Probab. Theory Relat. Fields 99, 237–250 (1994)

Probability Theory and Related Fields

Asymptotic limit law for the close approach of two trajectories in expanding maps of the circle

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Received: 21 May 1993/In revised form: 8 December 1993

Summary. Given two points $x, y \in S^1$ randomly chosen independently by a mixing absolutely continuous invariant measure μ of a piecewise expanding and smooth map f of the circle, we consider for each $\varepsilon > 0$ the point process obtained by recording the times $n > 0$ such that $|f^n(x) - f^n(y)| \leqq \varepsilon$. With the further assumption that the density of μ is bounded away from zero, we show that when ε tends to zero the above point process scaled by ε^{-1} converges in law to a marked Poisson point process with constant parameter measure. This parameter measure is given explicitly by an average on the rate of expansion of f.

Mathematics Subject Classification: 60F05, 60G55, 34C35

I Introduction

Consider the times when two orbits of a circle mapping get ε close. If the base points are randomly chosen independently by an invariant probability measure of the map considered, we obtain a point process. We are interested in the asymptotic limit law when ε tends to zero for the above process (when scaled by ε^{-1}). In the following we give the precise definitions and statement of results.

Let $f: S^1 \to S^1$ be a piecewise expanding and smooth map of the circle, i.e. there exists a partition $\mathscr{A}$ of the circle given by $0 \leqq a_0 < \cdots < a_r < 1$, such that f is smooth on each open interval (a_{j-1}, a_j) and there exist a power $m \geqq 1$ and a number $\rho > 1$ satisfying ess sup $|(f^m)'|^{1/m} > \rho$. We shall assume f is topologically mixing and hence f admits a unique absolutely continuous invariant measure μ (cf. [LY, HK]). The density of μ will be denoted by $h(x)$ and we recall that h is a function of bounded variation.

* Partially supported by FAPESP grant number 90/3918-5

Let the torus be denoted by $\mathcal{T}^2 = S^1 \times S^1$ and let $\bar{\mu}$ be the product measure $\mu \times \mu$. Define the point process $\tau^\varepsilon : \mathcal{T}^2 \to \mathcal{M}_\sigma[0, \infty)$ by

$$\tau^\varepsilon(x, y) = \sum_{\{n>0: |f^n(x) - f^n(y)| \leqq \varepsilon\}} \delta_{n\varepsilon} ,$$

where $\mathcal{M}_\sigma[0, \infty)$ denotes the σ-finite measures on $[0, \infty)$ and $\delta_{n\varepsilon}$ denotes Dirac measure at the point $n\varepsilon$. If we consider the product map $F : \mathcal{T}^2 \to \mathcal{T}^2$ given by $F(x, y) = (f(x), f(y))$ and define the ε-neighbourhood of the diagonal by

$$\Delta_\varepsilon = \{(x, y) \in \mathcal{T}^2 : |x - y| \leqq \varepsilon\} ,$$

then τ^ε can be written as

$$\tau^\varepsilon(\omega) = \sum_{n>0} \chi_{\Delta_\varepsilon}(F^n(\omega)) \delta_{n\varepsilon} ,$$

where $\chi_{\Delta_\varepsilon}$ denotes the indicator function of Δ_ε. The latter expression defines the process of visits to an ε-neighbourhood of the diagonal. Similarly

$$\tau_e^\varepsilon(\omega) = \sum_{n>0} \chi_{\Delta_\varepsilon}(F^n(\omega)) \chi_{\Delta_\varepsilon^c}(F^{n-1}(\omega)) \delta_{n\varepsilon} ,$$

defines the process of entrances to an ε-neighbourhood of the diagonal.
Let g be a continuous function with compact support on $[0, \infty)$. Following [Ne], we integrate g by the point process τ^ε to obtain the random variable

$$N_\varepsilon(g)(\omega) = \sum_{n>0} \chi_{\Delta_\varepsilon}(F^n(\omega)) g(n\varepsilon) .$$

We recall that convergence in law of $N_\varepsilon(g)$ for every g implies converge in law of the point process τ^ε (cf. [Ne]). We shall be using the following notation for the expectation with respect to the measure $\bar{\mu}$,

$$\langle N_\varepsilon(g) \rangle = \int N_\varepsilon(g)(\omega)\, d\bar{\mu}(\omega) .$$

We now formulate our main result.

Theorem. *Let f be a piecewise expanding and smooth map of the circle with a unique and mixing absolutely continuous invariant measure which has a density bounded away from zero. There are a positive number λ and a probability measure on the positive integers $\pi = \{\pi_k\}_{k>0}$ such that for any continuous non-negative function g with compact support on $[0, \infty)$, the random variable $N_\varepsilon(g)$ converges in law when ε tends to zero to a random variable $X(g)$ whose characteristic function is given by*

$$\langle e^{i\xi X(g)} \rangle = e^{\lambda \sum_{k=1}^{\infty} \pi_k \int_0^\infty (e^{i\xi k g(y)} - 1) dy} . \tag{1}$$

The general expression of λ and π_k in terms of the density of the absolutely continuous invariant measure and in terms of the expansion of f is given at the end of the paper. For the special case when f is uniformly expanding (i.e. ess sup $|f'| > \rho > 1$) the expressions simplify to

$$\lambda = 2 \int h^2(x) \left[1 - \frac{1}{|f'(x)|} \right] dx ;$$

for $k = 1$,

$$\lambda \pi_1 = 2 \int h^2(x) \left[1 - \frac{2}{|f'(x)|} + \frac{1}{|(f^2)'(x)|} \right] dx ;$$

and for $k>1$,

$$\lambda\pi_k=2\int h^2(x)\left[\frac{1}{|(f^{k-1})'(x)|}-\frac{2}{|(f^k)'(x)|}+\frac{1}{|(f^{k+1})'(x)|}\right]dx\,.$$

The proof of the result implies, through an indirect argument, that the above quantities are non-negative. It would be interesting to find a direct proof of this fact using basic properties of piecewise expanding maps of the circle.

In the particular case of the β-transformations, i.e. $f(x)=\beta x(\text{mod }1)$ with $\beta>1$, which are used in most "random" number generators, we obtain

$$\lambda=2c(\beta-1)\beta^{-1} \quad \text{and} \quad \pi_k=(\beta-1)\beta^{-k}$$

for $k\geqq 1$, where $c=\int h^2(x)\,dx$.

Now in order to interpret the limiting process we recall from [DV] the definition of a process whose characteristic function is given by (1). Consider an independent sequence of non-negative random variables $X_1, X_2, \ldots$ such that X_1 and $X_{n+1}-X_n$ for every $n\geqq 1$ have Poisson distribution of density λ and consider an independent sequence of positive integer valued random variables $K_1, K_2, \ldots$ such that $\mathbb{P}\{K_n=k\}=\pi_k$ for every $n, k\geqq 1$ and K_n independent of X_j for every n and j. Then (1) is the characteristic function of the point process

$$\tau(\omega)=\sum_{n>0} K_n(\omega)\,\delta_{X_n(\omega)}\,.$$

The above is the definition of a marked Poisson point process with constant parameter measure (cf. [DV]). Therefore we obtain the following immediate consequence of the Theorem.

Corollary I *Under the hypotheses of the Theorem, the process of successive visits to an* ε*-neighbourhood of the diagonal scaled by* ε^{-1} *converges in law to a marked Poisson point process with constant parameter measure* $\lambda\pi$.

Using a general description of the limiting process given in [DV], the next result also follows from the Theorem.

Corollary II *Under the hypotheses of the Theorem, the process of successive entrances to an* ε*-neighbourhood of the diagonal scaled by* ε^{-1} *converges in law to a Poisson point process with density* λ.

We should note that Poisson limit laws have been established for processes of visits to a set (when the measure of the set tends to zero) in various contexts. For Markov chains (and hyperbolic automorphisms of the torus) Pitskel [Pi] proves that given a sequence of cylinder sets in a neighbourhood basis of a given point with the measure tending to zero, the process of visits to each cylinder set converges in law to a Poisson point process of density 1 when the process is normalised by the measure of the cylinder (for almost every base point). Independently, Hirata [Hi] proves this result for a shift of finite type with a stationary equilibrium state of a Hölder continuous function. In the context of the present paper, i.e. for a piecewise expanding and smooth map of the circle with an absolutely continuous invariant measure, Collet and Galves [CG] prove a Poisson limit law of density 1 for the process of visits to a sequence of intervals with diverging time of self-intersection, with the process being normalised by the measure of the interval.

Here we prove that a similar (but different) regime occurs when we consider the close approach of two trajectories of such a map.

The ideas involved in the proof of the theorem follow the technique developed in [CG]. First we show convergence of the factorial moments of $N_\varepsilon(g)$ and then identify the limit as the derivatives at the origin of an analytic function. This analytic function is shown to have an analytic extension to a half-plane containing the origin which is then the Laplace transform of the desired marked Poisson point process.

II Convergence of factorial moments

The main property of such a piecewise expanding map f, which we will use in the sequel, is the exponential decay of correlations, i.e. there exist $C>0$ and $0<\gamma<1$ such that for every $u, v \in L^1(\mu)$ with u of bounded variation we have

$$\left|\int u\, v\circ f^n\, d\mu - \int u\, d\mu \int v\, d\mu\right| \leqq C\gamma^n\left(\bigvee(u)+\int |u(x)|\, dx\right)\int |v(x)|\, dx\,, \qquad (2)$$

where $\bigvee(u)$ denotes the variation of u (cf. [HK]).

Let $\mathscr{A}$ be the defining partition of f and denote by $\mathscr{C}$ the critical set of f, i.e. the set of points $x\in S^1$ such that x or $f(x)$ belongs to $\{a_0, \ldots, a_r\}$. For $n>0$, define the partition $\mathscr{A}_n$ whose atoms are sets of the form $\bigcap_{j=0}^n f^{-j}(I_{i_j})$, where each I_{i_j} belongs to $\mathscr{A}$.

Let $\delta_1(n)$ be the smallest diameter of the atoms of the partition $\mathscr{A}_n$. Let $\delta_2(n)$ be the smallest distance between the points in $\bigcup_{j=0}^n f^j(\mathscr{C})$. These two functions δ_1 and δ_2 are nonincreasing and δ_1 tends to zero when n tends to infinity. For ε small enough, we denote by $l(\varepsilon)$ the largest integer $N<\sqrt{-\log\varepsilon}$ such that

$$\min\{\delta_1(N), \delta_2(N)\} > \sqrt{\varepsilon}\,.$$

Note that this implies that $l(\varepsilon)$ diverges when ε tends to zero.

We will first estimate the variation in the horizontal (and vertical) direction of the characteristic function of sets of the form

$$\bigcap_{j=0}^{s} F^{-i_j}\Delta_\varepsilon\,,$$

where $i_0=0<i_1<\cdots<i_s$ is an increasing sequence of numbers which will appear in the proof of the main result. If φ is a function on the torus we shall denote respectively by

$$\bigvee_1 \varphi(\cdot, y) \quad \text{and} \quad \bigvee_2 \varphi(x, \cdot)\,,$$

the variation of the function $\varphi_y(x)=\varphi(x,y)$ for fixed y and the variation of the function $\varphi_x(y)=\varphi(x,y)$ for fixed x.

Lemma 1 *If the increasing sequence of numbers $i_0=0<i_1<\cdots<i_s$ satisfies $i_{j+1}-i_j<l(\varepsilon)/s$ for $0\leqq j<s$, and ε is sufficiently small, then*

$$\sup_y \bigvee_1 \chi_{\bigcap_{j=0}^s F^{-i_j}\Delta_\varepsilon}(\cdot, y) \leqq 6(s+1)$$

and

$$\sup_x \bigvee_2 \chi_{\bigcap_{j=0}^{s} F^{-i_j}\Delta_\varepsilon}(x,\cdot) \leqq 6(s+1) .$$

Proof. If suffices to prove the first inequality since the second one is analogous. For $s=0$ the result is obvious and we observe that for $s \geqq 1$

$$\chi_{\bigcap_{j=0}^{s} F^{-i_j}\Delta_\varepsilon} = \prod_{j=1}^{s} \chi_{\Delta_\varepsilon \cap F^{-i_j}\Delta_\varepsilon} .$$

Since

$$\bigvee (g_1 g_2) \leqq \| g_2 \|_\infty \bigvee g_1 + \| g_1 \|_\infty \bigvee g_2 ,$$

the result will follow from the estimate

$$\bigvee_1 \chi_{\Delta_\varepsilon \cap F^{-i}\Delta_\varepsilon}(\cdot, y) \leqq 6 ,$$

provided $i < l(\varepsilon)$ and ε is small enough. The following is devoted to the proof of this inequality.

First we divide the diagonal into the upper and lower parts, namely

$$\Delta_\varepsilon = \Delta_\varepsilon^+ \cup \Delta_\varepsilon^- ,$$

where

$$\Delta_\varepsilon^+ = \Delta_\varepsilon \cap \{(x,y) | x \leqq y\} \quad \text{and} \quad \Delta_\varepsilon^- = \Delta_\varepsilon \cap \{(x,y) | x \geqq y\} .$$

We have

$$\Delta_\varepsilon \cap F^{-i}\Delta_\varepsilon = \bigcup_{I, J \in \mathscr{A}_i} ((I \times J) \cap (\Delta_\varepsilon^+ \cup \Delta_\varepsilon^-) \cap F^{-i}\Delta_\varepsilon) .$$

We now observe that because of our choice of $l(\varepsilon)$ any horizontal line meets at most three sets of the form $(I \times J) \cap \Delta_\varepsilon$ with I and J in $\mathscr{A}_i$. It is easy to verify that the nonempty sets $(I \times J) \cap \Delta_\varepsilon^-$ and $(I \times J) \cap \Delta_\varepsilon^+$ are either triangles or trapezes (the latter with $I = J$). We shall need a precise description of the sets $(I \times J) \cap \Delta_\varepsilon^- \cap F^{-i}\Delta_\varepsilon$ (similar arguments can be developed for the sets $(I \times J) \cap \Delta_\varepsilon^+ \cap F^{-i}\Delta_\varepsilon$).

An important remark is that F^i restricted to any of the above triangles or trapezes is a diffeomorphism onto its image.

We will first analyse the triangles. The triangles contained in Δ_ε^- are of the form

$$T_z = \{(x,y) | x - \varepsilon \leqq y \leqq z \leqq x\} ,$$

where z is a boundary point of the partition $\mathscr{A}_i$. For each triangle T_z there are several possibilities according to the behaviour of the map f^i at the point z.

When f^i is discontinuous at z, it follows from our choice of $l(\varepsilon)$ that

$$T_z \cap F^{-i}\Delta_\varepsilon = \emptyset .$$

When f^i is continuous at z the most involved case is when the two branches of f^i which meet at z have slopes of opposite signs. We will discuss in detail the case where f^i is increasing on the left of z (and decreasing on the right). The opposite case can be treated similarly.

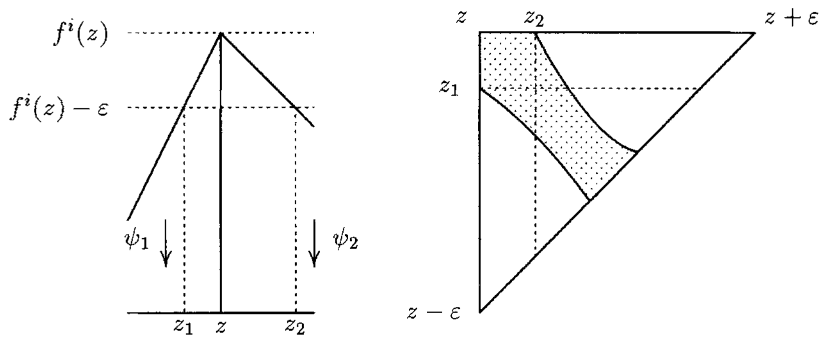

Fig. 1

We will denote by ψ_1 and ψ_2 the local inverse maps of f^i on the left and the right of z respectively (see the picture). Let $z_1=\psi_1(f^i(z)-\varepsilon)$, and $z_2=\psi_2(f^i(z)-\varepsilon)$.
We first describe the set $T_z\cap F^{-i}\Delta_\varepsilon\cap\{(x,y)|z\leqq x\leqq z_2\}$ which is equal to

$$T_z\cap\{(x,y)|z\leqq x\leqq z_2,\quad f^i(x)-\varepsilon\leqq f^i(y)\leqq f^i(z)\}\ .$$

Since ψ_1 is monotone increasing the above set is equal to

$$T_z\cap\{(x,y)|z\leqq x\leqq z_2,\quad \psi_1(f^i(x)-\varepsilon)\leqq y\leqq z\}\ . \tag{3}$$

For $z\leqq x\leqq z_2$, the curve $x\mapsto\psi_1(f^i(x)-\varepsilon)$ is monotone decreasing with a slope bounded away from zero and infinity. Therefore the intersection of the set (3) with any horizontal line is a compact interval if it is nonempty.
We next describe the set $T_z\cap F^{-i}\Delta_\varepsilon\cap\{(x,y)|z_2\leqq x\leqq z+\varepsilon\}$ which is equal to

$$T_z\cap\{(x,y)|z_2\leqq x\leqq z+\varepsilon,\quad f^i(x)-\varepsilon\leqq f^i(y)\leqq f^i(x)+\varepsilon\}\ .$$

As above, this set is equal to

$$T_z\cap\{(x,y)|z_2\leqq x\leqq z+\varepsilon,\quad \psi_1(f^i(x)-\varepsilon)\leqq y\leqq\psi_1(f^i(x)+\varepsilon)\}\ . \tag{4}$$

For $z_2\leqq x\leqq z+\varepsilon$, the curves $x\mapsto\psi_1(f^i(x)-\varepsilon)$ and $x\mapsto\psi_1(f^i(x)+\varepsilon)$ are monotone decreasing with slopes bounded away from zero and infinity. Therefore the intersection of the set (4) with any horizontal line is also a compact interval if it is nonempty.

The cases where the slopes of the branches of f^i are of the same sign on both sides of z require a similar and simpler argument, as does the case of trapezes. Thus the Lemma follows. □

For the next result we need a lower bound on the density $h(x)$ of μ. Therefore we shall assume throughout that ess sup $h(x)^{-1}>0$.

Lemma 2 *There is a positive constant C and a number γ with $0<\gamma<1$ such that if the increasing sequence of numbers $i_0=0<i_1<\dots<i_s$ satisfies $i_{j+1}-i_j<l(\varepsilon)/s$ for $0\leqq j<s$, and ε is sufficiently small, then for any measurable set B in $\mathscr{T}^2$, and for any integer q*

$$|\langle\chi_{\bigcap_{j=0}^s F^{-i_j}\Delta_\varepsilon}\chi_B\circ F^{q+i_s}\rangle-\langle\chi_{\bigcap_{j=0}^s F^{-i_j}\Delta_\varepsilon}\rangle\langle\chi_B\rangle|\leqq C(s+1)\langle\chi_B\rangle\gamma^{q+i_s}\ .$$

Proof. We shall prove that for any measurable set A in $\mathscr{T}^2$ such that $\sup_x \bigvee_2 \chi_A(x,\cdot)$ and $\sup_y \bigvee_1 \chi_A(\cdot, y)$ are finite we have

$$|\langle \chi_A \chi_B \circ F^q \rangle - \langle \chi_A \rangle \langle \chi_B \rangle| \leqq$$

$$C\left(2+\sup_x \bigvee_2 \chi_A(x,\cdot)+\sup_y \bigvee_1 \chi_A(\cdot, y)\right)\langle \chi_B \rangle \gamma^q .$$

Then this Lemma follows from Lemma 1. In order to prove the above inequality we use the decay of correlations of f (see (2)) in the horizontal and vertical direction as follows:

$$\langle \chi_A \chi_B \circ F^q \rangle = \int\int \chi_A(x,y)\chi_B(f^q(x), f^q(y))h(x)h(y)\,dx\,dy$$

$$= \int h(x)\left[\int h(y)\chi_A(x,y)\int h(z)\chi_B(f^q(x),z)\,dz\,dy\right.$$

$$\left. + r_q(x)\int \chi_B(f^q(x),z)\,dz\right]dx ,$$

where

$$|r_q(x)| \leqq C\left(\bigvee_2 \chi_A(x,\cdot)+1\right)\gamma^q .$$

Similarly we have

$$\int\int h(y)h(z)\left[\int h(x)\chi_A(x,y)\chi_B(f^q(x),z)\,dx\right]dy\,dz =$$

$$\int\int h(y)h(z)\left[\int h(t)\chi_B(t,z)\,dt\int h(x)\chi_A(x,y)\,dx + s_q(y)\int \chi_B(t,z)\,dt\right]dy\,dz ,$$

where

$$|s_q(y)| \leqq C\left(\bigvee_1 \chi_A(\cdot, y)+1\right)\gamma^q .$$

The above would give the result except that in the remainder term we get the L^1-norm of χ_B instead of the expectation. Since the density is bounded below away from zero we get the desired estimate. □

We now prove a Lemma about the convergence of the mean intermediate n times of visit to the ε-neighbourhood of the diagonal which happens within the scale $l(\varepsilon)$.

Lemma 3 *For any positive integer n the following limit exists*

$$C_n = \lim_{\varepsilon \to 0} \varepsilon^{-1} \sum_{\substack{0=q_0<q_1<\cdots<q_{n-1} \\ q_s-q_{s-1}\leqq l(\varepsilon)/(n-1)}} \left\langle \prod_{s=0}^{n-1} \chi_{A_\varepsilon} \circ F^{q_s} \right\rangle .$$

Moreover there are two positive numbers C and θ such that for $n \geqq 1$,

$$0 < C_n \leqq C\theta^n .$$

We have also

$$C_1 = 2\int h^2(x)\,dx ,$$

and for $n \geqq 2$,

$$C_n = 2 \sum_{0<q_1<\cdots<q_{n-1}} \int h^2(x) I_{q_1,\ldots,q_{n-1}}(x)\,dx\,,$$

where $I_{q_1,\ldots,q_{n-1}}$ *is the positive function defined by*

$$I_{q_1,\ldots,q_{n-1}}(x) = \min\left\{1, \frac{1}{|(f^{q_1})'(x)|}, \ldots, \frac{1}{|(f^{q_{n-1}})'(x)|}\right\}.$$

Proof. We will first prove a uniform bound on each term of the sum. We have obviously from our choice of $l(\varepsilon)$

$$\left\langle \prod_{s=0}^{n-1} \chi_{\Delta_\varepsilon} \circ F^{q_s} \right\rangle \leqq \langle \chi_{\Delta_\varepsilon} \chi_{\Delta_\varepsilon} \circ F^{q_{n-1}} \rangle \leqq \langle \chi_{B \cup \Delta_{\varepsilon K^{-1} \rho^{-q_{n-1}}}} \rangle\,,$$

where B is the intersection of the set $\Delta_\varepsilon \cap F^{-q_{n-1}} \Delta_\varepsilon$ with the union of the triangles defined in Lemma 1, and K appears due to the fact that f is not necessarily uniformly expanding. It is easy to verify that for each triangle T

$$\bar{\mu}(T \cap \Delta_\varepsilon \cap F^{-q_{n-1}} \Delta_\varepsilon) \leqq \mathcal{O}(1) \varepsilon^2 \rho^{-q_{n-1}}\,.$$

We now observe that the number of triangles is at most $(2|\mathscr{C}|)^{q_{n-1}}$, and since $q_{n-1} \leqq l(\varepsilon)$ we obtain for ε small enough

$$\left\langle \prod_{s=0}^{n-1} \chi_{\Delta_\varepsilon} \circ F^{q_s} \right\rangle \leqq \mathcal{O}(1) \varepsilon \rho^{-q_{n-1}}\,.$$

It is now enough to prove the convergence of

$$\varepsilon^{-1} \left\langle \prod_{s=0}^{n-1} \chi_{\Delta_\varepsilon} \circ F^{q_s} \right\rangle$$

for fixed integers $0 \leqq q_0 < q_1 < \cdots < q_{n-1}$. For $n \geqq 2$, let $I_{q_1,\ldots,q_{n-1}}$ be defined as in the statement of the lemma. It is easy to verify that away from the ε-neighbourhood D_ε of the boundary points of $\mathscr{A}_{q_{n-1}}$ (which contribute $\mathcal{O}(\varepsilon^2)$ to the measure), we have

$$D_\varepsilon^c \bigcap_{s=0}^{n-1} F^{-q_s} \Delta_\varepsilon =$$

$$D_\varepsilon^c \cap \{(x,y) \,|\, x - \varepsilon I_{q_1,\ldots,q_{n-1}}(x) - \mathcal{O}(\varepsilon^2) \leqq y \leqq x + \varepsilon I_{q_1,\ldots,q_{n-1}}(x) + \mathcal{O}(\varepsilon^2)\}\,.$$

This implies that

$$\varepsilon^{-1} \left\langle \prod_{s=0}^{n-1} \chi_{\Delta_\varepsilon} \circ F^{q_s} \right\rangle = \int h(x) \varepsilon^{-1} \int_{x-\varepsilon I_{q_1,\ldots,q_{n-1}}(x) - \mathcal{O}(\varepsilon^2)}^{x+\varepsilon I_{q_1,\ldots,q_{n-1}}(x) + \mathcal{O}(\varepsilon^2)} h(y)\,dy\,dx + \mathcal{O}(\varepsilon)\,,$$

and the convergence follows from the fact that almost surely

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{x-\varepsilon I_{q_1,\ldots,q_{n-1}}(x) - \mathcal{O}(\varepsilon^2)}^{x+\varepsilon I_{q_1,\ldots,q_{n-1}}(x) + \mathcal{O}(\varepsilon^2)} h(y)\,dy = 2h(x) I_{q_1,\ldots,q_{n-1}}(x)\,.$$

Replacing $I_{q_1,\ldots,q_{n-1}}$ by the constant function 1 in the above argument we also obtain the proof of the convergence for the case $n = 1$.

We finally prove the bound on C_n. We have from the uniform bound on the terms of the sum

$$C_n \leqq \mathcal{O}(1) \sum_{0=q_0<q_1<\cdots q_{n-1}} \rho^{-q_{n-1}} = \mathcal{O}(1) \sum_{q=n-1}^{\infty} \frac{(q-1)\cdots(q-n+2)}{(n-2)!} \rho^{-q}$$

$$= \mathcal{O}(1) \left(\frac{1}{\rho-1} \right)^{n-1} . \quad \square$$

In the next result we obtain the convergence of the factorial moments of the process of visits to the ε-neighbourhood of the diagonal, and we also get an explicit formulae for the limit.

Proposition 4 *For any continuous non-negative function g with compact support on $[0, \infty)$, the moments of the random variable $N_\varepsilon(g)$ converge, and for any integer $k>0$, we have*

$$\lim_{\varepsilon \to 0} \langle N_\varepsilon(g)^k \rangle = \sum_{p=1}^{k} \sum_{\substack{0<t_1,\ldots,0<t_p \\ t_1+\cdots+t_p=k}} \frac{k!}{t_1! \cdots t_p!} \times$$

$$\sum_{b=1}^{p} \sum_{\substack{0<n_1,\ldots,0<n_b \\ n_1+\cdots+n_b=p}} \prod_{i=1}^{b} \left(C_{n_i} \int g(t)^{\sum_{s=0}^{n_i-1} t_s + n_1 + \cdots + n_{i-1}+1} dt \right),$$

where the numbers C_n are defined in Lemma 3.

Proof. From the definition of $N_\varepsilon(g)$ it follows at once that

$$\langle N_\varepsilon(g) \rangle = \langle \chi_{\Delta_\varepsilon} \rangle \sum_{n=0}^{\infty} g(n\varepsilon)$$

which converges to

$$2 \int h^2(x)\, dx \int g(y)\, dy ,$$

since g is continuous with compact support and Lemma 3.
Let now k be an integer larger than 1. We have

$$\langle N_\varepsilon(g)^k \rangle = \sum_{0 \leqq j_1, \ldots, 0 \leqq j_k} \left\langle \prod_{s=1}^{k} g(j_s \varepsilon) \chi_{\Delta_\varepsilon} \circ F^{j_s} \right\rangle .$$

We now rearrange the sum into a sum over different indices, obtaining

$$\langle N_\varepsilon(g)^k \rangle = \sum_{p=1}^{k} \sum_{\substack{0<t_1,\ldots,0<t_p \\ t_1+\cdots+t_p=k}} \frac{k!}{p!\, t_1! \cdots t_p!} \sum_{\substack{0 \leqq j_1, \ldots, 0 \leqq j_p \\ j_q \neq j_r \text{ for } q \neq r}} \left\langle \prod_{s=1}^{p} g^{t_s}(j_s \varepsilon) \chi_{\Delta_\varepsilon} \circ F^{j_s} \right\rangle .$$

Now ordering the indices j_s we get

$$\langle N_\varepsilon(g)^k \rangle = \sum_{p=1}^{k} \sum_{\substack{0<t_1,\ldots,0<t_p \\ t_1+\cdots+t_p=k}} \frac{k!}{t_1! \cdots t_p!} \sum_{0 \leqq j_1 < j_2 < \ldots < j_p} \left\langle \prod_{s=1}^{p} g^{t_s}(j_s \varepsilon) \chi_{\Delta_\varepsilon} \circ F^{j_s} \right\rangle .$$

We will prove convergence of

$$\sum_{0\leqq j_1<j_2<\ldots<j_p}\left\langle\prod_{s=1}^{p}g^{t_s}(j_s\varepsilon)\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle$$

for fixed integers $t_1,\ldots,t_p$. We decompose the above sum into clusters of consecutive indices differing by at most $l(\varepsilon)/k$

$$\sum_{0\leqq j_1<j_2<\ldots<j_p}\left\langle\prod_{s=1}^{p}g^{t_s}(j_s\varepsilon)\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle$$

$$=\sum_{b=1}^{p}\sum_{\substack{0<n_1,\ldots,0<n_b\\ n_1+\cdots+n_b=p}}\sum_{(j_1,\ldots,j_p)\in\mathscr{Q}(n_1,\ldots,n_b)}\left\langle\prod_{s=1}^{p}g^{t_s}(j_s\varepsilon)\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle,$$

where

$$\mathscr{Q}(n_1,\ldots,n_b)=\{(j_1,\ldots,j_{n_1+\cdots+n_b})\,|\,j_1<j_2<\cdots<j_{n_1+\cdots+n_b},$$
$$j_{q+1}-j_q\leqq l(\varepsilon)/k\quad\text{if } q\notin\{n_1,n_1+n_2,\ldots,n_1+\cdots+n_{b-1}\}$$
$$\text{and else } j_{q+1}-j_q>l(\varepsilon)/k\}\,.$$

We now use the decay of correlations between the different clusters. We first fix the positive numbers $n_1,\ldots,n_b$ and then we fix the numbers $(j_1,\ldots,j_{n_1+\cdots+n_b})$ in $\mathscr{Q}(n_1,\ldots,n_b)$. We then write

$$\left\langle\prod_{s=1}^{n_1+\cdots+n_b}\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle=\left\langle\prod_{s=1}^{n_1}\chi_{\Delta_\varepsilon}\circ F^{j_s}\prod_{s=n_1+1}^{n_1+n_2}\chi_{\Delta_\varepsilon}\circ F^{j_s}\cdots\prod_{s=n_1+\cdots+n_{b-1}+1}^{n_1+\cdots+n_b}\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle.$$

The estimate will be done recursively and we need only to do one step. The above expression can be written as

$$\left\langle\prod_{s=1}^{n_1+\cdots+n_b}\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle=\left\langle\prod_{s=1}^{n_1}\chi_{\Delta_\varepsilon}\circ F^{j_s-j_1}\chi_B\circ F^{j_{n_1+1}-j_1}\right\rangle.$$

Applying Lemma 2, we get the estimate

$$\left\langle\prod_{s=1}^{n_1}\chi_{\Delta_\varepsilon}\circ F^{j_s-j_1}\chi_B\circ F^{j_{n_1+1}-j_1}\right\rangle=\left(\left\langle\prod_{s=1}^{n_1}\chi_{\Delta_\varepsilon}\circ F^{j_s-j_1}\right\rangle+r_{j_1,\ldots,j_{n_1+1}}\right)\langle\chi_B\rangle,$$

where

$$|r_{j_1,\ldots,j_{n_1+1}}|\leqq Cn_1\gamma^{j_{n_1+1}-j_1}\,. \tag{5}$$

Finally we obtain for $b>1$,

$$\left\langle\prod_{s=1}^{n_1+\cdots+n_b}\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle=\left(\left\langle\prod_{s=1}^{n_1}\chi_{\Delta_\varepsilon}\circ F^{j_s-j_1}\right\rangle+r_{j_1,\ldots,j_{n_1+1}}\right)\cdots \tag{6}$$
$$\left(\left\langle\prod_{s=n_1+\cdots+n_{b-2}+1}^{n_1+\cdots+n_{b-1}}\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle+r_{j_{n_1+\cdots+n_{b-2}+1},\ldots,j_{n_1+\cdots+n_{b-1}+1}}\right)$$
$$\times\left\langle\prod_{s=n_1+\cdots+n_{b-1}+1}^{n_1+\cdots+n_b}\chi_{\Delta_\varepsilon}\circ F^{j_s}\right\rangle.$$

From the above expression we see that $\langle N_\varepsilon(g)^k \rangle$ can be written as a sum of two quantities. The first one is

$$\sum_{p=1}^{k} \sum_{\substack{0<t_1,\dots,0<t_p \\ t_1+\dots+t_p=k}} \frac{k!}{t_1!\cdots t_p!} \sum_{b=1}^{p} \sum_{\substack{0<n_1,\dots,0<n_b \\ n_1+\dots+n_b=p}} \sum_{(j_1,\dots,j_p)\in\mathcal{Q}(n_1,\dots,n_b)}$$

$$\times \prod_{m=1}^{b} \left(\left\langle \prod_{s=n_1+\dots+n_{m-1}+1}^{n_1+\dots+n_m} \chi_{\Delta_\varepsilon} \circ F^{j_s} \right\rangle \prod_{s=n_1+\dots+n_{m-1}+1}^{n_1+\dots+n_m} g^{t_s}(j_s\varepsilon) \right),$$

where we defined $n_0=0$, and the second quantity is essentially a remainder. For fixed indices p, $t_1,\dots,t_p$, b, $n_1,\dots,n_b$ and for $(j_1,\dots,j_p)\in\mathcal{Q}(n_1,\dots,n_b)$ we define a double sequence of integers $(q_{m,s})$ with $1\leqq m\leqq b$ and $0\leqq s\leqq n_m-1$ by

$$q_{m,s}=j_{s+n_1+\dots+n_{m-1}+1}-j_{n_1+\dots+n_{m-1}+1}\,.$$

We have

$$\sum_{(j_1,\dots,j_p)\in\mathcal{Q}(n_1,\dots,n_b)} \prod_{m=1}^{b} \left(\left\langle \prod_{s=n_1+\dots+n_{m-1}+1}^{n_1+\dots+n_m} \chi_{\Delta_\varepsilon} \circ F^{j_s} \right\rangle \prod_{s=n_1+\dots+n_{m-1}+1}^{n_1+\dots+n_m} g^{t_s}(j_s\varepsilon) \right)$$

$$= \sum_{\substack{0<q_{m,1}<\dots<q_{m,n_m-1} \\ m=1,\dots,b \\ q_{m,s+1}-q_{m,s}\leqq l(\varepsilon)/k}} \varepsilon^{-b} \prod_{i=1}^{b} \left\langle \prod_{s=0}^{n_i-1} \chi_{\Delta_\varepsilon} \circ F^{q_{i,s}} \right\rangle \tag{7}$$

$$\times \sum_{\substack{j_1<j_{n_1+1}<\dots<j_{n_1+\dots+n_{b-1}+1} \\ j_{n_1+\dots+n_{r+1}+1}-j_{n_1+\dots+n_{r+1}+1}>q_{r,n_r-1}+l(\varepsilon)/k}} \varepsilon^b \prod_{i=1}^{b}$$

$$\times \prod_{s=0}^{n_i-1} g^{t_{s+n_1+\dots+n_{i-1}+1}}((j_{n_1+\dots+n_{i-1}+1}+q_{i,s})\varepsilon)\,.$$

Using Lemma 3 and elementary properties of the Riemann integral it is easy to prove that the above quantity converges to

$$\prod_{i=1}^{b} C_{n_i} \int_0^\infty dy_b\, g(y_b)^{\sum_{s=0}^{n_b-1} t_{s+n_1+\dots+n_{b-1}+1}} \int_0^{y_b} dy_{b-1}\, g(y_{b-1})^{\sum_{s=0}^{n_{b-1}-1} t_{s+n_1+\dots+n_{b-2}+1}}$$

$$\cdots \int_0^{y_2} dy_1\, g(y_1)^{\sum_{s=0}^{n_1-1} t_{s+1}}\,.$$

Now we must estimate the remainder. From Eq. (6) we see that this remainder is a sum of products of b terms each of the form $\langle \Pi\chi_{\Delta_\varepsilon}\circ F^{\cdot}\rangle$ or $r_{\dots}$ and there is at least one of the latter type. The summation over the indices will be performed as before introducing the indices m, s and $q_{m,s}$. We then obtain an expression similar to Eq. (7) except that we multiply and divide by a power of ε which is equal to the number of factors of the form $\langle \Pi\chi_{\Delta_\varepsilon}\circ F^{\cdot}\rangle$. Using the estimate (5) for the terms $r_{\dots}$ one can readily see that the remainder tends to zero with ε. This finishes the proof of the proposition. □

III Proof of the Theorem

Let φ_g be the function defined by

$$\varphi_g(z)=e^{\sum_{n=1}^{\infty} C_n \int (e^{zg(t)}-1)^n dt},$$

which is analytic on a disc around the origin (whose radius depends only on the number θ in Lemma 3 and on $\| g \|_{C^0}$). This implies that the k-th derivative μ_k of this function at the origin satisfies

$$\limsup_{k\to\infty} \frac{|\mu_k|^{1/k}}{k} < \infty .$$

One can verify that

$$\mu_k = \sum_{p=1}^{k} \sum_{\substack{0<t_1,\dots,0<t_p \\ t_1+\cdots+t_p=k}} \frac{k!}{t_1!\cdots t_p!} \sum_{b=1}^{p} \sum_{\substack{0<n_1,\dots,0<n_b \\ n_1+\cdots+n_b=p}} \prod_{i=1}^{b} C_{n_i}$$

$$\times \int_0^{\infty} dy_b\, g(y_b)^{\sum_{s=0}^{n_b-1} t_{s+n_1+\cdots+n_{b-1}+1}} \cdots \int_0^{y_2} dy_1\, g(y_1)^{\sum_{s=0}^{n_1-1} t_{s+1}},$$

and by Theorem 8.48 and Proposition 8.49 of [Br], this implies convergence in law of the sequence $N_\varepsilon(g)$.

Now we proceed to interpret the limiting process. Let Φ below be the analytic function defined on a small disc around the origin,

$$\Phi(u) = \sum_{n=1}^{\infty} C_n u^n .$$

Consider the function $I_{q_1,\dots,q_{n-1}}$ defined in Lemma 3. For $0<p_1<\cdots<p_n$ define

$$D_{p_1,\dots,p_n} = 2\int h^2(x) I_{p_1,\dots,p_n}(x)\,dx .$$

Recall that $m>0$ is the smallest $k>0$ such that $\operatorname{ess\,sup}|(f^k)'|^{1/k}>\rho$. Then for $n>m$ we have

$$C_n = \sum_{0<q_1<\dots<q_{n-m}<q_{n-m+1}<\cdots<q_{n-1}} 2\int h^2(x) I_{q_1,\dots,q_{n-1}}(x)\,dx$$

$$= \sum_{n-m-1<p_1<\cdots<p_m} D_{p_1,\dots,p_m} \frac{(p_1-1)\cdots(p_1-n+m+1)}{(n-m-1)!} .$$

Hence the function

$$\Phi(u) - \sum_{n=1}^{m} C_n u^n = \sum_{n=m+1}^{\infty} C_n u^n$$

can be written as

$$\sum_{n=m+1}^{\infty} u^{m+1} \sum_{n-m-1<p_1<\cdots<p_m} D_{p_1,\dots,p_m} u^{n-m-1} \frac{(p_1-1)!}{(n-m-1)!(p_1-n+m)!}$$

$$= u^{m+1} \sum_{p_1=1}^{\infty} \tilde{D}_{p_1} (1+u)^{p_1-1},$$

where we have defined for $p>0$,

$$\tilde{D}_p = \sum_{p<p_2<\cdots<p_m} D_{p,p_2,\ldots,p_m}.$$

Now let λ and π_k (such that $\sum \pi_k = 1$) be a formal solution of

$$\begin{aligned}\Phi(u) &= \sum_{n=1}^{m} C_n(1+u-1)^n + (1+u-1)^{m+1} \sum_{0<p_1<\cdots<p_m} (1+u)^{p_1-1} D_{p_1,\ldots,p_m} \\ &= \lambda\left(\sum_{k=1}^{\infty} \pi_k(1+u)^k - 1\right).\end{aligned} \tag{8}$$

Since $D_{p_1,\ldots,p_m} \leqq \mathcal{O}(1)\rho^{-p_m}$ we conclude that

$$\tilde{D}_p = \sum_{p<p_2<\cdots<p_m} D_{p,p_2,\ldots,p_m} \leqq \mathcal{O}(1)\rho^{-p}.$$

Therefore we obtain the estimate

$$|\pi_k| \leqq \mathcal{O}(1)\rho^{-k}.$$

This shows that the function

$$\Psi(u) = \lambda\left(\sum_{k=1}^{\infty} \pi_k(1+u)^k - 1\right)$$

is analytic on a disc centered at -1 and of radius $1+\delta$ $(\delta>0)$. From (8) we know that Φ and Ψ coincide on a small disc around the origin. Therefore, taking $u = e^{zg(y)} - 1$ we deduce that the function

$$\psi_g(z) = e^{\lambda \sum_{k=1}^{\infty} \pi_k \int_0^{\infty} (e^{zkg(y)} - 1)\,dy}$$

coincides with φ_g on a small disc around the origin. Since φ_g is positive definite, we conclude that ψ_g is positive definite and hence necessarily we must have $\pi_k \geqq 0$ for all $k>0$, and $\lambda>0$ (cf. [Do]). This finishes the proof of the Theorem. □

Here we give the explicit solutions for λ and π_k defined by Eq. (8). In the case $m>1$, we have

$$\begin{aligned}\lambda &= 2\int h^2(x)\,dx + \sum_{n=1}^{m-1} (-1)^n \sum_{0<q_1<\cdots<q_n} D_{q_1,\ldots,q_n} \\ &\quad + (-1)^m \sum_{1<p_2<\cdots<p_m} D_{1,p_2,\ldots,p_m}.\end{aligned}$$

For $k=1$,

$$\begin{aligned}\lambda\pi_1 &= 2\int h^2(x)\,dx + \sum_{n=1}^{m-1} \binom{n+1}{1} (-1)^n \sum_{0<q_1<\cdots<q_n} D_{q_1,\ldots,q_n} \\ &\quad + \sum_{j=m}^{m+1} \binom{m+1}{j} (-1)^j \sum_{j-m+1<p_2<\cdots<p_m} D_{j-m+1,p_2,\ldots,p_m};\end{aligned}$$

for $1<k\leqq m$,

$$\lambda\pi_k=\sum_{n=k-1}^{m-1}\binom{n+1}{k}(-1)^{n-k+1}\sum_{0<q_1<\cdots<q_n}D_{q_1,\ldots,q_n}$$

$$+\sum_{j=m-k+1}^{m+1}\binom{m+1}{j}(-1)^j\sum_{k-m+j<p_2<\cdots<p_m}D_{k-m+j,p_2,\ldots,p_m};$$

and for $k>m$,

$$\lambda\pi_k=\sum_{j=0}^{m+1}\binom{m+1}{j}(-1)^j\sum_{k-m+j<p_2<\cdots<p_m}D_{k-m+j,p_2,\ldots,p_m}.$$

In the case $m=1$, we obtain the expressions

$$\lambda=2\int h^2(x)\left[1-\frac{1}{|f'(x)|}\right]dx;$$

for $k=1$,

$$\lambda\pi_1=2\int h^2(x)\left[1-\frac{2}{|f'(x)|}+\frac{2}{|(f^2)'(x)|}\right]dx;$$

and for $k>1$,

$$\lambda\pi_k=2\int h^2(x)\left[\frac{1}{|(f^{k-1})'(x)|}-\frac{2}{|(f^k)'(x)|}+\frac{1}{|(f^{k+1})'(x)|}\right]dx.$$

Acknowledgements. This work is part of Projeto Temático de Equipe "Transição de Fase Dinâmica em Sistemas Evolutivos" financially supported by FAPESP grant number 90/3918-5. Pierre Collet thanks the Instituto de Matemática e Estatística da Universidade de São Paulo (São Paulo) for their hospitality and the financial support from FAPESP. Zaqueu Coelho thanks the Centre de Physique Théorique at Ecole Polytechnique (Palaiseau-Paris) and the Instituto de Matemática Pura e Aplicada (Rio de Janeiro) for their hospitality and financial support.

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