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Linear Fourier and stochastic analysis

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Summary. To extend the traditional Fourier theory of stationary processes, some new boundedness notions, for processes and for random measures, are introduced. This leads, for these processes and measures, to Plancherel and Hausdorff-Young type formulae and to a decomposition theory via dilations and multiplications. Various applications of our methods are also presented.

1 Introduction

The object of this work is to present some advances in the representation of stochastic processes as Fourier integrals. These advances provide a Plancherel and Hausdorff-Young theory for stochastic processes and random measures and, in particular, a framework in which to develop a Fourier theory for the ubiquitous white noise model. Several applications of these methods, when combined with a dilation theory, are also presented. Among others, existence results for linear stochastic differential equations.

The use of Fourier techniques to study linear problems in stochastic processes is hardly recent. It goes back to Slutsky [S] and Cramér [C] who obtained Fourier integral representations for (weakly) stationary processes and extends to Phillips ([Ph] and Kluvánek [K] who characterized processes as Fourier transforms of vector measures. Common assumptions made to derive these representations are the (norm) continuity of the process, the σ-additivity of the representing measure or the orthogonality of its increments. These assumptions are severe and for basic classes of processes, cannot be satisfied. Moreover, these developments do not recover classical scalar results such as, for example, the L^2-Fourier theory.

In the work below, we relax these various assumptions. In particular, we develop a "stochastic" Fourier theory in which expressions such as $\int_{\mathbb{R}^+} e^{it\xi} dW_\xi$, where W is the Wiener process on $\mathbb{R}^+$, make senses. With this approach, we

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also study some linear problems for the corresponding stochastic processes and random measures.

We now come to a brief description of the contents of this paper. In the forthcoming section, we set the stage with some definitions and examples, and also obtain a basic representation theorem. In the third section we relax the convergence assumptions and develop a summability theory for processes and random measures. Via finitely additive vector measures, we obtain in Sect. 4, a "stochastic" Plancherel and Hausdorff-Young theories. Our fifth section analyzes the relationships between "correlated and uncorrelated" stochastic measures, and it is shown that the later can be decomposed in terms of dilations and multiplications of the former. The last section is devoted to some applications of the previous sections. First to obtain a decomposition of random measures into continuous and discrete parts, then to give existence results for autonomous linear stochastic differential equations. Most of our results have been announced in [H2], while some of the discrete time versions are in [H1].

2 Preliminary results

Let $(\Omega, \mathscr{B}, \mathscr{P})$ be a probability space and for $1 \leqq \alpha \leqq 2$, let $L^\alpha(\Omega, \mathscr{B}, \mathscr{P})$ (($L^\alpha(\mathscr{P})$ for short) be the space of random variables with finite α^{th}-moment. On $L^\alpha(\mathscr{P})$, the norm is denoted by $\|\cdot\|_\alpha$ and for Y and Z in $L^\alpha(\mathscr{P})$, the covariation of Y and Z is $\langle Y, Z\rangle_\alpha = \mathscr{E} Y Z^{\langle \alpha-1\rangle}$, where $Z^{\langle\alpha\rangle} = |Z|^{\alpha-1}\bar{Z}$ and where $\mathscr{E}$ is the expectation. For $1 \leqq \beta \leqq +\infty$, $L^\beta(\mathbb{R})$ is the usual Lebesgue space with associated norm $\|\cdot\|_\beta$. We also denote by $C_0(\mathbb{R})$ (resp. $C_c(\mathbb{R})$) the space of continuous functions vanishing at infinity (resp. with compact support) and the uniform norm by $\|\cdot\|_\infty$. Finally, for $1 \leqq \beta < +\infty$, $L^\beta(\mathbb{R})^\vee$ is the space of functions $f \in L^1(\mathbb{R})$ such that $\hat{f}(t) = \int_{\mathbb{R}} e^{it\xi} f(\xi)\, d\xi \in L^\beta(\mathbb{R})$, while for $\beta = +\infty$, we set $L^\beta(\mathbb{R})^\vee = L^1(\mathbb{R})$, since in this case $\hat{f} \in C_0(\mathbb{R})$. A last notation, throughout this work K denotes a generic absolute constant whose value might change from an expression to another.

Let $\mathscr{B}(\mathbb{R})$ be the Borel σ-ring of $\mathbb{R}$ and let $\mathscr{B}_0(\mathbb{R})$ be the δ-ring of elements of $\mathscr{B}(\mathbb{R})$ which are bounded. A random measure is a finitely additive set function μ: $\mathscr{B}_0(\mathbb{R}) \to L^0(\mathscr{P})$, where $L^0(\mathscr{P})$ is the vector space of random variables. If χ_A denotes the indicator function of the set A, we now introduce a concept originating in Bochner [B1], [B2].

Definition 2.1. A random measure μ has finite (α, β)-variation whenever

$$\|\mu\| = \sup\{\|\mu\|(A), \quad A \in \mathscr{B}_0(\mathbb{R})\} < +\infty,$$

where

$$\|\mu\|(A) = \sup\left\{\left\|\sum_{i=1}^N a_i\, \mu(A_i)\right\|_\alpha : \{A_i\} \subset \mathscr{B}_0(\mathbb{R}) \text{ finite partition of } A,\ a_i \in \mathbb{C}, \left\|\sum_{i=1}^N a_i\, \chi_{A_i}\right\|_\beta \leqq 1\right\}.$$

The definition of (α, β)-boundedness makes implicit use of the Lebesgue measure, this is somehow superfluous. Most of our results continue to hold if we replace $L^\beta(\mathbb{R})$ by $L^\beta(\lambda)$ where λ is a σ-finite regular Borel measure on some locally compact Hausdorff space, replacing also $\mathscr{B}_0(\mathbb{R})$ by the corresponding δ-ring of sets with finite λ measure. Generally, $L^\alpha(\mathscr{P})$ can also be replaced by an arbitrary Banach space. To be consistent with the results of Section 4 and for probabilistic considerations (in which instance the case $\alpha > 2$ is of little interest, furthermore included in $L^2(\mathscr{P})$) we will however keep, with a few exceptions (mainly examples), our "restricted" framework.

For $\beta = +\infty$, Definition 2.1 reduces to the usual definition of semi- (or Fréchet) variation, but for $1 \leqq \beta < +\infty$, it differs from the definition of β-variation (see [Di]), in that it involves β and not its conjugate exponent. It is assumed in the (α, β)-boundedness requirement that $\mu(A) \in L^\alpha(\mathscr{P})$, hence if μ has finite (α, β)-variation it also has finite (γ, β)-variation, $\gamma < \alpha$. Typically, μ is only finitely additive, and in fact, if μ is σ-additive on $\mathscr{B}(\mathbb{R})$, then it has finite (α, ∞)-variation (see [DS, IV.10]).

The following properties of the (α, β)-variation are now easy to verify.

$$\|\mu(A)\|_\alpha \leqq \|\mu\|(A) \leqq \|\mu\|, \qquad A \in \mathscr{B}_0(\mathbb{R}).$$
$$\|\mu\|(A) \leqq \|\mu\|(B), \qquad A \subset B \in \mathscr{B}_0(\mathbb{R}).$$
$$\|\mu\|(A \cup B) \leqq \|\mu\|(A) + \|\mu\|(B), \qquad A, B \in \mathscr{B}_0(\mathbb{R}).$$

Even for A and B are disjoints, the last inequality is usually strict and $\|\mu\|(\cdot)$ is additive if and only if

$$\|\mu\|(A) = \sup\left\{\sum_{i=1}^N \|a_i\, \mu(A_i)\|_\alpha : \left\|\sum_{i=1}^N a_i\, \chi_{A_i}\right\|_\infty \leqq 1\right\},$$

for each $A \in \mathscr{B}_0(\mathbb{R})$.

The integration with respect to μ can now be defined as follows: for a simple function

$$f: \mathbb{R} \to \mathbb{C}, \qquad f = \sum_{i=1}^N a_i\, \chi_{A_i}, \qquad A_i \in \mathscr{B}_0(\mathbb{R}),$$

as usual, the integral (of f with respect to μ) is

$$\int_{\mathbb{R}} f\, d\mu = \sum_{i=1}^N a_i\, \mu(A_i). \tag{2.1}$$

If μ has finite (α, β)-variation, then

$$\left\| \int_{\mathbb{R}} f\, d\mu \right\|_\alpha \leqq \|\mu\|(A)\, \|f\|_\beta, \tag{2.2}$$

where $A = \bigcup_{i=1}^N A_i$. Since $\|\mu\|(A) \leqq \|\mu\|$, and by the density of the simple functions in $L^\beta(\mathbb{R})$ $1 \leqq \beta < +\infty$, (resp. in $C_0(\mathbb{R})$), the integral can be extended to $L^\beta(\mathbb{R})$ ((resp. to $C_0(\mathbb{R})$), this is done so, and integration with respect to μ is always taken in that sense. On $\mathscr{B}(\mathbb{R})$, and for σ-additive μ, the above integral is the Bartle-Dunford and Schwartz integral [DS, IV.10], while for $d\mu(t) = \mu(t)\, dt$, it is just the strong, i.e., the Bochner, integral.

Our first result identifying random measures with operators, is certainly known. For (α, ∞)-bounded measures and in a more general framework, it is in [DS], [P] and [K], replacing of course $L^\beta(\mathbb{R})$ below, by $C_0(\mathbb{R})$. For the sake of completeness, a sketch of proof in the case $\beta < +\infty$ is included. Let us say that a random measure μ is *regular* if for every $A \in \mathscr{B}_0(\mathbb{R})$ and $\varepsilon > 0$ there exist O open $\in \mathscr{B}_0(\mathbb{R})$ and C compact $\in \mathscr{B}_0(\mathbb{R})$ such that $C \subset A \subset O$ and $\|\mu(B)\|_\alpha < \varepsilon$, for every $B \subset O \setminus C$.

Theorem 2.2. *Let $\mathscr{D}$ be a dense linear subspace of $L^\beta(\mathbb{R})$, $1 \leqq \beta < +\infty$, and let $T: \mathscr{D} \to L^\alpha(\mathscr{P})$ be a linear mapping such that there exists a constant $K > 0$ with*

$$\|Tf\|_\alpha \leqq K \|f\|_\beta, \tag{2.3}$$

for all $f \in \mathscr{D}$. Then, there exists a unique regular random measure μ with finite (α, β)-variation such that

$$Tf = \int_{\mathbb{R}} f \, d\mu \tag{2.4}$$

for all $f \in \mathscr{D}$. Moreover, if there exists a random measure μ with finite (α, β)-variation such that (2.4) *holds, then* (2.3) *is also satisfied. In either case, $\|T\| = \|\mu\|$.*

Proof. From Definition 2.1, the "if" part is immediate. For the converse, since $\mathscr{D}$ is dense in $L^\beta(\mathbb{R})$, T can be extended to all of $L^\beta(\mathbb{R})$ without change of norm. We do extend it that way and still denote the extension by T. Now, if we set $\mu(A) = T(\chi_A)$, $A \in \mathscr{B}_0(\mathbb{R})$, then μ is well defined, additive, and has finite (α, β)-variation. Finally, the regularity of the random measure μ follows from the regularity of the Lebesgue measure, and the equality of the norms is also clear. □

Remark 2.3. As already mentioned, it is well known, that every σ-additive μ on $\mathscr{B}(\mathbb{R})$ has finite (α, ∞)-variation ([DS, IV.10]), and furthermore that there exist finitely additive (α, ∞)-bounded measures on $\mathscr{B}_0(\mathbb{R})$ which are not σ-additive on $\mathscr{B}(\mathbb{R})$, e.g., a Hahn-Banach extension of a Dirac measure. However, when given via an operator $T: \mathscr{D} \subset C_0(\mathbb{R}) \to L^\alpha(\mathscr{P})$ and since $L^\alpha(\mathscr{P})$, $1 \leqq \alpha \leqq 2$, is weakly complete the corresponding μ is σ-additive on $\mathscr{B}(\mathbb{R})$ (see [DS, VI.7] for the compact case, which can be modified to give the result for $C_0(\mathbb{R})$). Since all the (α, ∞)-bounded measures appearing in this work are given by, obtained from, or extended to bounded linear operators from $C_0(\mathbb{R})$ to $L^\alpha(\mathscr{P})$, we identify throughout the (α, ∞)-bounded case with the σ-additive one, i.e., with the bounded linear operator from $C_0(\mathbb{R})$ to $L^\alpha(\mathscr{P})$ case.

Remark 2.4. In Theorem 2.2, $L^\alpha(\mathscr{P})$ can be replaced by a Banach space B as long as (2.3) is replaced by the relative weak compactness of B of the set $S = \{T(f): \|f\|_\beta \leqq 1, f \in \mathscr{D}\}$. As in [K], for B weakly complete, S is relatively weakly compact whenever $T: \mathscr{D} \to B$, is linear and bounded.

From the above result, it is clear that the set of (α, β)-bounded measures μ from a Banach space under the norm $\|\mu\|$. It is denoted by $\mathscr{M}^{\alpha,\beta}$.

To complete this introductory section, we provide a few examples which illustrates the scope of our method. First, a stochastic measure μ is called *orthogonally scattered* if $\langle \mu(A), \mu(B) \rangle_2 = 0$, whenever $A, B \in \mathscr{B}_0(\mathbb{R})$, $A \cap B = \phi$. As well known, μ is orthogonally scattered if and only if there exists a finitely additive positive function m on $\mathscr{B}_0(\mathbb{R})$ such that $\langle\mu(A), \mu(B)\rangle_2 = m(A\cap B)$. For $1\leqq\alpha<2$, a similar rôle is played by the *isotropic independently scattered symmetric α-stable* ($S\alpha S$) (in short, independently scattered $S\alpha S$) random measures, namely, for each $A\in\mathscr{B}_0(\mathbb{R})$, $\mu(A)$ is an isotropic $S\alpha S$ random variable such that for any pairwise disjoint $\{A_1, A_2, \ldots, A_n\}\subset\mathscr{B}_0(\mathbb{R})$, the random variables $\mu(A_1)$, $\mu(A_2), \ldots, \mu(A_n)$ are independent. If μ is independently scattered and $S\alpha S$, there also exists a positive finitely additive m such that $[\mu(A), \mu(B)]_\alpha = m(A\cap B)$, $A, B\in\mathscr{B}_0(\mathbb{R})$, where $[\cdot,\cdot]_\alpha$ is the covariation of the corresponding random variables (see Weron [W] and Cambanis [C] for more details on the properties of $S\alpha S$ processes and measures used here). Furthermore, for any $1\leqq\gamma<\alpha$, $[\mu(A), \mu(B)]_\alpha = K\langle\mu(A), \mu(B)\rangle_\gamma$, $A, B\in\mathscr{B}_0(\mathbb{R})$, where K depends only on α and γ. With these notations, we now characterize the independently scattered $S\alpha S$ (resp. orthogonally scattered) elements of $\mathscr{M}^{\gamma,\beta}$ (resp. $\mathscr{M}^{2,\beta}$). Our proof is similar to the discrete time case (see [H1]) and so only sketched.

Proposition 2.5. *Let $\alpha\leqq\beta<+\infty$ (resp. $2\leqq\beta<+\infty$), and let μ be an independently scattered $S\alpha S$ (resp. orthogonally scattered) random measure μ with control measure m. Then, μ is (γ,β)-bounded, $1\leqq\gamma<\alpha$, (resp. $(2,\beta)$-bounded) if and only $dm = m\,dt$, $m\in L^{\beta/\beta-\alpha}(\mathbb{R})$ (resp. $m\in L^{\beta/\beta-2}(\mathbb{R})$).*

Proof. We first treat the case $1\leqq\alpha<2$. Let μ have the stated properties, with $dm = m\,dt$, $m\in L^{\beta/\beta-\alpha}(\mathbb{R})$. Then, stability and Hölder's inequality give $\mu\in\mathscr{M}^{\gamma,\beta}$. Conversely, if μ is independently scattered $S\alpha S$ and in $\mathscr{M}^{\gamma,\beta}$, then for $f = \sum_{i=1}^{N} a_i\chi_{A_i}$, $A_i\in\mathscr{B}_0(\mathbb{R})$ (and more generally for any $f\in L^\beta(\mathbb{R})$),

$$\Big\|\int_{\mathbb{R}} f\,d\mu\Big\|_\gamma = K\Big(\int_{\mathbb{R}} |f|^\alpha dm\Big)^{1/\alpha} \leqq \|\mu\|\,K\Big(\int_{\mathbb{R}} |f|^\beta dt\Big)^{1/\beta},$$

hence $dm = m\,dt$, with $m\geqq 0$ and locally in $L^1(\mathbb{R})$. Now as in [H1], in the above inequality we take $f = m^{1/\beta-\alpha}\chi_{]-n,n[}$, $\alpha<\beta$, get $\left(\int_{-n}^{n} m^{\beta/\beta-\alpha}dt\right)^{\beta-\alpha/\alpha\beta} \leqq K$, and monotone convergence gives the result. The case $\alpha=\beta$ can again be obtained essentially as in [H1], while for $\alpha=2$ the techniques are identical, replacing independence by orthogonality. □

For $\beta=+\infty$, comparable arguments easily show that $\mu\in\mathscr{M}^{\gamma,\infty}$ (resp. $\mathscr{M}^{2,\infty}$) if and only if m defines a positive bounded linear functional $C_c(\mathbb{R})$ which is uniquely extended to $C_0(\mathbb{R})$, i.e., m is a positive finite measure on $\mathscr{B}(\mathbb{R})$. Replacing, in the definition of variation, the Lebesgue measure by some σ-finite λ, we similarly get, for independently scattered μ, $dm = (dm/d\lambda)\,d\lambda$, with $dm/d\lambda\in L^{\beta/\beta-\alpha}(\lambda)$, this case corresponds, for example, to infinitely divisible random measures with control measure λ. As in [H1], for $1\leqq\beta<\alpha$, the only $S\alpha S$ independently scattered element of $\mathscr{M}^{\gamma,\beta}$ is the zero random measure and similarly for $\alpha=2$.

The archetype of orthogonally scattered random measure is the *white measure* (or *noise*) W, i.e., for each $A\in\mathscr{B}_0(\mathbb{R})$, $W(A)$ is a zero mean Gaussian random variable and $\langle W(A), W(B)\rangle_2 = |A\cap B|$ where $|A\cap B|$ denotes the Lebesgue measure of $A\cap B$. Then W has finite (2, 2)-variation, in fact finite $(\alpha, 2)$-variation $1\leqq\alpha<+\infty$. For $1\leqq\alpha<2$, we have the corresponding *α-white measure* V, i.e.,

V is $S\alpha S$ independently scattered with $[V(A), V(B)]_\alpha = |A \cap B|$. Moreover, V (and similarly W) is *stationarily scattered*, i.e., for any $t \in \mathbb{R}$ and any $A_1, A_2, \ldots, A_n \in \mathscr{B}_0(\mathbb{R})$, the law of $\{V(A_1+t), V(A_2+t), \ldots, V(A_n+t)\}$ does not depend on t, where $A+t = \{a+t, a \in A]$.

Two last properties worth mentioning: Firstly if μ is absolutely continuous, i.e., $\mu(A) = \int_A \mu(t)\,dt$, with $\mu(\cdot) \in L^1(\mathbb{R}, \alpha)$ (the Lebesgue-Bochner space of $L^\alpha(\mathscr{P})$-valued integrable functions), then by Hölder's inequality, $\mu \in \mathscr{M}^{\alpha,\beta}$ whenever $\mu \in L^{\beta/\beta-1}(\mathbb{R}, \alpha)$. Secondly, if $\mu \in \mathscr{M}^{\alpha,\beta}$ and if Q is a bounded linear operator on $L^\alpha(\mathscr{P})$ then $v = Q\mu$ defined by $v(A) = Q\mu(A)$, $A \in \mathscr{B}_0(\mathbb{R})$, is also and element of $\mathscr{M}^{\alpha,\beta}$, furthermore Q commutes with the integral.

3 Processes as Fourier integrals

Unless the measure $\mu\colon \mathscr{B}_0(\mathbb{R}) \to L^0(\mathscr{P})$ has finite (α, ∞)-variation, the exponentials are not μ-integrable, and expressions such as $\int_{\mathbb{R}} e^{it\xi}\,d\mu(\xi)$ do not make sense. To bypass this handicap, we need to extend our definition of the Fourier integral. A first natural step in that direction is to use summability methods and such is done below. Throughout this work we assume that a *process* X is a function $X\colon \mathbb{R} \to L^0(\mathscr{P})$.

Definition 3.1. A bounded (with respect to the $L^\alpha(\mathscr{P})$-norm) process X is (α, β)-*bounded* if X is strongly measurable and if there exists a constant $K > 0$ such that

$$\left\| \int_{\mathbb{R}} f(t) X_t\,dt \right\|_\alpha \leqq K \|\hat{f}\|_\beta, \tag{3.1}$$

for all $f \in L^\beta(\mathbb{R})^\nu$.

In (3.1), $\int_{\mathbb{R}} f(t) X_t\,dt$ is a well defined Bochner integral since $\|X_t\|_\alpha \leqq K$. The above notion of boundedness has been introduced, for discrete time processes and for $\alpha = 2$, in [H1], and is directly inspired by Bochner's V-boundedness, i.e., the case $\beta = +\infty$. If $L^\alpha(\mathscr{P})$ is replaced by a Banach space B and for X such that $\|X_t\|_B \leqq K$, the definition of (α, β)-boundedness can be translated into: the B-valued function X is strongly measurable and the set $\{\| \int_{\mathbb{R}} f(t) X_t\,dt\|_B\colon \|\hat{f}\|_\beta \leqq 1, f \in L^\beta(\mathbb{R})^\nu\}$ is relatively weakly compact in B. When B is weakly complete, this last requirement is equivalent to (3.1).

We now present some examples of processes satisfying (3.1). First, it is clear that if $Y = \{Y_t\}_{t\in\mathbb{R}}$ is a continuous second order stationary process or a harmonizable stable (of order α) process, it does satisfy (3.1) with $\beta = +\infty$. Hence for $X_t = QY_t$, $t \in \mathbb{R}$, where Q is a bounded linear operator on $L^2(\mathscr{P})$ (when Y is stationary) or on $L^\gamma(\mathscr{P})$, $1 \leqq \gamma < \alpha$, (when Y is stable harmonizable), we have:

$$\left\| \int_{\mathbb{R}} f(t) X_t\,dt \right\|_\gamma \leqq \|Q\| \left\| \int_{\mathbb{R}} f(t) Y_t\,dt \right\|_\gamma \leqq K \|\hat{f}\|_\infty, \quad 1 \leqq \gamma < \alpha,$$

and similarly for $\alpha=2$. Less immediate examples are ARMA processes or solutions to linear autonomous stochastic differential equations (see [H1] or Sect. 6). For a more atypical example, let $\alpha \geqq 2$ and let X be a discrete time L^α-bounded martingale difference process. For any $P(t)=\sum_{j=1}^{N} P_j e^{in_j\theta}$ Burkholder's square inequality as well as Minkowski's inequality give

$$\begin{aligned}\left\|\sum_{j=1}^{N} P_j X_{n_j}\right\|_\alpha^\alpha &\leqq K\left\|\sum_{j=1}^{N} |P_j|^2 |X_{n_j}^2|\right\|_{\alpha/2}^{\alpha/2}\\ &\leqq K\left\{\sum_{j=1}^{N} |P_j|^2 \|X_{n_j}^2\|_{\alpha/2}\right\}^{\alpha/2}\\ &\leqq K \sup \|X_{n_j}\|_\alpha^\alpha \left\{\sum_{j=1}^{N} |P_j|^2\right\}^{\alpha/2}\\ &= K\left\{\int_{-\pi}^{\pi} |P(t)|^2 dt\right\}^{\alpha/2}.\end{aligned}$$

Hence, X is $(\alpha, 2)$-bounded.

We are now ready to characterize (α, β)-bounded processes as Fourier integrals.

Theorem 3.2. *A bounded process X is strongly continuous and (α, β)-bounded if and only if there exists a (unique) regular random measure μ of finite (α, β)-variation such that*

$$X_t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1-\frac{|\xi|}{\lambda}\right) e^{it\xi} d\mu(\xi), \tag{3.2}$$

in $L^\alpha(\mathscr{P})$, uniformly on compact subsets of $\mathbb{R}$.

Proof. Let X be strongly continuous and (α, β)-bounded and let $\hat{L}^1(\mathbb{R})=\{\hat{f}: f \in L^1(\mathbb{R})\}$. Then, by the uniqueness of the Fourier transform, the mapping T: $\hat{L}^1(\mathbb{R}) \cap L^\beta(\mathbb{R}) \to L^\alpha(\mathscr{P})$ such that $T(\hat{f})=\int_{\mathbb{R}} f(t) X_t\, dt$ is well defined and also linear. Since X is (α, β)-bounded, T is bounded on $\hat{L}^1(\mathbb{R}) \cap L^\beta(\mathbb{R})$ which is dense in $L^\beta(\mathbb{R})$. Hence by Theorem 2.2, there exists a unique regular random measure μ of finite (α, β)-variation such that

$$\int_{\mathbb{R}} f(t) X_t\, dt = \int_{\mathbb{R}} \hat{f}(t)\, d\mu(t), \qquad f \in L^\beta(\mathbb{R})^\vee. \tag{3.3}$$

Let

$$K_\lambda(t) = \int_{-\lambda}^{\lambda} \left(1-\frac{|\xi|}{\lambda}\right) e^{-it\xi} d\xi = \lambda\left(\frac{\sin \lambda t/2}{\lambda t/2}\right)^2$$

be the Fejér kernel, and let $\hat{K}_\lambda(\xi)=\left(1-\frac{|\xi|}{\lambda}\right)\chi_{]-\lambda,\lambda[}(\xi)$ be its Fourier transform. By (3.3), and since $K_\lambda \in L^1(\mathbb{R})$, $\hat{K}_\lambda \in L^\beta(\mathbb{R})$,

$$\int_{\mathbb{R}} K_\lambda(t-\tau) X_\tau\, d\tau = \int_{-\lambda}^{\lambda} \left(1-\frac{|\xi|}{\lambda}\right) e^{it\xi} d\mu(\xi). \tag{3.4}$$

Now, using (3.4) and since K_λ is an approximate identity,

$$\left\|X_t-\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)e^{it\xi}d\mu(\xi)\right\|_\alpha=\|\int_{\mathbb{R}}K_\lambda(\tau)(X_t-X_{t-\tau})\,d\tau\|_\alpha$$

$$\leqq\sup_{|\tau|\leqq\delta}\|X_t-X_{t-\tau}\|_\alpha\int_{|\tau|\leqq\delta}|K_\lambda(\tau)|\,d\tau+2\sup_{\mathbb{R}}\|X_t\|_\alpha\int_{|\tau|>\delta}|K_\lambda(\tau)|\,d\tau$$

$$\leqq\sup_{|\tau|\leqq\delta}\|X_t-X_{t-\tau}\|_\alpha+K\int_{|\tau|>\delta}|K_\lambda(\tau)|\,d\tau.$$

The norm continuity of X on the compact set $|\tau|\leqq\delta$ gives $\lim_{\tau\to 0}\sup_{|\tau|\leqq\delta}\|X_t-X_{t-\tau}\|_\alpha=0$ and since again K_λ is an approximate identity, $\lim_{\lambda\to+\infty}\int_{|\tau|>\delta}|K_\lambda(\tau)|\,d\tau=0$. For the converse, X is the uniform limit of the continuous $X_t^\lambda=\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)e^{it\xi}d\mu(\xi)$, hence is (norm) continuous. Let $f\in L^\beta(\mathbb{R})^v$ have compact support, then by the uniform convergence on compacts, we have $\int_{\mathbb{R}}f(t)X_t\,dt=\lim_{\lambda\to+\infty}\int_{\mathbb{R}}f(t)X_t^\lambda\,dt$ in $L^\alpha(\mathscr{P})$. But, $\int_{\mathbb{R}}f(t)X_t^\lambda\,dt=\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)\hat{f}(\xi)\,d\mu(\xi)$, with μ of finite (α,β)-variation, and since $\hat{K}_\lambda(\xi)\leqq 1$ we get $\|\int_{\mathbb{R}}f(t)X_t\,dt\|_\alpha\leqq\|\mu\|(\int_{\mathbb{R}}|\hat{f}(\xi)|^\beta d\xi)^{1/\beta}$. For an arbitrary $f\in L^\beta(\mathbb{R})^v$, the functions $f_n(t)=\left(1-\frac{|t|}{n}\right)\cdot\chi_{]-n,n[}(t)f(t)$ have compact support, converge to f in $L^1(\mathbb{R})$, with moreover $\lim_{n\to+\infty}\|\hat{f}-\hat{f}_n\|_\beta=0$. Hence,

$$\|\int_{\mathbb{R}}f(t)X_t\,dt\|_\alpha\leqq\|\int_{\mathbb{R}}(f-f_n)(t)X_t\,dt\|_\alpha+\|\int_{\mathbb{R}}f_n(t)X_t\,dt\|_\alpha$$

$$=\sup_{\mathbb{R}}\|X_t\|_\alpha\int_{\mathbb{R}}|f(t)-f_n(t)|\,dt+\|\mu\|(\int_{\mathbb{R}}|\hat{f}_n(t)|^\beta dt)^{1/\beta},$$

from which the result easily follows. □

For $\beta=+\infty$, i.e., replacing $L^\beta(\mathbb{R})$ by $C_0(\mathbb{R})$, Theorem 3.2 is due, in a more general framework, to Phillips and Kluvánek. When X is (α,∞)-bounded, equivalently, when μ is σ-additive (see Remark 2.3), the exponentials are μ-integrable and $X_t=\int_{\mathbb{R}}e^{it\xi}d\mu(\xi)=\lim_{\lambda\to+\infty}\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)e^{it\xi}d\mu(\xi)$, by dominated convergence [DS, IV.10]. Theorem 3.2 continue to hold when X is B-valued as long as we replace our boundedness conditions by the equivalent, and now familiar, compactness conditions. Except for being an approximate identity, no particular property of the Fejér kernel has been used in the proof of Theorem 3.2. Hence, in (3.2) the Fejér kernel can be replaced by any approximate identity.

A process X is called *Cesáro harmonizable* if $X_t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} d\mu(\xi)$, for some $S\alpha S$ independently scattered random measure μ (orthogonally scatteredd μ for $\alpha = 2$). With this definition as well as Proposition 2.5 and its notations, we have:

Corollary 3.3. *For* $\alpha \leqq \beta < +\infty$ (resp. $2 \leqq \beta < +\infty$), *a strongly continuous Cesáro harmonizable process is* (γ, β)*-bounded* (resp. $(2, \beta)$*-bounded*) *if and only if* $dm = m\,dt$, $m \in L^1(\mathbb{R}) \cap L^{\beta/\beta-\alpha}(\mathbb{R})$ (resp. $L^{\beta/\beta-2}(\mathbb{R})$).

Proof. In view of Proposition 2.5, only the additional requirement $m \in L^1(\mathbb{R})$ needs some justification. Since the process X is a strongly continuous $S\alpha S$ process or since μ is a $S\alpha S$ random measure, we must have

$$\mathscr{E} e^{isX_t} = \lim_{\lambda \to +\infty} e^{is \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} d\mu(\xi)} = \lim_{\lambda \to +\infty} e^{-|s|^\alpha \int_{-\lambda}^{\lambda} \left|1 - \frac{|\xi|}{\lambda}\right|^\alpha m(\xi)\, d\xi}$$

$$= e^{-|s|^\alpha \int_{\mathbb{R}} m(\xi)\, d\xi},$$

where the last equality follows from Fatou's lemma and since

$$e^{-|s|^\alpha \int_{-\lambda}^{\lambda} \left|1 - \frac{|\xi|}{\lambda}\right|^\alpha m(\xi)\, d\xi} \geqq e^{-|s|^\alpha \int_{\mathbb{R}} m(\xi)\, d\xi}.$$

In other words, $m \in L^1(\mathbb{R})$ (of course, the process X is not identically zero). □

The case $\beta = +\infty$ is trivial: a strongly continuous Cesáro harmonizable process is always (γ, ∞)-bounded ($(2, \infty)$-bounded when $\alpha = 2$) since the control measure m is finite. The measure μ and the process X are dual of one another, and μ can be recovered from X (see Theorem 4.3 for a more general result).

Although Theorem 3.2 allows us to extend the definition of the Fourier transform beyond Bochner's V-bounded class, it retains some disadvantages. Very apparent disadvantages are the L^α-boundedness and the continuity requirements. These can be partially removed as shown by the following result, which for $\beta = +\infty$ is due to Phillips and which clearly is not a characterization. First, a process X is said to be *essentially bounded* if $\|X_t\|_\alpha \leqq K$, for almost all t (Lebesgue).

Theorem 3.4. *Let X be essentially bounded, strongly measurable and* (α, β)*-bounded, then there a (unique) regular random measure μ of finite* (α, β)*-variation such that for almost all t* (Leb.)

$$X_t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} d\mu(\xi), \tag{3.5}$$

in $L^\alpha(\mathscr{P})$.

Proof. Let X be strongly measurable and (α, β)-bounded. Then, as in the proof of Theorem 3.2, we have $\int_{\mathbb{R}} f(t)\, X_t\, dt = \int_{\mathbb{R}} \hat{f}(t)\, d\mu(t)$, for all $f \in L^\beta(\mathbb{R})^\nu$. Hence,

$$\left\| X_t - \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi}\, d\mu(\xi) \right\|_\alpha = \left\| \int_{\mathbb{R}} K_\lambda(\tau)\,(X_t - X_{t-\tau})\, d\tau \right\|_\alpha. \tag{3.6}$$

To prove the result, we need to show that for almost all t, the right hand side of (3.6) converges to zero. Since $K_\lambda(\tau) \leqq K/\lambda\tau^2$ and since $\|X_t\|_\alpha \leqq K$ for almost all t, it follows that $\|\int_{\mathbb{R}} K_\lambda(\tau)\,(X_t - X_{t-\tau})\, d\tau\|_\alpha$ is dominated by

$$\begin{aligned}
&\Big\| \int_{|\tau| \leqq \lambda^{-3/4}} K_\lambda(\tau)\,(X_t - X_{t-\tau})\, d\tau \Big\|_\alpha + \Big\| \int_{|\tau| > \lambda^{-3/4}} K_\lambda(\tau)\,(X_t - X_{t-\tau})\, d\tau \Big\|_\alpha \\
&\leqq \Big\| \int_{|\tau| \leqq \lambda^{-3/4}} K_\lambda(\tau)\,(X_t - X_{t-\tau})\, d\tau \Big\|_\alpha + 2K \int_{|\tau| > \lambda^{-3/4}} 1/\lambda\tau^2\, d\tau \\
&= \Big\| \int_{|\tau| \leqq \lambda^{-3/4}} K_\lambda(\tau)\,(X_t - X_{t-\tau})\, d\tau \Big\|_\alpha + 4K\lambda^{-1/4}.
\end{aligned}$$

To prove the result, it is thus enough that, in the above expression, the first term converges to zero as λ becomes infinite. It is clear that except at the origin, K_λ is absolutely continuous. By integrating by parts and since X is essentially bounded we get:

$$\begin{aligned}
&\Big\| \int_{|\tau| \leqq \lambda^{-3/4}} K_\lambda(\tau)\,(X_t - X_{t-\tau})\, d\tau \Big\|_\alpha \\
&\leqq \left\{ \left(\int_0^\tau \|X_t - X_{t-u}\|_\alpha\, du \right) K_\lambda(\tau) \right\}_{-\lambda^{-3/4}}^{\lambda^{-3/4}} \\
&\quad + \int_{|\tau| \leqq \lambda^{-3/4}} \left(\int_0^\tau \|X_t - X_{t-u}\|_\alpha\, du \right) K'_\lambda(\tau)\, d\tau \\
&\leqq 4K\lambda^{-3/4} K_\lambda(\lambda^{-3/4}) + 4K\lambda^{-3/4} K_\lambda(\lambda^{-3/4}).
\end{aligned}$$

Finally, $\lambda^{-3/4} K_\lambda(\lambda^{-3/4}) = 4\lambda^{-1/4}(\sin(\lambda^{1/4}/2))^2$ and the result follows. □

There is no hope with the methods presented above to go beyond Theorem 3.4. In other words, these methods can produce, at best, convergence results for almost all $t \in \mathbb{R}$, and so even under "weaker requirements". For example, if $X \in L^p(\mathbb{R}, \alpha)$, i.e., X is strongly measurable and $\|X_t\|_\alpha \in L^p(\mathbb{R})$, $1 \leqq p < +\infty$, it easily follows from the same methods that: X is (α, β)-bounded if and only if there exists a (unique) regular μ of finite (α, β)-variation such that

$$\lim_{\lambda \to +\infty} \int_{\mathbb{R}} \left\| X_t - \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi}\, d\mu(\xi) \right\|_\alpha^p dt = 0.$$

Similarly, at any point t where

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_0^{\varepsilon} \|X_{t-\tau}-X_t\| \, d\tau=0,$$

we have

$$\lim_{\lambda\to+\infty} \int_{-\lambda}^{\lambda} \left(1-\frac{|\xi|}{\lambda}\right) e^{it\xi} d\mu(\xi)=X_t \qquad (\text{in } L^{\alpha}(\mathscr{P})).$$

These almost sure results are somehow restrictive since there exist important random measures for which they do not hold. A case at hand is a white measure W, where for no $t\in\mathbb{R}$, does

$$\int_{-\lambda}^{\lambda} \left(1-\frac{|\xi|}{\lambda}\right) e^{it\xi} dW$$

converge, either in $L^2(\mathscr{P})$ or in $L^p(\mathbb{R}, 2)$, $1\leqq p\leqq +\infty$. In order to deal with such a case, we need to extend our definition of the Fourier transform, and such is done in the next section.

4 Random measures as Fourier integrals

For discrete time processes, (α, β)-boundedness implies (α, ∞)-boundedness and (α, β)-bounded processes are Fourier integrals; and similarly when μ has compact support. However, outside of these two cases things happen quite differently, since the Lebesgue measure of $\mathbb{R}$ is infinite, a random measure or a random process which is (α, β)-bounded is not necessarily (α, ∞)-bounded. In order to clear this obstacle and to extend our notion of Fourier transform, our strategy is to think of $\hat{\mu}$ not as a stochastic process, but more globally as a stochastic measure.

Again, for μ of finite (α, ∞)-variation, i.e., $\mu\in\mathscr{M}^{\alpha,\infty}$ the exponentials are μ-integrable, hence $\hat{\mu}=\{\hat{\mu}(t)\}_{t\in\mathbb{R}}$ makes perfect sense and is a (α, ∞)-bounded process, But, $\hat{\mu}$ can also be looked at in a different way, namely, as the absolutely continuous element of $\mathscr{M}^{\alpha,1}$, $d\hat{\mu}(t)=\hat{\mu}(t)\,dt$. This follows from the simple calculation given below, where we make use of a Fubini type theorem which is justified, since $\mu\in\mathscr{M}^{\alpha,\infty}$.

$$\left\|\int_{\mathbb{R}} f(t)\,\hat{\mu}(t)\,dt\right\|_{\alpha}=\left\|\int_{\mathbb{R}} \hat{f}(t)\,d\mu(t)\right\|_{\alpha}\leqq\|\hat{f}\|_{\infty}\leqq\int_{\mathbb{R}} |f(t)|\,dt, \qquad f\in L^1(\mathbb{R}).$$

Since for μ in $\mathscr{M}^{\alpha,\infty}$, $\hat{\mu}$ belongs to $\mathscr{M}^{\alpha,1}$, it becomes natural to try to define the Fourier transform of a random measure as being itself a random measure. This is done below, where we first tackle the case $\beta=2$, and where we drop in our notation the reference to α, i.e., we write $\mathscr{M}^2$ for $\mathscr{M}^{\alpha,2}$ and $\mathscr{M}^{\infty}$ for $\mathscr{M}^{\alpha,\infty}$. The relevant results is a Plancherel type theorem in which μ and $\hat{\mu}$ play identical rôles.

Theorem 4.1. *There exists a unique bounded linear operator $\mathscr{F}$ from $\mathscr{M}^2$ onto itself such that*

(i) $\mathscr{F}\mu=\hat{\mu}$, for $\mu\in\mathscr{M}^2\cap\mathscr{M}^{\infty}$.

(ii) $\|\mathscr{F}\mu\| = \|\mu\|$.

Proof. Let $\mu \in \mathscr{M}^2$ and let μ_n be defined via $\mu_n(A) = \mu(A \cap]-n, n[\,)$, $A \in \mathscr{B}_0(\mathbb{R})$. Clearly, for each n, μ_n is (2, 2)-bounded as well as (2, ∞)-bounded, with moreover $\int_{\mathbb{R}} f\, d\mu_n = \int_{-n}^{n} f\, d\mu$, $f \in C_c(\mathbb{R})$. Since, each $\mu_n \in \mathscr{M}^\infty$, $\hat{\mu}_n$ is the previously well defined Fourier transform, i.e., $\hat{\mu}_n$ is a (2, ∞)-bounded process or, $d\hat{\mu}_n = \hat{\mu}_n(t)\, dt \in \mathscr{M}^1$. Hence, for each n,

$$L_n\colon\ C_c(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathscr{P}), \qquad f\colon \to L_n(f) = \int_{\mathbb{R}} f\, d\hat{\mu}_n$$

is also well defined and linear. Furthermore, since $\int_{\mathbb{R}} f\, d\hat{\mu}_n = \int_{\mathbb{R}} \hat{f}\, d\mu_n$, ($f$ and μ_n have compact support so a Fubini theorem holds) L_n is also bounded. Now, the (2, 2)-boundedness of μ gives

$$\lim_{n,m\to\infty} \|L_n(f) - L_m(f)\|_2 \leqq \lim_{n,m\to\infty} \|\mu\|\, \|\hat{f}_n - \hat{f}_m\|_2 = 0,$$

since $f \in C_c(\mathbb{R})$ and by the classical Plancherel theory. Thus, the sequence $L_n(f)$ is a Cauchy sequence with limit denoted by $\hat{\mu}(f)$ also satisfying $\int_{\mathbb{R}} f\, d\hat{\mu} = \int_{\mathbb{R}} \hat{f}\, d\mu$, $f \in C_c(\mathbb{R})$. Clearly, $\hat{\mu}$ is linear and by the Banach-Steinhaus theorem it is also (2, 2)-bounded and, as such, has a unique extension (also denoted $\hat{\mu}$) to $L^2(\mathbb{R})$. It is also clear that for $\mu \in \mathscr{M}^\infty$, the two definitions of Fourier transform agree. Now, let $\mathscr{F}\colon \mathscr{M}^2 \to \mathscr{M}^2$, $\mu\colon \to \hat{\mu}$, then $\mathscr{F}$ is well defined, linear. By, the density of $C_c(\mathbb{R})$ in $L^2(\mathbb{R})$, by the above Parseval type formula as well as by the classical Plancherel theory, it follows that

$$\begin{aligned}\|\mathscr{F}\mu\| &= \sup_{\|f\|_2 \leqq 1} \Big\|\int_{\mathbb{R}} f\, d\hat{\mu}\Big\|_2 = \sup_{\substack{\|f\|_2 \leqq 1\\ f \in C_c(\mathbb{R})}} \Big\|\int_R f\, d\hat{\mu}\Big\|_2 \\ &= \sup_{\substack{\|f\|_2 \leqq 1\\ f \in C_c(\mathbb{R})}} \Big\|\int_R \hat{f}\, d\mu\Big\|_2 = \sup_{\substack{\|\hat{f}\|_2 \leqq 1\\ f \in C_c(\mathbb{R})}} \Big\|\int_R \hat{f}\, d\mu\Big\|_2 = \|\mu\|.\end{aligned}$$

Thus, $\mathscr{F}$ is norm preserving, hence bounded and one to one. It just remains to show that $\mathscr{F}$ is also onto. May be the simplest way to see that is to take f in a dense subset of $L^2(\mathbb{R})$, with $f = \hat{g}$ (the twice differentiable elements of $C_C(\mathbb{R})$ will do it, with $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$). Then,

$$\begin{aligned}\int_{\mathbb{R}} f(\xi)\, d\mu(\xi) &= \int_{\mathbb{R}} \hat{g}(\xi)\, d\mu(\xi) = \int_{\mathbb{R}} g(\xi)\, d\hat{\mu}(\xi) = \int_{\mathbb{R}} \hat{\hat{g}}(-\xi)\, d\hat{\mu}(\xi) \\ &= \int_{\mathbb{R}} \hat{g}(-\xi)\, d\hat{\hat{\mu}}(\xi) = \int_{\mathbb{R}} f(\xi)\, d\hat{\hat{\mu}}(-\xi),\end{aligned}$$

where the next to last equality is just the inversion formula for the classical Fourier transform. □

Let us mention a few immediate consequences of the arguments given above. Firstly, the definition of $\hat{\mu}$ is independent of the sequence, i.e., we can replace μ_n by any sequence

$$\nu_n \in \mathscr{M}^2 \cap \mathscr{M}^\infty \quad \text{with} \quad \lim_{n \to +\infty} \nu_n(f) = \mu(f), \qquad f \in C_c(\mathbb{R})$$

(setting $\nu_n(f) = \int_{\mathbb{R}} f \, d\nu_n$), and define $\hat{\mu}$ as the limit of the $\hat{\nu}_n$. Since $\mathscr{F}$ is invertible we have

$$\mu(f) = \lim_{n \to +\infty} \int_{-n}^{n} \hat{f} \, d\hat{\mu}, \qquad f \in L^2(\mathbb{R}).$$

Finally, we state Parseval's formula: $\int_{\mathbb{R}} \hat{f} \, d\mu = \int_{\mathbb{R}} f \, d\hat{\mu}$, for all $f \in L^2(\mathbb{R})$.

We will come back, in a short while, to the preceding theorem. In the meantime, let us study the case $\beta \neq 2$. As in the scalar case, a dichotomy occurs. First, for $2 < \beta < +\infty$, our approach can be extended.

Theorem 4.2. *For $2 < \beta < +\infty$, the Fourier transform operator is a contraction from $\mathscr{M}^\beta$ into $\mathscr{M}^\gamma$, $1/\beta + 1/\gamma = 1$.*

Proof. For $\beta = 2$, $\mathscr{F}$ is an isometry from $\mathscr{M}^2$ to $\mathscr{M}^2$. For $\beta = +\infty$, $\mathscr{F}$ is a contraction from $\mathscr{M}^\infty$ to $\mathscr{M}^1$. For $2 < \beta < +\infty$, the complex or the real interpolation method applied to the Banach spaces $\mathscr{M}^2$, $\mathscr{M}^1$ and $\mathscr{M}^\infty$ (see, for example, Bergh and Löfström [BL]) gives the result. □

For $2 < \beta < +\infty$, and for $\mu \in \mathscr{M}^\beta$, we can also define $\hat{\mu}$ by the method used in Theorem 4.1. For $f \in L^\beta(\mathbb{R})$, $\lim_{n \to +\infty} \|\mu_n(f) - \mu(f)\|_\alpha = 0$ and for $f \in C_c(\mathbb{R}) \subset L^\gamma(\mathbb{R})$, $1/\beta + 1/\gamma = 1$, $\int_{\mathbb{R}} f \, d\hat{\mu}_n = \int_{\mathbb{R}} \hat{f} \, d\mu_n$. Hence, by (α, β)-boundedness and the classical Hausdorff-Young inequality, it follows that

$$\begin{aligned}
\lim_{m,n \to +\infty} \left\| \int_{\mathbb{R}} f \, d\hat{\mu}_n - \int_{\mathbb{R}} f \, d\hat{\mu}_m \right\|_\alpha &= \lim_{m,n \to +\infty} \left\| \int_{\mathbb{R}} \hat{f} \, d\mu_n - \int_{\mathbb{R}} \hat{f} \, d\mu_m \right\|_\alpha \\
&\leqq \lim_{m,n \to +\infty} \left(\int_{\mathbb{R}} |\hat{f}_n(t) - \hat{f}_m(t)|^\beta \, dt \right)^{1/\beta} \\
&\leqq \lim_{m,n \to +\infty} \cdot \left(\int_{\mathbb{R}} |f_n(t) - f_m(t)|^\gamma \, dt \right)^{1/\gamma} = 0.
\end{aligned}$$

As in theorem 4.1, this method uniquely defines μ. It is then clear that a Parseval's formula holds, i.e., $\int_{\mathbb{R}} \hat{f} \, d\mu = \int_{\mathbb{R}} f \, d\hat{\mu}$, for all $f \in L^\gamma(\mathbb{R})$, and that the range of $\mathscr{F}$ is a proper subset of $\mathscr{M}^\gamma$. Again, if $\mu \in \mathscr{M}^2$, if $\mu \in \mathscr{M}^\infty$ or if as in Theorem 3.2, the Cesáro averages of μ converge, all the definitions of Fourier transform agree, and we have in the last instance

$$d\mathscr{F}\mu(t) = \left(\lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} \, d\mu(\xi) \right) dt.$$

Another important property shared by the Fourier transform on $\mathscr{M}^\beta$ is the inversion formula. Since $\mathscr{F}$ is not invertible, this is not as immediate as in the case $\beta=2$.

Theorem 4.3. *Let $\mu\in\mathscr{M}^\beta$, $2\leqq\beta\leqq+\infty$. Then, for any $A\in\mathscr{B}_0(\mathbb{R})$*

$$\lim_{n\to+\infty}\left\|\mu(A)-\int_{-n}^{n}\hat{\chi}_A(t)\,d\hat{\mu}(t)\right\|_\alpha.$$

Proof. The cases $\beta=2$ and $\beta=+\infty$ have already been obtained. For h in $L^\gamma(\mathbb{R})$, $1<\gamma<2$, the classical inversion formula gives

$$\lim_{n\to+\infty}\int_{\mathbb{R}}\left|h(t)-\int_{-n}^{n}\hat{h}(\xi)\,e^{it\xi}d\xi\right|^\gamma dt=0,\qquad 1/\beta+1/\gamma=1.$$

If $f\in C_c(\mathbb{R})$ is twice differentiable, then $f=\hat{g}$ with $g\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\subset L^\gamma(\mathbb{R})$, and

$$\left\|\int_{\mathbb{R}} f\,d\mu-\int_{-n}^{n}\hat{f}\,d\hat{\mu}\right\|_\alpha=\left\|\int_{\mathbb{R}} g\,d\hat{\mu}-\int_{-n}^{n}\hat{f}\,d\hat{\mu}\right\|_\alpha$$
$$\leqq\left(\int_{\mathbb{R}}|g(t)-\chi_{]-n,n[}(t)\hat{f}(t)|^\gamma dt\right)^{1/\gamma},$$

since $\mu\in\mathscr{M}^\beta$. But, by the classical $L^1(\mathbb{R})$-inversion formula,

$$\chi_{]-n,n[}(t)\hat{f}(t)=\int_{-n}^{n}\hat{g}(\xi)\,e^{-it\xi}d\xi,$$

and

$$\lim_{n\to+\infty}\left(\int_{\mathbb{R}}\left|g(t)-\int_{-n}^{n}\hat{g}(\xi)\,e^{-it\xi}d\xi\right|^\gamma dt\right)^{1/\gamma}=0.$$

Given any $f\in L^\beta(\mathbb{R})$, we can now find a sequence of compactly supported functions $\{f_n\}$ which are twice differentiable with

$$\lim_{n\to+\infty}\|f_n-f\|_\beta=0\quad\text{and}\quad\lim_{n\to+\infty}\|\hat{f}_n-\hat{f}\|_\infty=0.$$

Hence,

$$\lim_{n\to+\infty}\left\|\int_{\mathbb{R}} f\,d\mu-\int_{-n}^{n}\hat{f}\,d\hat{\mu}\right\|_\alpha=0,$$

for all $f\in L^\beta(\mathbb{R})$, from which the result follows. □

In the above proof, we obtained a stronger result than the one actually stated: χ_A can be replaced by any $f\in L^\beta(\mathbb{R})$. The version stated here, that we often cryptically rewrite as $\mu(A)=\hat{\mu}(\hat{A})$, is the most practical since it permits to recover μ from $\hat{\mu}$. A completely symmetrical formula which permits to recover $\hat{\mu}$ from μ is also valid. Theorem 4.3 can be "extended" to the (α,β)-bounded case under the conditions of Theorem 3.2 or 3.4, with $\hat{\mu}$ as defined there. It is also readily seen from the inversion formula the μ is Gaussian, $S\alpha S$ or infinitely divisible if and only if $\hat{\mu}$ has the same property. As a simple application of the above results we also have.

Corollary 4.4. *Let $\alpha=2$ and $2\leqq\beta\leqq+\infty$, and let μ be a random measure with finite (α, β)-variation. Then, μ is orthogonally scattered (resp. stationarily scattered) if and only if $\hat{\mu}$ is stationarily scattered (resp. orthogonally scattered).*

Proof. Let μ be orthogonally scattered with control measure m. By the inversion or Parseval's formula, and since $\hat{\chi}_{A+s}(t)=e^{its}\hat{\chi}_A(t)$ we have, for any $s\in\mathbb{R}$,

$$\langle\hat{\mu}(A+s), \hat{\mu}(B+s)\rangle_2=\langle\mu(\hat{\chi}_{A+s}), \mu(\hat{\chi}_{B+s})\rangle_2=\int_{\mathbb{R}}\hat{\chi}_{A+s}(t)\,\overline{\hat{\chi}_{B+s}(t)}\,dm(t)$$
$$=\int_{\mathbb{R}}\hat{\chi}_A(t)\,\overline{\hat{\chi}_B(t)}\,dm(t)=\langle\hat{\mu}(A+s), \hat{\mu}(B+s)\rangle_2.$$

Let $\hat{\mu}$ be stationarily scattered, then for any $A\in\mathscr{B}_0(\mathbb{R})$ with $|A|>0$, $S:\mathbb{R}\to L^2(\mathscr{P})$, $s:\to\hat{\mu}(A+s)$ is shift invariant, and since $\mu\in\mathscr{M}^\beta$, it is also continuous. Thus, S is a stationary process and there exists a random measure ν_A which is orthogonally scattered and $(2, \infty)$-bounded such that

$$\hat{\mu}(A+s)=\int_{\mathbb{R}}e^{its}d\nu_A(t), \quad \text{but} \quad \hat{\mu}(A+s)=\int_{\mathbb{R}}\chi_{A+s}(t)\,d\hat{\mu}(t)$$
$$=\int_{\mathbb{R}}\hat{\chi}_{A+s}(t)\,d\mu(t)=\int_{\mathbb{R}}e^{its}\hat{\chi}_A(t)\,d\mu(t).$$

Finally, the uniqueness of the Fourier transform gives $d\nu_A=\hat{\chi}_A\,d\mu$, and $d\mu=\hat{\chi}_A^{-1}\,d\nu_A$ is orthogonally scattered. □

For $\alpha=\beta=2$, the above corollary was already known to Bochner. Since a white measure is both orthogonally and stationarily scattered as well as Gaussian, so is $\mathscr{F}W=\hat{W}$. For $1\leqq\beta<2$, the approach presented here does not work, and this is not surprising, in view of the scalar case. The methods developed by Gel'fand (see [GV]) or Dudley [D1–2], i.e., generalized processes and random Schwartz distributions, do provide a way of defining the Fourier transform of random measures with finite (α, β)-variation, $1\leqq\beta<2$. In particular, and in contrast to the Brownian motion case, the Fourier transform of the Lévy motion cannot be defined by our methods. However, such a definition is possible via random Schwartz distribution (see [D1–2]). Another approach to this problem is also presented in [CH].

5 Dilation theory

In this section, our first task is to obtain another characterization of random measures with bounded (α, β)-variation. This characterization, via dominating measures and Grothendieck's type inequality, leads to a few important applications. In particular, it clarifies the rôle played by the orthogonally scattered elements in $\mathscr{M}^{\alpha,\beta}$. The cases $\beta<+\infty$ and $\beta=+\infty$ read slightly differently, because of well known duality problems. We first have:

Theorem 5.1. *Let $1\leqq\alpha\leqq2\leqq\beta<+\infty$ (resp. let $\beta=+\infty$). A measure μ has finite (α, β)-variation (resp. (α, ∞)-variation) if and only if there exists a non-negative function h in $L^{\beta/\beta-2}(\mathbb{R})$ (resp. a finite positive Borel measure h) such that $\|\int_{\mathbb{R}} f\,d\mu\|_\alpha\leqq(\int_{\mathbb{R}}|f|^2 h\,dt)^{1/2}$, for all $f\in L^\beta(\mathbb{R})$ (resp. $\|\int_{\mathbb{R}} f\,d\mu\|_\alpha\leqq(\int_{\mathbb{R}}|f|^2\,dh)^{1/2}$,* for all *$f\in C_0(\mathbb{R})$).*

Proof. The "if part" simply follows from Hölder's inequality. For the converse: the case $\alpha=2, \beta=+\infty$ is just the positive definite version of Grothendieck's inequality (see Pisier [Pi] for extensive developments on Grothendieck's inequality and its ramifications); while the case $1\leqq\alpha<2, \beta=+\infty$ is essentially contained, using also Pietsch domination theorem, in Lindenstrauss and Pelczynski [LP]. For $1\leqq\alpha<2\leqq\beta<+\infty$, we only sketch the proof indicating the major differences with the Hilbert space case, i.e., $\alpha=2$, already given, for μ with compact support, in [H1]. First, we need to show that there exists a constant $K>0$ such that

$$\sum_{i=1}^{n} \Big\| \int_{\mathbb{R}} f_i \, d\mu \Big\|_\alpha^2 \leqq K \left\| \sum_{i=1}^{n} |f_i|^2 \right\|_{\beta/2}, \tag{5.1}$$

for all $f_1, f_2, \ldots, f_n \in C_c(\mathbb{R})$. To do so, let $X_1, X_2, \ldots, X_n$ be independent $N(0,1)$ random variables on some probability space $(\Omega', \mathscr{B}', \mathscr{P}')$. Then, by Minkowski's inequality and by the independence and Gaussian assumptions we have,

$$\left\{\sum_{i=1}^{n} \Big\| \int_{\mathbb{R}} f_i \, d\mu \Big\|_\alpha^2\right\}^{1/2} \leqq \left\{\int_\Omega \left(\sum_{i=1}^{n} |(\int_{\mathbb{R}} f_i \, d\mu)(\omega)|^2\right)^{\alpha/2} d\mathscr{P}(\omega)\right\}^{1/\alpha}$$

$$=(\mathscr{E}|X_1|)^{-1}\left\{\int_\Omega\left(\int_{\Omega'}\left|\sum_{i=1}^{n}(\int_{\mathbb{R}} f_i\, d\mu)(\omega)\, X_i(\omega')\right| d\mathscr{P}'(\omega')\right)^\alpha d\mathscr{P}(\omega)\right\}^{1/\alpha}$$

$$\leqq(\mathscr{E}|X_1|)^{-1}\int_{\Omega'}\left\{\int_\Omega\left|\sum_{i=1}^{n}(\int_{\mathbb{R}} f_i\, d\mu)(\omega)\, X_i(\omega')\right|^\alpha d\mathscr{P}(\omega)\right\}^{1/\alpha} d\mathscr{P}'(\omega')$$

(Minkowski's inequality)

$$=(\mathscr{E}|X_1|)^{-1}\int_{\Omega'}\left\|\sum_{i=1}^{n}\int_{\mathbb{R}} f_i\, d\mu\, X_i(\omega')\right\|_\alpha d\mathscr{P}'(\omega')$$

$$\leqq \|\mu\|\,(\mathscr{E}|X_1|)^{-1}\int_{\Omega'}\left\|\sum_{i=1}^{n} f_i\, X_i(\omega')\right\|_\beta d\mathscr{P}'(\omega') \qquad ((\alpha,\beta)\text{-boundedness})$$

$$\leqq \|\mu\|\,(\mathscr{E}|X_1|)^{-1}\left\{\int_{\Omega'}\left\|\sum_{i=1}^{n} f_i\, X_i(\omega')\right\|_\beta^\beta d\mathscr{P}'(\omega')\right\}^{1/\beta} \qquad (\text{Hölder's inequality})$$

$$=\|\mu\|\,(\mathscr{E}|X_1|)^{-1}\left\{\int_{\mathbb{R}}\int_{\Omega'}\left|\sum_{i=1}^{n} f_i(t)\, X_i(\omega')\right|^\beta d\mathscr{P}'(\omega')\, dt\right\}^{1/\beta}$$

$$=\|\mu\|\,(\mathscr{E}|X_1|)^{-1}(\mathscr{E}|X_1|^\beta)^{1/\beta}\left\{\int_{\mathbb{R}}\left(\sum_{i=1}^{n}|f_i(t)|^2\right)^{\beta/2} dt\right\}^{1/\beta}$$

$(N(0,1)$, independence)

$$=\|\mu\|\,(\mathscr{E}|X_1|)^{-1}(\mathscr{E}|X_1|^\beta)^{1/\beta}\left\|\sum_{i=1}^{n}|f_i|^2\right\|_{\beta/2}^{1/2}.$$

This gives 5.1. Then, for f real valued and in $C_c(\mathbb{R})$, and for any finite set $\{f_1, \ldots, f_n\} \subset C_c(\mathbb{R})$, let

$$S(f, f_1, \ldots, f_n)=\left\{\int_{\mathbb{R}}\left|f+K\left(\sum_{i=1}^{n}|f_i|^2\right)\right|^{\beta/2} dt\right\}^{2/\beta}-\sum_{i=1}^{n}\Big\|\int_{\mathbb{R}} f_i\, d\mu\Big\|_\alpha^2,$$

where K is the infimum of the constants appearing in (5.1); and let also $Q(f) = \inf S(f, f_1, \ldots, f_n)$, where the infimum is taken over all finite sets of elements of $C_c(\mathbb{R})$. Then, as in [H1], Q is a subadditive homogeneous functional on $C_c^r(\mathbb{R})$ (the real valued elements of $C_c(\mathbb{R})$) such that:

$$|Q(f)| \leqq \|f\|_{\beta/2}. \tag{5.2}$$

Again, by applying Hahn-Banach twice, there exists a linear functional L such that for all $f \in C_c(\mathbb{R})$ $|L(f)| \leqq 2^{1/2} \|f\|_{\beta/2}$. Now, by the density of $C_c(\mathbb{R})$ in $L^{\beta/2}(\mathbb{R})$ and the Riesz representation theorem, there exists $h_0 \in L^{\beta/(\beta-2)}(\mathbb{R})$ such that $L(f) = \int_{\mathbb{R}} f h_0 \, dt$, $f \in L^{\beta/2}(\mathbb{R})$. Finally, and exactly as in [H1], L is a positive functional and furthermore,

$$\left\| \int_{\mathbb{R}} f \, d\mu \right\|_\alpha^2 \leqq K \int_{\mathbb{R}} |f|^2 h_0 \, dt. \tag{5.3}$$

Taking $h = K h_0$ and by the density of $C_c(\mathbb{R})$ in $L^\beta(\mathbb{R})$, the result follows. □

A major difference between the above theorem and the classical Grothendieck's inequality is the unboundedness of the dominating measure. A case at hand is, again, a white measure which has finite (2, 2)-variation and for which a dominating measure is Lebesgue measure. The dominating h above is clearly not unique, however (see [H1]), there exists a unique h such that $\|h\|_{\beta/\beta-2} = \inf K$ appearing in (5.1). The orthogonally (resp. independently) scattered case, clearly shows that for $1 \leqq \beta < 2$(resp. $1 \leqq \beta < \alpha$) the only h satisfying the conclusions of Theorem 5.1 is $h = 0$. It is also clear that in Theorem 5.1, $L^\alpha(\mathscr{P})$ and $L^\beta(\mathbb{R})$ can be respectively replaced by $L^\alpha(v)$ and $L^\beta(\eta)$, for some σ-finite v and η; the corresponding h is then in $L^{\beta/\beta-2}(\eta)$.

A consequence of Theorem 5.1 is the following dilation and multiplication result.

Theorem 5.2. *Let $1 \leqq \alpha \leqq 2 \leqq \beta \leqq +\infty$. A random measure μ: $\mathscr{B}_0(\mathbb{R}) \to L^0(\mathscr{P})$ has finite (α, β)-variation if and only if there exist a probability space $(\tilde{\Omega}, \tilde{\mathscr{B}}, \tilde{\mathscr{P}})$ with $L^2(\mathscr{P}) \subset L^2(\tilde{\mathscr{P}})$, an orthogonally scattered v: $\mathscr{B}_0(\mathbb{R}) \to L^0(\tilde{\mathscr{P}})$ with finite $(2, \beta)$-variation and λ in $L^{2\alpha/2-\alpha}(\mathscr{P})$ such that $\mu = \lambda P v$, i.e., $\mu(A) = \lambda P v(A)$, $A \in \mathscr{B}_0(\mathbb{R})$, where P is the orthogonal projection from $L^2(\tilde{\mathscr{P}})$ to $L^2(\mathscr{P})$.*

Proof. Hölder's inequality and the property of the (α, β)-variation mentioned at the end of Sect. 2 give the sufficiency part. For the necessity part, in order to find λ, it is enough to apply Maurey's criterion (see [M, Théor. 8]) after having identify μ with the operator given by Theorem 2.2. But, in Theorem 5.1, we proved that:

$$\begin{aligned}
&\left\{ \int_\Omega \left(\sum_{i=1}^n \left| \left(\int_{\mathbb{R}} f_i \, d\mu \right)(\omega) \right|^2 \right)^{\alpha/2} d\mathscr{P}(\omega) \right\}^{1/\alpha} \\
&\leqq \|\mu\| \, (\mathscr{E}|X_1|)^{-1} (\mathscr{E}|X_1|^\beta)^{1/\beta} \left\{ \int_{\mathbb{R}} \left(\sum_{i=1}^n |f_i(t)|^2 \right)^{\beta/2} dt \right\}^{1/\beta} \\
&\leqq \|\mu\| (\mathscr{E}|X_1|)^{-1} (\mathscr{E}|X_1|^\beta)^{1/\beta} \left(\sum_{i=1}^n \|f_i\|_\beta^2 \right)^{1/2},
\end{aligned}$$

where the last inequality is again Minkowski's since $2 \leqq \beta < +\infty$. We also get, using methods corresponding to the case

$$\beta = +\infty, \quad \left\{ \int_{\Omega} \left(\sum_{i=1}^{n} \left| \left(\int_{\mathbb{R}} f_i \, d\mu \right)(\omega) \right|^2 \right)^{\alpha/2} d\mathscr{P}(\omega) \right\}^{1/\alpha} \leqq K \left(\sum_{i=1}^{n} \|f_i\|_{\infty}^2 \right)^{1/2}.$$

Thus Maurey's criterion is verified and this gives $\lambda \in L^{2\alpha/2-\alpha}(\mathscr{P})$. For the rest of the proof, let h be as in Theorem 5.1 and let f and $g \in C_c(\mathbb{R})$. As in [H1], and after having identified the functions f such that $\|\int_{\mathbb{R}} f \, d\mu\|_2^2 = \int_{\mathbb{R}} |f|^2 h \, dt$ (resp. $= \int_{\mathbb{R}} |f|^2 dh$ when $\beta = +\infty$), we can define a inner product by

$$\langle f, g \rangle_0 = \int_{\mathbb{R}} f \bar{g} h \, dt - \langle \int_{\mathbb{R}} f \, d\mu, \int_{\mathbb{R}} g \, d\mu \rangle_2$$

$$\left(\int_{\mathbb{R}} f \bar{g} \, dh - \langle \int_{\mathbb{R}} f \, d\mu, \int_{\mathbb{R}} g \, d\mu \rangle_2 \quad \text{when} \quad \beta = +\infty \right).$$

Completing the corresponding space with respect to the norm $\|\cdot\|_0$, we get a Hilbert space $L^2(\mathscr{P}_0)$. Then, and again as in [H1], $L^2(\tilde{\mathscr{P}}) = L^2(\mathscr{P}_0) \oplus L^2(\mathscr{P})$ and for each $A \in \mathscr{B}_0(\mathbb{R})$, $\nu(A) = \nu_0(A) + \mu(A)$. As constructed, ν has the desired properties. □

Combining Theorem 3.2 and 5.2, the result below is clear.

Corollary 5.3. *Let $1 \leqq \alpha \leqq 2 \leqq \beta \leqq +\infty$. A bounded process X is strongly continuous and (α, β)-bounded if and only if there exists a bounded $(2, \beta)$-bounded Cesáro harmonizable strongly continuous process Y defined on $L^2(\tilde{\mathscr{P}})$, with $L^2(\mathscr{P}) \subset L^2(\tilde{\mathscr{P}})$, and a random variable λ in $L^{2\alpha/2-\alpha}(\mathscr{P})$ such that $X_t = \lambda P Y_t$, $t \in \mathbb{R}$, where P is the orthogonal projection from $L^2(\tilde{\mathscr{P}})$ to $L^2(\mathscr{P})$.*

For $\alpha = 2$ and $\beta = +\infty$, Theorem 5.1 is due to Niemi [N] (when $\alpha = 2$, λ can be taken constant hence $\mu = P\nu_1$, where $\nu_1 = \lambda\nu$). Combining Theorem 3.4 and 5.2, a measurable version of Corollary 5.3 readily follows.

For $1 \leqq \beta < \alpha$, Proposition 2.5 and its remark show that no version of Theorem 5.2 can hold. On the other hand, for $1 \leqq \alpha \leqq \beta < 2$, and by taking for μ the Lévy motion it is clear that with 2 replaced by α, (5.3) does not hold. Furthermore, as alluded to by the independently scattered $S\alpha S$ random measure, a potential range for the values of the parameters is $1 \leqq \gamma < \alpha \leqq \beta < 2$. However, no similar $1 \leqq \gamma < \alpha \leqq \beta < 2$ version of the domination theorem 5.1 do hold, since the ideals of α-summing operators from L^β to L^γ are increasing for $1 \leqq \gamma < 2$, $\beta' < \alpha < 2$ and $2 < \beta < +\infty$ (see Pietsch [P, 22.4]). Nevertheless, by replacing the independent $N(0, 1)$ by i.i.d. $S\alpha S$ random variables, it can be shown (see [M]) that

$$\left\| \left(\sum_{i=1}^{n} \left| \left(\int_{\mathbb{R}} f_i \, d\mu \right)(\omega) \right|^\alpha \right)^{1/\alpha} \right\|_\gamma \leqq K \left(\sum_{i=1}^{n} \|f_i\|_\beta^\alpha \right)^{1/\alpha}.$$

Thus, the corresponding version of Maurey's criterion is verified and μ can be decomposed as $\mu = \lambda\nu$, with $\lambda \in L^{\alpha\gamma/\alpha-\gamma}(\mathscr{P})$, ν itself cannot be further decomposed as in theorem 5.2.

6 Some applications

The rest of this paper is devoted to some applications of methods developed to this point and will be devoted to two problems (see [H3] for other types of applications). Firstly, the more theoretical problem of decomposing random measures, secondly, the more practical problem of finding existence results for linear stochastic differential equations.

Definition 6.1. Let $1\leqq\alpha\leqq 2$ and let $1\leqq\beta\leqq+\infty$. A random measure μ of bounded (α, β)-variation is *continuous* if $\mu(A)=0$, for all finite sets A and *discrete* if there exists an increasing sequence $\{A_n\}$ of finite sets such that $\lim_{n\to+\infty}\|\mu-\mu_n\|=0$, where μ_n is the restriction of μ to A_n, i.e., $\mu_n(A)=\mu(A\cap A_n)$, $A\in\mathscr{B}_0(\mathbb{R})$.

Although the following looks well known, we believe it is original to this work. It has, however, its origins in the bilinear case version, i.e., $\alpha=2$ and $\beta=+\infty$, as given by Graham and Schreiber [GS].

Theorem 6.2. *Let $1\leqq\alpha\leqq 2\leqq\beta\leqq+\infty$. A random measure μ of bounded (α, β)-variation admits a unique decomposition $\mu=\mu_c+\mu_d$ into continuous and discrete parts, with both μ_c and μ_d of bounded (α, β)-variation.*

Proof. The uniqueness follows directly from the above definition. Let h be any of the positive dominating measure in Theorem 5.1 and let $A=\{t\in\mathbb{R}\colon h(t)>0\}$, then A is at most countable and $A=\bigcup_{n=1}^{\infty}A_n$ where $A_1\subset A_2\subset\ldots\subset A_n\subset\ldots$ are finite sets. Let μ_d be the restriction of μ to A and let μ_n be the restriction of μ to A_n. It is also clear from elementary properties of the (α, β)-variation that μ_d and μ_n have bounded (α, β)-variation with $\|\mu_d\|\leqq\|\mu\|$ and $\|\mu_n\|\leqq\|\mu\|$. We now show that μ_d is discrete by showing that $\lim_{n\to+\infty}\|\mu_d-\mu_n\|=0$. Let $B\in\mathscr{B}_0(\mathbb{R})$ and $\{B_i\}\subset\mathscr{B}_0(\mathbb{R})$ be a finite partition of B, let $f=\sum_{i=1}^{N}b_i\chi_{B_i}$, with $b_i\in\mathbb{C}$ and let A_n^c denote the complement of A_n in A. Then,

$$\begin{aligned}
&\left\|\sum_{i=1}^{N}b_i\mu_d(B_i)-\sum_{i=1}^{N}b_i\,\mu_n(B_i)\right\|_\alpha\\
&=\left\|\sum_{i=1}^{N}b_i\,\mu(B_i\cap A)-\sum_{i=1}^{N}b_i\,\mu(B_i\cap A_n)\right\|_\alpha\\
&=\left\|\sum_{i=1}^{N}b_i\,\mu(B_i\cap A_n^c)\right\|_\alpha\\
&\leqq\|f\|_\beta\,h(A_n^c),
\end{aligned}$$

by Theorem 5.1 and Hölder's inequality. Since $A=\{t\in\mathbb{R}\colon h(t)>0\}$, we have $\lim_{n\to+\infty}h(A_n^c)=0$, hence $\lim_{n\to+\infty}\|\mu_d-\mu_n\|\leqq\lim_{n\to+\infty}h(A_n^c)=0$ and μ_d is discrete. Let $\mu_c=\mu-\mu_d$, then μ_c has bounded (α, β)-variation since both μ and μ_d share this property. Let B be any finite set, let A be as above, then $\mu_c(B)=\mu(B)-\mu_d(B)$

$=\mu(A\cap B)+\mu(A^c\cap B)-\mu_d(B)$, and by the domination theorem, $\|\mu(A^c\cap B)\|_\alpha \leqq h(A^c\cap B)=0$ since the discrete part of h is supported on A. Since μ_d is the restriction of μ to A, we have $\mu_c(B)=\mu(A\cap B)-\mu_d(B)=\mu(A\cap B)-\mu(A\cap B)=0$ and μ_c is continuous. □

As immediate consequences of the definition of boundedness Theorem 5.1 as well as Theorem 5.2, 6.2 and its proof (using also the notations of Sect. 5) we get:

Corollary 6.3. *For $1\leqq\alpha\leqq 2$ and $1\leqq\beta<+\infty$, a random measure μ of bounded (α,β)-variation is continuous.*

Corollary 6.4. *Let $1\leqq\alpha\leqq 2\leqq\beta\leqq+\infty$ and let μ have finite (α,β)-variation. Then, μ is discrete (resp. continuous) if and only if $\mu=\lambda P v$, with v discrete (resp. continuous).*

It is clear that, using Corollary 5.3, stochastic processes versions of our corollaries easily follows; but it is also well known that, in general, the measure μ does not have a Radon-Nikodym derivative, i.e., μ_c cannot be further decomposed into an L^1-function and a singular continuous part.

To end this work, we now present some results on linear stochastic differential equations (the derivatives are taken in the norm-sense). Once again, the case $\alpha=2$ and $\beta=+\infty$ was already known to Bochner [B2], the case $\alpha<2$ and $\beta=+\infty$ is omitted since essentially identical, and we thus assume that $1\leqq\alpha<2$ and $1\leqq\beta<+\infty$.

Theorem 6.5. *Let Y be a norm bounded strongly continuous (α,β)-bounded Cesáro harmonizable process, and let*

$$L=\sum_{k=0}^{n} a_k\, d^k/d\,t^k$$

be a linear autonomous differential operator. Then, the stochastic differential equation $LX=Y$ has a bounded strongly continuous (α,β)-bounded solution if and only if it has a bounded strongly continuous (α,β)-bounded Cesáro harmonizable solution.

Proof. The "if" part is trivial. For the "only if" part, let $L(\xi)=\sum_{k=0}^{n} a_k\,\xi^k$, $\xi\in\mathbb{R}$, be the characteristic polynomial of the operator L, let $N=\{\xi\colon L(\xi)=0\}$ and let

$$X_t=\lim_{\lambda\to+\infty}\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)e^{it\xi}\,d\mu_x(\xi)$$

be a strongly continuous (α,β)-bounded solution of $LX=Y$. Let $A\in\mathscr{B}_0(\mathbb{R})$ and let also

$$X_t^A=\lim_{\lambda\to+\infty}\int_{-\lambda}^{\lambda}\left(1-\frac{|\xi|}{\lambda}\right)e^{it\xi}\chi_A(\xi)\,d\mu_x(\xi).$$

Since a strong solution of the equation is also a weak solution and since L commutes with the integral (dominated convergence), we have

$$LX_t^A = L \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} \chi_A(\xi)\, d\mu_x(\xi)$$
$$= \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} L(\xi)\, \chi_A(\xi)\, d\mu_x(\xi) = Y_t^A$$
$$= \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} \chi_A(\xi)\, d\mu_y(\xi).$$

Since $A \in \mathscr{B}_0(\mathbb{R})$, this becomes

$$\int_{\mathbb{R}} e^{it\xi} L(\xi)\, \chi_A(\xi)\, d\mu_x(\xi) = \int_{\mathbb{R}} e^{it\xi} \chi_A(\xi)\, d\mu_y(\xi),$$

and the uniqueness of the Fourier transform gives, for each $A \in \mathscr{B}_0(\mathbb{R})$, $\chi_A L d\mu_x = \chi_A d\mu_y$, i.e., $L d\mu_x = d\mu_y$. Now, by Theorem 5.1 or by the very definition of boundedness, we have $h(N) = 0$, and

$$Z_t^\lambda = \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) L(\xi)^{-1} \chi_{\mathbb{R}\setminus N}(\xi)\, e^{it\xi} d\mu_y(\xi) = \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) L(\xi)^{-1} e^{it\xi} d\mu_y(\xi)$$

is well defined with $LZ_t^\lambda = Y_t^\lambda$. Again, since $\chi_A L d\mu_x = \chi_A d\mu_y$ it directly follows that $L^{-1}\mu_y$ has bounded (α, β)-variation and thus we can set

$$Z_t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} L^{-1}(\xi)\, d\mu_y(\xi),$$

uniformly on compact sets. As defined, Z has the required properties. □

Theorem 6.5, can roughly be restated in the following way: under the same hypotheses on Y, any (α, β)-bounded solution of $LX = Y$ must be stationary. In the above result, L can also take a more general form such as a differential-convolution operator. The resulting theorem admits a similar proof since what is needed is the interchangeability of the operator and the integral. In fact, this is the type of operators studied in [B2]. Also, only assuming the existence of a weak solution is enough. For $\beta = +\infty$, the result takes a slightly different form, since $h(N)$ might not be null. So, in the proof of the results, Z above should be replaced by

$$Z_t = \lim_{\lambda \to +\infty} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) e^{it\xi} L^{-1}(\xi)\, \chi_{\mathbb{R}\setminus N}(\xi)\, d\mu_y(\xi),$$

with the corresponding adjustments in the conclusions. To finish, it is clear that, for $\beta = +\infty$, if μ_y is discrete (resp. continuous) so is μ_z.

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