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Markov branching processes with instantaneous immigration

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Summary. Markov branching processes with instantaneous immigration possess the property that immigration occurs immediately the number of particles reaches zero, i.e. the conditional expectation of sojourn time at zero is zero. In this paper we consider the existence and uniqueness of such a structure. We prove that if the sum of the immigration rates is finite then no such structure can exist, and we provide a necessary and sufficient condition for existence for the case in which this sum is infinite. Study of the uniqueness problem shows that for honest processes the solution is unique.

1. Introduction

In this paper the continuous time homogeneous Markov branching process is considered exclusively and is abbreviated to CMBP. A CMBP is a temporally homogeneous Markov chain whose states are nonnegative integers and whose transition probabilities are a solution to the forward equations

$$dp_{ij}(t)/dt = \sum_{k=1}^{j+1} p_{ik}(t)\, k b_{j-k+1} \qquad (i \geqq 0, j \geqq 0, t \geqq 0) \tag{1.1}$$

where $\{b_j; j \geqq 0\}$ are constants and satisfy the conditions

$$b_1 \leqq 0, \quad b_j \geqq 0 \ (j \neq 1) \quad \text{and} \quad \sum_{j=0}^{\infty} b_j = 0. \tag{1.2}$$

Details of definitions and related results can be seen both in Harris (1963) and Athreya and Ney (1972). In particular, the transition probability of CMBP is uniquely determined by its infinitesimal generator and is the Feller minimal solution. The associated infinitesimal generator, the so-called Q-matrix $Q = (q_{ij}; i, j \geqq 0)$, is given by

$$q_{ij} = \begin{cases} i b_{j-i+1} & \text{if } j \geqq i-1 \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

where $\{b_j; j \geqq 0\}$ satisfy (1.2).

Models with state-dependent immigration were first considered by Foster (1971) and Pakes (1971, 1975, 1978). They studied a modification of the Galton-Watson process which admits an immigration component only in the state zero. The continuous time analogue of this process in the Markov case was investigated by Yamazato (1975). Further discussion is contained in Mitov et al. (1984).

In Yamazato's model, the transition probability is uniquely determined by its infinitesimal generator and is the Feller minimal solution. The generator $Q=(q_{ij}\,;\, i, j \geqq 0)$ now becomes

$$q_{ij}=\begin{cases}\alpha_j & \text{if } i=0 \text{ and } j\geqq 0\\ i b_{j-i+1} & \text{if } i\geqq 1 \text{ and } j\geqq i-1\\ 0 & \text{otherwise}\end{cases} \tag{1.4}$$

where

$$b_1\leqq 0, \quad b_j\geqq 0 \ (j\neq 1) \quad \text{and} \quad \sum_{j=0}^{\infty} b_j=0, \tag{1.5}$$

$$\alpha_0\leqq 0, \quad \alpha_j\geqq 0 \ (j\geqq 1) \quad \text{and} \quad \sum_{j=0}^{\infty} \alpha_j=0. \tag{1.6}$$

Note that $\{\alpha_j\,;\, j\geqq 1\}$ denotes the immigration rates, and condition (1.6) requires

$$\alpha_0>-\infty \tag{1.7}$$

or

$$\sum_{j=1}^{\infty} \alpha_j<+\infty. \tag{1.8}$$

Intuitively, condition (1.8) means that the sum of the immigration rates is "small"; the process will stay in state 0 for some positive time. It is therefore both natural and interesting to ask what will happen if the sum of the immigration rates is not "small", i.e. if

$$\sum_{j=1}^{\infty} \alpha_j=+\infty. \tag{1.9}$$

Note that (1.9) automatically forces the condition

$$\alpha_0=-\infty, \tag{1.10}$$

which means that immigration will occur immediately the number of particles reaches zero. More precisely, let $Z(t, \omega)$ denote the number of particles at time t (assuming that such a process exists). Since the transition function of $Z(t, \omega)$ is standard, we may assume that the process $Z(t, \omega)$ is separable. Hence by a well-known theorem (see, for example, Theorem 5.5, II, of Chung 1967) for any $s\geqq 0$ and $t>0$ we have

$$\Pr\{Z(u,\omega)\equiv 0, s<u<s+t \mid Z(s,\omega)=0\}=\exp(-\alpha_0 t) \tag{1.11}$$

irrespective of whether α_0 is finite or not. Let τ be the sojourn time at zero, and $E_0(.)$ be the conditional expectation of staying at state zero. Expressions (1.10) and (1.11) then lead to

$$E_0[\tau]=0. \tag{1.12}$$

We shall call such a process a continuous time homogeneous Markov branching process with instantaneous immigration and abbreviate it to CMBP-II. An exact definition will be given in Sect. 2.

Note that the case of instantaneous immigration produces a new phenomenon since the powerful result of Feller's minimal solution is no longer valid. Not only do we not now know whether there exists a transition function $P(t)$ which satisfies the branching property, but we also do not even know if the Q satisfying (1.9) is now an infinitesimal generator for general Markov chains (not necessarily branching chains). Indeed, from the point of view of Markov chain theory (see, for example, Chung 1967), zero is now an instantaneous state and, as is well known, few results have been obtained for such instantaneous chains. Although there is a long history of development associated with construction theory for Markov chains, i.e. existence, uniqueness and the construction of the transition function for a given Q-matrix, most of the results and methods are concerned only with the so-called totally stable case, i.e. the case where there exists no instantaneous state. Moreover, generalization of these results to cover the instantaneous case seems to be very difficult. As far as we know, only two examples have been considered for the mixed-state case, i.e. the case where there are both instantaneous and stable states, to which our CMBP-II belongs. (See the discussion of Kendall and Reuter (1954), Reuter (1969), Chung (1967) and Williams (1967)). Thus our CMBP-II problem cannot be covered by known general theorems of Markov chain theory, and so we need to proceed from a fresh start.

2. Definitions and basic results

Let $E=\{0, 1, 2, \ldots\}$ and $N=E\setminus\{0\}=\{1, 2, 3, \ldots\}$.

Definition 2.1. A matrix $Q=(q_{ij})$ defined on $E\times E$ is called a *CMBP-II pre-generator*, or a *CMBP-II pre-Q-matrix*, if

$$q_{ij}=\begin{cases}-\infty & \text{if } i=j=0\\ \alpha_j & \text{if } i=0, j\geqq 1\\ ib_{j-i+1} & \text{if } i\geqq 1, j\geqq i-1\\ 0 & \text{otherwise,}\end{cases} \tag{2.1}$$

where

$$\alpha_j\geqq 0 \quad (j\geqq 1), \tag{2.2}$$

$$-\infty<b_1\leqq 0, \quad b_j\geqq 0\ (j\neq 1) \quad \text{and} \quad \sum_{j=0}^{\infty} b_j=0. \tag{2.3}$$

Remark. $\{\alpha_j;\ j\geqq 1\}$ denotes the "immigration rate" whilst $\{b_j;\ j\neq 1\}$ denotes the "branching rate". Note that both the cases

$\sum_{j=1}^{\infty}\alpha_j=+\infty$ and $\sum_{j=1}^{\infty}\alpha_j<+\infty$ are allowed for the time being.

Definition 2.2. A continuous time homogeneous Markov branching process with instantaneous immigration (CMBP-II) is a Markov chain whose state space is the set E of nonnegative integers and whose transition function $P(t)=\{p_{ij}(t);\ i,j\in E\}$ satisfies the following two conditions:

$$\lim_{t\to 0^+}\frac{p_{00}(t)-1}{t}=-\infty, \tag{2.4}$$

$$dp_{ij}(t)/dt=\sum_{k\in E}p_{ik}(t)\,q_{kj}\qquad (i\geqq 0, j\geqq 1, t\geqq 0), \tag{2.5}$$

where $Q=\{q_{ij}\}$ is a CMBP-II pre-generator.

Remark 1. As in Harris (1963), we have defined the CMBP-II only in the sense that we have defined the transition function $P(t)$. We shall therefore call the transition function $P(t)$ a "process".

Remark 2. Note that the inequality $P(t)1\leqq 1$ is allowed for transition functions. If the equality $P(t)1=1$ holds, then we call the transition function (or process) "honest".

Remark 3. The customary method of studying ordinal branching processes (with or without immigration) by introducing a generating function through (2.5) cannot be applied to our case, at least for the time being, since the immigration generating function $\sum_j \alpha_j s^j$ ($|s|<1$) is not well-defined because $\sum_j \alpha_j=+\infty$.

For each CMBP-II pre-generator Q defined in (2.1), we associate two other matrices $Q^*=\{q^*_{ij}\}$ and $\tilde{Q}=\{\tilde{q}_{ij}\}$ defined on $E\times E$ and $N\times N$, respectively, as follows:

$$q^*_{ij}=\begin{cases}0 & \text{if } i=0\\ q_{ij} & \text{if } i\geqq 1\end{cases} \tag{2.6}$$

$$\tilde{q}_{ij}=q_{ij}\qquad (i\geqq 1, j\geqq 1). \tag{2.7}$$

That is, $\tilde{Q}$ is the restriction of Q on $N\times N$, whilst Q^* is the corresponding branching generator without immigration related to Q.

Since both Q^* and $\tilde{Q}$ are totally stable, they are certainly generators. Let $F^*(t)$ and $\tilde{F}(t)$ denote the Feller minimal transition functions (see Feller 1940) of Q^* and $\tilde{Q}$, respectively. Note that $F^*(t)$ is the transition function of a branching process without immigration, so we know its properties quite well (see, for example, Harris 1963 or Athreya and Ney 1972). We shall call $F^*(t)$ the corresponding branching process without immigration related to the original CMBP-II.

For a transition function $P(t)$, we can introduce the resolvent, i.e. its Laplace transform $R(\lambda)=\{r_{ij}(\lambda);\ i,j\in E\}$, where

(2.8) $$r_{ij}(\lambda)=\int_0^\infty e^{-\lambda t}p_{ij}(t)\,dt \qquad (\lambda>0).$$

It is well-known that there is a 1–1 correspondence between the transition function $P(t)$ and its resolvent $R(\lambda)$. Thus, following Reuter (1957), we shall call a resolvent $R(\lambda)$ a "process" just as we call a transition function $P(t)$ a "process".

Similarly, for a CMBP-II pre-generator Q, we shall use $\phi^*(\lambda)$ and $\tilde{\phi}(\lambda)$ to denote the Feller minimal resolvent of Q^* and $\tilde{Q}$, respectively. We call $\phi^*(\lambda)$ the corresponding branching process without immigration related to the original CMBP-II.

The main results of this paper are contained in the following three theorems which discuss the existence and uniqueness of CMBP-II.

Theorem 2.1. *For a given pre-generator defined in Definition* 2.1, *if*

(2.9) $$\sum_{j=1}^{\infty}\alpha_j<+\infty,$$

then there exists no branching process with instantaneous immigration (*CMBP-II*).

Theorem 2.2. *For a given pre-generator defined in Definition* 2.1, *if*

(2.10) $$\sum_{j=1}^{\infty}\alpha_j=+\infty,$$

then there exists a CMBP-II if and only if

(2.11) $$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\alpha_j\,\phi^*_{jk}(\lambda)<+\infty \qquad (\lambda>0)$$

where $\phi^*(\lambda)=\{\phi^*_{ij}(\lambda);\ i,j\in E\}$ *is the corresponding branching process without immigration. Furthermore, if* (2.11) *holds then there exists an honest CMBP-II.*

A direct consequence of Theorem 2.2 is:

Corollary 2.3. *If there exists a CMBP-II, then the corresponding CMBP without immigration must be an explosive one, i.e.*

(2.12) $$\lambda\sum_{j=0}^{\infty}\phi^*_{ij}(\lambda)<1 \qquad (\forall i\neq 0,\ \forall\lambda>0);$$

or, equivalently,

(2.13) $$\sum_{j=0}^{\infty}f^*_{ij}(t)<1 \qquad (\forall i\neq 0,\ \forall t>0).$$

Here $F^*(t)=\{f^*_{ij}(t);\ i,j\in E\}$ *denotes the Feller minimal transition function of* Q^*.

As regards the uniqueness problem, we have:

Theorem 2.4. *In the case of* (2.10), *if the existence condition* (2.11) *holds, then*

(i) *there exist infinitely many CMBP-IIs,*
(ii) *there exists only one honest CMBP-II,*

i.e. uniqueness holds true for the honest process.

The proofs of Theorem 2.1, Theorem 2.2 and Corollary 2.3 are given in Sect. 3, whilst the proof of Theorem 2.4 is given in Sect. 6.

3. Proof of existence theorem

Before reading Sects. 3 to 6, readers are requested to skim over the terminology, notation and basic conclusions of Sect. 7, especially Theorems 7.7, 7.8 and 7.10. We shall use these basic results without further explanation.

The main task of this section is to prove the existence theorem of CMBP-II, i.e. Theorems 2.1 and 2.2. Let Q, Q^* and $\tilde{Q}$ be given as in (2.1), (2.6) and (2.7).

Lemma 3.1. *The equation*

$$\begin{cases} v(\lambda I-\tilde{Q})=0 \\ 0\leqq v\in l \end{cases} \tag{3.1}$$

has only the zero solution where l, as usual, denotes the space of summable vectors. Hence $v\in l$ means $v=\{v_j; j\in N\}$ and $\Sigma|v_j|<\infty$.

Remark 1. Although the idea surrounding Lemma 3.1 has been considered by Harris (1963) using generating functions, here we shall give a direct proof.

Remark 2. By Reuter (1957), we know that the dimension of the solution space of Eq. (3.1) is independent of $\lambda>0$, i.e. Eq. (3.1) has only the zero solution for each $\lambda>0$ if and only if it has only the zero solution for some $\lambda_0>0$.

Proof of Lemma 3.1. Suppose (3.1) has a non-zero solution $v=\{v_j;\ j\geqq 1\}$ for some $\lambda_0>0$, then

$$\lambda_0 v_j=\sum_{k=1}^{j+1} v_k k b_{j-k+1} \qquad (j\geqq 1);$$

or, if we let $c_i=b_i(i+1)$ and $c_1=-b_1$,

$$(\lambda_0+2jc_1)v_j=\sum_{k=1}^{j+1} k v_k c_{j+1-k} \qquad (j\geqq 1). \tag{3.2}$$

Now on noting that $c_i\geqq 0$ $(\forall i\geqq 0)$ and $v_i\geqq 0$ $(\forall i\geqq 1)$, we have from (3.2) that

$$\left(\frac{\lambda_0}{j}+2c_1\right)v_j=\sum_{k=1}^{j}\frac{k}{j}v_k c_{j+1-k}+\frac{j+1}{j}v_{j+1}c_0 \qquad (j\geqq 1),$$

whence

$$\left(\frac{\lambda_0}{j}+2c_1\right)v_j \leqq \sum_{k=1}^{j} v_k c_{j+1-k}+\frac{j+1}{j}v_{j+1}c_0 \qquad (j\geqq 1). \tag{3.3}$$

Summing over $j\geqq 1$ in (3.3) then gives

$$\begin{aligned}\sum_{j=1}^{\infty}\left(\frac{\lambda_0}{j}+2c_1\right)v_j &\leqq \sum_{j=1}^{\infty}\sum_{k=1}^{j} v_k c_{j+1-k}+\sum_{j=1}^{\infty}\left(1+\frac{1}{j}\right)v_{j+1}c_0 \\ &\leqq\left(\sum_{j=1}^{\infty} v_j\right)\left(\sum_{j=0}^{\infty} c_j\right)+c_0\left(\sum_{j=1}^{\infty}\frac{v_{j+1}}{j}\right).\end{aligned} \tag{3.4}$$

As $\sum_{j=1}^{\infty} v_j<+\infty$ (see (3.1), $v\in l$) and $\sum_{j=0}^{\infty} c_j=2c_1<\infty$ (see (2.3)), we therefore have

$$\lambda_0\sum_{j=1}^{\infty}\frac{v_j}{j}\leqq c_0\sum_{j=1}^{\infty}\frac{v_{j+1}}{j}\leqq 2c_0\sum_{j=1}^{\infty}\frac{v_j}{j}. \tag{3.5}$$

Since $\sum_{j=1}^{\infty} v_j<+\infty$, it follows that $\sum_{j=1}^{\infty}\frac{v_j}{j}<+\infty$, and hence by (3.5) that $\lambda_0\leqq 2c_0$, contrary to the fact that the dimension of the solution space of (3.1) is independent of λ. □

We shall now prove Theorems 2.1 and 2.2 simultaneously.

Proof. Suppose there exists a CMBP-II, say $R(\lambda)=\{r_{ij}(\lambda)\}$. Then according to Theorem 7.7 it can be uniquely decomposed into

$$R(\lambda)=\begin{bmatrix}0 & 0\\ 0 & \psi(\lambda)\end{bmatrix}+r_{bb}(\lambda)\begin{bmatrix}1\\ \xi(\lambda)\end{bmatrix}[1,\eta(\lambda)], \tag{3.6}$$

and all the other conclusions of Theorem 7.7 are also satisfied. In particular, it follows from (7.32) that

$$\eta(\lambda)\in H_{\psi}. \tag{3.7}$$

Since $R(\lambda)$ is a CMBP-II, we have, by (2.5) of Definition 2.2, that

$$\lambda r_{ij}(\lambda)-\delta_{ij}=\sum_{k=0}^{\infty} r_{ik}(\lambda)\,q_{kj} \qquad (i\geqq 0, j\geqq 1, \lambda>0) \tag{3.8}$$

where $Q=(q_{ij};\ i,\ j\geqq 0)$ is the given CMBP-II pre-generator. (Note that (3.8) is the Laplace transform version of (2.5)). Substituting (3.6) into (3.8) yields

$$\lambda\psi_{ij}(\lambda)-\delta_{ij}=\sum_{k=1}^{\infty}\psi_{ik}(\lambda)\,q_{kj} \qquad (i\geqq 1, j\geqq 1, \lambda>0), \tag{3.9}$$

and so $\psi(\lambda)=\{\psi_{ij}(\lambda);\ i,j\geqq 1\}$ satisfies the forward equation for $\tilde{Q}$. Now according to Lemma 3.1, Eq. (3.1) has only a zero solution. Hence by Theorem 6.10 of Reuter (1957) – which implies that if Eq. (3.1) has only a zero solution then there is exactly one process, namely the Feller minimal process $\tilde{\phi}(\lambda)$, which satisfies (3.9) – we know that $\psi(\lambda)$ must be the Feller minimal process $\tilde{\phi}(\lambda)$, i.e.

$$\psi(\lambda)\equiv\tilde{\phi}(\lambda). \tag{3.10}$$

Thus (3.7) becomes $\eta(\lambda)\in H_{\tilde{\phi}}$, or by (7.17)

$$\begin{cases}\eta(\lambda)-\eta(\mu)=(\mu-\lambda)\,\eta(\lambda)\,\tilde{\phi}(\mu) & (\lambda,\mu>0)\\ 0\leqq\eta(\lambda)\in l.\end{cases} \tag{3.11}$$

Note that by Lemma 2.2 of Reuter (1959), (3.11) is true if and only if $\eta(\lambda)$ has the form

$$\eta(\lambda)=\tilde{\alpha}\tilde{\phi}(\lambda)+\bar{\eta}(\lambda) \qquad (\lambda>0)$$

where $\tilde{\alpha}=\lim_{\lambda\to\infty}\lambda\eta(\lambda)$ and $\bar{\eta}(\lambda)$ is a solution of Eq. (3.1).

Using Lemma 3.1 once again, we obtain $\bar{\eta}(\lambda)\equiv 0$, and so $\eta(\lambda)=\tilde{\alpha}\tilde{\phi}(\lambda)$. Now Theorem 7.7 (see (7.34)) and $\lim_{\lambda\to\infty}\lambda\eta(\lambda)=\tilde{\alpha}$ combine to show that $\tilde{\alpha}$ is the first row excluding the first element of CMBP-II pre-generator Q. So if we let $\alpha=\{\alpha_j;\ j\geqq 1\}$ stand for the vector in Definition 2.1, then $\tilde{\alpha}\equiv\alpha$. Hence

$$\eta(\lambda)=\alpha\tilde{\phi}(\lambda). \tag{3.12}$$

Moreover, if $\sum_{j=1}^{\infty}\alpha_j<+\infty$ as in (2.9), then it is easy to show that

$$\lim_{\lambda\to\infty}\lambda\eta(\lambda)1=\lim_{\lambda\to\infty}\lambda\alpha\tilde{\phi}(\lambda)1=\sum_{j=1}^{\infty}\alpha_j<+\infty$$

which contradicts Theorem 7.7 (see (7.38)). This ends the proof of Theorem 2.1.

To prove Theorem 2.2 we note that if there is a CMBP-II then $\eta(\lambda)\in l$ (see (3.11)) is just $\alpha\tilde{\phi}(\lambda)\in l$ (see (3.12)), i.e.

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\alpha_j\,\tilde{\phi}_{jk}(\lambda)<+\infty \qquad (\lambda>0). \tag{3.13}$$

Now it is easy to show by the existence theorem of Feller (1940) that the Feller minimal processes $\phi^*(\lambda)$ and $\tilde{\phi}(\lambda)$ (related to Q^* and $\tilde{Q}$, respectively) have the following relations because of the special form of Q^* and $\tilde{Q}$:

$$\phi^*_{ij}(\lambda)=\begin{cases}1/\lambda & \text{if } i=j=0\\ 0 & \text{if } i=0; j\geqq 1\\ \tilde{\phi}_{ij}(\lambda) & \text{if } i\geqq 1, j\geqq 1\\ b_0\,\tilde{\phi}_{i1}(\lambda)/\lambda & \text{if } i\geqq 1, j=0.\end{cases} \tag{3.14}$$

Thus (3.13) holds true if and only if

$$(3.15)\qquad \sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\alpha_j\,\phi^*_{jk}(\lambda)<+\infty\qquad(\lambda>0).$$

This ends the necessity part of Theorem 2.2. We shall now prove sufficiency.

Suppose (2.11) is true. Then by (3.14) we have

$$(3.16)\qquad \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\alpha_j\,\tilde{\Phi}_{jk}(\lambda)<+\infty\qquad(\lambda>0).$$

If we define $\eta(\lambda)=\alpha\tilde{\Phi}(\lambda)$ where $\tilde{\Phi}(\lambda)$ is the Feller minimal process related to $\tilde{Q}$ and $\alpha=\{\alpha_j;\,j\geqq 1\}$ is the first row of Q excluding the diagonal element, then it is easy to show that

$$\eta(\lambda)-\eta(\mu)=(\mu-\lambda)\,\eta(\lambda)\,\tilde{\Phi}(\mu).$$

Combining this result with (3.16) yields

$$(3.17)\qquad \eta(\lambda)\in H_{\tilde{\Phi}}.$$

By Reuter (1962) we know that

$$(3.18)\qquad \lim_{\lambda\to\infty}\lambda\eta(\lambda)=\lim_{\lambda\to\infty}\lambda\alpha\tilde{\Phi}(\lambda)=\alpha.$$

By (3.18) and Fatou's lemma we have

$$\lim_{\lambda\to\infty}\inf\lambda\eta(\lambda)1\geqq\sum_{j=1}^{\infty}\lim_{\lambda\to\infty}\inf\lambda\eta_j(\lambda)=\sum_{j=1}^{\infty}\alpha_j=+\infty.$$

Hence

$$(3.19)\qquad \lim_{\lambda\to\infty}\lambda\eta(\lambda)1=+\infty.$$

Note also that $\tilde{\Phi}(\lambda)$ is the Feller minimal solution related to $\tilde{Q}$. Following Reuter (1962), we therefore have

$$\lim_{\lambda\to\infty}\lambda\left[1-\lambda\sum_{j=1}^{\infty}\tilde{\Phi}_{ij}(\lambda)\right]=d_i\qquad(i\geqq 1)$$

where d_i is the "deficiency" of $\tilde{Q}$ in its i-th row. Thus

$$(3.20)\qquad \lim_{\lambda\to\infty}\lambda\left[1-\lambda\sum_{j=1}^{\infty}\tilde{\Phi}_{ij}(\lambda)\right]=\begin{cases}b_0 & (i=1)\\ 0 & (i\geqq 2)\end{cases}$$

is just the first column (excluding r_{00}) of the given Q.

As expressions (3.17)–(3.20) show that all the conditions of Theorem 7.10 are satisfied, there exists an honest Q-process where Q is just the given CMBP-II pre-generator. By the same theorem, this honest Q-process $R(\lambda)$ is constructed (using the Feller minimal $\tilde{Q}$-process $\tilde{\Phi}(\lambda)$ and $\eta(\lambda)=\alpha\tilde{\Phi}(\lambda)$) as follows:

(3.21) $$r_{00}(\lambda)=[\lambda+\lambda\eta(\lambda)1]^{-1}$$

(3.22) $$R(\lambda)=\begin{bmatrix}0 & 0\\ 0 & \tilde{\Phi}(\lambda)\end{bmatrix}+r_{00}(\lambda)\begin{bmatrix}1\\ 1-\lambda\tilde{\Phi}(\lambda)1\end{bmatrix}[1,\eta(\lambda)].$$

It is easy to verify that this honest Q-process $R(\lambda)=\{r_{ij}(\lambda);\ i,\ j\geqq 0\}$ is a CMBP-II. The Laplace transform version of (2.4) is

(3.23) $$\lim_{\lambda\to\infty}\lambda(\lambda r_{00}(\lambda)-1)=-\infty.$$

This follows since, by (3.19), (3.21) and Lemma 7.2 (ii), $\lim_{\lambda\to\infty}\lambda(\lambda r_{00}(\lambda)-1)$ $=\lim_{\lambda\to\infty}\dfrac{-\lambda\eta(\lambda)1}{1+\eta(\lambda)1}=-\infty$, whilst (2.5) is equivalent to

(3.24) $$\lambda r_{ij}(\lambda)-\delta_{ij}=\sum_{k=0}^{\infty} r_{ik}(\lambda)\,q_{kj}\qquad (i\geqq 0, j\geqq 1, \lambda>0).$$

The truth of (3.24) may be shown through the following argument (here $Q=(q_{ij})$ is the given CMBP-II pre-generator Q).

Since $\tilde{\Phi}(\lambda)$ is the Feller minimal $\tilde{Q}$-process, by Feller (1940) or Reuter (1957) it must satisfy the forward equation related to $\tilde{Q}$, viz.

(3.25) $$\lambda\tilde{\Phi}_{ij}(\lambda)-\delta_{ij}=\sum_{k=1}^{\infty}\tilde{\Phi}_{ik}(\lambda)\,\tilde{q}_{kj}\qquad (i\geqq 1, j\geqq 1, \lambda>0).$$

Moreover, since $\eta(\lambda)=\alpha\tilde{\Phi}(\lambda)$, we have by (2.14) of Reuter (1957) that

(3.26) $$\eta(\lambda)(\lambda I-\tilde{Q})=\alpha.$$

Note that the component form of (3.26) is

(3.27) $$\lambda\eta_j(\lambda)=\alpha_j+\sum_{k=1}^{\infty}\eta_k(\lambda)\,\tilde{q}_{kj}.$$

Substituting both (3.25) and (3.27) into (3.22) and considering (2.7), then yields

$$\lambda r_{ij}(\lambda)-\delta_{ij}=\sum_{k=0}^{\infty} r_{ik}(\lambda)\,q_{kj}\qquad (i\geqq 0, j\geqq 1, \lambda>0),$$

where $Q=(q_{ij})$ is the given CMBP-II pre-generator. Thus (3.24) is proved to be true. □

Remark 1. In the proof of Theorem 2.2 we note that the existence condition (2.11) is equivalent to

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\alpha_j\tilde{\phi}_{jk}(\lambda)<+\infty\qquad(\lambda>0), \tag{3.28}$$

due to the form of (3.14). This representation is sometimes easier to use than (2.11).

Remark 2. It is easy to prove the following results by the resolvent equation:

$$\forall\lambda>0,\quad\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\alpha_j\tilde{\phi}_{jk}(\lambda)<+\infty\Leftrightarrow\exists\lambda_0>0,\quad\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\alpha_j\tilde{\phi}_{jk}(\lambda_0)<+\infty,$$

$$\forall\lambda>0,\quad\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\alpha_j\phi^*_{jk}(\lambda)<+\infty\Leftrightarrow\exists\lambda_0>0,\quad\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\alpha_j\phi^*_{jk}(\lambda_0)<+\infty.$$

Thus in the existence conditions (2.11) and (3.28) λ can be replaced by some fixed $\lambda_0>0$.

Proof of Corollary 2.3. Suppose (2.12), or equivalently (2.13), is not true. Then there exists $t_0>0$ and $i_0>0$ such that

$$\sum_{j=0}^{\infty}f^*_{i_0j}(t_0)=1, \tag{3.29}$$

where $F^*(t)=\{f^*_{ij}(t);\ i,j\geqq 0\}$ denotes the Feller minimal transition function of Q^*. By Lévy's (1952) dichotomy, we conclude from (3.29) that for this fixed i_0 and all $t>0$

$$\sum_{j=0}^{\infty}f^*_{i_0j}(t)=1\qquad(\forall t>0). \tag{3.30}$$

Note that $F^*(t)=\{f^*_{ij}(t)\}$ is an ordinary branching process without immigration, so

$$\sum_{j=0}^{\infty}f^*_{kj}(t)=\left(\sum_{j=0}^{\infty}f^*_{1j}(t)\right)^k\qquad(\forall k\geqq 0,\forall t>0). \tag{3.31}$$

Thus, by combining (3.30) and (3.31), we get

$$\sum_{j=0}^{\infty}f^*_{1j}(t)=1,$$

which on using (3.31) again yields

$$\sum_{j=0}^{\infty}f^*_{kj}(t)=1\qquad(\forall k\geqq 0,\forall t\geqq 0).$$

Hence

$$\lambda \sum_{j=0}^{\infty} \phi_{kj}^*(\lambda)=1 \qquad (\forall k \geqq 0, \forall \lambda \geqq 0).$$

As $\sum_{j=1}^{\infty} \alpha_j = +\infty$, we therefore have

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \alpha_j \phi_{jk}^*(\lambda) = \sum_{j=1}^{\infty} \alpha_j \left(\sum_{k=0}^{\infty} \phi_{jk}^*(\lambda) \right) = \sum_{j=1}^{\infty} \alpha_j / \lambda = +\infty$$

which contradicts the existence condition (2.11). Corollary 2.3 now follows. □

4. Feasibility

In Sect. 3 we proved that there exists a CMBP-II if and only if

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \alpha_j \phi_{jk}^*(\lambda) < +\infty \qquad (\lambda > 0), \tag{4.1}$$

where $\alpha = \{\alpha_j\}$ is the immigration rate satisfying

$$\sum_{j=1}^{\infty} \alpha_j = +\infty, \tag{4.2}$$

and $\phi^*(\lambda) = \{\phi_{ij}^*(\lambda)\}$ denotes the corresponding CMBP without immigration. We have proved that as a direct consequence of (4.1)

$$\lambda \phi^*(\lambda) 1 < 1 \qquad (\lambda > 0), \tag{4.3}$$

i.e. the corresponding CMBP without immigration is an explosive one. Here (4.3) means that $\lambda \sum_{j=0}^{\infty} \phi_{ij}^*(\lambda) < 1$ for all $i \in E$ except state 0.

However, until now, we have not shown that there exists a sequence $\alpha = \{\alpha_j\}$ which can satisfy both (4.1) and (4.2). Indeed, given a CMBP $\phi^*(\lambda)$ without immigration, does there exist a sequence $\{\alpha_j\}$ which represents the instantaneous immigration rates of some CMBP-II whose corresponding CMBP without immigration is precisely $\phi^*(\lambda)$? Let us call this the "feasibility problem". Note that this is different from the existence problem, since in the latter the immigration rates $\{\alpha_j\}$ are fixed, whilst in the former we are trying to determine the $\{\alpha_j\}$. If we cannot solve the feasibility problem then the existence condition is meaningless. Fortunately, we have

Theorem 4.1. *There exists a sequence* $\{\alpha_j\}$ *which satisfies both* (4.1) *and* (4.2) *if and only if* (4.3) *holds.*

Proof. The necessity follows directly from Corollary 2.3, so we need only to prove sufficiency. Suppose (4.3) is true. Then if we denote $\sigma(t)=\sum_{j=0}^{\infty} f_{1j}^*(t)$ where $F^*(t)=\{f_{ij}^*(t);\ i, j\geqq 0\}$ is the Feller minimal transition function of the CMBP without immigration, we have

$$0<\sigma(t)<1 \qquad (\forall t>0). \tag{4.4}$$

On using (3.31),

$$\begin{aligned}\sum_{j=0}^{\infty} \phi_{kj}^*(\lambda)&=\sum_{j=0}^{\infty}\int_0^{\infty} e^{-\lambda t} f_{kj}^*(t)\,dt=\int_0^{\infty} e^{-\lambda t}\left(\sum_{j=0}^{\infty} f_{kj}^*(t)\right)dt \\ &=\int_0^{\infty} e^{-\lambda t}\left(\sum_{j=0}^{\infty} f_{1j}^*(t)\right)^k dt=\int_0^{\infty} e^{-\lambda t}(\sigma(t))^k\,dt.\end{aligned} \tag{4.5}$$

Now, for every fixed $t>0$, we have from (4.4) that

$$(\sigma(t))^k \downarrow 0 \qquad (k\to+\infty). \tag{4.6}$$

So by the monotone convergence theorem, we obtain in virtue of (4.5)

$$\lim_{k\to\infty}\sum_{j=0}^{\infty}\phi_{kj}^*(\lambda)=\lim_{k\to\infty}\int_0^{\infty} e^{-\lambda t}(\sigma(t))^k\,dt=\int_0^{\infty} e^{-\lambda t}[\lim_{k\to\infty}(\sigma(t))^k]\,dt=0. \tag{4.7}$$

Result (4.7), combined with

$$\sum_{j=0}^{\infty}\phi_{kj}^*(\lambda)\geqq\phi_{kk}^*(\lambda)>0 \qquad (\forall k\geqq 0, \forall\lambda>0), \tag{4.8}$$

shows that there exists an infinite set $\tilde{N}\subset N\equiv\{1, 2, 3, \ldots\}$ such that

$$\sum_{k\in\tilde{N}}\sum_{j=0}^{\infty}\phi_{kj}^*(\lambda)<+\infty. \tag{4.9}$$

Let

$$\alpha_j=\begin{cases}1 & \text{if } j\in\tilde{N}\\ 0 & \text{if } j\in N\setminus\tilde{N}.\end{cases}$$

Then

$$\sum_{j=1}^{\infty}\alpha_j=\sum_{j\in\tilde{N}}\alpha_j=\sum_{j\in\tilde{N}}1=+\infty$$

and

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\alpha_j\phi_{jk}^*(\lambda)&=\sum_{j=1}^{\infty}\alpha_j\left(\sum_{k=0}^{\infty}\phi_{jk}^*(\lambda)\right)\\ &=\sum_{j\in\tilde{N}}\left(\sum_{k=0}^{\infty}\phi_{jk}^*(\lambda)\right)<+\infty \qquad (\text{by (4.9)}). \quad \square\end{aligned}$$

Theorem 4.1 shows that for any given explosive Markov branching process without immigration, and only for explosive ones, there exists a Markov branching process with instantaneous immigration whose corresponding CMBP without immigration is the given one. Thus the discussion of Markov branching processes with instantaneous immigration is both meaningful and necessary. Note that we know when a Markov branching process is explosive, since it is easy to check whether (4.3) holds. We may use, for example, the "canonical" condition of Harris (1963) or the more recent results of Doney (1984) and Schuh (1982) (see next section for details).

5. Some corollaries

Although the existence criterion for CMBP-II has been given in Sect. 2, it is not easy to apply in certain cases. Let us therefore provide some necessary conditions which are more convenient to use.

Suppose the CMBP-II pre-generator Q is given by (2.1) with

$$\sum_{j=1}^{\infty} \alpha_j = +\infty. \tag{5.1}$$

Corollary 5.1. *If there exists a CMBP-II then*

$$\sum_{j=1}^{\infty} \alpha_j \sigma^j(t) < +\infty \qquad (\forall t > 0), \tag{5.2}$$

where

$$\sigma(t) = \sum_{j=0}^{\infty} f_{1j}^*(t) \qquad (\forall t > 0), \tag{5.3}$$

and $F^*(t) = \{f_{ij}^*(t)\}$ is the transition function of the corresponding branching process without immigration.

Remark. From Corollary 2.3, we know that

$$\sigma(t) < 1 \qquad (\forall t > 0). \tag{5.4}$$

Proof of Corollary 5.1. Suppose there exists a CMBP-II. Then by Theorem 2.2

$$\sum_{j=1}^{\infty} \alpha_j \sum_{k=0}^{\infty} \phi_{jk}^*(\lambda) < +\infty \qquad (\forall \lambda > 0)$$

i.e.

$$\sum_{j=1}^{\infty} \alpha_j \int_0^{\infty} e^{-\lambda t} \left(\sum_{k=0}^{\infty} f_{jk}^*(t) \right) dt < +\infty \qquad (\forall \lambda > 0).$$

But

$$\sum_{k=0}^{\infty} f_{jk}^*(t) = \left(\sum_{k=0}^{\infty} f_{1k}^*(t) \right)^j = (\sigma(t))^j,$$

and so

$$\int_0^\infty e^{-\lambda t}\left[\sum_{j=1}^\infty \alpha_j(\sigma(t))^j\right]dt<+\infty \qquad (\forall \lambda>0).$$

Thus

$$\sum_{j=1}^\infty \alpha_j(\sigma(t))^j<+\infty \qquad (\text{a.e. } t>0). \tag{5.5}$$

Since it is easy to show that $\sigma(t)$ is nonincreasing, we obtain

$$\sum_{j=1}^\infty \alpha_j\sigma^j(t)<+\infty \qquad (\forall t>0). \quad \square$$

Corollary 5.2. *If there exists a CMBP-II then*

$$\sum_{j=1}^\infty \alpha_j s^j<+\infty \qquad (|s|<1). \tag{5.6}$$

So if we define the immigration generating function as $h(s)=\sum_{j=1}^\infty \alpha_j s^j$ *then* $h(s)$ *is well-defined for each* $|s|<1$, *whence* $\limsup_{n\to\infty}(\alpha_n)^{1/n}=1$.

Proof. This result follows directly from Corollary 5.1 and the fact that $\sigma(t)\to 1$ as $t\to 0$. That $\limsup_{n\to\infty}(\alpha_n)^{1/n}=1$ then follows from (5.6), the Cauchy-Hadamard formula, and the fact that $\sum_{j=1}^\infty \alpha_j=+\infty$. $\square$

Remark. In Remark 3 following Definition 2.2 we pointed out that the immigration generating function could not be used straightaway. However, since we have now proved that the immigration generating function $h(s)$ is well-defined (as long as the CMBP-II exists), we are now justified in using the generating function approach to study the CMBP-II.

Corollary 5.3. *If there exists a CMBP-II then the corresponding CMBP without immigration is explosive, i.e.*

$$\sum_{j=0}^\infty f_{ij}^*(t)<1 \qquad (\forall t>0, \forall i>0). \tag{5.7}$$

Condition (5.7) *is equivalent to each of the following statements*

(i) *the integral* $\int_{1-\varepsilon}^1 \frac{ds}{g(s)-s}$ *converges for each small* $\varepsilon>0$, *i.e.*

$$\int_{1-\varepsilon}^1 \frac{ds}{g(s)-s}<+\infty; \tag{5.8}$$

in particular, by Harris (1963) *we have* $g'(1) = +\infty$ *where* $g(s)$ *is the generating function of the corresponding CMBP without immigration.*

(ii) $\sum_{n=1}^{\infty} \{nl(n)\}^{-1} < +\infty$ *where* $l(n) = \sum_{r=0}^{n} \Pr(M > r)$ *and* M *denotes the typical family size of the corresponding CMBP without immigration.*

(iii) *The equation* $\begin{cases}(\lambda I - \tilde{Q})U = 0 \\ 0 \leqq U \leqq 1\end{cases}$ *has a non-zero solution.*

Proof. Result (5.7) is just (2.13) of Corollary 2.3. That (5.7) is equivalent to (i) follows from Harris (1963), to (ii) from Doney (1984) and Schuh (1982), and to (iii) from Reuter (1959). □

Corollary 5.4. *If there exists a CMBP-II then*

(5.9) $$\sum_{n=1}^{\infty} \frac{\alpha_n}{n} < +\infty.$$

In particular,

(5.10) $$\liminf_{n \to \infty} (\alpha_n) = 0.$$

So if $\lim_{n \to \infty} \alpha_n$ *exists then it must be zero.*

Proof. From the existence condition (2.11), we have $\sum_{j=1}^{\infty} \alpha_j \phi_{jj}^*(\lambda) < \infty$. Now from the well-known inequality $p_{ii}(t) \geqq \exp(-q_i t)$ we know that

$$\phi_{jj}^*(\lambda) \geqq [\lambda + q_j]^{-1} = [\lambda + j(-b_1)]^{-1},$$

and so

(5.11) $$\sum_{j=1}^{\infty} \frac{\alpha_j}{\lambda + (-b_1)j} < +\infty.$$

It is easy to see that (5.11) is true if and only if

(5.12) $$\sum_{j=1}^{\infty} \frac{\alpha_j}{j} < +\infty.$$

Hence (5.9) is proved true, whence (5.10) follows as a direct consequence. □

Remark. Result (5.10) means there exists an infinite subset $K \subset E$ such that $\sum_{n \in K} \alpha_n < +\infty$. Thus although $\sum_{n=1}^{\infty} \alpha_n$ is divergent, it cannot diverge "too much". For example, if $\forall n\, \alpha_n \geqq C > 0$ (where C is a constant) then there will be no CMBP-II. Therefore Williams's (1979) famous "S" condition holds for CMBP-II, though, as is well known, it need not hold for general uni-instantaneous state Markov chains.

6. Proof of uniqueness theorem

In this section we prove the uniqueness criterion reported in Theorem 2.4 under the assumption that the existence condition (2.11) holds.

Proof of Theorem 2.4. We prove (ii) first. Suppose there are two honest CMBP-IIs $R(\lambda)$ and $R'(\lambda)$ having the same generator Q. Then by resolvent decomposition Theorem 7.7 we must have

(6.1) $$R(\lambda)=\begin{bmatrix}0 & 0\\ 0 & \psi(\lambda)\end{bmatrix}+r_{00}(\lambda)\begin{bmatrix}1\\ \xi(\lambda)\end{bmatrix}[1,\eta(\lambda)],$$

(6.2) $$R'(\lambda)=\begin{bmatrix}0 & 0\\ 0 & \psi'(\lambda)\end{bmatrix}+r'_{00}(\lambda)\begin{bmatrix}1\\ \xi'(\lambda)\end{bmatrix}[1,\eta'(\lambda)].$$

Because both $R(\lambda)$ and $R'(\lambda)$ are honest, applying Theorem 7.7 yields (see (7.39) and (7.40)):

(6.3) $$\xi(\lambda)=1-\lambda\psi(\lambda)1 \quad\text{and}\quad r_{00}(\lambda)=(\lambda+\lambda\langle\eta(\lambda),1\rangle)^{-1},$$

(6.4) $$\xi'(\lambda)=1-\lambda\psi'(\lambda)1 \quad\text{and}\quad r'_{00}(\lambda)=(\lambda+\lambda\langle\eta'(\lambda),1\rangle)^{-1}.$$

From the proofs of Theorems 2.1 and 2.2, and considering the fact that the decomposition forms of (6.1) and (6.2) are unique, we must have (see (3.10) and (3.12)) $\psi(\lambda)\equiv\tilde{\Phi}(\lambda)\equiv\psi'(\lambda)$ and $\eta(\lambda)\equiv\alpha\tilde{\Phi}(\lambda)\equiv\eta'(\lambda)$. Thus from (6.3) and (6.4) we further obtain $\xi(\lambda)=\xi'(\lambda)$ and $r_{00}(\lambda)=r'_{00}(\lambda)$, and so $R(\lambda)\equiv R'(\lambda)$. Result (ii) follows.

Let us now prove (i). First, for a given CMBP-II pre-generator Q, we see by Corollary 7.9 and the remark immediately preceding it that there exist infinitely many such Q-processes each one of which corresponds to a choice of the constant C (see the proof of Theorem 7.8).

Second, it is easy to prove that each of these Q-processes is actually a CMBP-II. This proof is the same as the sufficiency part of Theorem 2.2. Indeed, when we prove (3.24) we use only (3.25) and (3.27) neither of which depends upon "honesty"; i.e. (3.25) and (3.27) are true for each of the above Q-processes. Result (i) then follows.

Remark. Theorem 2.4 shows that uniqueness is true in the sense of honest CMBP-II, though there exist infinitely many non-honest CMBP-II. Since we are mostly interested in honest processes, the conclusion of Theorem 2.4 is perfectly satisfactory. We shall discuss further properties involving recurrence, positive-recurrence and the use of generating functions for this unique honest CMBP-II in a subsequent paper.

7. Resolvent decomposition theorem

This section digresses from the main topic, but it serves to introduce a fundamental tool, namely the resolvent decomposition theorem. This theorem is useful not only in CMBP-II, but also in general continuous time Markov chains. The basic idea of this powerful result, due to J. Neveu, K.L. Chung and D. Williams concerns exit and entrance decomposition (see, for example, Williams 1979).

Here we shall provide some important refinements, and a new proof which is analytic and algebraic rather than probabilistic.

Following Reuter (1957), define a process as a set $P(t)=\{p_{ij}(t)\}$ of real-valued functions defined on $[0, \infty)$, where i and j range over the countable set E, which satisfy the following four conditions:

$$P(t) \geqq 0; \tag{7.1}$$

$$P(t)1 \leqq 1; \tag{7.2}$$

$$P(t+s)=P(t)P(s); \tag{7.3}$$

$$\lim_{t\to 0^+} P(t)=P(0)=I. \tag{7.4}$$

The process $P(t)$ is called honest if (7.2) becomes an equality, i.e. if

$$P(t)1=1. \tag{7.5}$$

It is well-known (Doob 1945; Kolmogorov 1957 and Kendall 1955) that if $P(t)$ is a process then the limit

$$\lim_{t\to 0^+} \frac{P(t)-I}{t}=Q \tag{7.6}$$

exists, and that $Q=(q_{ij};\, i, j\in E)$ satisfies

$$0 \leqq q_{ij} < +\infty \qquad (i\neq j;\, i, j\in E); \tag{7.7}$$

$$-\infty \leqq q_{ii} \leqq 0 \qquad (i\in E); \tag{7.8}$$

$$\sum_{j\neq i} q_{ij} \leqq -q_{ii} \qquad (i\in E). \tag{7.9}$$

Q is called the density matrix of $P(t)$ and, conversely, $P(t)$ is called a Q-process if its density matrix is Q. We shall let q_i denote $-q_{ii}$.

Definition 7.1. A matrix $Q=(q_{ij})$ defined on $E\times E$ is called a *pre-generator* if (7.7), (7.8) and (7.9) hold. Furthermore, a state $i\in E$ is called stable if $q_i<+\infty$, and instantaneous if $q_i=+\infty$. If all states are stable then Q is called totally stable.

Definition 7.2. A matrix $Q=(q_{ij})$ defined on $E\times E$ is called a *generator* if there exists a process such that (7.6) holds.

A generator must therefore be a pre-generator, but the converse is not always true. For a process $P(t)$ denote the resolvent

$$\psi(\lambda)=\int_0^\infty e^{-\lambda t}P(t)\,dt. \tag{7.10}$$

Then the following theorem can be proved by using the Hille-Yosida theorem (Hille 1948; Yosida 1948).

Theorem 7.1. *$\psi(\lambda)$ is a resolvent if and only if the following four conditions hold simultaneously*:

(7.11) $\psi(\lambda) \geqq 0 \quad (\lambda > 0)$ *(non-negative condition)*,

(7.12) $\lambda\psi(\lambda)1 \leqq 1 \quad (\lambda > 0)$ *(norm condition)*,

(7.13) $\psi(\lambda) - \psi(\mu) = (\mu - \lambda)\psi(\lambda)\psi(\mu) \quad (\lambda, \mu > 0)$ *(resolvent equation)*,

(7.14) $\lim_{\lambda \to \infty} \lambda\psi(\lambda) = I$ *(standard condition)*.

Moreover, (7.5) *is true if and only if*

(7.15) $$\lambda\psi(\lambda)1 = 1;$$

whilst the density matrix of $P(t)$ is Q if and only if

(7.16) $$\lim_{\lambda \to \infty} \lambda(\lambda\psi(\lambda) - I) = Q.$$

Since there is a 1–1 correspondence between the transition function $P(t)$ and resolvent $\psi(\lambda)$, we shall call $\psi(\lambda)$ a process from now on; again following Reuter if conditions (7.11)–(7.14) hold. Similarly, we shall call the process $\psi(\lambda)$ honest if (7.15) holds, and $\psi(\lambda)$ a Q-process if (7.16) holds.

We shall now give some simple lemmas which are extensions of Reuter's (1962) result. These results are the Laplace transform versions of exit and entrance laws (Chung 1970).

Suppose $\psi(\lambda)$ is a process, i.e. it satisfies conditions (7.11)–(7.14). Let

(7.17) $$H_\psi = \{\eta(\lambda); 0 \leqq \eta(\lambda) \in l, \eta(\lambda) - \eta(\mu) = (\mu - \lambda)\eta(\lambda)\psi(\mu); \lambda, \mu > 0\}$$

(7.18) $$K_\psi = \{\xi(\lambda); 0 \leqq \xi(\lambda) \leqq 1, \xi(\lambda) - \xi(\mu) = (\mu - \lambda)\psi(\lambda)\xi(\mu); \lambda, \mu > 0\}.$$

Note that for each $\lambda > 0$, $\eta(\lambda)$ is a non-negative summable row vector whilst $\xi(\lambda)$ is a non-negative bounded column vector on E. Moreover, $\eta(\lambda)\psi(\mu) = \eta(\mu)\psi(\lambda)$ and $\psi(\lambda)\xi(\mu) = \psi(\mu)\xi(\lambda)$.

Lemma 7.2. *Suppose $\eta(\lambda) \in H_\psi$. Then*

(i) $\eta(\lambda) \equiv 0 \ (\forall \lambda > 0) \Leftrightarrow \eta(\lambda_0) = 0 \ (\exists \lambda_0 > 0)$;

(ii) $\eta(\lambda) \downarrow 0 \ (\lambda \uparrow \infty)$; $\eta(\lambda)1 \downarrow 0 \ (\lambda \uparrow \infty)$;

(iii) $\lambda\eta(\lambda)1 \uparrow \ (\lambda \uparrow \infty)$.

Thus $\lim_{\lambda \to \infty} \lambda\eta(\lambda)1$ *exists but may be infinite.*

Proof. (i) follows directly from Definition (7.17). That $\eta(\lambda)\downarrow$ also follows from (7.17), together with the non-negativity of $\eta(\lambda)$ and $\psi(\lambda)$. Hence $\eta(\lambda)1\downarrow$.

Again using (7.17) we have

(7.19) $$\eta(\lambda)1 - \eta(\mu)1 = [(\mu - \lambda)/\mu]\,\eta(\lambda)\,\mu\psi(\mu)1.$$

Let $\mu \to +\infty$. Then by the dominated convergence theorem we obtain

$$\eta(\lambda)1 - \lim_{\mu \to \infty} \eta(\mu)1 = \eta(\lambda)(\lim_{\mu \to \infty} \mu\psi(\mu)1) = \eta(\lambda)1.$$

Thus

$$\eta(\lambda)1\downarrow 0 \qquad (\lambda\uparrow\infty)$$

and so

$$\eta(\lambda)\downarrow 0 \qquad (\lambda\uparrow\infty).$$

This proves (ii).

By (7.19) we have

$$\lambda\eta(\lambda)1-\mu\eta(\mu)1=(\lambda-\mu)\,\eta(\lambda)(1-\mu\psi(\mu)1). \tag{7.20}$$

Combining (7.20) and the fact that $(1-\mu\psi(\mu)1)$ is non-negative, we have (iii) $\lambda\eta(\lambda)1\uparrow(\lambda\uparrow\infty)$, as required. □

Lemma 7.3. *Suppose* $\xi(\lambda)\in K_\psi$. *Then*

(i) $\xi(\lambda)\equiv 0\ (\forall\lambda>0)\Leftrightarrow\xi(\lambda_0)=0\ (\exists\lambda_0>0)$;
(ii) $\xi(\lambda)\downarrow 0\ (\lambda\uparrow\infty)$;
(iii) $\lim_{\lambda\to 0}\xi(\lambda)=\xi$ *exists and* $0\leqq\xi\leqq 1$;
(iv) $\xi-\xi(\lambda)=\lambda\psi(\lambda)\xi$.

Proof. Both (i) and the conclusion that $\xi(\lambda)$ is a decreasing function when $\lambda\to\infty$ are obvious. Result (iii) therefore follows immediately. We only need to prove $\lim_{\lambda\to\infty}\xi(\lambda)=0$ and (iv).

By definition (7.18) we have

$$\xi_i(\lambda)-\xi_i(\mu)=[(\mu-\lambda)/\lambda]\sum_{k\in E}\lambda\psi_{ik}(\lambda)\,\xi_k(\mu). \tag{7.21}$$

But since $\xi(\lambda)\leqq 1$, we have that $\forall i\in E$

$$\begin{aligned}\xi_i(\mu)&=\lim_{\lambda\to\infty}\lambda\psi_{ii}(\lambda)\,\xi_i(\mu)\\ &\leqq\liminf_{\lambda\to\infty}\sum_{k\in E}\lambda\psi_{ik}(\lambda)\,\xi_k(\mu)\leqq\limsup_{\lambda\to\infty}\sum_{k\in E}\lambda\psi_{ik}(\lambda)\,\xi_k(\mu)\\ &\leqq\lim_{\lambda\to\infty}\lambda\psi_{ii}(\lambda)\,\xi_i(\mu)+\limsup_{\lambda\to\infty}\sum_{k\neq i}\lambda\psi_{ik}(\lambda)\\ &\leqq\xi_i(\mu)+\limsup_{\lambda\to\infty}(1-\lambda\psi_{ii}(\lambda))=\xi_i(\mu).\end{aligned}$$

Thus $\lim_{\lambda\to\infty}\sum_{k\in E}\lambda\psi_{ik}(\lambda)\,\xi_k(\mu)=\xi_i(\mu)$. Hence on letting $\lambda\to+\infty$ in (7.21), we obtain

$$\lim_{\lambda\to\infty}\xi_i(\lambda)-\xi_i(\mu)=-\xi_i(\mu).$$

Thus

$$\lim_{\lambda\to\infty}\xi_i(\lambda)=0 \qquad (\forall i\in E)$$

or

$$\xi(\lambda)\downarrow 0 \qquad (\lambda\uparrow\infty).$$

By letting $\mu \to 0$ in

$$\xi(\lambda) - \xi(\mu) = (\mu - \lambda)\,\psi(\lambda)\,\xi(\mu),$$

and using the monotone convergence theorem together with conclusion (iii), we obtain (iv). □

Lemma 7.4. *Suppose* $\eta(\lambda) \in H_\psi$, $\xi(\lambda) \in K_\psi$. *Let* $\xi = \lim_{\lambda \to 0} \xi(\lambda)$.

Then

(i) $(\lambda - \mu)\langle \eta(\mu), \xi(\lambda)\rangle = \lambda\langle \eta(\lambda), \xi\rangle - \mu\langle \eta(\mu), \xi\rangle$;

(ii) $\lambda\langle \eta(\lambda), 1\rangle$ *and* $\lambda\langle \eta(\lambda), \xi\rangle$ *are increasing functions of* $\lambda > 0$. *Under the additional assumption* $\xi(\lambda) \leqq 1 - \lambda\psi(\lambda)1$,

(iii) $\lambda\langle \eta(\lambda), 1 - \xi\rangle$ *is increasing.*

Hence the limits

$$\lim_{\lambda \to \infty} \lambda\langle \eta(\lambda), 1\rangle, \qquad \lim_{\lambda \to \infty} \lambda\langle \eta(\lambda), \xi\rangle \quad \text{and} \quad \lim_{\lambda \to \infty} \lambda\langle \eta(\lambda), 1 - \xi\rangle$$

exist but may be infinite.

Note. Here and elsewhere $\langle \cdot, \cdot \rangle$ denotes the product of a row and a column vector, in order to emphasise that the result is scalar.

Proof. (i) follows from the fact that

$$\begin{aligned}
\lambda\langle \eta(\lambda), \xi\rangle &= \lambda\langle \eta(\mu) + (\mu - \lambda)\,\eta(\mu)\,\psi(\lambda), \xi\rangle \qquad \text{(see (7.17) and its note)}\\
&= \lambda\langle \eta(\mu), (I + (\mu - \lambda)\,\psi(\lambda))\,\xi\rangle\\
&= \lambda\langle \eta(\mu), \xi\rangle + (\mu - \lambda)\langle \eta(\mu), \lambda\psi(\lambda)\,\xi\rangle\\
&= \lambda\langle \eta(\mu), \xi\rangle + (\mu - \lambda)\langle \eta(\mu), \xi - \xi(\lambda)\rangle \qquad \text{(see Lemma 7.3 (iv))}\\
&= \mu\langle \eta(\mu), \xi\rangle - (\mu - \lambda)\langle \eta(\mu), \xi(\lambda)\rangle.
\end{aligned}$$

We shall now prove (ii) and (iii). For the increasing property of $\lambda\langle \eta(\lambda), 1\rangle$ see Lemma 7.2. That $\lambda\langle \eta(\lambda), \xi\rangle$ increases follows from the result we have just proved in (i), namely

$$\lambda\langle \eta(\lambda), \xi\rangle - \mu\langle \eta(\mu), \xi\rangle = (\lambda - \mu)\langle \eta(\mu), \xi(\lambda)\rangle. \tag{7.22}$$

Note that (7.20) can be re-written as

$$\lambda\langle \eta(\lambda), 1\rangle - \mu\langle \eta(\mu), 1\rangle = (\lambda - \mu)\langle \eta(\mu), 1 - \lambda\psi(\lambda)1\rangle. \tag{7.23}$$

By subtracting (7.22) from (7.23) we get

$$\lambda\langle \eta(\lambda), 1 - \xi\rangle - \mu\langle \eta(\mu), 1 - \xi\rangle = (\lambda - \mu)\langle \eta(\mu), 1 - \lambda\psi(\lambda)1 - \xi(\lambda)\rangle. \tag{7.24}$$

The increasing property of $\lambda\langle \eta(\lambda), 1 - \xi\rangle$ then follows from (7.24) and the additional condition $\xi(\lambda) \leqq 1 - \lambda\psi(\lambda)1$. □

Remark. In proving this lemma we have used the associative and distributive laws for the infinite matrix, the permissibility of which needs to be verified since they do not hold in general. Verification under our conditions can be obtained by using Propositions 1.2 and 1.5 of Kemeny et al. (1966).

Lemma 7.5. *Suppose $\psi(\lambda)$ is a process and $\eta(\lambda)\in H_\psi$. Let*

$$\xi^0(\lambda)=1-\lambda\psi(\lambda)1.$$

Then

(i) $\xi^0(\lambda)\in K_\psi$ *and so* $\xi^0=\lim_{\lambda\to 0}\xi^0(\lambda)$ *exists,*

(ii) $\lambda\langle\eta(\lambda),\ 1-\xi^0\rangle$ *is independent of* λ *and finite, i.e.* $\lambda\langle\eta(\lambda),\ 1-\xi^0\rangle$ *is a finite constant.*

Proof. Since $\psi(\lambda)$ is a process it follows that $0\leqq 1-\lambda\psi(\lambda)1\leqq 1$, i.e.

$$0\leqq\xi^0(\lambda)\leqq 1. \tag{7.25}$$

Moreover, $\psi(\lambda)$ is a process satisfying the resolvent equation (7.13), i.e.

$$\psi(\lambda)-\psi(\mu)=(\mu-\lambda)\,\psi(\mu)\,\psi(\lambda).$$

Hence

$$(1-\lambda\psi(\lambda)1)-(1-\mu\psi(\mu)1)=(\mu-\lambda)\,\psi(\mu)(1-\lambda\psi(\lambda)1),$$

or

$$\xi^0(\lambda)-\xi^0(\mu)=(\mu-\lambda)\,\psi(\mu)\,\xi^0(\lambda). \tag{7.26}$$

Combining (7.25) and (7.26) then shows that

$$\xi^0(\lambda)\in K_\psi.$$

Now by Lemma 7.3, $\xi^0(\lambda)$ enjoys each property of K_ψ. In particular

$$\xi^0=\lim_{\lambda\to 0}(1-\lambda\psi(\lambda)1) \quad \text{exists and } 0\leqq\xi^0\leqq 1.$$

By (7.24), it is easy to see that $\lambda\langle\eta(\lambda),\ 1-\xi^0\rangle$ is a constant. □

In order to prove the resolvent decomposition theorem below, we require certain inequalities.

Lemma 7.6. *Suppose $R(\lambda)=\{r_{ij}(\lambda);\ i,j\in E\}$ is a process. Then*

(i) $r_{ij}(\lambda)\,r_{kk}(\lambda)\geqq r_{ik}(\lambda)\,r_{kj}(\lambda)$ $(i,j,\ k\in E,\ \lambda>0)$,

(ii) $r_{kk}(\lambda)\,d_i(\lambda)\geqq r_{ik}(\lambda)\,d_k(\lambda)$ $(i,\ k\in E,\ \lambda>0)$

where $d_i(\lambda)=1-\lambda\sum_{j\in E}r_{ij}(\lambda)$ $(i\in E,\ \lambda>0)$, *and*

(iii) $r_{ik}(\lambda)\leqq r_{kk}(\lambda)$ $(i,\ k\in E,\ \lambda>0)$.

Proof. If $i=k$ or $j=k$ then (i) is trivial. Otherwise (i) follows from well-known properties of transition functions (see, for example, Lemma 11.8 (II) of Chung (1967), or (83.13) of Williams (1979)), i.e.

$$p_{ij}(t)\geqq\int_0^t p_{kj}(t-s)\,dF_{ik}(s) \qquad (i\neq k,\ t>0) \tag{7.27}$$

and

$$p_{ik}(t)=\int_0^t p_{kk}(t-s)\,dF_{ik}(s) \qquad (i\neq k, t>0) \tag{7.28}$$

where $F_{ik}(t)$ is the conditional distribution of the hitting-time of k. Note that, for $i\neq k$, there exists a finite continuous function $f_{ik}(.)$ on $[0, \infty)$ such that

$$F_{ik}(t)=\int_0^t f_{ik}(s)\,ds.$$

(ii) is a corollary of (i).

By summing $j\in E$ in (i) and using (ii), we get (iii). □

We shall now prove the basic resolvent decomposition theorem. Suppose $Q=\{q_{ij};\ i, j\in E\}$ is a pre-generator defined on $E\times E$ (see Definition 7.1), and let $b\in E$ be a singleton state with $N=E\backslash\{b\}$. $\tilde{Q}$ denotes the restriction of Q on $N\times N$. We shall further assume that $q_b=+\infty$ since this is the case of interest.

Theorem 7.7 (resolvent decomposition theorem). *Suppose $R(\lambda)=\{r_{ij}(\lambda);\ i, j\in E\}$ is a Q-process defined on $E\times E$ where $E=N\cup\{b\}$ and the generator Q satisfies*

$$q_b\equiv -q_{bb}=+\infty. \tag{7.29}$$

Then $R(\lambda)$ can be uniquely decomposed into

$$R(\lambda)=\begin{bmatrix}0 & 0\\ 0 & \psi(\lambda)\end{bmatrix}+r_{bb}(\lambda)\begin{bmatrix}1\\ \xi(\lambda)\end{bmatrix}[1, \eta(\lambda)] \tag{7.30}$$

where:

$$\psi(\lambda) \text{ is a } \tilde{Q}\text{-process}; \tag{7.31}$$

$$\eta(\lambda)\in H_\psi \quad \text{and} \quad \xi(\lambda)\in K_\psi; \tag{7.32}$$

$$\xi(\lambda)\leqq 1-\lambda\psi(\lambda)1; \tag{7.33}$$

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda)=\alpha \quad \text{and} \quad \lim_{\lambda\to\infty}\lambda\xi(\lambda)=\beta \tag{7.34}$$

where

$$\alpha=\{q_{bj};j\in N\} \text{ and } \beta=\{q_{jb};j\in N\};$$

$$r_{bb}(\lambda)=(C+\lambda+\lambda\langle\eta(\lambda), \xi\rangle)^{-1} \tag{7.35}$$

where $\xi=\lim_{\lambda\to 0}\xi(\lambda)$ and C is a finite constant such that

$$C\geqq \lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), 1-\xi\rangle; \tag{7.36}$$

and

$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), \xi\rangle = +\infty \quad \text{or, equivalently,} \tag{7.37}$$

$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), 1\rangle = +\infty. \tag{7.38}$$

If $R(\lambda)$ is honest then:

$$\xi(\lambda) = 1 - \lambda\psi(\lambda)1; \tag{7.39}$$

$$r_{bb}(\lambda) = (\lambda + \lambda\langle\eta(\lambda), 1\rangle)^{-1}; \tag{7.40}$$

and

$$C \equiv \lambda\langle\eta(\lambda), 1-\xi\rangle, \tag{7.41}$$

i.e. $\lambda\langle\eta(\lambda), 1-\xi\rangle$ *is independent of* λ.

Proof. Suppose $R(\lambda) = \{r_{ij}(\lambda);\ i, j\in E\}$ is a Q-process. Then since $r_{bb}(\lambda)$ is strictly greater than zero we may let $\xi_i(\lambda) = \dfrac{r_{ib}(\lambda)}{r_{bb}(\lambda)}$ and $\eta_j(\lambda) = \dfrac{r_{bj}(\lambda)}{r_{bb}(\lambda)}$. Now, if we denote $\xi(\lambda) = \{\xi_i(\lambda);\ i\in N\}$ and $\eta(\lambda) = \{\eta_j(\lambda);\ j\in N\}$, then we have

$$0 \leqq \eta(\lambda) \in l \qquad (\lambda > 0) \tag{7.42}$$

and, by (iii) of Lemma 7.6,

$$0 \leqq \xi(\lambda) \leqq 1. \tag{7.43}$$

Let $\psi_{ij}(\lambda) = r_{ij}(\lambda) - \xi_i(\lambda)\, r_{bb}(\lambda)\, \eta_j(\lambda)$ and $\psi(\lambda) = \{\psi_{ij}(\lambda);\ i, j\in N,\ \lambda > 0)\}$. Then (i) of Lemma 7.6 guarantees that

$$\psi(\lambda) \geqq 0. \tag{7.44}$$

Now $R(\lambda)$ has been written as

$$R(\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & \psi(\lambda) \end{bmatrix} + r_{bb}(\lambda) \begin{bmatrix} 1 \\ \xi(\lambda) \end{bmatrix} [1, \eta(\lambda)]. \tag{7.45}$$

Whence on substituting (7.45) into the resolvent equation (see (7.13))

$$R(\lambda) - R(\mu) = (\mu - \lambda)\, R(\lambda)\, R(\mu), \tag{7.46}$$

we obtain, by some cumbersome but easy algebra, the following four equations:

$$r_{bb}(\lambda) - r_{bb}(\mu) = (\mu - \lambda)\, r_{bb}(\lambda)\, r_{bb}(\mu) + (\mu - \lambda)\, r_{bb}(\lambda)\, r_{bb}(\mu)\, \eta(\lambda)\, \xi(\mu), \tag{7.47}$$

$$\eta(\lambda) - \eta(\mu) = (\mu - \lambda)\, \eta(\lambda)\, \psi(\mu), \tag{7.48}$$

$$\xi(\lambda) - \xi(\mu) = (\mu - \lambda)\, \psi(\lambda)\, \xi(\mu), \tag{7.49}$$

$$\psi(\lambda) - \psi(\mu) = (\mu - \lambda)\, \psi(\lambda)\, \psi(\mu). \tag{7.50}$$

From $\lim_{\lambda\to\infty} \lambda R(\lambda)=I$, we easily obtain $\lim_{\lambda\to\infty} \xi(\lambda)=\lim_{\lambda\to\infty} \eta(\lambda)=0$, and so

(7.51) $$\lim_{\lambda\to\infty} \lambda\psi(\lambda)=I.$$

Substituting (7.45) into the norm condition $\lambda R(\lambda)1\leqq 1$ (see (7.12)), yields

(7.52) $$\lambda r_{bb}(\lambda)+\lambda r_{bb}(\lambda)\,\eta(\lambda)1\leqq 1$$

and

(7.53) $$\lambda\psi(\lambda)1+\lambda r_{bb}(\lambda)\,\xi(\lambda)+\lambda r_{bb}(\lambda)\,\xi(\lambda)\,\eta(\lambda)1\leqq 1.$$

Whilst on introducing $d_i(\lambda)=1-\lambda\sum_{j\in E} r_{ij}(\lambda)$ $(i\in N)$, $d_b(\lambda)=1-\lambda r_{bb}(\lambda)-\lambda r_{bb}(\lambda)\,\eta(\lambda)1$ and $D(\lambda)=\{d_i(\lambda);\ i\in N\}$, we see that inequalities (7.52) and (7.53) can be written as the equalities

(7.54) $$\lambda r_{bb}(\lambda)+\lambda r_{bb}(\lambda)\,\eta(\lambda)1+d_b(\lambda)=1$$

and

(7.55) $$\lambda\psi(\lambda)1+\lambda r_{bb}(\lambda)\,\xi(\lambda)+\lambda r_{bb}(\lambda)\,\xi(\lambda)\,\eta(\lambda)1+D(\lambda)=1.$$

Substituting (7.54) into (7.55), and using (ii) of Lemma 7.6, yields

(7.56) $$\xi(\lambda)+\lambda\psi(\lambda)1\leqq 1$$

and so

(7.57) $$\lambda\psi(\lambda)1\leqq 1.$$

Combining (7.44), (7.50), (7.51) and (7.57) shows that $\psi(\lambda)$ is a process on $N\times N$ (see (7.11)–(7.14)). Whilst (7.32) and (7.33) follow from (7.42), (7.43), (7.48), (7.49) and (7.56).

On substituting (7.45) into the condition $\lim_{\lambda\to\infty} \lambda(\lambda R(\lambda)-I)=Q$, we easily get (7.34) together with

(7.58) $$\lim_{\lambda\to\infty} \lambda(\lambda\psi_{ij}(\lambda)-\delta_{ij})=q_{ij}\qquad (i,j\in N).$$

This shows that $\psi(\lambda)$ is a $\tilde{Q}$-process, i.e. the density matrix of $\psi(\lambda)$ is just the restriction of Q on $N\times N$.

Since we have proved (7.31)–(7.33), we have by (i) of Lemma 7.4 that

(7.59) $$(\lambda-\mu)\langle\eta(\lambda),\xi(\mu)\rangle=\lambda\langle\eta(\lambda),\xi\rangle-\mu\langle\eta(\mu),\xi\rangle.$$

On substituting (7.59) into (7.47), dividing by $r_{bb}(\lambda)\,r_{bb}(\mu)>0$, and using (7.59), we obtain

$$r_{bb}^{-1}(\mu)-r_{bb}^{-1}(\lambda)=(\mu-\lambda)-\lambda\langle\eta(\lambda),\xi\rangle+\mu\langle\eta(\mu),\xi\rangle,$$

or

$$r_{bb}^{-1}(\mu)-\mu-\mu\langle\eta(\mu),\xi\rangle=r_{bb}^{-1}(\lambda)-\lambda-\lambda\langle\eta(\lambda),\xi\rangle. \tag{7.60}$$

Equation (7.60) shows that $r_{bb}^{-1}(\lambda)-\lambda-\lambda\langle\eta(\lambda),\xi\rangle$ is independent of λ, i.e. $r_{bb}^{-1}(\lambda)-\lambda-\lambda\langle\eta(\lambda),\xi\rangle$ is a finite constant. On denoting this constant by C we obtain (7.35).

Using (7.35), result (7.52) can be re-written as

$$C+\lambda+\lambda\langle\eta(\lambda),\xi\rangle\geqq\lambda+\lambda\langle\eta(\lambda),1\rangle,$$

i.e.

$$C\geqq\lambda\langle\eta(\lambda),1-\xi\rangle. \tag{7.61}$$

By (iii) of Lemma 7.4 we know that $\lambda\langle\eta(\lambda),1-\xi\rangle$ is an increasing function of $\lambda>0$. Noting that (7.61) is true for each $\lambda>0$ for constant C then shows that $\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),1-\xi\rangle$ is finite and

$$C\geqq\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),1-\xi\rangle. \tag{7.62}$$

Thus we have now proved (7.36).

Using result (7.35), the condition $\lim_{\lambda\to\infty}\lambda(1-\lambda r_{bb}(\lambda))=q_b$ can be re-written as

$$\lim_{\lambda\to\infty}\frac{C+\lambda\langle\eta(\lambda),\xi\rangle}{\frac{C}{\lambda}+1+\langle\eta(\lambda),\xi\rangle}=q_b. \tag{7.63}$$

By (ii) of Lemma 7.2 we know that $\lim_{\lambda\to\infty}\langle\eta(\lambda),1\rangle=0$, and since

$$0\leqq\langle\eta(\lambda),\xi\rangle\leqq\langle\eta(\lambda),1\rangle$$

we see that

$$\lim_{\lambda\to\infty}\langle\eta(\lambda),\xi\rangle=0.$$

Hence (7.63) shows that

$$C+\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\xi\rangle=q_b.$$

But we have assumed that $q_b=+\infty$ (see (7.29)) and C is a constant, hence

$$\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\xi\rangle=+\infty. \tag{7.64}$$

Since we have proved in (7.62) that $\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\ 1-\xi\rangle$ is finite, (7.64) is true if and only if

$$\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda), 1\rangle=+\infty. \tag{7.65}$$

This proves (7.37) and (7.38).

If $R(\lambda)$ is an honest process, i.e. $\lambda R(\lambda)1=1$, then both (7.52) and (7.53) become equalities, viz.

$$\lambda r_{bb}(\lambda)+\lambda r_{bb}(\lambda)\,\eta(\lambda)1=1, \tag{7.66}$$

and

$$\lambda\psi(\lambda)1+\lambda r_{bb}(\lambda)\,\xi(\lambda)+\lambda r_{bb}(\lambda)\,\xi(\lambda)\,\eta(\lambda)1=1. \tag{7.67}$$

Substituting (7.66) into (7.67) yields

$$\lambda\psi(\lambda)1+\xi(\lambda)=1. \tag{7.68}$$

Thus (7.39) and (7.40) follow from (7.68) and (7.66), respectively, whilst (7.41) follows from (7.35) and (7.66).

The proof of the theorem has now been completed apart from showing the uniqueness of the decomposition form (7.30). Suppose there are two forms of decomposition

$$R(\lambda)=\begin{bmatrix}0 & 0\\ 0 & \psi(\lambda)\end{bmatrix}+r_{bb}(\lambda)\begin{bmatrix}1\\ \xi(\lambda)\end{bmatrix}[1,\eta(\lambda)]=\begin{bmatrix}0 & 0\\ 0 & \tilde{\psi}(\lambda)\end{bmatrix}+r_{bb}(\lambda)\begin{bmatrix}1\\ \tilde{\xi}(\lambda)\end{bmatrix}[1,\tilde{\eta}(\lambda)].$$

Then since $r_{bb}(\lambda)>0$ we see that $\xi(\lambda)=\tilde{\xi}(\lambda)$ and $\eta(\lambda)=\tilde{\eta}(\lambda)$, whence $\psi(\lambda)=\tilde{\psi}(\lambda)$ and so the uniqueness property is true. □

Theorem 7.7 is particularly useful for determining properties provided we know that a process exists. In many cases, however, all we are given is a pre-generator Q and we do not know whether a Q-process exists. In such situations the following theorem, which can be seen as the converse of Theorem 7.7, plays an important role.

Theorem 7.8. *Suppose Q is a given pre-generator defined on $E\times E$ where $E=N\cup\{b\}$, satisfying*

$$q_b\equiv -q_{bb}=+\infty. \tag{7.69}$$

Suppose there exists a $\tilde{Q}$-process $\psi(\lambda)$ and a pair of $\eta(\lambda)$ and $\xi(\lambda)$ which satisfy

$$\eta(\lambda)\in H_\psi \quad \textit{and} \quad \xi(\lambda)\in K_\psi, \tag{7.70}$$

$$\xi(\lambda)\leqq 1-\lambda\psi(\lambda)1, \tag{7.71}$$

$$\lim_{\lambda\to\infty}\lambda\eta(\lambda)=\alpha \quad \textit{and} \quad \lim_{\lambda\to\infty}\lambda\xi(\lambda)=\beta \tag{7.72}$$

where $\alpha=\{q_{bj}; j\in N\}$ *and* $\beta=\{q_{jb}; j\in N\}$,

(7.73)
$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), 1\rangle=+\infty,$$

and

(7.74)
$$\lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), 1-\xi\rangle<+\infty$$

where $\xi=\lim_{\lambda\to 0}\xi(\lambda)$. *Then* Q *is a generator* (*see Definition* 2.2), *i.e. there exists a* Q*-process.*

Note. In the above theorem $\tilde{Q}$ denotes the restriction of Q on $N\times N$.

Proof. To construct a Q-process suppose that all the required conditions are satisfied. We first choose a constant C such that

(7.75)
$$C\geqq \lim_{\lambda\to\infty} \lambda\langle\eta(\lambda), 1-\xi\rangle,$$

and then let

(7.76)
$$r_{bb}(\lambda)=(C+\lambda+\lambda\langle\eta(\lambda), \xi\rangle)^{-1} \qquad (\lambda>0)$$

and

(7.77)
$$R(\lambda)=\begin{bmatrix}0 & 0\\ 0 & \psi(\lambda)\end{bmatrix}+r_{bb}(\lambda)\begin{bmatrix}1\\ \xi(\lambda)\end{bmatrix}[1, \eta(\lambda)] \qquad (\lambda>0)$$

where $\psi(\lambda)$, $\xi(\lambda)$ and $\eta(\lambda)$ are taken from the ones given by the conditions. Condition (7.74) guarantees that we can choose a constant C to satisfy (7.75), and so (7.76) and (7.77) are well-defined for each $\lambda>0$.

We shall prove that $R(\lambda)$ defined in (7.77) is a Q-process. First it is easy to show that

(7.78)
$$R(\lambda)\geqq 0.$$

Second, by conditions (7.70), (7.71) and Lemma 7.4, we know that

$$\lambda\langle\eta(\lambda), 1-\xi\rangle\uparrow(\lambda\to\infty).$$

Thus by (7.75) we have

$$C\geqq\lambda\langle\eta(\lambda), 1-\xi\rangle \qquad (\forall\lambda>0),$$

whence

$$C+\lambda+\lambda\langle\eta(\lambda), \xi\rangle\geqq\lambda+\lambda\langle\eta(\lambda), 1\rangle.$$

Comparing this result with (7.76) we see that

$$r_{bb}(\lambda)\leqq(\lambda+\lambda\langle\eta(\lambda), 1\rangle)^{-1},$$

or

$$\lambda r_{bb}(\lambda)+\lambda r_{bb}(\lambda)\,\eta(\lambda)1\leqq 1. \tag{7.79}$$

Condition (7.71), together with (7.79), shows that

$$\lambda R(\lambda)1\leqq 1. \tag{7.80}$$

Third, Lemma 7.4 (i) and (7.76) shows that

$$r_{bb}(\lambda)-r_{bb}(\mu)=(\mu-\lambda)\,r_{bb}(\lambda)\,r_{bb}(\mu)+(\mu-\lambda)\,r_{bb}(\lambda)\,r_{bb}(\mu)\,\eta(\lambda)\,\xi(\mu).$$

This result, together with condition (7.70) and the condition that $\psi(\lambda)$ is a process, yields the following three equations

$$\begin{aligned}\psi(\lambda)-\psi(\mu)&=(\mu-\lambda)\,\psi(\lambda)\,\psi(\mu),\\ \eta(\lambda)-\eta(\mu)&=(\mu-\lambda)\,\eta(\lambda)\,\psi(\mu),\\ \xi(\lambda)-\xi(\mu)&=(\mu-\lambda)\,\psi(\lambda)\,\xi(\mu).\end{aligned}$$

A little algebra then shows that $R(\lambda)$ (defined in (7.77)) satisfies

$$R(\lambda)-R(\mu)=(\mu-\lambda)\,R(\lambda)\,R(\mu). \tag{7.81}$$

Fourth, by (ii) of Lemma 7.2, (iii) of Lemma 7.3, and condition (7.70) we have $\lim_{\lambda\to\infty}\eta(\lambda)1=0$ and $\lim_{\lambda\to\infty}\langle\eta(\lambda),\xi\rangle=0$, and so

$$\lim_{\lambda\to\infty}\lambda r_{bb}(\lambda)=\lim_{\lambda\to\infty}[(C/\lambda)+1+\langle\eta(\lambda),\xi\rangle]^{-1}=1 \tag{7.82}$$

and

$$\lim_{\lambda\to\infty} r_{bb}(\lambda)=0. \tag{7.83}$$

On using these results together with condition (7.72) we obtain

$$\lim_{\lambda\to\infty}\lambda r_{bj}(\lambda)=\lim_{\lambda\to\infty} r_{bb}(\lambda)\lim_{\lambda\to\infty}\lambda\eta_j(\lambda)=0\times q_{bj}=0 \quad (j\in N), \tag{7.84}$$

$$\lim_{\lambda\to\infty}\lambda r_{jb}(\lambda)=\lim_{\lambda\to\infty} r_{bb}(\lambda)\lim_{\lambda\to\infty}\lambda\xi_j(\lambda)=0\times q_{jb}=0 \quad (j\in N), \tag{7.85}$$

$$\lim_{\lambda\to\infty}\lambda^2 r_{bj}(\lambda)=\lim_{\lambda\to\infty}\lambda r_{bb}(\lambda)\lim_{\lambda\to\infty}\lambda\eta_j(\lambda)=1\times q_{bj}=q_{bj} \quad (j\in N), \tag{7.86}$$

$$\lim_{\lambda\to\infty}\lambda^2 r_{jb}(\lambda)=\lim_{\lambda\to\infty}\lambda r_{bb}(\lambda)\lim_{\lambda\to\infty}\lambda\xi_j(\lambda)=1\times q_{jb}=q_{jb} \quad (j\in N), \tag{7.87}$$

$$\begin{aligned}\lim_{\lambda\to\infty}\lambda r_{ij}(\lambda)&=\lim_{\lambda\to\infty}(\lambda\psi_{ij}(\lambda)+\lambda\xi_i(\lambda)\,r_{bb}(\lambda)\,\eta_j(\lambda))=\lim_{\lambda\to\infty}\lambda\psi_{ij}(\lambda)\\ &=\delta_{ij} \quad (i,j\in N),\end{aligned} \tag{7.88}$$

and

(7.89) $$\lim_{\lambda\to\infty}\lambda(\lambda r_{ij}(\lambda)-\delta_{ij})=\lim_{\lambda\to\infty}\lambda(\lambda\psi_{ij}(\lambda)-\delta_{ij})+\lim_{\lambda\to\infty}\lambda\xi_i(\lambda)\,r_{bb}(\lambda)\,\lambda\eta_j(\lambda)$$

$$=q_{ij}+(q_{ib}\times 0\times q_{bj})=q_{ij}\qquad (i,j\in N).$$

We know on combining (7.73) with (7.74) that

$$\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\xi\rangle=+\infty.$$

So

(7.90) $$\lim_{\lambda\to\infty}\lambda(1-\lambda r_{bb}(\lambda))=\lim_{\lambda\to\infty}\frac{C+\lambda\langle\eta(\lambda),\xi\rangle}{\frac{C}{\lambda}+1+\langle\eta(\lambda),\xi\rangle}=\lim_{\lambda\to\infty}\lambda\langle\eta(\lambda),\xi\rangle=+\infty.$$

Thus (7.82), (7.84), (7.85) and (7.88) together show that

(7.91) $$\lim_{\lambda\to\infty}\lambda R(\lambda)=I,$$

whilst (7.86)–(7.90) together show that

(7.92) $$\lim_{\lambda\to\infty}\lambda(\lambda R(\lambda)-I)=Q.$$

Now (7.78), (7.80), (7.81), (7.91) and (7.92) show that $R(\lambda)$, as constructed in (7.77), is a process and its density matrix is the given Q (see (7.11)–(7.14) and (7.16)). Thus $R(\lambda)$ is a Q-process. □

Remark. Note that in proving Theorem 7.8 we found that the base from which we construct the Q-process is to choose a constant C such that (7.75) holds. Apparently, there are infinitely many ways to choose such a constant C, and different choices of C lead to different Q-processes (at least $r_{bb}(\lambda)$ is different because of (7.76)). We therefore have the following corollary.

Corollary 7.9. *If all the conditions of Theorem* 7.8 *hold, then there exist infinitely many Q-processes.*

The Q-process constructed in Theorem 7.8 may not be an honest one. However, in many applications we are only interested in honest processes, and for these we have the following theorem.

Theorem 7.10. *Suppose Q is a given pre-generator defined on $E\times E$ where $E=N\cup\{b\}$ and satisfies $q_b\equiv -q_{bb}=+\infty$. If there exists a $\tilde{Q}$-process $\psi(\lambda)$ and a row-vector $\eta(\lambda)$ $(\lambda>0)$ which satisfy*

(7.93) $$\eta(\lambda)\in H_\psi,$$

(7.94) $$\lim_{\lambda\to\infty}\lambda\eta(\lambda)=\alpha\qquad\text{where }\alpha=(q_{bj};j\in N),$$

(7.95) $$\lim_{\lambda\to\infty}\lambda(1-\lambda\psi(\lambda)1)=\beta\qquad\text{where }\beta=(q_{jb};j\in N),$$

and

$$\lim_{\lambda\to\infty} \lambda\eta(\lambda)1 = +\infty, \tag{7.96}$$

then there exists an honest Q-process.

Proof. Let $\xi^0(\lambda)=1-\lambda\psi(\lambda)1$. Then by Lemma 7.5 we know that $\xi^0(\lambda)\in K_\psi$ and $\lambda\langle\eta(\lambda), 1-\xi^0\rangle$ is a finite constant, say C. In other words, if we let

$$C=\lambda\langle\eta(\lambda), 1-\xi^0\rangle \quad \text{where } \xi^0=\lim_{\lambda\to 0}\xi^0(\lambda)=\lim_{\lambda\to 0}(1-\lambda\psi(\lambda)1), \tag{7.97}$$

$$r_{bb}(\lambda)=(C+\lambda+\lambda\langle\eta(\lambda),\xi\rangle)^{-1}=(\lambda+\lambda\langle\eta(\lambda),1\rangle)^{-1}, \tag{7.98}$$

and

$$R(\lambda)=\begin{bmatrix}0 & 0\\ 0 & \psi(\lambda)\end{bmatrix}+r_{bb}(\lambda)\begin{bmatrix}1\\ 1-\lambda\psi(\lambda)1\end{bmatrix}[1,\eta(\lambda)], \tag{7.99}$$

then $R(\lambda)$, $r_{bb}(\lambda)$ and C are all well-defined.

Now the proof of Theorem 7.8 works well since $\xi^0(\lambda)=1-\lambda\psi(\lambda)1$ is a special case of general $\xi(\lambda)$. Hence $R(\lambda)$ constructed in (7.99) is a Q-process.

In order to complete our proof we need to show that $R(\lambda)$ constructed in (7.99) is honest. This follows since we know that $R(\lambda)$ is honest if and only if the following two conditions hold:

$$\lambda r_{bb}(\lambda)+\lambda r_{bb}(\lambda)\,\eta(\lambda)1=1, \tag{7.100}$$

$$\xi^0(\lambda)+\lambda\psi(\lambda)1=1. \tag{7.101}$$

Now (7.100) is equivalent to (7.98), whilst (7.101) is just the equality $\xi^0(\lambda)=1-\lambda\psi(\lambda)1$. This completes the proof. □

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