\left| \int_0^t D_\pm(\psi_1, \psi_2)\, d(v(s; \psi'_2, \varphi_1) - v(s; \psi'_2, \varphi'_1)) \right|$$
$$\leqq C((1/\delta(\varepsilon)) \| v(\cdot; \psi'_2, \varphi_1) - v(\cdot; \psi'_2, \varphi'_1) \|_{0,t} + \varepsilon)$$
$$\leqq C((1/\delta(\varepsilon)) \| \varphi_1 - \varphi'_1 \|_{0,t} + \varepsilon) . \tag{A10}$$

It is of principle importance that there is no difference $\|\psi_2 - \psi_2'\|_{0,t}$ on the right-hand side of (A10), so that (A9) and (A10) imply

$$|\psi_2(t) - \psi_2'(t)| \leqq C\left(\int_0^t \|\psi_2 - \psi_2'\|_{0,s}\,ds \right.$$

$$\left. + (1/\delta(\varepsilon))\,\|\varphi_1 - \varphi_1'\|_{0,t} + \varepsilon + \|\varphi_2 - \varphi_2'\|_{0,t} \right),$$

and we get from Gronwall's lemma

$$|\psi_2(t) - \psi_2'(t)| \leqq e^{Ct}((1/\delta(\varepsilon))\,\|\varphi_1 - \varphi_1'\|_{0,t} + \|\varphi_2 - \varphi_2'\|_{0,t} + \varepsilon).$$

The lemma follows from the last inequality and Lemma 3.1. □

Acknowledgement. We are grateful to the referee for many useful and constructive comments.

References

1. Borovkov, A.A.: Boundary problems for random walks and large deviations in functional spaces (in Russian). Theory Probab. Appl. **12**, 635–654 (1967)
2. Deuschel, J.-D., Strook, D.W.: Large deviations. San Diego: Academic Press 1989
3. Dupuis, P., Ellis, R.S.: Large deviations for Markov processes with discontinuous statistics, II: Random walks. Probab. Theory Relat. Fields **91**, 153–194 (1992)
4. Dupuis, P., Ellis, R.S., Weiss, A.: Large deviations for Markov processes with discontinuous statistics, I: General upper bounds. Ann. Probab. **19**, 1280–1297 (1991)
5. Freidlin, M.I.: Action functional for a class of random processes (in Russian). Theory Probab. Appl. **17**, 536–541 (1972)
6. Freidlin, M.I., Wentzell, A.D.: On small random perturbations of dynamical systems (in Russian). Usp. Mat. Nauk **25**, (1) 3–55 (1970)
7. Freidlin, M.I., Wentzell, A.D.: Random perturbations of dynamical systems. Berlin Heidelberg New York: Springer 1984
8. Girsanov, I.V.: On Ito stochastic integral equation (in Russian). Dokl. Akad. Nauk **138**, 18–21 (1961).
9. Korostelev, A.P., Leonov, S.L.: Action functional for diffusion process with discontinuous drift (in Russian). Theory Probab. Appl. **37**, 570–576 (1992)