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Shock fluctuations in the asymmetric simple exclusion process

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Summary. We consider the one dimensional nearest neighbors asymmetric simple exclusion process with rates q and p for left and right jumps respectively; $q < p$. Ferrari et al. (1991) have shown that if the initial measure is $\nu_{\rho,\lambda}$, a product measure with densities ρ and λ to the left and right of the origin respectively, $\rho < \lambda$, then there exists a (microscopic) shock for the system. A shock is a random position X_t such that the system as seen from this position at time t has asymptotic product distributions with densities ρ and λ to the left and right of the origin respectively, uniformly in t. We compute the diffusion coefficient of the shock $D = \lim_{t\to\infty} t^{-1}(E(X_t)^2 - (EX_t)^2)$ and find $D = (p-q)(\lambda-\rho)^{-1}(\rho(1-\rho)+\lambda(1-\lambda))$ as conjectured by Spohn (1991). We show that in the scale $\sqrt{t}$ the position of X_t is determined by the initial distribution of particles in a region of length proportional to t. We prove that the distribution of the process at the average position of the shock converges to a fair mixture of the product measures with densities ρ and λ. This is the so called dynamical phase transition. Under shock initial conditions we show how the density fluctuation fields depend on the initial configuration.

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1 Introduction

Let $\eta_t \in \{0,1\}^{\mathbb{Z}}$ be the asymmetric nearest neighbors one dimensional simple exclusion process (Spitzer (1970), Liggett (1985)). Its generator is given by

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} \sum_{y=x\pm 1} p(x,y)\eta(x)(1-\eta(y))[f(\eta^{x,y}) - f(\eta)],$$

where f is a continuous function on $\{0,1\}^{\mathbb{Z}}$, the configuration $\eta^{x,y}(z)$ is defined by

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if} \quad z \neq x, y \\ \eta(x) & \text{if} \quad z = y \\ \eta(y) & \text{if} \quad z = x \end{cases}$$

and

$$p(x,y)=\begin{cases} p & \text{if} \quad y=x+1 \\ q & \text{if} \quad y=x-1 \\ 0 & \text{otherwise} \end{cases},$$

$p+q=1$. We consider without loss of generality, $p>q\geqq 0$. Let $S(t)$ denote the corresponding semigroup. In words, the process describes the evolution of particles in $\mathbb{Z}$ under the constriction that at most one particle is allowed at each site. Particles jump to left and right nearest neighbor empty sites at rate q and p respectively. No jumps are allowed to occupied sites. We consider as initial measure $\nu_{\rho,\lambda}$, the product measure with densities ρ and λ to the left and right of the origin respectively. We fix $\rho<\lambda$. We say that a random position X_t is a microscopic shock if the distribution of $\tau_{X_t}\eta_t$ has asymptotic distributions ν_ρ and ν_λ to the left and right of the origin respectively, uniformly in t. The operator τ_x is translation by x; ν_ρ and ν_λ stand for product measures with density ρ and λ respectively. Ferrari et al. (1991) showed that there exists a shock for this system. Ferrari (1992) showed that Z_t, the position at time t of a second class particle with respect to η_t is a shock. The motion of the second class particle can be described as follows. It jumps to empty left and right sites at rate q and p respectively and interchanges positions with left and right particles at rate p and q respectively. The process $\tau_{Z_t}\eta_t$ is Markovian and under initial distribution $\nu_{\rho,\lambda}$ converges weakly to an invariant measure with asymptotic product distributions with densities ρ and λ. Write E and P for the expectation and the probability induced by the process with initial distribution $\nu_{\rho,\lambda}$. Our main result is the following

Theorem 1.1 *Assume that the process η_t has initial distribution $\nu_{\rho,\lambda}$. Let Z_t be the position of the shock given by a second class particle initially put at the origin. Then*

$$EZ_t=(p-q)(1-\lambda-\rho)t. \tag{1.1}$$

Diffusion coefficient

$$D:=\lim_{t\to 0}\frac{E(Z_t)^2-(EZ_t)^2}{t}=(p-q)\frac{\rho(1-\rho)+\lambda(1-\lambda)}{\lambda-\rho}. \tag{1.2}$$

Dependence on the initial configuration.

Let $N_t(\eta)=\sum_{x=0}^{(p-q)(\lambda-\rho)t}(1-\eta(x))-\sum_{x=-(p-q)(\lambda-\rho)t}^{0}\eta(x)$. *Then*

$$\lim_{t\to 0}\frac{1}{t}E[Z_t-(\lambda-\rho)^{-1}N_t(\eta_0)]^2=0. \tag{1.3}$$

In Chap. 5 of Spohn (1991) (1.1) was proven and (1.2) conjectured. Boldrighini et al. (1989) performed computer simulations confirming (1.2). Gärtner and Presutti (1989) showed (1.3) for $\rho=0$ and $p=1$. Ferrari (1992) showed that (1.2) and (1.3) are equivalent and that the right hand side of (1.2) is a lower bound for D and (1.3) for $\rho=0$ and all $p>q$. We show (1.1) and (1.2) using recent results relating the expected value and the variance of a tagged particle with the variance of the current of particles through a fixed or travelling position (Ferrari and Fontes (1993)). The following Theorem is a corollary to (1.3).

Theorem 1.2 *Convergence to the finite dimensional distributions of Brownian motion. Let $W(t)$ be Brownian motion with diffusion coefficient D. Then under the conditions of Theorem* 1.1

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2}(Z_{\varepsilon^{-1}.} - EZ_{\varepsilon^{-1}.}) = W(.)$$

weakly, in the sense of the finite dimensional distributions.

It is well known that this process is related to the unviscous Burgers equation. This is the non linear partial differential equation for $u(r,t) \in [0, 1]$,

$$\frac{\partial u}{\partial t} = -\theta \frac{\partial[u(1-u)]}{\partial r}. \tag{1.4}$$

We are particularly interested in solutions to this equation when the initial condition assumes only two values: $\rho < \lambda$ to the left and right of the origin respectively (shock initial condition). The solutions of (1.4) are constant along the *characteristics.* For shock initial condition the characteristic emanating from a is the straight line $a + (1 - 2u(a,0))t$ that equal $a + (1 - 2\rho)t$ for $a < 0$ and $a + (1 - 2\lambda)t$ for $a > 0$. Since $\lambda > \rho$, the characteristics emanating from the right of the origin meet those emanating from the left of it producing the shock (Lax (1972).) The (weak) solution of the Burgers equation (1.4) with initial shock condition $u(r,0) = \lambda 1\{r \geqq 0\} + \rho 1\{r < 0\}$ is just this shock translated by vt: $u(r,t) = \lambda 1\{r \geqq vt\} + \rho 1\{r < vt\}$, where $v = (p - q)(1 - \lambda - \rho)$.

The hydrodynamic limit plus local equilibrium give the following: Let $u_0(r)$ be a piecewise continuous function, and let $\nu^\varepsilon_{u_0}$ be a family of product measures with marginals $\nu^\varepsilon_{u_0}(\eta(\varepsilon^{-1}r)) = u_0(r)$. Then

$$\lim_{\varepsilon\to 0} \nu^\varepsilon_{u_0} S(\varepsilon^{-1}t)\tau_{\varepsilon^{-1}r} = \nu_{u(r,t)} \tag{1.5}$$

in the continuity points of $u(r,t)$, the unique entropy solution of (1.4) with initial condition $u(r,0) = u_0(r)$. For general initial conditions this theorem is a consequence of the law of large numbers of Rezakhanlou (1990) and the proof of local equilibrium of Landim (1992). Before them and for shock initial condition, Liggett (1975, 1977) has shown this result for the case $r = 0$. Rost (1982), Benassi and Fouque (1987) showed (1.5) for decreasing initial profiles. Andjel and Vares (1987) proved the hydrodynamical limit for the increasing case. Then Benassi et al. (1991) computed the limit for monotone initial profiles. In the shock case (1.5) means that under initial distribution $\nu_{\rho,\lambda}$, a traveller moving at deterministic velocity r observes asymptotically that the particles are distributed as ν_ρ for $r > v$ and ν_λ for $r < v$, where $v = (p - q)(1 - \lambda - \rho)$. Indeed $u(r,t) = \rho 1\{r < vt\} + \lambda 1\{r > vt\}$ is the entropy solution of the Burgers equation when $u_0(r) = \lambda$ for $r > 0$ and ρ for $r \leqq 0$. It was conjectured that when $r = v$ the system converges to a fair mixture of ν_ρ and ν_λ. This was proven by Wick (1985) and De Masi et al. (1988) for $\rho = 0$ and by Andjel et al. (1988) for $\lambda + \rho = 1$. Our next result shows the conjecture for all cases. The proof is based on the central limit theorem for Z_t established in Theorem 1.2. Let $w(r,t) = P(W(t) \leqq r) = (1/\sqrt{2\pi Dt})\int_{-\infty}^r \exp(-s^2/2Dt)ds$, the normal distribution with variance Dt.

Theorem 1.3 *Dynamical phase transition. Let $v = (p - q)(1 - \lambda - \rho)$. Then*

$$\lim_{\varepsilon\to 0} \nu_{\rho,\lambda} S(t)\tau_{vt + at^{1/2}} = (1 - w(a,1))\nu_\rho + w(a,1)\nu_\lambda\,. \tag{1.6}$$

Let Y_t^ε be the fluctuations fields defined by

$$Y_t^\varepsilon(\Phi) = \varepsilon^{1/2} \sum_{x \in \mathbb{Z}} \Phi(\varepsilon x) [\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)] , \tag{1.7}$$

where Φ is the indicator of an interval. If the initial configuration η_0 is distributed according to $\nu_{\rho,\lambda}$,

$$\lim_{\varepsilon \to 0} Y_0^\varepsilon(\Phi) = Y(\Phi) , \tag{1.8}$$

where Y is Gaussian white noise with mean zero and covariance

$$E(Y(\Psi) Y(\Phi)) = \int u_0(r)(1 - u_0(r)) \Psi(r) \Phi(r) dr, \tag{1.9}$$

where $u_0(r) = \lambda 1\{r \geqq 0\} + \rho 1\{r < 0\}$. A more intuitive description of the field Y is the following. Let $W_1(r)$ and $W_2(r)$ be Brownian motion with variances $\rho(1 - \rho)$ and $\lambda(1 - \lambda)$ respectively, where r plays the role of the time parameter. Assume that Φ is the indicator of the interval (b_1, b_2), namely $\Phi(r) = 1\{b_1 < r < b_2\}$, then

$$Y(\Phi) = \begin{cases} W_2(b_2) - W_2(b_1) & \text{if } \ 0 < b_1 < b_2 \\ W_1(-b_2) - W_1(-b_1) & \text{if } \ b_1 < b_2 < 0 \\ -W_1(-b_1) + W_2(b_2) & \text{if } \ b_1 < 0 < b_2 \end{cases} .$$

So if we define the process $W(r) = W_1(-r)1\{r \leqq 0\} + W_2(r)1\{r > 0\}$, the field Y of the interval (a_1, a_2) is just the increment of $W(r)$ in the interval (a_1, a_2).

To describe $Y_t(.)$, the limiting fields at time t, let us introduce some notation. Assume that Φ is the indicator of the interval (a_1, a_2). For $i = 1, 2$ let $b_i(t) = b_i(t, a_i)$ be defined by

$$b_i(t) = \begin{cases} a_i - (p - q)(1 - 2\rho)t & \text{if } \ a_i < vt \\ a_i - (p - q)(1 - 2\lambda)t & \text{if } \ a_i > vt \end{cases} . \tag{1.10}$$

So that $b_i(t)$ is the starting point of the characteristic that arrives at a_i at time t. Let Φ be the indicator of the interval (a_1, a_2) and assume $a_i \neq vt$. Define $B_t\Phi$ as the indicator of the interval $(b_1(t), b_2(t))$. The field at time t is defined by

$$Y_t(\Phi) = Y_0(B_t \Phi) . \tag{1.11}$$

So that the field Y_t of the interval (a_1, a_2) is the increment of $W(r)$ in the interval $(b_1(t), b_2(t))$.

Theorem 1.4 *Convergence of the fluctuation fields. Assume that the initial distribution of the process is $\nu_{\rho,\lambda}$. As $\varepsilon \to 0$, the finite dimensional distributions of the fluctuation fields $Y_\cdot^\varepsilon$ defined in* (1.7) *converge in distribution to the field $Y_\cdot$, where Y_t is defined by* (1.11). *Namely, for all $n \geqq 1$, $0 \leqq t_1 < \dots < t_n$, and Φ_i indicator of an interval whose extremes do not equal vt_i, $i = 1, \dots, n$,*

$$\lim_{\varepsilon \to 0} (Y_{t_1}^\varepsilon(\Phi_1), \dots, Y_{t_n}^\varepsilon(\Phi_n)) = (Y_{t_1}(\Phi_1), \dots, Y_{t_n}(\Phi_n))$$

in distribution.

Remark. We can interpret the result by saying that the fluctuations translate rigidly along the characteristics of the Burgers equation. When $vt \in (a_1, a_2)$ the characteristics to the right of the origin meet those coming from the left and then the fluctuations present in the interval $(-(p - q)(\lambda - \rho)t, (p - q)(\lambda - \rho)t)$ at time

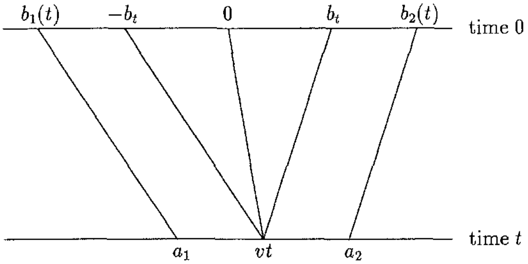

Fig. 1. The fluctuations on intervals to the right of $b_t = (p - q)(\lambda - \rho)t$ and to the left of $-b_t$ translate rigidly up to time t. Those on $(-b_t, b_t)$ concentrate at vt

zero concentrate in the point $vt = (p - q)(1 - \rho - \lambda)t$ at time t. Figure 1 illustrates this point.

Let $u(r,t) = \lambda 1\{r > vt\} + \rho 1\{r < vt\}$, the solution of the Burgers equation (1.4). Formally, Theorem 1.4 says that the fluctuation fields (1.7) converge to a weak solution Y_t of the nonhomogeneous linear equation

$$\frac{\partial}{\partial t} Y_t(r) = \frac{\partial}{\partial r}(1 - 2u(r,t))\, Y_t(r)\,, \tag{1.12}$$

with initial condition Y, the Gaussian field with zero mean and covariance given by (1.9), as conjectured by Spohn (1991).

For $p = 1$ and $\rho = 0$ the convergence away from the shock have been obtained by Benassi and Fouque (1992). Theorem 1.4 is a consequence of the L_2 convergence of the fluctuation fields established in the next theorem where we also study the fluctuations that concentrate in the point vt. Formula (1.14) below says that these fluctuations are present in the scale $\sqrt{t}$. Indeed they reflect the shock fluctuations that occur in this scale.

Theorem 1.5 *Let E be the expected value with respect to the process with initial measure $\nu_{\rho,\lambda}$. Let $A_\varepsilon = \mathbb{Z} \cap (\varepsilon^{-1}a_1, \varepsilon^{-1}a_2)$, $B_\varepsilon(t) = \mathbb{Z} \cap (\varepsilon^{-1}b_1(t), \varepsilon^{-1}b_2(t))$. Then*

$$\lim_{\varepsilon \to 0} \varepsilon E\Bigg(\sum_{x \in A_\varepsilon} [\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)] - \sum_{x \in B_\varepsilon(t)} [\eta_0(x) - E\eta_0(x)]\Bigg)^2 = 0. \tag{1.13}$$

Let $c > 0$, $C_\varepsilon(t) = \mathbb{Z} \cap (\varepsilon^{-1}vt - \varepsilon^{-1/2}c,\ \ \varepsilon^{-1}vt + \varepsilon^{-1/2}c)$ and $K_\varepsilon(t) = \mathbb{Z} \cap (-\varepsilon^{-1}t(p - q)(\lambda - \rho), \varepsilon^{-1}t(p - q)(\lambda - \rho))$. Then

$$\lim_{\varepsilon \to 0} \varepsilon E\Bigg(\sum_{x \in C_\varepsilon(t)} [\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)] - T_{\varepsilon^{-1/2}c} \sum_{x \in K_\varepsilon(t)} [\eta_0(x) - E\eta_0(x)]\Bigg)^2 = 0\,, \tag{1.14}$$

where T_c is truncation by c:

$$T_c F(.) = \begin{cases} F(.) & \text{if } |F(.)| \leqq c \\ c & \text{if } F(.) > c \\ -c & \text{if } F(.) < -c\,. \end{cases}$$

Note that $C_\varepsilon(t)$ is an interval of length proportional to $\varepsilon^{-1/2}$ around the macroscopic point vt. When $c \to \infty$, (1.14) says that the fluctuations at time t in a region of length proportional to $\sqrt{t}$ around vt are given by the fluctuations at time 0 in a region of length proportional to t.

2 Graphical construction and coupling

The main tool to deal with this process is coupling, the joint realization of two versions of the process with different initial configurations. One way to define a coupling is via the joint generator (Liggett (1976), (1985)). Another way is by a graphical construction of the process. This is something like to use the same random numbers for different initial configurations. To describe the graphical construction attach two Poisson processes to each pair of sites $(x, x+1)$. One of rate p and the other of rate q. A Poisson process is a sequence of random times. To each time of the Poisson process of rate p an arrow going from x to $x+1$ is drawn and for the times of the process of rate q an arrow is drawn from $x+1$ to x. The product of these Poisson processes induces a probability space $(\Omega, \mathscr{F}, P)$. We discard the null event "two arrows occur at the same time". Given an initial configuration η, the configuration at time t for the set of arrows ω, starting from η is denoted $\eta_t^{\eta,\omega}$ and is constructed in the following way. When an arrow appears from site x to y, if there is a particle at x and no particle at y then, after the arrow the particle will be at y and x will be empty. We denote η_t^η the random process defined on $(\Omega, \mathscr{F}, P)$ with initial configuration η.

Consider now two initial configurations η^0 and η^1 and write $\eta_t^i = \eta_t^{\eta^i}$, for the configurations at time t. Use the same structure of arrows for η_t^0 and η_t^1. In this case (η_t^0, η_t^1) is the "basic coupling" (Liggett (1985)). If $\eta^0(x) \leqq \eta^1(x)$ for all $x \in \mathbb{Z}$ (in this case we say $\eta^0 \leqq \eta^1$) then for all times $\eta_t^0 \leqq \eta_t^1$. This property is called attractivity. Let ν_ρ be the product measure with density ρ. Take $\rho < \lambda$ and realize jointly the measures ν_ρ and ν_λ in the following way. Let $U(x) \in [0, 1]$ be i.i.d. uniformly distributed random variables. Then define $\eta^0(x) = 1\{U(x) \leqq \rho\}$, $\eta^1(x) = 1\{U(x) \leqq \lambda\}$. Hence, η^0 is distributed according to ν_ρ, η^1 is distributed according to ν_λ and $\eta^0 \leqq \eta^1$. Define $\sigma(x) = \eta^0(x)$ and $\xi(x) = \eta^1(x) - \eta^0(x)$. We say that the distribution of (σ, ξ) has the good marginals if the σ marginal is ν_ρ and the $\sigma + \xi$ marginal is ν_λ. Calling π_2 the distribution of (σ, ξ), we have that

$$\pi_2 \text{ is a product measure with the good marginals .} \tag{2.1}$$

Define $\sigma_t(x) = \eta_t^0(x)$ and $\xi_t(x) = \eta_t^1(x) - \eta_t^0(x)$. The motion of (σ_t, ξ_t) obeys the following rule. The σ particles have priority over the ξ particles: when an arrow from a σ particle to a ξ particle appears, then after the arrow the particles interchange positions. Otherwise the particles interact by exclusion. We say that the ξ particles behave as "second class particles". If the distribution of (σ_0, ξ_0) has the good marginals, the same is true for the distribution of (σ_t, ξ_t). We call $S_2(t)$ the corresponding semigroup.

Let ν_2 be a translation invariant measure with the good marginals and $\nu_2' = \nu_2(.|\xi(0) = 1)$. Let X_t be the position of the ξ particle initially at the origin. Let $S_2'(t)$ be the semigroup of the process as seen from the second class particle $(\tau_{X_t}\sigma_t, \tau_{X_t}\xi_t)$. The key tool in Ferrari et al. (1991) to show that X_t is a microscopic shock is the following. If ν_2 is translation invariant and has the good marginals, then

$$(\nu_2 S_2(t))' = \nu_2' S_2'(t) . \tag{2.2}$$

In words, the law of the process as seen from the tagged second class particle looks as the law of the process seen from the origin conditioned to have a second class particle at the origin. Ferrari (1992) showed the following law of large numbers. Let ν_2 have the good marginals, then under initial measure ν'_2,

$$\lim_{t\to 0} \frac{X_t}{t} = v \text{ almost surely .} \tag{2.3}$$

Let X^1 denote the position of the first ξ particle to the right of the origin. The following technical lemma will be used later on.

Lemma 2.1 *There exist positive constants c', c'' such that*

$$\sup_{\nu_2} \nu'_2 (X^1 > n) \leqq c' \exp(-c'' n)$$

where the sup *is taken over $\{\nu_2$: ν_2 is translation invariant and has the good marginals$\}$.*

Proof. Since $\nu_2(\xi(0) = 1) = \lambda - \rho$,

$$\begin{aligned}
\nu'_2(X^1 > n)(\lambda - \rho) &\leqq \nu_2\left(\sum_{x=1}^{n} \xi(x) = 0\right) \\
&\leqq \nu_2\left(\sum_{x=1}^{n} \xi(x) = 0, \left|\sum_{x=1}^{n} \sigma(x) - n\rho\right| \leqq \varepsilon n, \left|\sum_{x=1}^{n} (\sigma(x) + \xi(x)) - n\lambda\right| \leqq \varepsilon n\right) \\
&\quad + \nu_2\left(\left|\sum_{x=1}^{n} \sigma(x) - n\rho\right| > \varepsilon n\right) + \nu_2\left(\left|\sum_{x=1}^{n} (\sigma(x) + \xi(x)) - n\lambda\right| > \varepsilon n\right)
\end{aligned} \tag{2.4}$$

For $\varepsilon < (\lambda - \rho)/2$, the first term in the right hand side of (2.4) vanishes. The second and third term depend only on the marginals ν_ρ and ν_λ respectively. The result follows from large deviations of Bernoulli measures. ♣

Using the same arrows there is a natural coupling between (σ_t, ξ_t) with initial measure π'_2 and η_t with initial measure $\nu_{\rho,\lambda}$. To describe it one let (σ, ξ) to be a configuration taken from the distribution π'_2. Now mark independently the i-th ξ particle as γ with probability $(p/q)^i/(1 + (p/q)^i)$, otherwise as ζ. Then consider the process $(\sigma_t, \gamma_t, \zeta_t)$ with priorities σ over γ over ζ. In this way σ_t has distribution ν_ρ for all t, $\eta_t = \sigma_t + \gamma_t$ has distribution (absolutely continuous with respect to) $\nu_{\rho,\lambda} S(t)$ and $\sigma_t + \gamma_t + \zeta_t$ has distribution ν_λ. See Ferrari et al. (1991) and Ferrari (1992) for details.

3 Tagged second class particles and currents

Consider the joint process (σ_t, ξ_t) described in the previous section. Define the current of ξ particles as $J_{2,t} :=$ number of ξ particles to the left of the origin at time 0 and to the right of the origin at time t minus number of ξ particles to the right of the origin at time 0 and to the left of the origin at time t. Analogously define $J_{0,t}$ for the current of σ particles and $J_{1,t}$ for the total current of $\sigma + \xi$ particles.

Consider a configuration (σ, ξ) taken from π'_2, the measure π_2 conditioned to have a ξ particle at the origin. This configuration has $\xi(0) = 1$ and $\sigma(0) = 0$, i.e., it has a ξ particle at the origin. Let $\sigma^*(x) = 1\{x \neq 0\}\sigma(x) + 1\{x = 0\}(1 - \sigma(x))$ and analogously ξ^*. Now, using the same arrows, couple (σ_t, ξ_t) with (σ_t, ξ^*_t). At time t the two processes will differ at only one site whose position is called R_t. Similarly, coupling (σ_t, ξ_t) with (σ^*_t, ξ^*_t) we get only one discrepancy located at a position denoted $\bar{R}_t$. In words, R_t is like a third class particle, while $\bar{R}_t$ is a second class particle with respect to σ_t but has priority over ξ_t.

Theorem 3.1 *Let (σ_t, ξ_t) be the joint process of first and second class particles with initial product measure π_2 defined in* (2.1). *Let X_t be the position of the tagged second class particle put initially at the origin. Then it holds that*

$$EJ_{2,t} = (\lambda - \rho)EX_t \tag{3.1}$$

where the expected values are taken with respect to the process with initial distribution π_2. Furthermore, denoting the variance by V,

$$\begin{aligned} VJ_{2,t} = (\lambda-\rho)^2 VX_t - (\lambda-\rho)(1-(\lambda-\rho))E(X_t) \\ + 2(\lambda-\rho)(1-\lambda)(E(R_t)^+ - E(R_t - X_t)^+) \\ + 2(\lambda-\rho)\rho(E(\bar{R}_t)^+ - E(\bar{R}_t - X_t)^+)\,. \end{aligned} \tag{3.2}$$

Proof. The proof or (3.1) is the same as the proof of (3.2) in Ferrari and Fontes (1993). The proof of (3.2) is very similar to the proof of (3.10) in the same paper, where the variance of the current of (first class) particles in simple exclusion is written as a function of moments of a (first class) tagged particle and a discrepancy. We just sketch it, pointing out the main different point, referring the reader to the mentioned paper for details. Write $J_{2,t} = (J_{2,t})^+ - (J_{2,t})^-$, where

$$(J_{2,t}(\sigma,\xi))^+ = \sum_{x \leqq 0} \xi(x) 1\{X_t^x(\sigma,\xi) > 0\}, \quad (J_{2,t}(\sigma,\xi))^- = \sum_{x>0} \xi(x) 1\{X_t^x(\sigma,\xi) \leqq 0\}.$$

Here $X_t^x(\sigma,\xi)$ is the position at time t of a tagged ξ particle starting at x, when the initial condition is (σ,ξ). The variance of $J_{2,t}$ is then expressed in terms of variances and expectations of $(J_{2,t})^+$ and $(J_{2,t})^-$. The main calculation which follows is that of $E((J_{2,t})^+)^2$ and $E((J_{2,t})^-)^2$. The first one is expressed in various terms one of which is

$$2(\lambda-\rho) \sum_{y<x\leqq 0} [P(X_t^y > 0, \xi(y) = 1 \mid \xi(x) = 1) - P(X_t^y > 0, \xi(y) = 1)]\,. \tag{3.3}$$

The sum in (3.3) can be rewritten as

$$\begin{aligned} &(1-\lambda) \sum_{y<x\leqq 0} [P(X_t^y > 0, \xi(y) = 1 \mid \xi(x) = 1) - P(X_t^y > 0, \xi(y) = 1 \mid \xi(x) = \sigma(x) = 0)] \\ &+ \rho \sum_{y<x\leqq 0} [P(X_t^y > 0, \xi(y) = 1 \mid \xi(x) = 1) - P(X_t^y > 0, \xi(y) = 1 \mid \sigma(x) = 1)]\,. \end{aligned} \tag{3.4}$$

These terms are reexpressed after a coupling argument as

$$(1-\lambda)(1-\lambda) \sum_{x\leqq 0} P(X_t^x > 0, R_t^x \leqq 0) + \rho(1-\lambda) \sum_{x\leqq 0} P(X_t^x > 0, \bar{R}_t^x \leqq 0)\,. \tag{3.5}$$

The expression (3.5), when combined with expressions obtained similarly in the calculation of $E((J_{2,t})^-)^2$, lead to the desired result in a straightforward manner. ♣

Theorem 3.2 *Under the conditions of Theorem 3.1, it holds that*

$$\lim_{t\to\infty} \frac{1}{t} E(J_{2,t} - N_{2,t}(\sigma_0, \xi_0) - (p-q)(\lambda^2 - \rho^2)t)^2 = 0\,, \tag{3.6}$$

where $N_{2,t}(\sigma, \xi)$ is a random variable that does not depend on ω. It depends only on the initial configurations σ and ξ and it is given below by (3.12).

Proof. By mass conservation:

$$J_{1,t} = J_{0,t} + J_{2,t}. \tag{3.7}$$

The current $J_{0,t}$ depends only on the σ marginal of the process, while $J_{1,t}$ depends on the $\sigma + \xi$ marginal. Hence, writing E for the expectation of the process with initial distribution π_2 and noting that the distribution of (σ_t, ξ_t) has the good marginals,

$$EJ_{2,t} = (p - q)(\lambda(1 - \lambda) - \rho(1 - \rho))t\,. \tag{3.8}$$

On the other hand, (1.5) of Ferrari and Fontes (1993) implies that

$$\begin{aligned} &\lim_{t\to\infty} \frac{E(J_{0,t} - N_{0,t}(\sigma_0, \xi_0) - (p - q)\rho^2 t)^2}{t} = 0\,, \\ &\lim_{t\to\infty} \frac{E(J_{1,t} - N_{1,t}(\sigma_0, \xi_0) - (p - q)\lambda^2 t)^2}{t} = 0\,, \end{aligned} \tag{3.9}$$

where

$$N_{0,t}(\sigma, \xi) = \begin{cases} \displaystyle\sum_{x=-(p-q)(1-2\rho)t}^{0} \sigma(x), & \text{when } 1 - 2\rho > 0, \\ \displaystyle -\sum_{x=0}^{(p-q)(2\rho-1)t} \sigma(x), & \text{when } 1 - 2\rho \leqq 0, \end{cases} \tag{3.10}$$

$$N_{1,t}(\sigma, \xi) = \begin{cases} \displaystyle\sum_{x=-(p-q)(1-2\lambda)t}^{0} (\sigma(x) + \xi(x)), & \text{when } 1 - 2\lambda > 0, \\ \displaystyle -\sum_{x=0}^{(p-q)(2\lambda-1)t} (\sigma(x) + \xi(x)), & \text{when } 1 - 2\lambda \leqq 0. \end{cases} \tag{3.11}$$

Define

$$N_{2,t}(\sigma, \xi) = N_{1,t}(\sigma, \xi) - N_{0,t}(\sigma, \xi). \tag{3.12}$$

The result follows from (3.7), (3.9) and (3.12). ♣

Proof of Theorem 1.1 We first show (1.1) and (1.2) for X_t instead of Z_t. It follows from (3.1) and (3.8),

$$EX_t = (p - q)(1 - \lambda - \rho)t\,. \tag{3.13}$$

From (3.10), (3.11) and (3.12), we have that $N_{2,t}(\sigma, \xi)$ equals

$$\sum_{x=-(p-q)(1-2\rho)t}^{-(p-q)(1-2\lambda)t} \sigma(x) + \sum_{x=-(p-q)(1-2\lambda)t}^{0} \xi(x), \text{ when } \lambda \leqq 1/2,$$

$$\sum_{x=-(p-q)(1-2\rho)t}^{0} \sigma(x) + \sum_{x=0}^{(p-q)(2\lambda-1)t} (\xi(x) + \sigma(x)), \text{ when } \rho \leqq 1/2 < \lambda,$$

$$\sum_{x=0}^{(p-q)(2\rho-1)t} \xi(x) + \sum_{x=(p-q)(2\rho-1)t}^{(p-q)(2\lambda-1)t} (\xi(x) + \sigma(x)), \text{ when } \rho > 1/2.$$

Hence $\lim_{t\to\infty}(VJ_{2,t}/t) = \lim_{t\to\infty}(VN_{2,t}/t)$ equals

$$\begin{aligned} &2(p-q)\rho(1-\rho)(\lambda-\rho) + (p-q)(\lambda-\rho)(1-\lambda+\rho)(1-2\lambda), \text{ when } \lambda \leqq 1/2, \\ &(p-q)(1-2\rho)\rho(1-\rho) + (p-q)(1-2\lambda)\lambda(1-\lambda), \text{ when } \rho \leqq 1/2 < \lambda, \\ &(p-q)(1-2\rho)(\lambda-\rho)(1-\lambda+\rho) + 2(p-q)(\lambda-\rho)\lambda(1-\lambda), \text{ when } \rho > 1/2. \end{aligned} \tag{3.14}$$

On the other hand, it is proven by Ferrari and Fontes (1993) that

$$\lim_{t\to\infty} \frac{E(R_t)^+}{t} = \begin{cases} (p-q)(1-2\lambda) & \text{if } \lambda < 1/2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.15}$$

$$\lim_{t\to\infty} \frac{E(\bar{R}_t)^+}{t} = \begin{cases} (p-q)(1-2\rho) & \text{if } \rho < 1/2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.16}$$

$$\lim_{t\to\infty} \frac{E(R_t - X_t)^+}{t} = 0 \tag{3.17}$$

and

$$\lim_{t\to\infty} \frac{E(\bar{R}_t - X_t)^+}{t} = (p-q)(\lambda-\rho)\,. \tag{3.18}$$

Substituting (3.13), (3.14), (3.15), (3.16), (3.17) and (3.18) in (3.2) we get

$$\lim_{t\to\infty} \frac{E(X_t)^2 - (EX_t)^2}{t} = (p-q)\frac{\rho(1-\rho) + \lambda(1-\lambda)}{\lambda-\rho}. \tag{3.19}$$

Now we show the theorem for Z_t. We consider the process (η_t, Z_t), where Z_t is a second class particle with respect to η_t. Ferrari (1992) has shown that it is possible to realize the processes (η_t, Z_t) and (σ_t, ξ_t, X_t) with initial distribution π'_2, in such a way that if one calls X_t^i the i-th ξ particle ($X_t^0 = X_t$), and let $\mathscr{F}_{2,t}$ be the sigma algebra generated by $\{(\sigma_s, \xi_s)\colon s \leqq t\}$, then for all times

$$P(Z_t = X_t^i \mid \mathscr{F}_{2,t}) = m(i), \text{ where}$$

$$m(i) = M\left(\left(1 + (p/q)^{i-1/2}\right)\left(1 + (q/p)^{i+1/2}\right)\right)^{-1} \tag{3.20}$$

and M is a normalizing constant making $\sum_{i\in\mathbb{Z}} m(i) = 1$. The symmetry of $m(i)$, (3.13) and (3.20) show (1.1). Since $m(i)$ is a probability with exponential decay, to show (1.2) it suffices to prove that

$$\lim_{t\to\infty} \frac{E(X_t - X_t^i)^2}{t} = 0, \quad \text{for all } i \in \mathbb{Z}\,. \tag{3.21}$$

But (3.21) follows from Lemma 2.1 and the fact that, by translation invariance, the law of $X_t^{i-1} - X_t^i$ is independent of i. Ferrari (1992) showed that (1.2) and (1.3) are equivalent. ♣

Remark 3.1 Note that (3.14) implies that $\lim_{t\to\infty}(VJ_{2,t}/t) = 0$ if $\lambda + \rho = 1$. In this case $v = 0$. We do not use this.

We finish this section with a lemma to be used in Sect. 5. Let $J_t^{b,a}$ be the number of η particles to the left of b at time zero and to the right of a at time t minus number of

η particles to the right of b at time zero and to the left of a at time t. Let $J_{i,t}^{b,a}$ be the analogous current for particles σ, $\sigma + \xi$ and ξ for $i = 0,1,2$ respectively.

Lemma 3.1 *Consider the process η_t with initial distribution $\nu_{\rho,\lambda}$ and the process (σ_t, ξ_t) under initial distribution π_2 coupled as described at the end of Sect. 2. If $b > 0$ and $a > v$ then*

$$\lim_{t\to\infty} \frac{E(J_t^{bt,at} - J_{1,t}^{bt,at})^2}{t} = 0 . \tag{3.22}$$

If $b < 0$ and $a < v$ then

$$\lim_{t\to\infty} \frac{E(J_t^{bt,at} - J_{0,t}^{bt,at})^2}{t} = 0 . \tag{3.23}$$

Proof. First consider $p = 1$. For $b > 0$ and $a > v$, $J_t^{bt,at} - J_{1,t}^{bt,at} = (J_{2,t}^{0,at})^+ \leqq (X_t - at)^+ \leqq (X_t - vt)^+$. By (2.3) $\lim_{t\to\infty} P(((X_t - at)^+)^2/t > s) = 0$ for all $s \geqq 0$. By Theorem 1.2, $\lim_{t\to\infty} P((X_t - vt)^2/t > s) = 2(1 - w(\sqrt{s}, 1))$. Write

$$\frac{E((X_t - at)^+)^2}{t} = \int_0^\infty P((X_t - at)^+/\sqrt{t} > \sqrt{s})ds .$$

Now $P((X_t - at)^+/\sqrt{t} > \sqrt{s}) \leqq P((X_t - vt)/\sqrt{t} > \sqrt{s})$ and $\lim_{t\to\infty} \int_0^\infty P((X_t - vt/\sqrt{t}) > \sqrt{s})ds = \int_0^\infty 2(1 - w(\sqrt{s}, 1))ds = D < \infty$ by Theorem 1. By dominated convergence we get (3.22). Analogously we get (3.23). If $1 > p > q$ one repeats the argument using D_t, the position of the rightmost ζ particle and the fact that $\lim_{t\to\infty} E(D_t - vt)^2/t < \infty$. For (3.23) one uses G_t, the position of the leftmost γ particle. ♣

Remark 3.2 Since for the process (σ_t, ξ_t) the σ marginal is ν_ρ and the $\sigma + \xi$ marginal is ν_λ, it follows from Ferrari and Fontes (1993) that

$$\lim_{t\to\infty} \frac{E(J_{i,t}^{bt,at} - EJ_{i,t}^{bt,at})^2}{t} = \begin{cases} (p-q)\rho(1-\rho)|1 - 2\rho - (a-b)| & \text{if } i = 0 \\ (p-q)\lambda(1-\lambda)|1 - 2\lambda - (a-b)| & \text{if } i = 1. \end{cases} \tag{3.24}$$

4 Dynamical phase transition

In order to prove Theorem 1.3 we need the following result.

Lemma 4.1 *Weak limits as $t \to \infty$ of $\nu_{\rho,\lambda} S(t)\tau_{vt+a\sqrt{t}}$ are translation invariant.*

Proof. Since $\tau_1 \nu_{\rho,\lambda} S(t) = \nu_{\rho,\lambda} S(t)\tau_1$, it suffices to show

$$\lim_{t\to\infty} |\tau_1 \nu_{\rho,\lambda} S(t)\tau_{vt+a\sqrt{t}} f - \nu_{\rho,\lambda} S(t)\tau_{vt+a\sqrt{t}} f| = 0 \tag{4.1}$$

for any cylinder f. Since the measures $\nu_{\rho,\lambda}$ and $\tau_1 \nu_{\rho,\lambda}$ are product, we can construct a product measure $\tilde{\nu}$ on $\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$ with marginals $\nu_{\rho,\lambda}$ and $\tau_1\nu_{\rho,\lambda}$ in such a way that if (η, η^*) is distributed according to $\tilde{\nu}$, then $\eta(x) = \eta^*(x)\ \forall x \neq 0$ and

$$\tilde{\nu}[\eta(0) = \eta^*(0)] = 1 - (\lambda - \rho), \quad \tilde{\nu}[\eta(0) = 1, \eta^*(0) = 0] = \lambda - \rho . \tag{4.2}$$

Then we construct the coupled process (η_t, η_t^*) with initial distribution $\tilde{\nu}$, using the same arrows. If $\eta(0) \neq \eta^*(0)$, then the processes η_t and η_t^* differ at most in one site. The position of this discrepancy behaves as a second class particle with respect to η. We call it Z_t. At time 0, $Z_0 = 0$. If f depends on sites $\{-k, \ldots, k\}$, the expression inside the limit in (4.1) is bounded above by

$$(\lambda - \rho)\|f\|_\infty P(|Z_t - vt - a\sqrt{t}| \leqq k) . \tag{4.3}$$

The probability in (4.3) converges to zero as $t \to \infty$, by the convergence of $(Z_t - vt)/\sqrt{t}$ to a normal random variable with nonzero diffusion coefficient D, proven in Theorem 1.2. ♣

Proof of Theorem 1.3 We first show the result for $p = 1$. In this case $Z_t \equiv X_t$. To avoid heavy notation we prove the theorem for $a = 0$. The extension is straightforward. Assume f depends on the sites $\{-k, \ldots, k\}$. Then by Lemma 4.1 we have (along convergent subsequences)

$$\lim_{t\to\infty} \nu_{\rho,\lambda} S(t)\tau_{vt} f = \frac{1}{2n+1} \lim_{t\to\infty} \nu_{\rho,\lambda} S(t)\tau_{vt} \sum_{x=-n}^{n} \tau_{(2k+1)x} f . \tag{4.4}$$

for all $n \geqq 0$. We choose to translate by $(2k+1)x$ because in that way the support of $\tau_{(2k+1)x} f$ is disjoint of the support of $\tau_{(2k+1)y} f$ if $x \neq y$. To compute the second limit in (4.4) write $x' = (2k+1)x$ and

$$\begin{aligned}
\frac{1}{2n+1} \nu_{\rho,\lambda} S(t)\tau_{vt} \sum_{x=-n}^{n} \tau_{(2k+1)x} f &= \frac{1}{2n+1} E\left[\tau_{vt} \sum_{x=-n}^{n} \tau_{x'} f(\eta_t)\right] \\
&= \frac{1}{2n+1} E\left[\tau_{vt} \sum_{x=-n}^{n} \tau_{x'} f(\eta_t) 1\{X_t - vt > t^{1/4}\}\right] \\
&\quad + \frac{1}{2n+1} E\left[\tau_{vt} \sum_{x=-n}^{n} \tau_{x'} f(\eta_t) 1\{X_t - vt < -t^{1/4}\}\right] \\
&\quad + \frac{1}{2n+1} E\left[\tau_{vt} \sum_{x=-n}^{n} \tau_{x'} f(\eta_t) 1\{|X_t - vt| \leqq t^{1/4}\}\right] \\
&= I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{4.5}$$

By Theorem 1.2 $\lim_{t\to\infty} I_3(t) = 0$. Couple η_t with initial distribution $\nu_{\rho,\lambda}$ and (σ_t, ξ_t) with initial distribution π_2 as described at the end of Sect. 2. For $t^{1/4} > n(2k+1)$, since $p = 1$,

$$I_1(t) = \frac{1}{2n+1} E\left[\tau_{vt} \sum_{x=-n}^{n} \tau_{x'} f(\sigma_t) 1\{X_t - vt > t^{1/4}\}\right] .$$

Now,

$$\begin{aligned}
&\left| I_1(t) - E[\nu_\rho f\, 1\{X_t - vt > t^{1/4}\}]\right|^2 \\
&\leqq \left| E\left[\left(\tau_{vt} \frac{1}{2n+1} \sum_{x=-n}^{n} \tau_{x'} f(\sigma_t) - \nu_\rho f\right) 1\{X_t - vt > t^{1/4}\}\right]\right|^2 \\
&\leqq E\left[\tau_{vt} \frac{1}{2n+1} \sum_{x=-n}^{n} (\tau_{x'} f(\sigma_t) - \nu_\rho f)\right]^2 .
\end{aligned} \tag{4.6}$$

But $\{\tau_{(2k+1)x} f(\sigma_t)\}_x$ are i.i.d. with distribution induced by ν_ρ, hence the r.h.s. of (4.6) does not depend on translations by vt and equals $\nu_\rho(f - \nu_\rho f)^2/(2n+1)$. By Theorem 1.2 (central limit theorem for X_t) $\lim_{t\to\infty} E[1\{X_t - vt > t^{1/4}\}] = 1/2$. Hence

$$\left| \lim_{t\to\infty} I_1(t) - \frac{1}{2}\nu_\rho f \right| \leqq O\left(\frac{1}{\sqrt{n}}\right).$$

Analogously,

$$\left| \lim_{t\to\infty} I_2(t) - \frac{1}{2}\nu_\lambda f \right| \leqq O\left(\frac{1}{\sqrt{n}}\right).$$

We get (1.6) for $p = 1$ and $a = 0$ by taking n to infinity. To obtain the result for $a \neq 0$ it suffices to make a partition inside the expectation in (4.5) according to $\{X_t - vt > at^{1/2} + t^{1/4}\}$, $\{X_t - vt < at^{1/2} - t^{1/4}\}$ and $\{|X_t - vt - at^{1/2}| \leqq t^{1/4}\}$ and observe that by Theorem 1.2, $P(X_t - vt < at^{1/2} - t^{1/4}) \to w(a, 1)$, the normal distribution with variance D defined before Theorem 1.3. The proof goes then along the same steps than in the case $a = 0$. In the case $p \in (1/2, 1)$ one uses the three particle representation of the system given at the end of Sect. 2. By (3.21), G_t, the position of the leftmost γ particle and D_t, the position of the rightmost ζ particle at time t satisfy (1.2), (1.3) and Theorem 1.2. From this it is not difficult to construct an argument similar to the case $p = 1$ to show (1.6) for all cases. ♣

5 Fluctuation fields

In this section we show the convergence of the density fluctuation fields in the case of a shock. We first prove Theorem 1.5 and then Theorem 1.4. The proof of (1.13) is based on the fact that the variance of the current through certain lines parallel to $(t(1-2\rho), t)$ and $(t(1-2\lambda), t)$ vanishes. The proof of (1.14) is based in Theorems 1.1–1.3.

Proof of Theorem 1.5 We first show (1.13). Since the number of particles can change only on the boundaries,

$$\sum_{x\in A_\varepsilon} \eta_{\varepsilon^{-1}t}(x) - \sum_{x\in B_\varepsilon} \eta_0(x) = J_t^{\varepsilon^{-1}b_1(t), \varepsilon^{-1}a_1} - J_t^{\varepsilon^{-1}b_2(t), \varepsilon^{-1}a_2}, \tag{5.1}$$

where $J_t^{\varepsilon^{-1}b_i(t), \varepsilon^{-1}a_i}$ has been defined before Lemma 3.1. By (3.24) and Lemma 3.1,

$$\lim_{\varepsilon\to 0} \varepsilon E(J_t^{\varepsilon^{-1}b_i(t), \varepsilon^{-1}a_i} - E J_t^{\varepsilon^{-1}b_i(t), \varepsilon^{-1}a_i})^2 = 0, \quad i = 1, 2. \tag{5.2}$$

Then (1.13) is a consequence of (5.1) and (5.2). Now we show (1.14). We prove below that

$$\lim_{\varepsilon\to 0} E\left(\varepsilon^{1/2} \sum_{x\in C_\varepsilon(t)} \eta_{\varepsilon^{-1}t}(x) - (\lambda(c - T_c W_\varepsilon(t)) + \rho(T_c W_\varepsilon(t) + c))\right)^2 = 0\,, \tag{5.3}$$

where $W_\varepsilon(t) = \varepsilon^{1/2}(Z_{\varepsilon^{-1}t} - \varepsilon^{-1}vt)$ and T_c is truncation by c defined in Theorem 1.5. Since $\sum_{x\in K_\varepsilon(t)} E\eta_0(x) = \varepsilon^{-1}vt$, (1.3) implies

$$\lim_{\varepsilon\to 0} E\left(\varepsilon^{1/2} T_{\varepsilon^{-1/2}c} \sum_{x\in K_\varepsilon(t)} (\eta_0(x) - E\eta_0(x)) - (\lambda - \rho)T_c W_\varepsilon(t)\right)^2 = 0\,. \tag{5.4}$$

By Theorem 1.3,

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2} \sum_{x\in C_\varepsilon(t)} E\eta_{\varepsilon^{-1}t}(x) = \int_{-c}^{c} (\rho(1-w(r,t)) + \lambda w(r,t))dr\,, \tag{5.5}$$

where $w(r,t)$ is defined in Theorem 1.3. Finally, by symmetry of $(r,\ w(r,t))$ with respect to $(0,\ 1/2)$,

$$\int_{-c}^{c} (\rho(1-w(r,t)) + \lambda w(r,t))dr - (\lambda(c - T_cW_\varepsilon(t)) + \rho(T_cW_\varepsilon(t)+c))$$
$$= (\lambda - \rho)T_cW_\varepsilon(t)\,. \tag{5.6}$$

Then (1.14) follows from (5.3), (5.4), (5.5) and (5.6). ♣

Proof of (*5.3*) We first show it for $p = 1$. Let $C_\varepsilon^-(t) = [-\varepsilon^{-1/2}(T_cW_\varepsilon(t)+c),\ 0]\cap\mathbb{Z}$, $C_\varepsilon^+(t) = [0, \varepsilon^{-1/2}(c - T_cW_\varepsilon(t))]\cap\mathbb{Z}$. Use the coupling described at the end of Sect. 2. Let $\eta'_t = \tau_{X_t}\eta_t$, $\sigma'_t = \tau_{X_t}\sigma_t, \xi'_t = \tau_{X_t}\xi_t$. Then,

$$\sum_{x\in C_\varepsilon(t)} \eta_{\varepsilon^{-1}t}(x) = \sum_{x\in C_\varepsilon^-(t)} \eta'_{\varepsilon^{-1}t}(x) + \sum_{x\in C_\varepsilon^+(t)} \eta'_{\varepsilon^{-1}t}(x)$$
$$= \sum_{x\in C_\varepsilon^-(t)} \sigma'_{\varepsilon^{-1}t}(x) + \sum_{x\in C_\varepsilon^+(t)} (\sigma'_{\varepsilon^{-1}t}(x) + \xi'_{\varepsilon^{-1}t}(x)). \tag{5.7}$$

The first marginal of $(\sigma_t,\ \xi_t)$ is ν_ρ for all t and $|C_\varepsilon^-(t)| \leqslant 2c\varepsilon^{-1}$. Hence, by (2.2),

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2} \sum_{x\in C_\varepsilon^-(t)} \sigma'_{\varepsilon^{-1}t}(x) - \rho(T_cW_\varepsilon(t)+c)) = 0 \text{ a.s.} \tag{5.8}$$

Then (5.8) and dominated convergence imply

$$\lim_{\varepsilon\to 0} E\left(\varepsilon^{1/2} \sum_{x\in C_\varepsilon^-(t)} \sigma'_{\varepsilon^{-1}t}(x) - \rho(T_cW_\varepsilon(t)+c)\right)^2 = 0\,.$$

Analogously

$$\lim_{\varepsilon\to 0} E\left(\varepsilon^{1/2} \sum_{x\in C_\varepsilon^+(t)} (\sigma'_{\varepsilon^{-1}t}(x) + \xi'_{\varepsilon^{-1}t}(x)) - \lambda(c - T_cW_\varepsilon(t))\right)^2 = 0\,.$$

We leave to the reader the proof for $p\in(1/2,\ 1)$. ♣

Proof of Theorem 1.4 Let Φ be the indicator of the interval $(a_1,\ a_2)$ and $B_t\Phi$ be the indicator of the interval $(b_1(t),\ b_2(t))$ as defined in (1.10). By Theorem (1.5) we have that for any fixed time t as $\varepsilon\to 0$, the fluctuation fields $Y_t^\varepsilon(\Phi)$ converges in $L^2(P)$ to $Y(B_t\Phi)$, the Gaussian field with covariance (1.9). It is immediate to extend this convergence to the finite dimensional distributions: Let $t_1 < \ldots < t_n$. Then (1.13) implies

$$\lim_{\varepsilon\to 0} \varepsilon E\left(\max_{i\in\{1,\ldots,n\}} |Y_{t_i}^\varepsilon(\Phi_i) - Y(B_{t_i}\Phi_i)|\right)^2 = 0.$$

This implies in particular the weak convergence claimed in Theorem 1.4. ♣

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