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Probability Theory and Related Fields

Dynkin's isomorphism theorem and the Ray-Knight theorems

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Summary. In the case of diffusions, we show that the isomorphism theorem of Dynkin and the Ray-Knight theorems can be derived from each other. Our proof uses additivity properties of squared Bessel processes and an absolute continuity relation between squared Bessel processes of dimensions one and three.

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Introduction

The isomorphism theorem of Dynkin [5] is an identity in law involving on one hand the local times of a given Markov process with a symmetric Green's function, and on the other hand the centered Gaussian process with this Green's function as its covariance. Using the properties of this Gaussian process, Sheppard [15] recovered parts of the Ray-Knight theorems on the local times of a diffusion (cf. [14, 9]). Here we adopt a different approach based on the additivity properties of squared Bessel processes, to prove in a new way the general Ray-Knight theorems (see, for example, Jeulin [7]) and also the Markov properties of the local times process. Conversely, our study allows us to establish Dynkin's isomorphism theorem as a simple consequence of the Ray-Knight theorems. Our arguments use the mutual absolute continuity between squared Bessel processes of dimensions one and three on every bounded time interval.

In Sect. I, we introduce the isomorphism theorem and describe the processes involved in this identity. In Sect. II, we establish the equivalence between the isomorphism theorem and the Ray-Knight theorems. Using the previous study, Sect. III translates the Ray-Knight theorems in terms of infinitesimal generators. In Sect. IV we give, in a particular case, an unconditional version of the isomorphism theorem.

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I The isomorphism theorem

We start by introducing Dynkin's isomorphism.

Let Y be a regular diffusion on an open interval E of R, admitting symmetric transition densities $p_t(x, y)$ with respect to a reference measure, with a finite Green's function $g(x, y) = \int_0^{+\infty} p_t(x, y)\, dt$ for $(x, y) \in E^2$ (Itô and McKean proved in [6, p. 149] that this hypothesis is always satisfied by a transient diffusion with its speed measure taken as reference measure).

Let the probability $\tilde{P}_{ab}$ be defined by

$$\tilde{P}_{ab}|_{\mathscr{F}_t} = \frac{g(Y_t, b)}{g(a, b)} P_a|_{\mathscr{F}_t}, \qquad a, b \in E,$$

where P_a is the distribution of Y starting at a, and $P_a|_{\mathscr{F}_t}$ is the restriction of P_a to $\mathscr{F}_t$ the σ-field generated by Y on the time interval $[0, t]$. We notice that, under $\tilde{P}_{ab}$, Y is killed at b. As Y is regular, $g(a, b)$ is strictly positive for every (a, b) in E^2.

Let $(L^x_\eta, x \in E)$ be the local time process of Y, evaluated at the life time η of Y. Let $(\phi_x, x \in E)$ be a Gaussian process with zero mean and with covariance equal to the Green's function $(g(x, y), (x, y) \in E^2)$, defined on an arbitrary probability space unrelated to the path space of Y. The expectation with respect to the probability on this probability space will be denoted by $\langle \cdot \rangle$. Note that ϕ is continuous (see Remark 1.3.1). Marcus and Rosen proved in [11] that the continuity of ϕ is equivalent to the continuity of the process $(L^x_\eta, x \in E)$.

The isomorphism theorem of Dynkin can be expressed as

$$\left\langle \frac{\phi_a \phi_b}{g(a, b)} F\left(\frac{1}{2}\phi^2\right)\right\rangle = \tilde{P}_{ab}\left\langle F\left(\frac{1}{2}\phi^2 + L_\eta\right)\right\rangle$$

for every positive measurable functional F.

One of the surprising consequences of this identity is that the law of $\frac{1}{2}\phi^2$ under the signed measure $\left\langle \frac{\phi_a \phi_b}{g(a, b)}; \right\rangle$ is a probability measure. Proposition 1.1 gives an explanation of this in our case.

From now we suppose that $a < b$.

$C(\mathrm{R}_+, \mathrm{R}_+)$ is the set of the continuous functions on R_+ to R_+.

Proposition 1.1 *For every positive measurable functional F, we have*

$$\tilde{P}_{ab}\left\langle F\left(\frac{1}{2}\phi^2 + L_\eta\right)\right\rangle = \left\langle \frac{|\phi_a \phi_b|}{g(a, b)} (1_A . F)\left(\frac{1}{2}\phi^2\right)\right\rangle$$

and

$$\langle \phi_a \phi_b | (|\phi_x|; x \in \mathrm{R}_+)\rangle = |\phi_a \phi_b|\, 1_A(|\phi|)$$

where

$$A = \{\psi \in C(\mathrm{R}_+, \mathrm{R}_+) : \forall u \in (a, b), \psi_u > 0\}.$$

Proof of Proposition 1.1 Let $B=A^C$. By the isomorphism theorem,

$$\left\langle \frac{\phi_a \phi_b}{g(a,b)} (1_B.F)\left(\frac{1}{2}\phi^2\right)\right\rangle = \tilde{P}_{ab}\left\langle (1_B.F)\left(\frac{1}{2}\phi^2 + L_\eta\right)\right\rangle.$$

But under $\tilde{P}_{ab}$, our diffusion takes almost surely all the values between a and b. Consequently the support of $L_\eta^{\cdot}$ contains (a, b) (see for example Jeulin [8]). Hence we have $\left\langle \frac{\phi_a \phi_b}{g(a,b)} (1_B.F)\left(\frac{1}{2}\phi^2\right)\right\rangle = 0$. As ϕ is continuous, if it does not reach zero between a and b, we have $\operatorname{sgn}(\phi_a) = \operatorname{sgn}(\phi_b)$. Thus

$$\left\langle \frac{\phi_a \phi_b}{g(a,b)} F\left(\frac{1}{2}\phi^2\right)\right\rangle = \left\langle \frac{|\phi_a \phi_b|}{g(a,b)} (1_A.F)\left(\frac{1}{2}\phi^2\right)\right\rangle.$$

Remark 1.2 If the original diffusion Y is obtained by killing a real Brownian motion B, at the first time it reaches zero, then, as: $g(x,y)=2(x\wedge y)$ for x, $y>0$, if B starts from a positive value, ϕ is equal to $\sqrt{2}\beta$ with β a Brownian motion starting from zero. The previous equality then becomes, for $s<t$,

$$E[\beta_s \beta_t 1_{(d_s<t)} | (|\beta_u|, u \geqq 0)] = 0$$

where $d_s = \inf\{u \geqq s : \beta_u = 0\}$. One can prove this equality directly by using the strong Markov property at time d_s.

Notation. We have to work with processes indexed on E. We can define the processes on R by taking them equal to zero when the index lies outside of E. We will use the canonical notation with probability measures on the space of continuous functions from R to R_+ and $(X_x, x \in \mathrm{R})$ will denote the coordinate process. Let P be the law of $(\frac{1}{2}\phi_x^2, x \in \mathrm{R})$, and ${}^{ab}\mathbb{L}$ the law of $(L_\eta^x, x \in \mathrm{R})$ under $\tilde{P}_{ab}$.

With this notation, the isomorphism theorem of Dynkin becomes

$${}^{ab}\mathbb{L} * \mathrm{P} = 2\frac{(X_a X_b)^{1/2}}{g(a,b)} 1_A \mathrm{P}$$

where $*$ denotes the convolution of two probability measures. Defining

$$\tilde{\mathrm{P}} = 2\frac{(X_a X_b)^{1/2}}{g(a,b)} 1_A \mathrm{P}$$

we finally get the following formulation of Dynkin's isomorphism

(I) $${}^{ab}\mathbb{L} * \mathrm{P} = \tilde{\mathrm{P}}.$$

From this it may be seen that, if we were able to describe precisely P and $\tilde{\mathrm{P}}$, we could deduce ${}^{ab}\mathbb{L}$ from (I) as the solution of a convolution equation. The Ray-Knight theorems would be obtained as a consequence of Dynkin's isomorphism.

Conversely, suppose that ${}^{ab}\mathbb{L}$ is known. As P is easy to describe (see Lemma 1.3), the computation of ${}^{ab}\mathbb{L}*\mathrm{P}$ must give us $\tilde{\mathrm{P}}$. Thus Dynkin's isomorphism appears as a consequence of the Ray-Knight theorems.

The claimed equivalence between the Ray-Knight theorems and Dynkin's isomorphism is based on a proper understanding of the probability measure $\tilde{\mathrm{P}}$. That is why, it will be studied below, independently of those theorems.

We need the following classical result (see for example Wong [17] or Neveu [12]).

Lemma 1.3 *Let $(\phi_x, x \in \mathrm{R})$ a Gaussian process with covariance $(g(x,y), (x,y) \in \mathrm{R}^2)$ and zero mean. Then: $(\phi_x, x \in \mathrm{R})$ is a Markov process if and only if there exist two functions f and h, such that $g(x,y) = f(x)\,h(y)$ for $x \leqq y$. When the last condition is satisfied, the function $\tau(x) = \frac{f(x)}{h(x)}$ is increasing and ϕ can be represented as $\phi_x = h(x)\,B_{\tau(x)}$, where B is a Brownian motion starting at zero. Consequently, for every $a \in E$, $\left(\frac{\phi_x}{g(a,x)}; x \geqq a\right)$ is a martingale.*

Remark 1.3.1 As Y is a transient diffusion, there exist two continuous functions f and h such that

$$g(x,y) = f(x)\,h(y), \qquad x \leqq y,$$

(see [6, p. 160]). Consequently ϕ is a Markov process. Moreover ϕ is continuous. Any other choice for the pair (f,h) is of the form $\left(\lambda f, \frac{h}{\lambda}\right)$ with $\lambda > 0$. Because of scaling properties of Bessel processes, our results are independent of the choice of (f,h).

Theorem 1.4 *Under $\tilde{\mathrm{P}}$, $(X_x, x \in \mathrm{R})$ is a Markov process. In particular we have*

$$(1) \qquad \tilde{\mathrm{P}}(X_{t+b}; t \geqq 0 \mid X_b = x) = \mathrm{P}(X_{t+b}; t \geqq 0 \mid X_b = x)$$

$$(2) \qquad \tilde{\mathrm{P}}(X_{a-t}; 0 \leqq t \leqq a \mid X_a = y) = \mathrm{P}(X_{a-t}; 0 \leqq t \leqq a \mid X_a = y).$$

Proof of Theorem 1.4 Since B is a real valued Brownian motion, B^2 is a Markov process, and by the above lemma $(X_x, x \in \mathrm{R})$ is also a Markov process under P.

For every $s \in \mathrm{R}$ and every functional F of $(X_x, x \in \mathrm{R})$, we have for every $\lambda > 0$

$$\tilde{\mathrm{P}}(e^{-\lambda X_s} F) = \mathrm{P}\left(\frac{2(X_a X_b)^{1/2}}{g(a,b)} 1_A\, e^{-\lambda X_s} F\right).$$

This is equivalent to

$$\int_0^{+\infty} \tilde{\mathrm{P}}(F \mid X_s = x)\, e^{-\lambda x}\, \tilde{\mathrm{P}}(X_s \in dx) = \int_0^{+\infty} e^{-\lambda x}\, \mathrm{P}\left(\frac{2(X_a X_b)^{1/2}}{g(a,b)} 1_A\, F \mid X_s = x\right) \mathrm{P}(X_s \in dx).$$

Using the above equality with F equals to 1, it is easy to verify that

$$\mathrm{P}\left(\frac{2(X_a X_b)^{1/2}}{g(a,b)} 1_A\, F \mid X_s = x\right) \mathrm{P}(X_s \in dx)$$

is equal to $\tilde{\mathrm{P}}(X_s \in dx)$.

Arguing as above one also obtains

$$\text{(i)} \qquad \tilde{P}(F \mid X_s = x) = \frac{P((X_a X_b)^{1/2} 1_A F \mid X_s = x)}{P((X_a X_b)^{1/2} 1_A \mid X_s = x)}.$$

Substitute in (i) $F = F_1 . F_2$ with F_1 measurable with respect to the σ-field generated by $(X_x; x \leqq s)$ and F_2 measurable with respect to $\sigma(X_x; x > s)$. It follows that for $s \leqq a$ or $s \geqq b$

$$\tilde{P}(F_1 F_2 \mid X_s = x) = \tilde{P}(F_1 \mid X_s = x)\, \tilde{P}(F_2 \mid X_s = x).$$

If $s \in (a, b)$, then

$$\begin{aligned} \tilde{P}(F_1 F_2 \mid X_s = x) &= \frac{P(X_a^{1/2} 1_{A_1} F_1 \mid X_s = x)}{P(X_a^{1/2} 1_A \mid X_s = x)} \times \frac{P(X_b^{1/2} 1_{A_2} F_2 \mid X_s = x)}{P(X_b^{1/2} 1_{A_2} \mid X_s = x)} \\ &= \tilde{P}(F_1 \mid X_s = x) \times \tilde{P}(F_2 \mid X_s = x) \end{aligned}$$

where
$$A_1 = \{\forall u \in (a, s];\, X_u > 0\}$$
$$A_2 = \{\forall u \in (s, b);\, X_u > 0\}.$$

This establishes the Markov property of $(X_x, x \in \mathrm{R})$ under $\tilde{P}$. (1) and (2) are immediate consequences of (i). □

Let Q_x^d denote the law on $\mathscr{C}(\mathrm{R}_+, \mathrm{R}_+)$ of the squared Bessel process of dimension $d \geqq 0$, starting at $x \geqq 0$.

The following theorem is the main tool used to prove the equivalence between the Ray-Knight theorems and the isomorphism theorem. It describes the law of $(X_x, x \in \mathrm{R})$ under $\tilde{P}$.

Theorem 1.5 *Under $\tilde{P}$ the law of $(X_x, x \in \mathrm{R})$ is given by:*

$$\text{(1)} \qquad (X_{t+b};\, t \geqq 0 \mid X_b = x) \overset{(d)}{=} Q^1_{\frac{2x}{h^2(b)}} \left(\frac{h^2(b+t)}{2} X_{\tau(b+t) - \tau(b)};\, t \geqq 0 \right)$$

$$\text{(2)} \qquad (X_{t+a};\, 0 \leqq t \leqq b - a) \overset{(d)}{=} Q_0^3 \left(\frac{h^2(a+t)}{2} X_{\tau(a+t)};\, 0 \leqq t \leqq b - a \right)$$

$$\text{(3)} \qquad (X_{a-t};\, t \geqq 0 \mid X_a = y) \overset{(d)}{=} Q^1_{\frac{2y}{h^2(a)}} \left(\frac{h^2(a-t)}{2} X_{\tau(a) - \tau(a-t)};\, t \geqq 0 \mid X_{\tau(a)} = 0 \right)$$

where h and τ are as defined in Lemma 1.3.

Proof of Theorem 1.5 By Lemma 1.3, $\frac{1}{2}\phi_t^2 = \frac{1}{2} h^2(t)\, B^2_{\tau(t)}$ where B is a real valued Brownian motion starting at zero. This means that

$$P(X_t;\, t \in \mathrm{R}) \overset{(d)}{=} Q_0^1 \left(\frac{h^2(t)}{2} X_{\tau(t)};\, t \in \mathrm{R} \right).$$

Under the conditions of Remark 1.2, this becomes

$$P(X_t;\, t \geqq 0) \overset{(d)}{=} Q_0^1 (X_t;\, t \geqq 0).$$

Since τ is an increasing continuous and positive function, it is enough to prove Theorem 1.5 for τ as in Remark 1.2 and the result will follow by a time change.

By Theorem 1.4(1), we have

$$\tilde{\mathrm{P}}(X_{t+b};t\geqq 0|X_b=x)=Q_x^1(X_t;t\geqq 0)$$

which proves (1).

By Theorem 1.4(2)

$$\begin{aligned}\tilde{\mathrm{P}}(X_{a-t};0\leqq t\leqq a|X_a=y)&=\mathrm{P}(X_{a-t};0\leqq t\leqq a|X_a=y)\\&=Q_0^1(X_{a-t};0\leqq t\leqq a|X_a=y)\\&=Q_y^1(X_t;0\leqq t\leqq a|X_a=0)\end{aligned}$$

which proves (3).

To prove (2), we use the absolute continuity of Q_x^3 with respect to Q_x^1 on $\mathscr{F}_t=\sigma(X_s;s\leqq t)$. It was proved by Knight [10, p. 124], Pitman and Yor [13] and Biance and Yor [1] that

$$(*)\qquad Q_x^3|_{\mathscr{F}_t}=\frac{(X_{t\wedge T(0)})^{1/2}}{x^{1/2}}\,Q_x^1|_{\mathscr{F}_t}$$

where $T(0)=\inf\{u\geqq 0;X_u=0\}$.

Let $F_{a,b}=F(X_{(u+a)\wedge b};u\geqq 0)$. Then $F_{a,b}=F_{0,b-a}\circ\theta_a$

$$\begin{aligned}\tilde{\mathrm{P}}(F_{a,b})&=\mathrm{P}\left(\frac{(X_a X_b)^{1/2}}{a}1_A F_{a,b}\right)\\&=Q_0^1\left(\frac{(X_a X_b)^{1/2}}{a}1_A F_{a,b}\right)\\&=Q_0^1\left(\frac{X_a^{1/2}}{a}Q_{X_a}^1(X_{(b-a)\wedge T(0)}^{1/2}F_{0,b-a})\right)\\&=Q_0^1\left(\frac{X_a}{a}Q_{X_a}^1\left(\frac{X_{(b-a)\wedge T(0)}^{1/2}}{X_a^{1/2}}F_{0,b-a}\right)\right)\\&=Q_0^1\left(\frac{X_a}{a}Q_{X_a}^3(F_{0,b-a})\right)\end{aligned}$$

where the third equality follows from the Markov property at time a and the fifth equality is thanks to $(*)$.

Since $\frac{x}{a}Q_0^1(X_a\in dx)=Q_0^3(X_a\in dx)$, it follows again from the Markov property at time a that

$$Q_0^1\left(\frac{(X_a X_b)^{1/2}}{a}1_A F_{a,b}\right)=Q_0^3(F_{a,b})$$

which proves (2).

II The equivalence between Dynkin's isomorphism theorem and the Ray-Knight theorems

Assuming the isomorphism theorem, we shall prove the Ray-Knight theorems in the following form

Theorem 2.1 *Under* ${}^{ab}\mathbb{L}$ *(or* ${}^{ba}\mathbb{L}$*)* $(X_x, x\in\mathrm{R})$ *is a Markov process. Its law is characterised by*

$$(1)\qquad (X_{b+s}; s\geqq 0 \mid X_b = x) \overset{(d)}{=} Q^0_{\frac{2x}{h^2(b)}}\left(\frac{h^2(b+s)}{2} X_{\tau(b+s)-\tau(b)}; s\geqq 0\right)$$

$$(2)\qquad (X_{a+s}; 0\leqq s\leqq b-a) \overset{(d)}{=} Q^2_0\left(\frac{h^2(a+s)}{2} X_{\tau(a+s)}; 0\leqq s\leqq b-a\right)$$

$$(3)\qquad (X_{a-s}; s\geqq 0 \mid X_a = y) \overset{(d)}{=} Q^0_{\frac{2y}{h^2(a)}}\left(\frac{h^2(a-s)}{2} X_{\tau(a)-\tau(a-s)}; s\geqq 0 \mid X_{\tau(a)}=0\right).$$

Proof of Theorem 2.1 As in the proof of Theorem 1.6, it is sufficient to establish (1), (2) and (3) for the special case of Remark 1.2. The Markov property will be proved at the end. Assertion (2) follows easely from the relation $Q^2_0 * Q^1_0 = Q^3_0$ and from the fact that if X, Y, Z are independent random variables and $X+Y \overset{(d)}{=} X+Z$ then $Y \overset{(d)}{=} Z$.

To prove (1), by the isomorphism theorem, we have

$$\begin{aligned}({}^{ab}\mathbb{L} * Q^1_0)(F(X_u; u\geqq 0)\circ\theta_b) &= Q^1_0\left(\frac{(X_a X_b)^{1/2}}{a} 1_A F(X_u; u\geqq 0)\right)\\ &= Q^1_0(F(X_u; u\geqq 0)\circ\theta_b)\end{aligned}$$

using the Markov property at time b.

The additivity property of squared Bessel processes

$$Q^0_x * Q^1_y = Q^1_{x+y} \qquad \forall x\in\mathrm{R}^*_+, \qquad \forall y\in\mathrm{R}_+$$

leads after integration with respect to the pair of probability measures (μ, ν) on R_+ to

$$Q^0_\mu * Q^1_\nu = Q^1_{\mu*\nu}.$$

Hence

$${}^{ab}\mathbb{L}(F(X_u; u\geqq 0) \mid X_b = x) = Q^0_x(F(X_u; u\geqq 0)).$$

Assertion (3) is obtained by the same arguments.

We have thus shown that under ${}^{ab}\mathbb{L}$, X is a Markov process on $(-\infty, a]$, on $[a, b]$ and on $[b, +\infty)$. This, however, is not enough to claim that X is a Markov process on R under ${}^{ab}\mathbb{L}$. To prove it, we shall need the following definition.

Definition. *A probability measure P on* $C(\mathbb{R}_+, \mathbb{R}_+)$ **has property (*)** *if for every* $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_n)$ $\lambda_i \geqq 0$, $t=(t_1, t_2, \ldots, t_n)$ *satisfying* $0 \leqq t_1 < t_2 < \ldots < t_n$, *and* $s \geqq 0$ *there exist* $\phi_s(t, \lambda)$ *and* $\psi_s(t, \lambda)$ *strictly positive such that*

$$P\left[\exp\left(-\sum_{i=1}^{n} \lambda_i X_{t_i+s}\right) \middle| X_s\right] = e^{-X_s \psi_s(t,\lambda)} \phi_s(t, \lambda).$$

Using the results of Pitman and Yor [13] the following lemma can be easely established.

Lemma 2.2 (i) *Squared Bessel processes satisfy property (*), all with the same* $\psi_s(t, \lambda)$.

(ii) *Squared Bessel bridges ending at zero satisfy property (*), all with the same* $\psi_s(t, \lambda)$.

Theorem 2.3 *Let P and Q be two probability laws on* $C(\mathbb{R}_+, \mathbb{R}_+)$ *satisfying property (*) with the same* $\psi_s(t, \lambda)$. *Let Y and Z be two independent stochastic processes governed by the laws P and Q respectively. Then for every* $s \geqq 0$

$$((Y+Z)(t+s), t \geqq 0 \mid Y(s), Z(s)) \stackrel{(d)}{=} ((Y+Z)(t+s), t \geqq 0 \mid Y(s)+Z(s)).$$

Proof. By the independence of Y and Z, we have

$$E\left[\exp\left(-\sum_{i=1}^{n} \lambda_i (Y+Z)(t_i+s)\right) \middle| Y(s), Z(s)\right]$$
$$= E\left[\exp\left(-\sum_{i=1}^{n} \lambda_i Y(t_i+s)\right) \middle| Y(s)\right] . E\left[\exp\left(-\sum_{i=1}^{n} \lambda_i Z(t_i+s)\right) \middle| Z(s)\right].$$

Applying property (*) to each term of the above product, it follows that

$$E\left[\exp\left(-\sum_{i=1}^{n} \lambda_i (Y+Z)(t_i+s)\right) \middle| Y(s), Z(s)\right]$$

is a function of $Y(s)+Z(s)$ only. □

We now return to the proof of the Markov property. Toward this end it is easy to check, using Lemma 2.2, that the laws of $(X_{a+s}, s \geqq 0)$ and $(X_{b-s}, s \geqq 0)$ under both $\tilde{\mathrm{P}}$, P satisfy property (*) with the same $\psi_s(t, \lambda)$. Since ${}^{ab}\mathbb{L} * \mathrm{P} = \tilde{\mathrm{P}}$, the same is true under ${}^{ab}\mathbb{L}$.

For the sake of clarity we shall return to the notation involving L_η^x and ϕ_x introduced at the beginning of the paper.

We first establish the Markov property at time a of $L_\eta^{\cdot}$ under $\tilde{P}_{ab}$.

For $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, let

$$F_1(f) = \exp\left(-\int_{-\infty}^{a} f_s \, d\mu(s)\right)$$

$$F_2(f) = \exp\left(-\int_{a}^{+\infty} f_s \, d\mu(s)\right)$$

and $F = F_1 . F_2$. We have to show that

$$\tilde{P}_{ab}(F(L_\eta) | L^a_\eta) = \tilde{P}_{ab}(F_1(L_\eta) | L^a_\eta) . \tilde{P}_{ab}(F_2(L_\eta) | L^a_\eta).$$

Using the Markov property of $L_\eta + \frac{1}{2}\phi^2$ obtained in Theorem 1.4, we have

$$\begin{aligned}
&\tilde{P}_{ab}\langle F(L_\eta + \tfrac{1}{2}\phi^2) | L^a_\eta + \tfrac{1}{2}\phi^2_a = x + y\rangle \\
&\quad = \tilde{P}_{ab}\langle F_1(L_\eta + \tfrac{1}{2}\phi^2) | L^a_\eta + \tfrac{1}{2}\phi^2_a = x + y\rangle . \\
&\qquad \cdot \tilde{P}_{ab}\langle F_2(L_\eta + \tfrac{1}{2}\phi^2) | L^a_\eta + \tfrac{1}{2}\phi^2_a = x + y\rangle \\
&\quad = \tilde{P}_{ab}(F_1(L_\eta) | L^a_\eta = x) . \tilde{P}_{ab}(F_2(L_\eta) | L^a_\eta = x) . \langle F(\tfrac{1}{2}\phi^2) | \tfrac{1}{2}\phi^2_a = y\rangle,
\end{aligned}$$

where the last equality follows from Theorem 2.3.

Set,

$$\begin{aligned}
f_1(x) &= \tilde{P}_{ab}(F_1(L_\eta) | L^a_\eta = x) . \tilde{P}_{ab}(F_2(L_\eta) | L^a_\eta = x) \\
f_2(x) &= \langle F(\tfrac{1}{2}\phi^2) | \tfrac{1}{2}\phi^2_a = y\rangle \\
f_3(x) &= \tilde{P}_{ab}(F(L_\eta) | L^a_\eta = x).
\end{aligned}$$

Then

$$\tilde{P}_{ab}\langle F(L_\eta + \tfrac{1}{2}\phi^2) | L^a_\eta + \tfrac{1}{2}\phi^2_a\rangle = f_1(L^a_\eta) f_2(\tfrac{1}{2}\phi^2_a).$$

On the other hand

$$\begin{aligned}
&\tilde{P}_{ab}\langle F(L_\eta + \tfrac{1}{2}\phi^2) | L^a_\eta + \tfrac{1}{2}\phi^2_a\rangle \\
&\quad = \tilde{P}_{ab}\langle \tilde{P}_{ab}\langle F(L_\eta + \tfrac{1}{2}\phi^2) | (L^a_\eta, \tfrac{1}{2}\phi^2_a\rangle | L^a_\eta + \tfrac{1}{2}\phi^2_a\rangle \\
&\quad = \tilde{P}_{ab}\langle f_3(L^a_\eta) f_2(\tfrac{1}{2}\phi^2_a) | L^a_\eta + \tfrac{1}{2}\phi^2_a\rangle.
\end{aligned}$$

Thus we have

$$\tilde{P}_{ab}\left\langle \frac{f_3(L^a_\eta)}{f_1(L^a_\eta)} \middle| L^a_\eta + \frac{1}{2}\phi^2_a \right\rangle = 1.$$

In particular, for every $\lambda > 0$, we have

$$\tilde{P}_{ab}\left\langle \frac{f_3(L^a_\eta)}{f_1(L^a_\eta)} e^{-\lambda(L^a_\eta + \frac{1}{2}\phi^2_a)} \right\rangle = \tilde{P}_{ab}\langle e^{-\lambda(L^a_\eta + \frac{1}{2}\phi^2_a)}\rangle,$$

which is equivalent to

$$\tilde{P}_{ab}\left(\frac{f_3(L^a_\eta)}{f_1(L^a_\eta)} \middle| L^a_\eta \right) = 1,$$

and consequently $f_3(L^a_\eta) = f_1(L^a_\eta)$.

The Markov property at time b can be established in the same manner.

To prove the global Markov property, let x be in $(-\infty, a]$ and for f in $C(\mathrm{R}_+, \mathrm{R}_+)$ let

$$F_1(f)=\exp\left(-\int_{-\infty}^{x} f_s\, d\mu(s)\right)$$

$$F_2(f)=\exp\left(-\int_{x}^{a} f_s\, d\mu(s)\right)$$

$$F_3(f)=\exp\left(-\int_{a}^{+\infty} f_s\, d\mu(s)\right)$$

and $F=F_1\, F_2\, F_3$. We have to show that

$$\tilde{P}_{ab}(F(L_\eta)\,|\,L_\eta^x)=\tilde{P}_{ab}(F_1(L_\eta)\,|\,L_\eta^x).\tilde{P}_{ab}(F_2\, F_3(L_\eta)\,|\,L_\eta^x).$$

Using the Markov property at time a

$$\begin{aligned}\tilde{P}_{ab}(F(L_\eta)\,|\,L_\eta^x)&=\tilde{P}_{ab}(\tilde{P}_{ab}(F(L_\eta)\,|\,\sigma(L_\eta^y, y<a))\,|\,L_\eta^x)\\&=\tilde{P}_{ab}(F_1\, F_2(L_\eta)\,\tilde{P}_{ab}(F_3(L_\eta)\,|\,\sigma(L_\eta^y\,;\, y<a))\,|\,L_\eta^x)\\&=\tilde{P}_{ab}(F_1\, F_2(L_\eta)\,\tilde{P}_{ab}(F_3(L_\eta)\,|\,L_\eta^a\,|\,L_\eta^x)\\&=\tilde{P}_{ab}(F_1(L_\eta)\,|\,L_\eta^x)\,\tilde{P}_{ab}(F_2(L_\eta)\,\tilde{P}_{ab}(F_3(L_\eta)\,|\,L_\eta^a)\,|\,L_\eta^x)\end{aligned}$$

where the last equality follows from the Markov property of $(L_\eta^{a-y}, y\geqq 0)$ at time $(a-x)$. The global Markov property at x in $[a, +\infty)$ is argued similarly. This finishes the proof of Theorem 2.1. □

For the other direction of the equivalence we assume now the Ray-Knight theorems and deduce Dynkin's isomorphism theorem. This amounts to showing that

$$\text{(I)} \qquad {}^{ab}\mathbb{L} * \mathrm{P}=\tilde{\mathrm{P}}$$

is a consequence of Theorem 2.1.

Proof of (I) On one hand, we described completely $\tilde{\mathrm{P}}$ in Theorem 1.4 (independently of the isomorphism theorem) and saw that it was markovian. On the other hand, using the equality $\mathrm{P}(X_t\,;\, t\in\mathrm{R}) \overset{(d)}{=} Q_0^1\left(\frac{h^2(t)}{2} X_{\tau(t)}\,;\, t\in\mathrm{R}\right)$, Theorem 2.1 and the additivity property of squared Bessel processes, it is easy to obtain the precise expression of ${}^{ab}\mathbb{L}*\mathrm{P}$ on $(-\infty, a]$, $[a, b]$ and $[b, +\infty)$. Thus we check that $\tilde{\mathrm{P}}$ and ${}^{ab}\mathbb{L}*\mathrm{P}$ are equal on each of those three intervals. Now to see that they are completely equal, it is sufficient to prove that ${}^{ab}\mathbb{L}*\mathrm{P}$ is Markovian. We know already that ${}^{ab}\mathbb{L}$ and P are Markovian and that under those two probability measures the laws of $(X_{a+s}, s\geqq 0)$ and $(X_{b-s}, s\geqq 0)$ have property (*) with the same $\psi_s(t,\lambda)$. This is enough, thanks to Theorem 2.3 to prove the global Markov property of ${}^{ab}\mathbb{L}*\mathrm{P}$. We detail for example the computations for the proof of the Markov property at time a. We take the same notations as in the proof of the Markov property of ${}^{ab}\mathbb{L}$ in Theorem 2.1. For f in $C(\mathrm{R}_+, \mathrm{R}_+)$ let

$$F_1(f)=\exp\left(-\int_{-\infty}^{a} f_s\,d\mu(s)\right)$$

$$F_2(f)=\exp\left(-\int_{a}^{+\infty} f_s\,d\mu(s)\right)$$

and $F=F_1.F_2$. We want to show that

$$\begin{aligned}&\tilde{P}_{ab}\langle F(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a+\tfrac{1}{2}\phi_a^2\rangle\\ &\quad=\tilde{P}_{ab}\langle F_1(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a+\tfrac{1}{2}\phi_a^2\rangle.\tilde{P}_{ab}\langle F_2(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a+\tfrac{1}{2}\phi_a^2\rangle.\end{aligned}$$

Using the independence of L_η and ϕ, we have

$$\begin{aligned}&\tilde{P}_{ab}\langle F(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a+\tfrac{1}{2}\phi_a^2\rangle.\\ &\quad=\tilde{P}_{ab}\langle F(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a,\tfrac{1}{2}\phi_a^2\,|\,L_\eta^a+\tfrac{1}{2}\phi_a^2\rangle\\ &\quad=\tilde{P}_{ab}\langle\tilde{P}_{ab}(F(L_\eta)\,|\,L_\eta^a)\langle F(\tfrac{1}{2}\phi^2)\,|\,\tfrac{1}{2}\phi_a^2\rangle\,|\,L_\eta^a+\tfrac{1}{2}\phi_a^2\rangle\\ &\quad=\tilde{P}_{ab}\langle\tilde{P}_{ab}\langle F_1(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a,\tfrac{1}{2}\phi_a^2\rangle\\ &\qquad\cdot\tilde{P}_{ab}\langle F_2(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a,\tfrac{1}{2}\phi_a^2\rangle\,|\,L_\eta^a+\tfrac{1}{2}\phi_a^2\rangle\end{aligned}$$

where the last equality follows from the Markov property of L_η and $\frac{1}{2}\phi^2$ at time a.

Now, by Theorem 2.3, we have for $k=1,2$

$$\tilde{P}_{ab}\langle F_k(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a,\tfrac{1}{2}\phi_a^2\rangle=\tilde{P}_{ab}\langle F_k(L_\eta+\tfrac{1}{2}\phi^2)\,|\,L_\eta^a+\tfrac{1}{2}\phi_a^2\rangle$$

which finishes to proof.

III Use of the Ray-Knight theorems under the form given by the isomorphism theorem

Using Theorem 2.1, we can immediately enunciate the following theorem

Theorem 3.1 *Assume that the functions f and h are continuously differentiable. Then the law of the processes*

$$(X_{b+s};0\leqq s<\sup E-b),\quad (X_s;a\leqq s\leqq b)\quad \textit{and}\quad (X_{a-s};a-\inf E<s\leqq a)$$

under ${}^{ab}\mathbb{L}$ or ${}^{ba}\mathbb{L}$ are respectively characterized by their strong generators defined as follows

$$A_{b+s}=\alpha(b+s)\,x\frac{d^2}{dx^2}+\beta(b+s)\,x\frac{d}{dx},\qquad 0\leqq s<\sup E-b$$

$$A_{a+s}=\alpha(a+s)\,x\frac{d^2}{dx^2}+(\beta(a+s)\,x+\alpha(a+s))\frac{d}{dx};\qquad a\leqq s\leqq b$$

$$A_{a-s}=\alpha(a-s)\,x\frac{d^2}{dx^2}+\gamma(a-s)\,x\frac{d}{dx},\qquad a-\inf E<s\leqq a$$

with

$$\begin{cases} \alpha(s) = h^2(s)\,\tau'(s) \\ \beta(s) = 2\,\dfrac{h'(s)}{h(s)} \\ \gamma(s) = -2\,\dfrac{f'(s)}{f(s)}. \end{cases}$$

Example. In the case of a real Brownian motion killed at an independent exponential time of parameter $\theta^2/2$, the Green's function of this diffusion is: $g(x, y) = 1/\theta\, e^{-\theta(y-x)}$ if $x \leqq y$ so: $\alpha(s)=2$, $\beta(s)=-2\theta$, $\gamma(s)=-2\theta$. The formulation given by Theorem 3.1 of the Ray-Knight theorem in that case, has been found by Borodin [4]. (See also Biane and Yor [3].)

Now setting: $I_\eta = \inf_{s \geqq 0} Y_s$, and $S_\eta = \sup_{s \geqq 0} Y_s$; where Y is our original diffusion, we have the following corollary of Theorem 3.1.

Corollary 3.2

$$\tilde{\mathbb{P}}_{ab}(I_\eta > m) = \frac{\tau(a)-\tau(m)}{2\tau(a)-\tau(m)} \qquad \text{for } m < a$$

$$\tilde{\mathbb{P}}_{ab}(S_\eta < \rho) = \frac{\tau(\rho)-\tau(b)}{\tau(\rho)} \qquad \text{for } \rho > b$$

with $(m, \rho) \in E^2$.

Proof of Corollary 3.2 By Theorem 3.1,

$$\begin{aligned} \tilde{\mathbb{P}}_{ab}(I_\eta > m \mid L^a_\eta = x) &= {}^{ab}\mathbb{L}(X_m = 0 \mid X_a = x) \\ &= Q^0_{\frac{2x}{h^2(a)}}\left(\frac{h^2(m)}{2} X_{\tau(a)-\tau(m)} = 0\right) \\ &= \exp\left(-\frac{x}{h^2(a)(\tau(a)-\tau(m))}\right) \end{aligned}$$

hence

$$\tilde{P}_{ab}(I_\eta > m) = {}^{ab}\mathbb{L}\left[\exp\left(-\frac{X_a}{h^2(a)(\tau(a)-\tau(m))}\right)\right]$$

and by Theorem 3.1(2), under ${}^{ab}\mathbb{L}$, X_a has an exponential law with parameter $(g(a, a))^{-1}$. For S_η, the proof is similar.

IV Unconditional version of the isomorphism theorem

Suppose that Y is a diffusion such that the upper boundary point of the state space is the only absorbing point. In this case $g(x, y) = h(x \vee y)$ (see [6, Chapt. 4]). The following theorem gives a simplified version of the isomorphism theorem.

Theorem 4.1 (1) *If* $g(x, y) = h(x \vee y)$ *for* (x, y) *in* E^2 *we set* $F_z = F(X_u; u \leqq z)$. *Then:* $\forall (a, z) \in E^2$

$$P_a\left\langle F_z\left(L_\eta + \frac{1}{2}\phi^2\right)\right\rangle = \left\langle \frac{\phi_a \phi_{a \vee z}}{g(a, z)} F_z\left(\frac{1}{2}\phi^2\right)\right\rangle.$$

(2) *If* $g(x, y) = f(x \wedge y)$, *for* (x, y) *in* E^2 *we set* $G_z = G(X_u; u \geqq z)$. *Then:* $\forall (a, z) \in E^2$:

$$P_a\left\langle G_z\left(L_\eta + \frac{1}{2}\phi^2\right)\right\rangle = \left\langle \frac{\phi_a \phi_{a \wedge z}}{g(a, z)} G_z\left(\frac{1}{2}\phi^2\right)\right\rangle.$$

Proof of (1) We show first that

$$\tilde{P}_{ab}(L^u_\eta; u \leqq b) \stackrel{(d)}{=} P_a(L^u_\eta; u \leqq b).$$

Let $\tilde{p}_t(x, y)$ denote the transition density of Y under $\tilde{P}_{ab}$, and $\tilde{g}(x, y)$ the Green's function under $\tilde{P}_{ab}$, then

$$\tilde{p}_t(x, y) = \frac{g(b, y)}{g(b, x)} p_t(x, y) = p_t(x, y) \quad \text{and} \quad \tilde{g}(x, y) = g(x, y)$$

for $x, y \leqq b$.

We use now the following formulas established by Marcus and Rosen in [11]. For $a \leqq x_1 \leqq x_2 \ldots \leqq x_n \leqq b$.

$$\begin{cases} \tilde{P}_{ab}\left[\prod_{i=1}^n L^{x_i}_\eta\right] = \sum_{s \in \mathscr{S}_n} \tilde{g}(a, x_{s(1)}) \times \tilde{g}(x_{s(1)}, x_{s(2)}) \ldots \times \tilde{g}(x_{s(n-1)}, x_{s(n)}) \\ P_a\left[\prod_{i=1}^n L^{x_i}_\eta\right] = \sum_{s \in \mathscr{S}_n} g(a, x_{s(1)}) \times g(x_{s(1)}, x_{s(2)}) \ldots \times g(x_{s(n-1)}, x_{s(n)}) \end{cases}$$

where $\mathscr{S}_n$ is the set of all the permutations of $\{1, 2, \ldots, n\}$.

Hence for all $(m_1, \ldots, m_n)$ in $\mathbf{N}^n$

$$\tilde{P}_{ab}\left[\prod_{i=1}^n (L^{x_1}_\eta)^{m_i}\right] = P_a\left[\prod_{i=1}^n (L^{x_i}_n)^{m_i}\right]$$

and thus

$$\tilde{P}_{ab}\left[\exp\left(-\sum_{i=1}^n \lambda_i L^{x_i}_\eta\right)\right] = P_a\left[\exp\left(-\sum_{i=1}^n \lambda_i L^{x_i}_\eta\right)\right].$$

On the other hand, by the isomorphism theorem

$$\tilde{P}_{ab}\left\langle F_z\left(L_\eta + \frac{1}{2}\phi^2\right)\right\rangle = \left\langle \frac{\phi_a \phi_b}{g(a, b)} F_z\left(\frac{1}{2}\phi^2\right)\right\rangle, \qquad z \in (a, b).$$

Since $(\phi_x, x \in E)$ is a Markov process and $\left(\frac{\phi_x}{g(a,x)}; x \geqq a\right)$ is a martingale (see Lemma 1.3), we have

$$\left\langle \frac{\phi_a \phi_b}{g(a,b)} F_z\left(\frac{1}{2}\phi^2\right)\right\rangle = \left\langle \phi_a F_z\left(\frac{1}{2}\phi^2\right)\left\langle \frac{\phi_b}{g(a,b)} \,|\, \sigma(\phi_u; u \leqq z)\right\rangle\right\rangle$$
$$= \left\langle \frac{\phi_a \phi_z}{g(a,z)} F_z\left(\frac{1}{2}\phi^2\right)\right\rangle$$

Assertion (2) is obtained by the same way. □

The equivalence between the isomorphism theorem formulated that way for those diffusions and Ray-Knight theorems under P_a can be established by the same arguments.

Example. We consider the process of the local times $(\tilde{L}^x_\infty, x \geqq 0)$ of a Bessel process of dimension $d = \alpha + 2$, $\alpha > 0$, with respect to its speed measure. The Green's function with respect to the speed measure is: $\tilde{g}(x,y) = \frac{2}{\alpha}(x \vee y)^{-\alpha}$. By Theorem 4.1

$$P_0(\tilde{L}^x_\infty; x > 0) \overset{(d)}{=} Q^2_0\left(\frac{1}{\alpha^2 x^{2\alpha}} X_{\alpha x^\alpha}; x > 0\right).$$

Let $(L^x_\infty, x \geqq 0)$ be the process of the local times with respect to the Lebesgue measure, then $(L^x_\infty, x \geqq 0) = (x^{\alpha+1}\tilde{L}^x_\infty, x \geqq 0)$. Therefore

$$P_0(L^x_\infty, x > 0) = Q^2_0\left(\frac{1}{\alpha x^{\alpha-1}} X_{x^\alpha}; x > 0\right).$$

This result has been established by Yor [18]. (See also Biane and Yor [2] for another description.)

We end by a remark generalising a well-known result

Theorem 4.2 *Let Z be a homogeneous symmetric Markov process with finite Green's function. Then, for every (a,b) such that $g(a,b) > 0$ and $g(a,a) > 0$, the variable $L^a_\eta(Z)$ has an exponential law with parameter $1/g(a,a)$ under $\tilde{P}_{ab}$ and $\tilde{P}_{ba}$.*

Proof of Theorem 4.2 We use once more the isomorphism theorem

$$\tilde{P}_{ab}[e^{-\lambda L^a_\eta}] \times \langle e^{-\frac{\lambda}{2}\phi^2_a}\rangle = \left\langle \frac{\phi_a \phi_b}{g(a,b)} e^{-\frac{\lambda}{2}\phi^2_a}\right\rangle.$$

On one hand, we have $\langle e^{-\frac{\lambda}{2}\phi^2_a}\rangle = (1 + \lambda g(a,a))^{-1/2}$ (ϕ_a has a variance equal to $g(a,a)$). On the other hand

$$\left\langle \frac{\phi_a \phi_b}{g(a,b)} e^{-\frac{\lambda}{2}\phi^2_a}\right\rangle = \left\langle \frac{\phi_a e^{-\frac{\lambda}{2}\phi^2_a}}{g(a,b)} \langle \phi_b | \phi_a\rangle\right\rangle$$
$$= \left\langle \frac{\phi^2_a}{g(a,a)} e^{-\frac{\lambda}{2}\phi^2_a}\right\rangle$$
$$= (1 + \lambda g(a,a))^{-3/2}.$$

(This last equality is obtained by taking the derivative of $\langle e^{-\frac{\lambda}{2}\phi_a^2}\rangle$.)

Consequently

$$\tilde{P}_{ab}(e^{-\lambda L_\eta^a}) = (1+\lambda g(a,a))^{-1}.$$

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