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Noncoalescence for the Skorohod equation in a convex domain of $\mathbb{R}^2$

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Summary. Given a convex domain of $\mathbb{R}^2$, we show that a.s. the paths of two solutions of the Skorohod equations driven by the same Brownian motion but starting at different points do not meet at the same time.

1. Introduction

Given a convex planar domain D and a Brownian motion B in $\mathbb{R}^2$, it is well known (cf. [1, 2, 4]) there is a strong solution to the Skorohod equation

$$d X_t(x) = d B_t + n(X_t(x))\, d\varphi_t^x, \qquad X_0 = x.$$

Here, φ_t^x is the local time of $X_t(x)$ on ∂D and $n(y)$ is the inward directed unit normal vector at $y \in \partial D$. Actually it has been shown by Tanaka [6] and Lions-Sznitman [5] one can construct a solution to this equation for any continuous function B_t. Also, one can note it is very easy (see Eq. (5) below) to prove pathwise uniqueness. Therefore, the existence of a strong solution follows from the existence of a weak solution, which can be obtained, by conformal mapping and time change, from a reflected Brownian motion in the half plane.

Now consider two particles undergoing such a motion

$$d X_t^l = d B_t + n(X_t^l)\, d\varphi_t^l, \qquad X_0^l = x_l, \qquad l = 1, 2$$

with $x_1 \neq x_2$. A moments thought reveals that there is much collapsing under the flow $x \to X_t(x)$. If D is a square or even has two perpendicular flat spots, Weerasinghe (Ph D. thesis) observed that $T = \inf\{t > 0 : X_t^1 = X_t^2\}$ will be finite a.s. In [3], we showed that T is infinite a.s. when D is a disc. The proof used the symmetry of the disc. In this work we show that the technique used in the disc can be extended to show

Theorem 1. *Let D be a convex planar domain with a C^2 boundary. Suppose there is some constant $K > 0$ so that $K(x) \geqq K$ for $x \in \partial D$, where $K(x)$ is the curvature of ∂D at x. Then $P(T = \infty) = 1$.*

The approach will be to approximate D near the points where X_t^1 or X_t^2 hits ∂D by a moving osculating circle. This leads to a moving frame in which a natural coordinate system for the disc is utilized together with the Frenet formulas for moving frames.

2. Notation and definitions

Following our previous work, we set

$$z_t = \tfrac{1}{2} \|X_t^2 - X_t^1\|$$
$$m_t = \tfrac{1}{2}(X_t^1 + X_t^2)$$

and for $t < T$,

$$i_t = \frac{X_t^2 - X_t^1}{2 z_t}$$
$$j_t = i_t^\perp$$

where $(x, y)^\perp = (-y, x)$.

Suppose $O \in D$ and that ∂D is represented by the polar equation $r = f(\theta)$. Set $\theta_t = \theta(m_t)$, the polar coordinate angle of m_t and $P_t = (f(\theta_t), \theta_t)$.

Define R_t to be the radius of curvature of ∂D at P_t. For the Frenet formulas it is convenient to introduce an arclength parameter s_t and use $P(s_t)$ in place of P_t. Next we denote the center of curvature at P_t (or s_t) by

$$\omega_t = P_t + R_t n(P_t) = P(s_t) + R_t n(s_t).$$

In what follows an important role is played by

$$x_t = \langle m_t - \omega_t, j_t \rangle$$

and

$$y_t = \langle m_t - \omega_t, i_t \rangle.$$

A fact that will often be used is that

$$y_t(d\varphi_t^2 - d\varphi_t^1) = |y_t|\, d\varphi_t, \qquad \text{where } \varphi_t = \varphi_t^1 + \varphi_t^2.$$

The idea of the proof of Theorem 1 is to show that $T < \infty$ can only occur if the segment $\overrightarrow{X_t^1 X_t^2}$ strikes ∂D (i.e. on supp $d\varphi_t^2$ or supp $d\varphi_t^1$) at a right angle to the tangent of ∂D and it must do so repeatedly when z_t is small. Next it is shown that in fact $\overrightarrow{X_t^1 X_t^2}$ will never strike ∂D at a right angle.

3. Calculations and proof of the theorem

When $X_t^2 \in \partial D$ and z is small, $\overrightarrow{X_t^1 X_t^2}$ will be at, or nearly at, a right angle to ∂D if and only if $L_t = x_t^2 + (y_t - R_t)^2$ will be $O(z_t^2)$. Similarly for $X_t^1 \in \partial D$, $\overrightarrow{X_t^1 X_t^2}$ is nearly perpendicular to ∂D if and only if $N_t = x_t^2 + (y_t + R_t)^2$ is $O(z_t^2)$.

We need to derive Itô expansions for x_t, y_t, z_t and R_t in order to examine the behavior of L_t and N_t. We begin with

Lemma 1. *On* $\operatorname{supp} d\varphi_t^l$, $l=1, 2$

$$\|R_t n(X_t^l)+(X_t^l-\omega_t)\| = O(z_t^2)$$

and $\|R_t n(s_t)+X_t^l-\omega_t\| = O(z_t)$.

Consequently, on $\operatorname{supp} d\varphi_t^l$, $l=1, 2$.

(1) $$\left\|\langle n(X_t^l), i_t\rangle + \frac{y_t+(-1)^l z_t}{R_t}\right\| = O(z_t^2)$$

(2) $$\left\|\langle n(X_t^l), j_t\rangle + \frac{x_t}{R_t}\right\| = O(z_t^2)$$

(3) $$\left\|\langle n(s_t), j_t\rangle + \frac{x_t}{R_t}\right\| = O(z_t)$$

(4) $$\left\|\langle n(s_t), i_t\rangle + \frac{y_t}{R_t}\right\| = O(z_t).$$

Proof. Introduce rectangular coordinates (u, v), centered at ω_t with the positive v-axis in the direction $-n(s_t)$. Represent ∂D near s_t in this coordinate system by $v=g(u)$. Then $g(0)=R_t$, $g'(0)=0$, $g''(0)=-\kappa_t$ so that

$$g(u)=R_t-\kappa_t\frac{u^2}{2}+O(u^3).$$

Also, the unit tangent vector at $(u, g(u))=(u, v)$ is given by

$$T(u, v)=\frac{(1, g'(u))}{\sqrt{1+(g'(u))^2}}$$

and $g'(u)=-\kappa_t u+O(u^2)$. This gives

$$n(u, v)=\frac{(g'(u), -1)}{\sqrt{1+(g'(u))^2}}$$

as the inward unit normal vector. Noticing that

$$\sqrt{1+(g'(u))^2}=1+\tfrac{1}{2}\kappa_t^2 u^2+O(u^3)$$

it follows that

$$n(u, v)=(g'(u), -1)+O(u^2).$$

On $\operatorname{supp} d\varphi_t^l$, for z_t small enough, the angle between PX^l and OP is bounded away from 0 so that $P_t X_t^l=O(z_t)$. Now,

$$X_t^l-\omega_t=(u_t, g(u_t)) \qquad \text{for some } |u_t|\leqq O(z_t).$$

Hence

$$R_t^l \equiv \|X_t^l-\omega_t\| = R_t+O(z_t^2),$$

$$\frac{\omega_t-X_t^l}{R_t^l} = -\frac{(u_t, g(u_t))}{R_t}+O(u_t^2)$$

and

$$\frac{\omega_t-X_t^l}{R_t^l}=\frac{\omega_t-X_t^l}{R_t}+O(z_t^2).$$

Therefore, on supp $d\varphi_t^l$,

$$\left\|n(X_t^l)-\frac{\omega_t-X_t^l}{R_t}\right\|\leqq\kappa_t^2\,\frac{u_t^2}{2}+O(z_t^2)=O(z_t^2)$$

and

$$\left\|n(s_t)-\frac{\omega_t-X_t^l}{R_t}\right\|=\left\|\left(\kappa_t u_t,\,\kappa_t^2\,\frac{u_t^2}{2}\right)\right\|+O(z_t^2)$$
$$=\kappa_t|u_t|+O(z_t^2)=O(z_t)$$

(1)~(4) follow immediately.

This enables us to derive formulas for dx_t, dy_t, dz_t which are the same as in the disc up to $O(z_t)$ terms except for terms which arise from the moving frame.

Considering z_t first, for $t<T$, and with

$$\begin{aligned}2z_t\,dz_t&=\tfrac{1}{2}\langle n(X_t^2)\,d\varphi_t^2-n(X_t^1)\,d\varphi_t^1,\,X_t^2-X_t^1\rangle\\&=z_t\langle n(X_t^2)\,d\varphi_t^2-n(X_t^1)\,d\varphi_t^1,\,i_t\rangle\\&=-z_t\left(\frac{|y_t|+z_t}{R_t}\right)d\varphi_t+O(z_t^3)\,d\varphi_t\end{aligned}$$

by (2) so that

$$dz_t=-\frac{1}{2}\left(\frac{|y_t|+z_t}{R_t}\right)d\varphi_t+O(z_t^2)\,d\varphi_t. \tag{5}$$

Next, $\langle i_t, j_t\rangle=0$ implies di_t must be parallel to j_t so

$$\begin{aligned}di_t&=\frac{1}{2z_t}\langle n(X_t^2)\,d\varphi_t^2-n(X_t^1)\,d\varphi_t^1,\,j_t\rangle\,j_t\\&=-\frac{x_t}{2R_t z_t}(d\varphi_t^2-d\varphi_t^1)\,j_t+O(z_t)\,d\varphi_t\,j_t,\end{aligned} \tag{6}$$

by (2) and automatically,

$$dj_t=\frac{x_t}{2R_t z_t}(d\varphi_t^2-d\varphi_t^1)\,i_t+O(z_t)\,d\varphi_t\,i_t. \tag{7}$$

Recall the Frenet formulas, with $T=T(s)$, the tangent vector at s,

$$\frac{dT}{ds}=\frac{n}{R},\qquad \frac{dn}{ds}=-\frac{T}{R}.$$

Thus, using Stratonovich differentials which will be denoted with o,

$$\begin{aligned}d\omega_t&=d(P(s_t)+R_t n(s_t))\\&=T(s_t)\,o\,ds_t+n(s_t)\,o\,dR_t-T(s_t)\,o\,ds_t\\d\omega_t&=n(s_t)\,o\,dR_t.\end{aligned} \tag{8}$$

Using (2), (7), (8) and setting

$$
(9)\qquad \begin{aligned}
dW_t^1 &= \langle dB_t, j_t\rangle \\
dx_t &= \langle d(m_t-\omega_t), j_t\rangle + \langle m_t-\omega_t, dj_t\rangle \\
&= dW_t^1 + \left(\frac{x_t|y_t|}{2R_t z_t} - \frac{x_t}{2R_t}\right) d\varphi_t - \langle n(s_t)\, o\, dR_t, j_t\rangle + O(z_t)\, d\varphi_t.
\end{aligned}
$$

Using (1), (6), (8) and setting

$$
(10)\qquad \begin{aligned}
dW_t^2 &= \langle dB_t, i_t\rangle \\
dy_t &= \langle d(m_t-\omega_t), i_t\rangle + \langle m_t-\omega_t, di_t\rangle \\
&= dW_t^2 - \frac{y_t}{2R_t}\, d\varphi_t - \left(\frac{x_t^2}{2R_t z_t} - \frac{z_t}{2R_t}\right)(d\varphi_t^2 - d\varphi_t^1) \\
&\quad - \langle n(s_t)\, o\, dR_t, i_t\rangle + O(z_t)\, d\varphi_t.
\end{aligned}
$$

Observe that (W_t^1, W_t^2) is a standard two-dimensional Brownian motion.

The formulas (5), (9), (10) are similar to those from our previous work [3]. In the case of a circle of radius R, one would have $R_t \equiv R$ and the $O(z_t^2)\, d\varphi_t$ missing in (5). For (9) and (10) in a circle of radius R, one would have $R_t \equiv R$ and the terms involving $n(s_t)\, o\, dR_t$ and $O(z_t)\, d\varphi_t$ would be missing. The trouble in extending the argument arises from the terms involving $n(s_t)\, o\, dR_t$ so we now devote a little time to discussing these.

First, $R_t = R(\theta_t)$, and $\theta_t = \theta(m_t)$ so

$$
(11)\qquad dR_t = R'\, d\theta_t + \tfrac{1}{2} R''\, d\langle\theta\rangle_t
$$

and

$$
(12)\qquad d\theta_t = \frac{m_t^\perp}{\|m_t\|^2}\, dB_t + \frac{1}{2}\sum_{l=1}^{2} \frac{\langle m_t^\perp, n(X_t^l)\rangle}{\|m_t\|^2}\, d\varphi_t^l.
$$

Now a simple stopping time argument can be applied to prevent $\|m_t\|$ from becoming too small (z_t doesn't decrease when z_t is small and $\|m_t\|$ is close to 0). Also, since ∂D is C^2, R' and R'' are bounded continuous functions of θ so that the martingale and bounded variation parts of R_t are well-behaved.

Next, $n(s_t) = F(\theta_t)$ where $F: S^1 \to S^1$ is a C^2-function, since ∂D is C^2. Thus

$$
(13)\qquad dn(s_t) = dF_t = F'\, d\theta_t + \tfrac{1}{2} F''\, d\langle\theta\rangle_t.
$$

By (11) and (13), we get

$$
(14)\qquad \begin{aligned}
n(s_t)\, o\, dR_t &= n(s_t)\, dR_t + \tfrac{1}{2} d\langle n(s_t), R_t\rangle \\
&= n(s_t)\, dR_t + \frac{1}{2}\frac{R' F'}{\|m_t\|^2}\, dt \\
&= n(s_t)\, dR_t + O(1)\, dt.
\end{aligned}
$$

These results are summarized by the next lemma which is implied by Lemma 1, (11), (12), and (14).

Lemma 2. *For times* $T>t>0$ *such that* $\|m_t\| \geqq \varepsilon > 0$

$$\langle n(s_t)\, o\, dR_t, i_t\rangle = O(1)\, dt + O(1)\left(\frac{y_t}{R_t} + O(z_t)\right) dR_t$$

$$\langle n(s_t)\, o\, dR_t, j_t\rangle = O(1)\, dt + O(1)\left(\frac{x}{R_t} + O(z_t)\right) dR_t$$

$$dR_t = O(1)\, dW_t + O(1)\, d\varphi_t + O(1)\, dt$$

where $dW_t = \frac{m_t^{\perp}}{\|m_t\|}\, dB_t$ *is a one-dimensional Brownian motion increment.*

Recall that $L_t = x_t^2 + (y_t - R_t)^2$, $N_t = x_t^2 + (y_t + R_t)^2$.

With this background we can now establish

Lemma 3. (a) $L_t N_t \neq 0$ *for* $t<T$

(b) *On* $\{T<\infty\}$, $\inf_{t<T} \operatorname{Log}(L_t N_t) > -\infty$ *a.s.*

(These assertions valid modulo the stopping time argument.)

Proof. Let us admit (a) for a while. Then, (b) follows from an inspection of the Itô expansion of $u_t = \operatorname{Log}(L_t N_t)$.

The argument goes as follows. We shall show that for some c_t, d_t, e_t and local martingale M_t

$$du_t = dM_t + c_t dt + d_t d\varphi_t + e_t\, dt,$$

where

$$c_t \geqq c_0 > -\infty, \qquad d_t \geqq d_0 > -\infty, \qquad \text{for all } t<T, \tag{15}$$

and on $\{T<\infty\}$, $\int_0^T e_t\, dt < \infty$.

Since (see [3]), $\lim_{t\to\infty} \frac{\varphi_t}{t} = \frac{\sigma(\partial D)}{m(D)}$ a.s. $\inf_{t<T} u_t = -\infty$ can only occur on $\{T<\infty\}$ when $\inf_{t<T} M_t = -\infty$. This last can only occur if $\sup_{t<T} M_t = \infty$ (M is a time-change of Brownian motion). But u_t is a quantity bounded from above so a contradiction arises since on $\{T<\infty\}$ the terms $\int_0^T d_t\, d\varphi_t$, $\int_0^T c_t\, dt$, $\int_0^T e_t\, dt$ can not cancel the arbitrarily large positive M_t values to keep u_t bounded.

Lemma 3 will thus be proved once the bounds at (15) are established. This is done by writing out Itô's expansion for $u_t = \operatorname{Log}(L_t N_t)$. In this expansion, regard $f(x_t, y_t + R_t) = \operatorname{Log}(L_t)$, $g(x_t, y_t - R_t) = \operatorname{Log}(N_t)$.

Then

$$f_{xx}=2\frac{(y-R)^2-x^2}{L^2}, \qquad g_{xx}=2\frac{(y+R)^2-x^2}{N^2}$$
$$f_{x,y-R}=-\frac{4x(y-R)}{L^2}, \qquad g_{x,y+R}=-\frac{4x(y+R)}{N^2}$$
$$f_{y-R,y-R}=2\frac{x^2-(y-R)^2}{L^2}, \qquad g_{y+R,y+R}=2\frac{x^2-(y-R)^2}{N^2}.$$

The quadratic terms are

$$d\langle x, x\rangle_t=\left[1+2\frac{xR'\langle m^\perp, j\rangle}{R\|m\|^2}+\left(\frac{xR'}{R\|m\|}\right)^2\right]dt$$
$$d\langle x, y-R\rangle_t=\left[\frac{(y-R)R'\langle m^\perp, j\rangle}{R\|m\|^2}+\frac{xR'\langle m^\perp, i\rangle}{R\|m\|^2}+\frac{x(y-R)(R')^2}{R^2\|m\|^2}\right]dt$$
$$d\langle y-R, y-R\rangle_t=\left[1+2\frac{(y-R)R'\langle m^\perp, i\rangle}{R\|m\|^2}+\left(\frac{(y-R)R'}{R\|m\|}\right)^2\right]dt$$
$$d\langle x, y+R\rangle_t=\left[\frac{(y+R)R'\langle m^\perp, j\rangle}{R\|m\|^2}+\frac{xR'\langle m^\perp, i\rangle}{R\|m\|^2}+\frac{x(y+R)R'^2}{R^2\|m\|}\right]dt$$
$$d\langle y+R, y+R\rangle_t=\left[1+2\frac{(y+R)R'\langle m^\perp, j\rangle}{R\|m\|^2}+\left(\frac{(y+R)R'}{R\|m\|}\right)^2\right]dt$$

(see (9), (12)).

Using the stopping time argument we have:

$$\begin{aligned} d\langle x, x\rangle &=(1+O(1)\,x)\,dt \\ d\langle x, y\pm R\rangle &=(O(1)(y\pm R)+O(1)\,x)\,dt \\ d\langle y\pm R, y\pm R\rangle &=(1+O(1)(y\pm R))\,dt. \end{aligned} \tag{16}$$

Hence we get the Itô expansion in the form

$$\begin{aligned} du_t=&L_t^{-1}(2x_t\,dx_t+2(y_t-R_t)\,d(y_t-R_t))+N_t^{-1}(2x_t\,dx_t+2(y_t+R_t)\,d(y_t+R_t)) \\ &+L^{-2}[((y-R)^2-x^2)(O(1)\,x+O(1)(y-R)) \\ &+4x(y-R)(O(1)(y-R)+O(1)\,x)]\,dt \\ &+N^{-2}[((y+R)^2-x^2)(O(1)\,x+O(1)(y+R)) \\ &+4x(y+R)(O(1)\,x+O(1)(y+R))]\,dt. \end{aligned} \tag{17}$$

Recalling the definitions of L_t and N_t, one sees immediately that $|x_t|\,L_t^{-1/2}$, $|y-R_t|\,L_t^{-1/2}$, $|x_t|\,N_t^{-1/2}$ and $|y_t+R_r|\,N_t^{-1/2}$ are bounded by 1. Hence the two last terms can be written $O(1)\,L_t^{-1/2}+O(1)\,N_t^{-1/2}$.

Moreover, using the same stopping time argument as before we may suppose that m_t lies at a distance from the boundary less than ε and that $z_t<\varepsilon$. Note that the angle ψ between the tangent at P and OP is uniformly bounded away from 0 as well as the radius of the osculatory circle.

Hence, for ε small enough, m_t will be inside the osculatory circle i.e.

$$x_t^2+y_t^2-R_t^2 \leqq 0. \tag{18}$$

Also, if B_ε is the band of width ε and axis OP, the portion of B_ε limited by the tangent at P and the osculatory circle will be bounded, and X_t^l will lie either in this portion or inside the osculatory circle.

Hence $\|\omega-X_t^l\|$ will be at most $\sqrt{R^2+(z/\cos\psi)^2}$.

Therefore $\sqrt{x^2+(|y|+z)^2} \leqq R+O(z^2)$ and $|y|+z \leqq R+O(z^2)$ which implies $z \leqq O(R-|y|)$. Thus $z_t L_t^{-1/2}$ and $z_t N_t^{-1/2}$ are also bounded, which proves (a) of Lemma 3.

We use $M_t^1 \dots M_t^4$ to denote local martingales.

First, by Lemma 2 and Eq. (9)

$$\begin{aligned}2xL^{-1}\,dx &= dM^1+L^{-1}\left(\frac{x^2|y|}{Rz}-\frac{x^2}{R}\right)d\varphi+xL^{-1}O(1)\,dt\\&\quad+(xL^{-1}O(z)+x^2L^{-1}O(1))\,d\varphi\\&=dM^1+L^{-1}\left(\frac{x^2|y|}{Rz}-\frac{x^2}{R}\right)d\varphi+\sqrt{L^{-1}}\,O(1)\,dt+O(1)\,d\varphi+O(1)\,dt.\end{aligned}$$

Using Lemma 2 and (10),

$$\begin{aligned}2(y-R)L^{-1}d(y-R) &= dM^2-L^{-1}\frac{y(y-R)}{R}\,d\varphi-L^{-1}(y-R)\cdot\left(\frac{x^2}{Rz}+\frac{z}{R}\right)\\&\quad\cdot(d\varphi^2-d\varphi^1)\\&\quad+(y-R)L^{-1}O(1)\,dt+O(1)\left(\frac{y-R}{R}+O(z)\right)(y-R)\,d\varphi\\&=dM^2-L^{-1}\frac{y(y-R)}{R}\,d\varphi-L^{-1}(y-R)\left(\frac{x^2}{Rz}+\frac{z}{R}\right)\\&\quad\cdot(d\varphi^2-d\varphi^1)\\&\quad+\sqrt{L^{-1}}\,O(1)\,dt+O(1)\,d\varphi+O(1)\,dt.\end{aligned}$$

Similarly,

$$2xN^{-1}\,dx=dM^3+N^{-1}\left(\frac{x^2|y|}{Rz}-\frac{x^2}{R}\right)d\varphi+\sqrt{N^{-1}}\,O(1)\,dt+O(1)\,d\varphi$$

$$\begin{aligned}2(y+R)N^{-1}\,d(y+R) &= dM^4-N^{-1}\frac{y(y+R)}{R}\,d\varphi-N^{-1}(y+R)\left(\frac{x^2}{Rz}+\frac{z}{R}\right)\\&\quad\cdot(d\varphi^2-d\varphi^1)\\&\quad+\sqrt{N^{-1}}\,O(1)\,dt+O(1)\,d\varphi+O(1)\,dt.\end{aligned}$$

On summing these terms and recalling that remaining (2nd order) terms are $O(1)\,L_t^{-1/2}+O(1)\,N_t^{-1/2})\,dt$, it arises, in view of $y(d\varphi^2-d\varphi^1)=|y|\,d\varphi$, that

$$du=dM+(L^{-1}+N^{-1})\left[\left(\frac{x^2|y|}{Rz}-\frac{x^2}{R}\right)-\frac{y^2}{R}\right]d\varphi-(L^{-1}+N^{-1})|y|\left[\frac{x^2}{Rz}+\frac{z}{R}\right]d\varphi$$
$$+(L^{-1}-N^{-1})\,y\,d\varphi+(L^{-1}-N^{-1})\left(\frac{x^2}{z}+z\right)(d\varphi^2-d\varphi^1)$$
$$+O(1)\,d\varphi+O(1)\,dt+(O(1)\,L^{-1/2}+O(1)\,N^{-1/2})\,dt$$

and since $\dfrac{1}{L}-\dfrac{1}{N}=\dfrac{4yR}{LN}$ on $\operatorname{Supp}(d\varphi)$

$$(19)\quad du=dM+\frac{4R^2}{LN}\left(\frac{y^2}{R}+\frac{x^2|y|}{Rz}+\frac{|y|z}{R}\right)d\varphi-(L^{-1}+N^{-1})\left(\frac{x^2+y^2+|y|z}{R}\right)d\varphi$$
$$+O(1)\,L^{-1/2}\,dt+O(1)\,N^{-1/2}\,dt+O(1)\,d\varphi+O(1)\,dt$$

$$(20)\quad du=dM+\frac{y^2+|y|z}{RLN}(4R^2-(L+N))\,d\varphi+\frac{4R|y|x^2}{LNz}\,d\varphi$$
$$-\frac{x^2}{R}(L^{-1}+N^{-1})\,d\varphi+(O(1)\,L^{-1/2}+O(1)\,N^{-1/2})\,dt+O(1)\,d\varphi+O(1)\,dt.$$

Now the two first $d\varphi_t$ terms have a nonnegative coefficient. (Recall (18)). Since $\dfrac{x_t^2}{L_t}\leqq 1$ and $\dfrac{x_t^2}{N_t}\leqq 1$ the third $d\varphi_t$ coefficient is bounded from below by some negative constant. Thus from these remarks

$$du_t=dM_t+d_t\,d\varphi_t+O(1)\,dt+\left(\frac{O(1)}{\sqrt{L_t}}+\frac{O(1)}{\sqrt{N_t}}\right)dt$$

where $d_t\geqq d_0>-\infty$ for all t.

We can apply the argument mentioned at the beginning of this proof once we show

Lemma 4. *On* $\{T<\infty\}$, $\int\limits_0^T \frac{1}{\sqrt{L_s}}\,ds<\infty$ *and* $\int\limits_0^T \frac{1}{\sqrt{N_s}}\,ds<\infty$. *a.s.*

Proof. It will do to consider $w=\sqrt{L}+\sqrt{N}$, w is clearly bounded. By Itô's formula, for $t<T$,

$$dw_t=\frac{x}{\sqrt{L}}\,dx+\frac{y-R}{\sqrt{L}}\,d(y-R)+\frac{x}{\sqrt{N}}\,dx+\frac{y-R}{\sqrt{N}}\,d(y-R)$$
$$+(y+R)^2\,\frac{L^{-3/2}}{2}+(y-R)^2\,\frac{N^{-3/2}}{2}\,d\langle x,x\rangle$$
$$+\frac{x^2}{2}L^{-3/2}\,d\langle y+R,y+R\rangle+\frac{x^2}{2}N^{-3/2}\,d\langle y-R,y-R\rangle$$
$$+L^{-3/2}\,x(y+R)\,d\langle x,y+R\rangle+N^{-3/2}\,x(y-R)\,d\langle x,y-R\rangle.$$

Comparing with (17), we see the first order terms are obtained by multiplying the analogous terms in there by $\sqrt{L}$ or $\sqrt{N}$.

Itô correction yields $\frac{1}{2}(N^{-1/2}+L^{-1/2})\,dt+O(1)\,dt$. Hence

$$\begin{aligned}
dw &= dM+\left(\frac{x}{\sqrt{L}}+\frac{x}{\sqrt{N}}\right)\left(\frac{x|y|}{2Rz}-\frac{x}{2R}\right)d\varphi \\
&\quad+\left(\frac{y-R}{\sqrt{L}}+\frac{y+R}{\sqrt{N}}\right)\left[\frac{y}{2R}\,d\varphi-\left(\frac{x^2}{2Rz}+\frac{z}{2R}\right)(d\varphi^2-d\varphi^1)\right] \\
&\quad+\tfrac{1}{2}(N^{-1/2}+L^{-1/2})\,dt+O(1)\,dt+O(1)\,d\varphi \\
&= dM+\frac{1}{2}\left(\frac{x^2(y)}{Rz}-\frac{x^2}{R}-\frac{y^2}{R}\right)d\varphi \\
&\quad-\tfrac{1}{2}(L^{-1/2}+N^{-1/2})|y|\left(\frac{x^2}{Rz}+\frac{z}{R}\right)d\varphi+\tfrac{1}{2}(L^{-1/2}-N^{-1/2})\,y\,d\varphi \\
&\quad+\tfrac{1}{2}(L^{-1/2}-N^{-1/2})\left(\frac{x^2}{z}+z\right)(d\varphi^2-d\varphi^1)+O(1)\,dt+O(1)\,d\varphi \\
&= dM+\frac{1}{2R}(L^{-1/2}+N^{-1/2})(-x^2-|y|\,z-y^2)\,d\varphi \\
&\quad+\left(\frac{x^2|y|}{z}+|y|\,z+y^2\right)\frac{2R}{(LN)^{-1/2}(L^{+1/2}+N^{1/2})}\,d\varphi \\
&\quad+\tfrac{1}{2}(N^{-1/2}+L^{-1/2})\,dt+O(1)\,d\varphi+O(1)\,dt.
\end{aligned}$$

Compare this expression with (19). Call $\alpha(\beta)$ the first two $d\varphi$ terms in $du(dw)$. Then,

$$\begin{aligned}
\beta &= \frac{(LN)^{1/2}}{2(L^{1/2}+N^{1/2})}\,\alpha \\
&\quad-\frac{1}{R}\left(\frac{L+N}{2(LN)^{-1/2}(L^{+1/2}+N^{1/2})}-\frac{L^{1/2}+N^{1/2}}{2L^{1/2}N^{1/2}}\right)(x^2+y^2+z|y|)\,d\varphi \\
&= \frac{(LN)^{1/2}}{2(L^{1/2}+N^{1/2})}\,\alpha+\frac{1}{R(L^{1/2}+N^{1/2})}(x^2+y^2+z|y|)\,d\varphi.
\end{aligned}$$

Since L and N cannot become small simultaneously, we can conclude that

$$dw_t = dM_t+c_t\,d\varphi_t+O(1)\,dt+\tfrac{1}{2}(L_t^{-1/2}+N^{-1/2})\,dt$$

with $c_t \geqq -c$.

Since w_t ifself is bounded, if $T<\infty$ and $\int_0^T (L_t^{-1/2}+N_t^{-1/2})\,dt=\infty$, then in view of the bound $c_t \geqq -c$ and the fact that the dt coefficient is $O(1)$, $\inf_{t<T} M_t = -\infty$ must happen. Then, since M is a local martingale $\sup_{t<T} M_t=\infty$ must occur. However, this violates the boundeness of w_t. Therefore $\int_0^T (L_t^{-1/2}+N^{-1/2})\,dt<\infty$

on $\{T<\infty\}$ and the proof is complete. Recall this also completes the proof of Lemma 3.

Proof of Theorem 1. Now examine the Itô expansion of $v_t=\operatorname{Log}(L_t N_t z_t^2)$ obtained from (5) and (20)

$$d v_t=d M_t+\frac{|y_t|}{R_t z_t}\left(\frac{4x_t^2 R_t^2-L_t N_t}{L_t N_t}\right) d\varphi_t-\frac{1}{R_t} d\varphi_t+O(z_t)\, d\varphi_t-\frac{1}{R_t}\left(\frac{x_t^2}{L_t}+\frac{y_t^2}{N_t}\right) d\varphi_t$$
$$+\frac{(y_t^2+|y_t| z_t)}{R_t L_t N_t}(4R_t^2-(L_t+N_t))\, d\varphi_t+(\text{remaining terms at (20)}).$$

The new and potentially troublesome term is

$$\frac{|y_t|}{R_t z_t}\left(\frac{4x_t^2 R_t^2-L_t N_t}{L_t N_t}\right) d\varphi_t. \tag{21}$$

However, this term only contributes on Supp $d\varphi$ and on this set of times

$$\begin{aligned}4x_t^2 R_t^2-L_t N_t&=4x_t^2 R_t^2-(x_t^2+y_t^2+R_t^2)^2+4y_t^2 R_t^2\\&=-(x_t^2+y_t^2-R_t^2)^2\\&=O(z_t^2)\quad\text{(from Lemma 1)}.\end{aligned}$$

Thus the expression (21) is $O(z_t)\, d\varphi_t$.

So proceeding as in the proof of Lemma 3, on $\{T<\infty)$,

$$d v_t=d M_t+c_t\, d\varphi_t+e_t\, dt+(\text{remaining terms at (20)})$$

we see by Lemma 3 that $c_t\geqq c_0>-\infty$ on $\{T<\infty\}$. The term (remaining terms at (20)) were already found to be well-behaved in Lemma 3.

Thus $\inf_{t<T} v_t=-\infty$ is impossible on $\{T<\infty\}$.

This proves the theorem since M_t cannot explode towards $-\infty$ only.

Appendix

The fact of noncoalescence may be used to derive an upper bound on the rate at which z_t tends of 0. Namely,

$$\lim_{t\to\infty}\frac{1}{t}\log z_t\leqq-\frac{1}{2}\left[K^2+\frac{1}{m(D)}\int_{\partial D} K(y)\,\sigma(dy)\right].$$

Begin by observing that z_t is nonincreasing. This and noncoalescence, together with

$$z_t=z_0-\frac{1}{2}\int_0^t\left(\frac{|y_s|+z_s}{R_s}\right) d\varphi_s+\int_0^t O(z_s^2)\, d\varphi_s$$

imply

$$\int_0^\infty \frac{|y_s|}{R_s}\, d\varphi_s < \infty. \tag{22}$$

From the formula for $d y_t$, it easily follows that

$$d(y_t^2) = 2 y_t\, d W_t^2 + dt - \frac{y_t^2}{R_t}\, d\varphi_t - \left[\frac{x_t^2 |y_t|}{R_t z_t} + \frac{|y_t| z_t}{R_t}\right] d\varphi_t$$

which implies, with (22),

$$\lim_{\gamma \to \infty} \frac{1}{t} \int_0^t \frac{x_s^2 |y_s|}{R_s z_s}\, d\varphi_s = 1. \tag{23}$$

Now the expression above for z_t makes it clear that

$$\log z_t = \log z_0 - \frac{1}{2} \int_0^t \left(\frac{|y_s| + z_s}{z_s R_s}\right) d\varphi_s + \int_0^t O(z_s)\, d\varphi_s$$

whence

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log z_t &= \frac{1}{2} \lim_{t\to\infty} \frac{1}{t} \int_0^t \left(\frac{|y_s|}{z_s R_s} + K_s\right) d\varphi_s \\ &\leqq -\frac{1}{2}\left[\lim_{t\to\infty} \frac{1}{t} K^2 \int_0^t \frac{x_s^2 |y_s|}{z_s R_s}\, d\varphi_s + \frac{1}{m(D)} \int_{\partial D} K(y)\, \sigma(dy)\right] \\ &\leqq -\frac{1}{2}\left[K^2 + \frac{1}{m(D)} \int_{\partial D} K(y)\, \sigma(dy)\right] \end{aligned}$$

where we have used (23), the convergence of the measure $\frac{1}{t}\varphi_t(\cdot)$ to $\frac{1}{m(D)}\sigma(\cdot)$ and the bound $1 \geqq \frac{x_s^2}{R_s^2} = K_s^2 x_s^2 \geqq K^2 x_s^2$.

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