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Probability Theory and Related Fields

Noncommutative Stochastic Processes with Independent and Stationary Increments Satisfy Quantum Stochastic Differential Equations*

Michael Schürmann

Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, D-6900 Heidelberg 1, Federal Republic of Germany

Summary. The notion of a unitary noncommutative stochastic process with independent and stationary increments is introduced, and it is proved that such a process, under a continuity assumption, can be embedded into the solution of a quantum stochastic differential equation in the sense of Hudson and Parthasarathy [8].

1. Introduction

A stochastic process taking values in the group U_d of complex unitary $d \times d$-matrices can be regarded as a family of unitary $d \times d$-matrices where the matrix elements are elements of the commutative von Neumann algebra consisting of L^∞-functions on the underlying probability space. A noncommutative generalisation of a time-indexed U_d-valued stochastic process is a family of time-indexed $d \times d$-matrices where the matrix elements are elements of a von Neumann algebra. In other words, we consider families $(U_t)_{t \in \mathbb{R}_+}$ of unitary operators on a Hilbert space $\mathbb{C}^d \otimes \mathscr{H}$ which is the tensor product of a d-dimensional space $\mathbb{C}^d$ and a Hilbert space $\mathscr{H}$. The probability measure of the underlying probability space is replaced by a state, represented by a unit vector Φ_0 in $\mathscr{H}$, on the algebra $B(\mathscr{H})$ of bounded linear operators on $\mathscr{H}$. The family $(U_t)_{t \in \mathbb{R}_+}$ may be interpreted as the time evolution of a d-dimensional physical system in contact with a heat bath represented by the 'big' Hilbert space $\mathscr{H}$. We call $(U_t)_{t \in \mathbb{R}_+}$ a d-dimensional unitary noncommutative stochastic process if Φ_0 is cyclic for the elements of the matrices U_t, $t \in \mathbb{R}_+$. The notions of independent and of stationary increments are translated from the classical commutative case of a U_d-valued stochastic process but with the additional assumption that the matrix elements of the increments $U_s^\dagger U_t$ and $U_{s'}^\dagger U_{t'}$ commute for disjoint open intervals (s, t) and (s', t'), an assumption which becomes trivial in the commutative case. The main result of this paper (the 'embedding theorem') is a characterisation of unitary processes with independent and stationary increments. It turns

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out that such processes, if for $t \downarrow 0$ the operators U_t converge weakly to $U_0 = \mathbf{1}$, can be 'embedded' into solutions of linear quantum stochastic differential equations in the sense of Hudson and Parthasarathy [8]. This means that $(U_t)_{t \in \mathbb{R}_+}$ is isomorphic to a solution of a quantum stochastic differential equation restricted to a certain invariant subspace, namely the subspace generated by the Fock space vacuum vector. For a finite-dimensional initial space this result is more general than the result of [9], because we do not assume $\mathscr{H}$ to be a Fock space but an arbitrary Hilbert space, the Fock space structure coming from the properties of $(U_t)_{t \in \mathbb{R}_+}$ itself.

The main method used for the proof of our embedding theorem is the theory of the noncommutative analogue $\mathscr{K}_d$ of the coefficient algebra of the group $\boldsymbol{U}_d$; see [6] for the notion of the coefficient algebra of a compact group and [3, 12, 14] for the noncommutative analogue in the case of $\boldsymbol{U}_d$. (Recently, a structure theorem for $\mathscr{K}_d$ has been obtained [4].) The algebra $\mathscr{K}_d$ is a *-algebra, that is an algebra with an involution. Moreover, $\mathscr{K}_d$ is a *-bialgebra [11], which means that there is a coalgebra [1, 13] structure on $\mathscr{K}_d$ compatible with the *-algebra structure. A theory of quantum stochastic independent, stationary increment processes over a graded *-bialgebra was developed in [3, 12]. It was proved that such a process, under a continuity assumption, is determined (up to a canonical equivalence) by its generator, a hermitian, conditionally positive linear functional on the *-bialgebra. In the present paper, we apply this general result to the special case of the *-bialgebra $\mathscr{K}_d$. It turns out that d-dimensional unitary noncommutative stochastic processes with independent and stationary increments are nothing else but quantum independent, stationary increment processes over $\mathscr{K}_d$. Thus a process $(U_t)_{t \in \mathbb{R}_+}$ of the kind we are interested in is determined by its generator, and two such processes with the same generator are equivalent. For the proof of the embedding theorem we proceed as follows. We start from an arbitrary hermitian, conditionally positive linear functional ψ on $\mathscr{K}_d$. We show that there is a *-representation of $\mathscr{K}_d$ associated to ψ in a canonical way. With the help of this *-representation we define the coefficients of a quantum stochastic differential equation. The solution of this equation, if restricted to the subspace generated by the vacuum vector, is a d-dimensional unitary process with independent and stationary increments. Finally, we show that the generator of this process is the hermitian, conditionally positive ψ we started from.

The organisation of this paper is as follows. In Sect. 2 we introduce the quantum stochastic differential equations relevant for our problem. In Sect. 3 we give the precise definition of unitary noncommutative stochastic processes with independent and stationary increments. We state the embedding theorem. Section 4 contains a review of results of [3, 12] and the new concept of *-representations associated to conditionally positive linear functionals on *-bialgebras. Section 5 is devoted to the proof of the embedding theorem. It is here where we explain how unitary processes with independent increments fit into the general concept of quantum independent increment processes developed in [3, 12]. This section also includes the calculation of the generator (Theorem 5.1.) of a process given by the solution of a quantum stochastic differential equation.

All vector spaces will be over the complex numbers. An algebra is always understood to be a complex associative unital algebra. For a *-algebra $\mathscr{A}$ we let $\mathbb{Z}_2=\{0,1\}$ operate on $\mathscr{A}$ by setting for $a\in\mathscr{A}$

$$a^{\varepsilon}=\begin{cases} a & \text{if } \varepsilon=0 \\ a^* & \text{if } \varepsilon=1. \end{cases}$$

We denote by $\mathbb{C}^d$, $d\in\mathbb{N}$, a d-dimensional complex Hilbert space with orthonormal basis $\{e_k\colon k=1,\ldots,d\}$ and by $M_d=B(\mathbb{C}^d)$ the *-algebra of complex $d\times d$-matrices. The matrix units $e_{kl}\in M_d$, $k,l=1,\ldots,d$, are given by the equations

$$e_{kl}e_n=\delta_{ln}e_k,$$

$k,l,n=1,\ldots,d$, where δ_{ln} is the Kronecker delta. For a vector space $\mathscr{W}$ an element $\underline{w}$ of $M_d\otimes\mathscr{W}$ can be written in a unique way in the form

$$\underline{w}=\sum_{k,l=1}^{d} e_{kl}\otimes w_{kl}$$

or

$$\underline{w}=(w_{kl})_{kl=1,\ldots,d}$$

with $w_{kl}\in\mathscr{W}$.

2. Unitary Solutions of Quantum Stochastic Differential Equations

We need a generalisation of a result of [8] on unitary solutions of quantum stochastic differential equations. This generalisation will be valid for the case of a finite-dimensional initial space whereas in [8] initial spaces of arbitrary dimension were considered. Let H be a Hilbert space. For $\underline{\xi}\in M_d\otimes H$ and $\underline{B}\in M_d\otimes B(H)\cong B(\mathbb{C}^d\otimes H)$ we write $\underline{B}\underline{\xi}$ for the element $\underline{\zeta}$ of $M_d\otimes H$ with

$$\zeta_{kl}=\sum_{n=1}^{d} B_{kn}\xi_{nl}.$$

Clearly $\underline{B}\underline{\xi}$ does not depend on the choice of the orthonormal basis $\{e_k\}$ of $\mathbb{C}^d$. For $\underline{\xi},\underline{\zeta}\in M_d\otimes H$ we denote by $\langle\underline{\xi},\underline{\zeta}\rangle$ the matrix $\underline{b}$ in M_d with

$$b_{kl}=\sum_{n=1}^{d}\langle\xi_{nk},\zeta_{nl}\rangle.$$

Let $L^2(\mathbb{R}_+,H)$ be the Hilbert space of almost everywhere defined measurable functions

$$f\colon\mathbb{R}_+\to H$$

such that

$$\int_0^\infty \|f(t)\|^2 \, dt < \infty.$$

For $\xi \in H$ and $t \in \mathbb{R}_+$ denote by ξ_t the element in $L^2(\mathbb{R}_+, H)$ with

$$\xi_t(s) = \xi \chi_{(0,t)}(s)$$

where $\chi_{(0,t)}$ is the characteristic function of the open interval $(0, t)$ in $\mathbb{R}_+$. For $B \in B(H)$ and $t \in \mathbb{R}_+$ denote by B_t the element in $B(L^2(\mathbb{R}_+, H))$ with

$$B_t(f)(s) = B(f(s)) \chi_{(0,t)}(s),$$

$f \in L^2(\mathbb{R}_+, H)$. Let $A(f)$ and $A^\dagger(f), f \in L^2(\mathbb{R}_+, H)$, be the annihilation and creation operators on the Bose Fock space $\Gamma(L^2(\mathbb{R}_+, H))$ over $(L^2(\mathbb{R}_+, H)$ with domain of definition the subspace $\mathscr{E}(H)$ of $\Gamma(L^2(\mathbb{R}_+, H))$ spanned by the exponential vectors

$$\mathrm{E}(f) = 1 \oplus f \oplus \frac{1}{\sqrt{2!}}(f \otimes f) \oplus \frac{1}{\sqrt{3!}}(f \otimes f \otimes f) \oplus \ldots,$$

$f \in L^2(\mathbb{R}_+, H)$, and for $F \in B(L^2(\mathbb{R}_+, H))$ denote by $\Lambda(F)$ the operator with domain of definition $\mathscr{E}(H)$ which is the differential second quantisation of F, that is

$$\Lambda(F)\mathrm{E}(f) = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}(\mathrm{e}^{tF} f)|_{t=0}.$$

The processes

$$A_t^\varepsilon(f) = A^\varepsilon(f \chi_{(0,t)}), \qquad \varepsilon = 0, 1$$
$$\Lambda_t(F) = \Lambda(F \chi_{(0,t)})$$

are the integrators for the quantum stochastic calculus of Hudson and Parthasarathy. For $\xi \in H$ and $B \in B(H)$ we set

$$A_t^\varepsilon(\xi) = A^\varepsilon(\xi_t)$$
$$\Lambda_t(B) = \Lambda(B_t)$$

and for $\underline{\xi} \in M_d \otimes H$, $\underline{B} \in M_d \otimes B(H)$ and $\underline{b} \in M_d$ we set

$$\mathrm{d}A_t^\varepsilon(\underline{\xi}) = \sum_{k,l=1}^{d} (e_{kl})^{\varepsilon+1} \otimes \mathrm{d}A_t^\varepsilon(\xi_{kl})$$

$$\mathrm{d}\Lambda_t(\underline{B}) = \sum_{k,l=1}^{d} e_{kl} \otimes \mathrm{d}\Lambda_t(B_{kl})$$

$$\underline{b}\,\mathrm{d}t = \sum_{k,l=1}^{d} e_{kl} \otimes b_{kl} \mathrm{d}t.$$

Theorem 2.1. *Let $\underline{\xi} \in M_d \otimes H$, $\underline{B} \in M_d \otimes B(H)$ unitary and $\underline{h} \in M_d$ selfadjoint. Then the quantum stochastic differential equation*

$$\mathrm{d}V_t = V_t(\mathrm{d}A_t(\underline{\xi}) - \mathrm{d}A_t^\dagger(\underline{B}\,\underline{\xi}) + \mathrm{d}\Lambda_t(\underline{B}-\mathbf{1}) + (\mathrm{i}\underline{h} - \tfrac{1}{2}\langle\underline{\xi},\underline{\xi}\rangle)\,\mathrm{d}t \tag{2.1}$$

with the initial condition $V_0 = \mathbf{1}$, has a unique solution $\mathscr{V} = (V_t)_{t\in\mathbb{R}_+}$ with V_t, $t \in \mathbb{R}_+$, unitary operators on $\mathbb{C}^d \otimes \Gamma(L^2(\mathbb{R}_+, H))$.

Proof. We have that $V_n(t)$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, given by

$$\begin{aligned} V_0(t) &= \mathbf{1} \\ V_{n+1}(t) &= \mathbf{1} + \int_0^t V_n(s)(\mathrm{d}A_s(\underline{\xi}) - \mathrm{d}A_s^\dagger(\underline{B}\,\underline{\xi}) + \mathrm{d}\Lambda_s(\underline{B}-\mathbf{1}) \\ &\qquad + (\mathrm{i}\underline{h} - \tfrac{1}{2}\langle\underline{\xi},\underline{\xi}\rangle)\,\mathrm{d}s) \end{aligned}$$

satisfy for $T \in \mathbb{R}_+$ and $f \in L^2(\mathbb{R}^+, H)$ locally bounded

$$\sup_{\substack{0 \leq t \leq T \\ 1 \leq m \leq d}} \|(V_n(t) - V_{n-1}(t))\, e_m \otimes \mathrm{E}(f)\| \leq \mathrm{e}^{\frac{1}{2}(T + \|f\|^2)} \frac{(2d\sqrt{6T}\,\alpha(T))^n}{\sqrt{n!}} \tag{2.2}$$

with

$$\begin{aligned} \alpha(T) = \sup_{0 \leq t \leq T} \max \Big\{ &|\langle f(t), (B_{kl} - \delta_{kl}\mathbf{1}) f(t)\rangle|, |\langle \xi_{lk}, f(t)\rangle|, |\langle f(t), \sum_{n=1}^d B_{kn}\xi_{nk}\rangle|, \\ &\|(B_{kl} - \delta_{kl}\mathbf{1}) f(t)\|^2, \Big\|\sum_{n=1}^d B_{kn}\xi_{nl}\Big\|^2, \Big|\mathrm{i}h_{kl} - \frac{1}{2}\sum_{n=1}^d \langle \xi_{nk}, \xi_{nl}\rangle\Big| : k, l = 1, \dots, d \Big\}. \end{aligned}$$

This follows from

$$\begin{aligned} &\|(V_n(t) - V_{n-1}(t))\, e_m \otimes \mathrm{E}(f)\| \\ &= \Big\| \sum_{k,l=1}^d \int_0^t \Big(\sum_{u=1}^d e_{ul} \otimes (V_{n-1}(s) - V_{n-2}(s))_{uk} \Big) (\mathrm{d}A_s(\xi_{lk}) - \mathrm{d}A_s^\dagger(\sum_n B_{kn}\xi_{nl}) \\ &\quad + \mathrm{d}\Lambda_s(B_{kl} - \delta_{kl}\mathbf{1}) + (\mathrm{i}h_{kl} - \tfrac{1}{2}\sum_n \langle \xi_{nk}, \xi_{nl}\rangle)\,\mathrm{d}s)\, e_m \otimes \mathrm{E}(f) \Big\| \\ &\leq 2\sqrt{6}\,\alpha(T) \sum_{k,l=1}^d \Big\{ \int_0^T \mathrm{e}^{t-s} \Big\| \Big(\sum_{u=1}^d e_{ul} \otimes (V_{n-1}(s) - V_{n-2}(s))_{uk} \Big) e_m \otimes \mathrm{E}(f) \Big\|^2 \mathrm{d}s \Big\}^{\frac{1}{2}} \\ &= 2\sqrt{6}\,\alpha(T) \sum_{k=1}^d \Big\{ \int_0^T \mathrm{e}^{t-s} \|(V_{n-1}(s) - V_{n-2}(s))\, e_k \otimes \mathrm{E}(f)\|^2 \mathrm{d}s \Big\}^{\frac{1}{2}} \\ &\leq 2d\sqrt{6}\,\alpha(T) \Big\{ \int_0^T \mathrm{e}^{t-s} \max_{1 \leq k \leq d} \|(V_{n-1}(s) - V_{n-2}(s))\, e_k \otimes \mathrm{E}(f)\|^2 \mathrm{d}s \Big\}^{\frac{1}{2}} \end{aligned}$$

where we made use of the estimate (4.9) of [8]. Based on the estimate (2.2), it can be shown, exactly as in the proof of Proposition 7.1 of [8], that $V_n(t)$ converge to a solution V_t of (2.1). The unitarity of the V_t follows by the same argument used for the proof of Theorem 7.1 of [8]. □

3. Unitary Noncommutative Stochastic Processes

Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{D}$ be a subset of $B(\mathscr{H})$. A vector $\xi \in \mathscr{H}$ is called cyclic for $\mathscr{D}$ if $N(\mathscr{D})\xi = \{B\xi : B \in N(\mathscr{D})\}$ is dense in $\mathscr{H}$, where $N(\mathscr{D})$ is the von Neumann algebra generated by $\mathscr{D}$. For $\mathscr{D} \subset B(\mathbb{C}^d \otimes \mathscr{H})$ we say that $\xi \in \mathscr{H}$ is cyclic for $\mathscr{D}$ if ξ is cyclic for the subset $\{B_{kl}: k, l = 1, \ldots, d, \underline{B} \in \mathscr{D}\}$ of $B(\mathscr{H})$.

Definition 3.1. A *d-dimensional unitary noncommutative stochastic process* is a triplet $(\mathscr{H}, \mathscr{U}, \Phi_0)$ consisting of

(i) a Hilbert space $\mathscr{H}$,
(ii) a family $\mathscr{U} = (U_t)_{t \in \mathbb{R}}$ of unitary operators on $\mathbb{C}^d \otimes \mathscr{H}$,
(iii) a unit vector $\Phi_0 \in \mathscr{H}$ which is cyclic for $\{U_t : t \in \mathbb{R}_+\}$.

Two d-dimensional unitary noncommutative stochastic processes $(\mathscr{H}^{(i)}, \mathscr{U}^{(i)}, \Phi_0^{(i)})$, $i = 1, 2$, are said to be *equivalent* if there exists a unitary operator

$$\mathfrak{U}: \mathscr{H}^{(1)} \to \mathscr{H}^{(2)}$$

such that

$$\mathfrak{U}\, \Phi_0^{(1)} = \Phi_0^{(2)}$$

and

$$U_t^{(1)} = (\mathbf{1} \otimes \mathfrak{U}^\dagger)\, U_t^{(2)} (\mathbf{1} \otimes \mathfrak{U}).$$

Let $(\Omega, \mathscr{F}, P)$ be a probability space consisting of a set Ω, a σ-algebra $\mathscr{F}$ of subsets of Ω, and a probability measure P on the measure space $(\Omega, \mathscr{F})$. Let $L^\infty(\Omega, \mathscr{F}, P)$ be the commutative von Neumann algebra of equivalence classes of bounded measurable functions on Ω. Let $\mathscr{X} = (X_t)_{t \in \mathbb{R}_+}$ be a family of random variables on $(\Omega, \mathscr{F}, P)$ taking values in the group U_d, that is $\mathscr{X}$ is a classical U_d-valued stochastic process. This process can be regarded as a family $\mathscr{U} = (U_t)_{t \in \mathbb{R}_+}$ of unitary elements in the von Neumann algebra $M_d \otimes L^\infty(\Omega, \mathscr{F}, P)$ by the equations

$$(U_t)_{kl}(\omega) = (X_t(\omega))_{kl} \tag{3.1}$$

$\omega \in \Omega$, $k, l = 1, \ldots, d$. Conversely, any family $\mathscr{U} = (U_t)_{t \in \mathbb{R}_+}$ of unitary elements in $M_d \otimes L^\infty(\Omega, \mathscr{F}, P)$ gives rise to a stochastic process $\mathscr{X} = (X_t)_{t \in \mathbb{R}_+}$ through equations (3.1). By setting $\mathscr{H} = L^2(\Omega, \mathscr{F}, P)$ and Φ_0 the constant function $\mathbf{1}$, we see that Definition 3.1 includes the classical U_d-valued stochastic processes, and that our notion of equivalence in this case is the usual notion of stochastic equivalence of stochastic processes. In the following a unitary noncommutative stochastic process will just be called a *unitary process.*

For a unitary process $(\mathcal{H}, \mathcal{U}, \Phi_0)$ denote by U_{st}, $0 \leqq s \leqq t$, the increment $U_s^\dagger U_t$ and by N_{st} the von Neumann algebra generated by $(U_{st})_{kl}$, $k, l = 1, \ldots, d$. We write $\boldsymbol{E}[A]$ for the expectation value $\langle \Phi_0, A\Phi_0 \rangle$ of an operator A on $\mathcal{H}$.

Definition 3.2. A unitary process $(\mathcal{H}, \mathcal{U}, \Phi_0)$ is called a *process with independent increments* if

(a) $N_{st} \subset (N_{s't'})'$ for all $0 \leqq s \leqq t$ and $0 \leqq s' \leqq t'$ such that (s, t) and (s', t') are disjoint,

(b) $\boldsymbol{E}[A_1 \ldots A_n] = \boldsymbol{E}[A_1] \ldots \boldsymbol{E}[A_n]$ for all $n \in \mathbb{N}$, $0 \leqq t_1 \leqq \ldots \leqq t_n \leqq t_{n+1}$, and $A_k \in N_{t_k, t_{k+1}}$, $k = 1, \ldots, n$.

The process is called a *process with stationary increments* if

(c) $\boldsymbol{E}[(U_{st})_{k_1 l_1}^{\varepsilon_1} \ldots (U_{st})_{k_n l_n}^{\varepsilon_n}] = \boldsymbol{E}[(U_{s+r,t+r})_{k_1 l_1}^{\varepsilon_1} \ldots (U_{s+r,l+r})_{k_n l_n}^{\varepsilon_n}]$ for all $r \in \mathbb{R}_+$, $0 \leqq s \leqq t$, $n \in \mathbb{N}$, $k_1, \ldots, k_n, l_1, \ldots, l_n = 1, \ldots, d$ and $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Z}_2$.

We always assume our processes to be *weakly continuous*, that is

(d) $U_t \to U_0 = \mathbf{1}$ weakly for $t \downarrow 0$.

In the classical case (b) together with (c) is the usual definition of a process with independent and stationary increments whereas (a) becomes trivial. Condition (d) means that for $t \downarrow 0$ the distributions μ_t of X_t converge weakly to the Dirac measure concentrated at the unit element of $\boldsymbol{U}_d$, so (d) is the continuity of the convolution semi-group $\{\mu_t : t \in \mathbb{R}_+\}$ of probability measures on $\boldsymbol{U}_d$ associated to a stochastic process with independent and stationary increments.

We omit the proof of the following theorem, because it goes along the same lines as that of Proposition 3.1 of [9]; see also [3, 12].

Theorem 3.1. *Let* $\mathcal{V} = (V_t)_{t \in \mathbb{R}_+}$ *be the solution of the quantum stochastic differential equation* (2.1). *Then* $(\mathcal{H}^{\mathcal{V}}, \mathcal{U}^{\mathcal{V}}, \Phi_0^{\mathcal{V}})$ *is a d-dimensional unitary process with independent and stationary increments where*

(i) $\Phi_0^{\mathcal{V}}$ *is the vacuum state in* $\Gamma(L^2(\mathbb{R}_+, H))$,

(ii) $\mathcal{H}^{\mathcal{V}}$ *is the closure of* $N\Phi_0^{\mathcal{V}}$ *with N the von Neumann algebra generated by* $(V_t)_{kl}$, $t \in \mathbb{R}_+$, $k, l = 1, \ldots, d$,

(iii) $\mathcal{U}^{\mathcal{V}} = (U_t^{\mathcal{V}})_{t \in \mathbb{R}_+}$ *with* $U_t^{\mathcal{V}}$, $t \in \mathbb{R}_+$, *the restriction of* V_t *to the subspace* $\mathcal{H}^{\mathcal{V}}$ *of* $\Gamma(L^2(\mathbb{R}_+, H))$. □

Here the question arises when $\mathcal{H}^{\mathcal{V}} = \Gamma(L^2(\mathbb{R}_+, H))$ does hold, that is when the vacuum state is cyclic for $\{V_t : t \in \mathbb{R}_+\}$. Actually, this is the case if $H = \mathbb{C}$, $\underline{B} = \mathbf{1}$ and ξ is selfadjoint (Weyl operator case). It is a conjecture that $\mathcal{H}^{\mathcal{V}} = \Gamma(L^2(\mathbb{R}_+, H))$ holds in general.

We are now ready to state our main result.

Theorem 3.2 (embedding theorem). *Let* $(\mathcal{H}, \mathcal{U}, \Phi_0)$ *be a d-dimensional unitary process with independent and stationary increments. Then there exist*

(i) *a Hilbert space* H,
(ii) *an element* ξ *in* $M_d \otimes H$,
(iii) *a unitary element* $\underline{B}$ *in* $M_d \otimes B(H)$,
(iv) *a selfadjoint element* $\underline{h}$ *in* M_d

such that $(\mathscr{H}, \mathscr{U}, \Phi_0)$ *is equivalent to* $(\mathscr{H}^{\mathscr{V}}, \mathscr{U}^{\mathscr{V}}, \Phi_0^{\mathscr{V}})$ *where* $\mathscr{V}=(V_t)_{t\in\mathbb{R}_+}$ *is the solution of the quantum stochastic differential equation*

$$\mathrm{d}V_t = V_t(\mathrm{d}A_t(\underline{\xi}) - \mathrm{d}A_t^\dagger(\underline{B}\,\underline{\xi}) + \mathrm{d}\Lambda_t(\underline{B}-\mathbf{1}) + (\mathrm{i}\underline{h} - \tfrac{1}{2}\langle\underline{\xi}, \underline{\xi}\rangle)\,\mathrm{d}t)$$

with the initial condition $V_0 = \mathbf{1}$.

A proof of this theorem will be given at the end of Sect. 5.

4. Involutive Bialgebras

Most of this section is a brief review of facts on involutive bialgebras; see [3, 12]. On the other hand, Proposition 4.1 introduces a simple new construction which will be important for the proof of the embedding theorem.

A coalgebra is a triplet $(\mathscr{C}, \Delta, \delta)$ consisting of a vector space $\mathscr{C}$, a linear map $\Delta: \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}$ and a linear functional δ on $\mathscr{C}$ such that

$$(\Delta \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \Delta) \circ \Delta \quad \text{(coassociativity)}$$

and

$$(\delta \otimes \mathbf{1}) \circ \Delta = (\mathbf{1} \otimes \delta) \circ \Delta = \mathbf{1} \quad \text{(counit property);}$$

see [1, 13]. The map Δ is called the *comultiplication* and the functional δ is called the *counit* of the coalgebra $\mathscr{C}$. Let $\mathscr{A}$ be an algebra with multiplication map $M: \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$. The space $L(\mathscr{C}, \mathscr{A})$ of linear maps from $\mathscr{C}$ to $\mathscr{A}$ is turned into an algebra by defining a product $R * R'$ for $R, R' \in L(\mathscr{C}, \mathscr{A})$ by

$$R * R' = M \circ (R \otimes R') \circ \Delta.$$

We call $R * R'$ the *convolution product* of R with R'. The unit of $L(\mathscr{C}, \mathscr{A})$ is the map $c \mapsto \delta(c)\mathbf{1}$. For the special case $\mathscr{A} = \mathbb{C}$ the algebraic dual space $\mathscr{C}^* = L(\mathscr{C}, \mathbb{C})$ of $\mathscr{C}$ becomes an algebra with convolution product

$$\phi * \psi = (\phi \otimes \psi) \circ \Delta,$$

$\phi, \psi \in \mathscr{C}^*$, and unit δ. For any ψ in $\mathscr{C}^*$ the convolution exponential $\exp_* \psi \in \mathscr{C}^*$ can be defined as the pointwise limit

$$(\exp_* \psi)(c) = \sum_{n=0}^{\infty} \frac{\psi^{*n}}{n!}(c),$$

$c \in \mathscr{C}$, where $\psi^{*0} = \delta$ and ψ^{*n}, $n \in \mathbb{N}$, is the n-fold convolution product of ψ with itself; see [11]. For an algebra $\mathscr{A}$ the vector space tensor product $\mathscr{A} \otimes \mathscr{A}$ is again an algebra with multiplication given by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb',$$

$a, a', b, b' \in \mathscr{A}$. If $\mathscr{A}$ is a *-algebra an involution on $\mathscr{A} \otimes \mathscr{A}$ is defined by

$$(a \otimes b)^* = a^* \otimes b^*,$$

$a, b \in \mathscr{A}$. If $(\mathscr{B}, \Delta, \delta)$ is a coalgebra and $\mathscr{B}$ is also a *-algebra such that Δ and δ are *-algebra homomorphisms, then $\mathscr{B}$ is called an involutive bialgebra or a *-bialgebra [11]. A linear functional ψ on a *-bialgebra $\mathscr{B}$ is called *conditionally positive* if

$$\psi(b^* b) \geqq 0 \quad \text{for all } b \in \mathscr{B} \quad \text{with } \delta(b) = 0.$$

For a linear functional ψ on a *-bialgebra $\mathscr{B}$ we define two sesquilinear forms K_ψ and L_ψ on $\mathscr{B}$ by

$$K_\psi(b, c) = \psi(b^* c)$$

and

$$\begin{aligned} L_\psi(b, c) &= \psi((b - \delta(b)\mathbf{1})^* (c - \delta(c)\mathbf{1})) \\ &= \psi(b^* c) - \overline{\delta(b)}\,\psi(c) - \psi(b^*)\,\delta(c) + \overline{\delta(b)}\,\delta(c)\,\psi(\mathbf{1}) \end{aligned}$$

$b, c \in \mathscr{B}$. A linear functional ψ on $\mathscr{B}$ is positive if and only if K_ψ is positive, and it is conditionally positive if and only if L_ψ is positive. Assume that $\psi \in \mathscr{B}^*$ is such that $\exp_*(t\psi)$ is positive for all $t \in \mathbb{R}_+$. Differentiating with respect to t at $t = 0$, yields that ψ must be hermitian and conditionally positive. The converse of this is the content of Theorem 4.2 in [11]. Moreover, if $\mathscr{C}$ is a coalgebra and $\{\phi_t : t \in \mathbb{R}_+\}$ is a convolution semi-group of linear functionals on $\mathscr{C}$ satisfying

$$\lim_{t \downarrow 0} \phi_t(c) = \delta(c) \quad \text{for all } c \in \mathscr{C} \tag{4.1}$$

then $\phi_t = \exp_*(t\psi)$ for some $\psi \in \mathscr{C}^*$; see [3, 12]. Thus the convolution semi-groups $\{\phi_t : t \in \mathbb{R}_+\}$ of states on a *-bialgebra, which are continuous in the sense of (4.1), are exactly the semi-groups of the form $\exp_*(t\psi)\}$ with ψ a hermitian, conditionally positive linear functional vanishing at $\mathbf{1}$.

Let $\mathscr{B}$ be a *-bialgebra and let π be a representation of $\mathscr{B}$ as an algebra of linear operators on a vector space $\mathscr{D}$. We turn $\mathscr{D}$ into a $\mathscr{B}$-bimodule (with respect to π) by defining a left and right action of $\mathscr{B}$ on $\mathscr{D}$ by

$$b \cdot \xi \cdot c = \pi(b)(\xi)\,\delta(c),$$

$b, c \in \mathscr{B}$, $\xi \in \mathscr{D}$. The vector space $\mathbb{C}$ will always be considered as a $\mathscr{B}$-bimodule with respect to the representation δ of $\mathscr{B}$ on $\mathbb{C}$. We use the language of the Hochschild algebra cohomology theory for $\mathscr{B}$ taking values in $\mathscr{D}$; see [7]. Thus a 1-cocycle with respect to a representation π of $\mathscr{B}$ on $\mathscr{D}$ is a linear map $\eta : \mathscr{B} \to \mathscr{D}$ such that

$$\eta(bc) = \pi(b)\,\eta(c) + \eta(b)\,\delta(c),$$

and a bilinear map $\mathscr{L} : \mathscr{B} \otimes \mathscr{B} \to \mathbb{C}$ is the coboundary of $\psi \in \mathscr{B}^*$ if

$$\mathscr{L}(b, c) = \delta(b)\,\psi(c) - \psi(bc) + \psi(b)\,\delta(c),$$

$b, c \in \mathscr{B}$. If $\mathscr{W}$ is a vector space and S is a positive sesquilinear form on $\mathscr{W}$, denote by $\mathscr{D}(\mathscr{W}, S)$ the pre-Hilbert space $\mathscr{W}/\mathscr{N}_S$, where $\mathscr{N}_S = \{w \in \mathscr{W}: S(w, w) = 0\}$, with the scalar product given by S. Denote by

$$\eta_S: \mathscr{W} \to \mathscr{D}(\mathscr{W}, S)$$

the canonical map.

Proposition 4.1. *Let $\mathscr{B}$ be a *-bialgebra and let ψ be a hermitian, conditionally positive linear functional on $\mathscr{B}$. Then the equation*

$$\pi^{(\psi)}(b)\,\eta_{L_\psi}(c) = \eta_{L_\psi}(b(c - \delta(c)\mathbf{1})) = \eta_{L_\psi}(bc) - \eta_{L_\psi}(b)\,\delta(c),$$

$b, c \in \mathscr{B}$, defines a-representation of $\mathscr{B}$ on the pre-Hilbert space $\mathscr{D}(\mathscr{B}, L_\psi)$. Moreover, η_{L_ψ} is a 1-cocycle with respect to $\pi^{(\psi)}$ and the bilinear form $\mathscr{L}_\psi$ on $\mathscr{B}$ given by*

$$\mathscr{L}_\psi(b, c) = -L_\psi(b^*, c) + \delta(b)\,\delta(c)\,\psi(\mathbf{1})$$

$b, c \in \mathscr{B}$, is the coboundary of ψ.

Proof. The last statement is just the definition of L_ψ. Let $b \in \mathscr{B}$ and $\eta_{L_\psi}(b) = 0$ that is $L_\psi(b, b) = 0$. Then for $c \in \mathscr{B}$ we have

$$\begin{aligned} &L_\psi(c(b - \delta(b)\mathbf{1}), c(b - \delta(b)\mathbf{1})) \\ &\quad = \psi((b - \delta(b)\mathbf{1})^* c^* c(b - \delta(b)\mathbf{1})) \\ &\quad = L_\psi(b, c^* c(b - \delta(b)\mathbf{1})) \end{aligned}$$

which is equal to 0 by Cauchy-Schwartz inequality. It follows that $\pi^{(\psi)}(b)$ is well-defined as a linear operator on $\mathscr{D}(\mathscr{B}, L_\psi)$. Using again the fact that δ is a *-algebra homomorphism, one checks that $\pi^{(\psi)}$ is a *-representation. The map η_{L_ψ} is a 1-cocycle with respect to $\pi^{(\psi)}$ by the definition of $\pi^{(\psi)}$. □

Note that we did not use the comultiplication of $\mathscr{B}$ and that Proposition 4.1 also can be formulated for the more general situation when $\mathscr{B}$ is a *-algebra and $\delta: \mathscr{B} \to \mathbb{C}$ is a *-algebra homomorphism.

The case of a hermitian, conditionally positive definite function on a group G (cf. [5, 10]) is included in the above proposition. $\mathscr{B}$ is the group *-algebra of G that is the free vector space spanned by G with multiplication given by the group multiplication and involution given by $x \mapsto x^{-1}$, $x \in G$. The comultiplication and the counit of $\mathscr{B}$ are given by linear extension of the maps $x \mapsto x \otimes x$ and $x \mapsto 1$, respectively.

We now turn to the concept of independent, stationary increment processes over a *-bialgebra as introduced in [3, 12]. In the sense of Accardi et al. [2] a quantum stochastic process over a *-algebra $\mathscr{B}$ is a triplet $(\mathscr{A}, j, \Phi)$ consisting of

(i) another *-algebra $\mathscr{A}$,
(ii) a family $j = (j_\alpha)_{\alpha \in I}$ of *-algebra homomorphisms $j_\alpha: \mathscr{B} \to \mathscr{A}$ indexed by a set I,
(iii) a state Φ on $\mathscr{A}$.

Two processes $(\mathscr{A}^{(i)}, j^{(i)}, \Phi^{(i)})$, $i=1,2$, over $\mathscr{B}$, indexed by the same set I, are said to be equivalent if for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in I$ and $b_1, \ldots, b_n \in \mathscr{B}$ the complex numbers

$$\Phi^{(i)}(j_{\alpha_1}^{(i)}(b_1) \ldots j_{\alpha_n}^{(i)}(b_n))$$

are the same for $i=1$ and $i=2$. In the following I will be the set of all ordered pairs of nonnegative real numbers. For a process $(\mathscr{A}, j, \Phi)$ let $\mathscr{A}_{st}$ be the subalgebra of $\mathscr{A}$ generated by the elements $j_{st}(b)$, $b \in \mathscr{B}$, and let ϕ_{st}, $0 \leqq s \leqq t$, be the state $\Phi \circ j_{st}$ on $\mathscr{B}$. An independent, stationary increment process over a *-bialgebra $\mathscr{B}$ is a process $(\mathscr{A}, j, \Phi)$ over $\mathscr{B}$ in the sense of Accardi, Frigerio and Lewis such that

(i) $j_{rs} * j_{st} = j_{rt}$; $j_{t,t} = \delta \mathbf{1}$ for all $0 \leqq r \leqq s \leqq t$ (increment property),

(ii) $\mathscr{A}_{st}$ and $\mathscr{A}_{s't'}$ commute for all $0 \leqq s \leqq t$ and $0 \leqq s' \leqq t'$ such that the intervalls (s, t) and (s', t') are disjoint, and $\Phi(\alpha_1 \ldots \alpha_n) = \Phi(\alpha_1) \ldots \Phi(\alpha_n)$ for all $n \in \mathbb{N}$, $0 \leqq t_1 \leqq \ldots \leqq t_n \leqq t_{n+1}$, and $a_k \in \mathscr{A}_{t_k, t_{k+1}}$, $k=1, \ldots, n$ (independents of increments),

(iii) $\phi_{st} = \phi_{s+r, t+r}$ for all $r \in \mathbb{R}_+$, $0 \leqq s \leqq t$ (stationarity of increments).

The process is called continuous if

$$\lim_{t \downarrow s} \phi_{st}(b) = \delta(b)$$

for all $s \in \mathbb{R}_+$, $b \in \mathscr{B}$. If we put $\phi_t = \phi_{0,t}$, for an independent, stationary increment process, $\{\phi_t : t \in \mathbb{R}_+\}$ is a continuous semi-group of states on $\mathscr{B}$, and if the process is also continuous we have

$$\phi_t = \exp_*(t\psi),$$

$t \in \mathbb{R}_+$, for some hermitian, conditionally positive $\psi \in \mathscr{B}^*$ vanishing at $\mathbf{1}$. The linear functional ψ is called the generator of the process. It was proved in [3, 12] that a continuous independent, stationary increment process is determined by its generator up to equivalence.

5. Proof of the Embedding Theorem

We give another interpretation of d-dimensional unitary processes. Let M_d^c be the complex conjugate vector space of M_d. The elements of M_d^c are $\underline{h}^c$ with $\underline{h} \in M_d$ and the vector space structure of M_d^c is given by

$$\lambda \underline{h}^c + \underline{k}^c = (\bar{\lambda} \underline{h} + \underline{k})^c,$$

$\lambda \in \mathbb{C}$, $\underline{h}, \underline{k} \in M_d$. We denote by $\mathscr{I}$ the tensor algebra

$$\mathbb{C}\mathbf{1} \oplus \mathscr{W} \oplus (\mathscr{W} \otimes \mathscr{W}) \oplus (\mathscr{W} \otimes \mathscr{W} \otimes \mathscr{W}) \oplus \ldots$$

over the vector space $\mathscr{W} = M_d^c \oplus M_d$. The multiplication in $\mathscr{I}$ is defined by

$$w_1 \ldots w_n = w_1 \otimes \ldots \otimes w_n,$$

$n\in\mathbb{N}$, $w_1, \ldots, w_n\in\mathscr{W}$. An involution on $\mathscr{I}$ is given by

$$(\underline{h})^* = \underline{h}^c$$

for $\underline{h}\in M_d$. We divide the *-algebra $\mathscr{I}$ by the ideal $\mathscr{J}$ generated by the elements

$$\sum_{n=1}^{d} e_{kn}\otimes(e_{ln})^* - \delta_{kl}$$

and

$$\sum_{n=1}^{d} (e_{nk})^*\otimes e_{nl} - \delta_{kl},$$

$k, l=1, \ldots, d$. Notice that $\mathscr{J}$ is a *-ideal and does not depend on the choice of the orthonormal basis $\{e_k\}$ of $B(\mathbb{C}^d)$. We denote by $\mathscr{K}_d$ the *-algebra $\mathscr{I}/\mathscr{J}$ which can be regarded as the noncommutative analogue of the coefficient algebra of U_d; see [12, 14]. We let $x_{kl}, x^*_{kl}\in\mathscr{K}_d$ be the equivalence class of $e_{kl}, (e_{kl})^*\in\mathscr{I}$ respectively.

Proposition 5.1. *Let π be a *-representation of $\mathscr{K}_d$ on a pre-Hilbert space $\mathscr{D}$. Then the operators $\pi(b)$, $b\in\mathscr{K}_d$, are bounded and the operator $((\pi(x_{kl}))_{k,l=1,\ldots d}$ on $\mathbb{C}^d\otimes\mathscr{D}$ extends to a unitary operator on $\mathbb{C}^d\otimes H$ where H is the Hilbert space which is the completion of $\mathscr{D}$*

Proof. The relations in $\mathscr{K}_d$ imply that the operator $\tilde{U}=(\pi(x_{kl}))_{k,l=1,\ldots,d}$ on $\mathbb{C}^d\otimes\mathscr{D}$ satisfies

$$\tilde{U}^\dagger\tilde{U} = \tilde{U}\tilde{U}^\dagger = \mathbf{1}.$$

So $\tilde{U}$ can be extended in a unique way to a unitary operator U on $\mathbb{C}^d\otimes H$. Then U_{kl}, $k, l=1, \ldots, d$, are bounded operators on H and the restriction of U_{kl} to $\mathscr{D}$ is equal to $\pi(x_{kl})$. $\square$

Let $\mathscr{A}$ be a *-algebra, Φ a state on $\mathscr{A}$ and $j\colon\mathscr{K}_d\to\mathscr{A}$ a *-algebra homomorphism. The left multiplication in $\mathscr{A}$ defines a *-representation ρ of $\mathscr{A}$, the Gelfand-Naimark-Segal representation, on the pre-Hilbert space $\mathscr{D}=\mathscr{D}(\mathscr{A}, K_\Phi)$ where as before K_Φ is the positive sesquilinear form on $\mathscr{A}$ with $K_\Phi(a, b) = \Phi(a^* b)$, $a, b\in\mathscr{A}$. Moreover, $\mathscr{D}=\rho(\mathscr{A})\Phi_0$ where Φ_0 is the unit vector $\eta_{K_\Phi}(\mathbf{1})$, and $\rho\circ j$ is a *-representation of $\mathscr{K}_d$ on $\mathscr{D}$. By Proposition 5.1 the operator $((\rho\circ j)(x_{kl}))_{k,l=1,\ldots,d}$ extends to a unitary operator on $\mathbb{C}^d\otimes H$, where H is the completion of $\mathscr{D}$, which we denote by $U(j)$. Conversely, let U be a unitary operator on $\mathbb{C}^d\otimes H$ with H an arbitrary Hilbert space. A *-algebra homomorphism

$$j(U)\colon\mathscr{K}_d\to B(H)$$

is determined by

$$j(U)(x_{kl}) = U_{kl},$$

$k, l=1, \ldots, d$. Thus stochastic processes $(\mathscr{A}, j, \Phi)$ over $\mathscr{K}_d$ and d-dimensional unitary processes $(\mathscr{H}, \mathscr{U}, \Phi_0)$ represent the same object.

We define a *-bialgebra structure on $\mathscr{K}_d$; see [3, 12]. First define the *-algebra homomorphisms

$$\tilde{\Delta}: \mathscr{I} \to \mathscr{I} \otimes \mathscr{I}$$

and

$$\tilde{\delta}: \mathscr{I} \to \mathbb{C}$$

by

$$\tilde{\Delta} e_{kl} = \sum_{n=1}^{d} e_{kn} \otimes e_{nl}$$

and

$$\tilde{\delta} e_{kl} = \delta_{kl}.$$

$\mathscr{I}$ is a *-bialgebra with comultiplication $\tilde{\Delta}$ and counit $\tilde{\delta}$, and $\mathscr{J}$ is not only a *-ideal but also a coideal in $\mathscr{I}$. Thus $\tilde{\Delta}$ and $\tilde{\delta}$ give rise to a comultiplication and a counit on $\mathscr{K}_d$. If $d=1$ the coefficient algebra $\mathscr{K}_d$ reduces to the group algebra of $\mathbb{Z}$.

The proof of the next proposition is now straightforward.

Proposition 5.2. *Let $(\mathscr{A}, j, \Phi)$ be a continuous independent, stationary increment process over $\mathscr{K}_d$. Then $(\mathscr{H}, \mathscr{U}, \Phi_0)$ is a d-dimensional unitary process with independent and stationary increments where*

(i) $\mathscr{H}$ is the completion of $\mathscr{D}(\mathscr{A}, \mathrm{K}_\Phi)$,
(ii) $\mathscr{U}=(U_t)_{t\in\mathbb{R}_+}$ with $U_t = U(j_{0,t})$,
(iii) $\Phi_0 = \eta_{\mathrm{K}_\psi}(\mathbf{1})$.

Conversely, let $(\mathscr{H}, \mathscr{U}, \Phi_0)$ be a d-dimensional unitary process with independent and stationary increments. Then $(\mathscr{A}, j, \Phi)$ is a continuous independent, stationary increment process over $\mathscr{K}_d$ where

(i) $\mathscr{A} = B(\mathscr{H})$,
(ii) $j = (j_{st})_{(s,t)\in I}$ with $j_{st} = j(U_{st})$,
(iii) $\Phi(A) = \boldsymbol{E}[A]$, $A \in B(\mathscr{H})$.

Moreover, the notions of equivalence for processes $(\mathscr{A}, j, \Phi)$ and $(\mathscr{H}, \mathscr{U}, \Phi_0)$ correspond. □

Using Theorem 3.1 and Theorem 4.1 of [3], it follows that a d-dimensional unitary process $(\mathscr{H}, \mathscr{U}, \Phi_0)$ with independent and stationary increments is determined up to equivalence by its generator ψ which is the hermitian, conditionally positive linear functional on $\mathscr{K}_d$ given by

$$\psi(x_{k_1 l_1}^{\varepsilon_1} \ldots x_{k_n l_n}^{\varepsilon_n}) = \frac{\mathrm{d}}{\mathrm{d}t} \langle \Phi_0, (U_t)_{k_1 l_1}^{\varepsilon_1} \ldots (U_t)_{k_n l_n}^{\varepsilon_n} \Phi_0 \rangle |_{t=0},$$

$n \in \mathbb{N}$, $k_1, \ldots, k_n, l_1, \ldots, l_n = 1, \ldots, d$, $\varepsilon_1, \ldots, \varepsilon_n \in \mathbb{Z}_2$. Before we come to the proof of Theorem 3.2, we treat an example of a class of hermitian, conditionally positive linear functionals on $\mathcal{K}_d$; see [15]. Let $\sigma: \mathcal{K}_d \to \mathbb{C}$ be a *-algebra homomorphism, that is σ is determined by the matrix $\underline{w}$ in M_d with $w_{km} = \sigma(x_{km})$. The relations in $\mathcal{K}_d$ imply that $\underline{w}$ is unitary. For $\underline{l} \in M_d$ we define the σ-derivation $D_{\underline{l}}$ on $\mathcal{K}_d$ as follows. We set

$$D_{\underline{l}}(x_{km}) = l_{km}$$

$$D_{\underline{l}}(x_{km}^*) = -\sum_{n=1}^{d} \bar{w}_{kn} l_{mn}$$

and require $D_{\underline{l}}$ to satisfy the equation

$$D_{\underline{l}}(bc) = D_{\underline{l}}(b)\,\sigma(c) + \delta(b)\,D_{\underline{l}}(c),$$

$b, c \in \mathcal{K}_d$. We define the hermitian linear functional $\psi_{\underline{w},\underline{l}}$ on $\mathcal{K}_d$ by

$$\psi_{\underline{w},\underline{l}}(\mathbf{1}) = 0$$

$$\psi_{\underline{w},\underline{l}}(x_{km}) = -\tfrac{1}{2} \sum_{n=1}^{d} l_{kn} \bar{l}_{mn}$$

and by requiring $\psi_{\underline{w},\underline{l}}$ to fulfill

$$\psi_{\underline{w},\underline{l}}(bc) = \psi_{\underline{w},\underline{l}}(b)\,\delta(c) + \delta(b)\,\psi_{\underline{w},\underline{l}}(c) + D_{\underline{l}}(b)\,\overline{D_{\underline{l}}(c^*)},$$

$b, c \in \mathcal{K}_d$. Using quantum Ito's formula, it can be shown by a long computation that $\psi_{\underline{w},\underline{l}}$ is the generator of $(\mathcal{H}^{\mathcal{V}}, \mathcal{U}^{\mathcal{V}}, \Phi_0^{\mathcal{V}})$ where $\mathcal{V}$ is the solution of the quantum stochastic differential equation

$$\mathrm{d}V_t = V_t(\underline{l} \otimes \mathrm{d}A_t - \underline{w}\,\underline{l}^\dagger \otimes \mathrm{d}A_t^\dagger + (\underline{w} - \mathbf{1}) \otimes \mathrm{d}\Lambda_t - \tfrac{1}{2}\underline{l}\,\underline{l}^\dagger \otimes \mathrm{d}t)$$

with the initial condition $V_0 = \mathbf{1}$ on $\mathbb{C}^d \otimes \Gamma(L^2(\mathbb{R}_+))$. This also follows from Theorem 3.2.

Proposition 5.3. *Let H be a Hilbert space and $d \in \mathbb{N}$. There is a one-to-one correspondence between pairs $(\underline{B}, \underline{\xi})$, where $\underline{B} \in M_d \otimes B(H)$ unitary and $\underline{\xi} \in M_d \otimes H$, and pairs (π, η), where π is a *-representation of $\mathcal{K}_d$ on H and η is a 1-cocycle with respect to π, given by the equations*

$$\pi(x_{kl}) = B_{kl} \tag{5.1}$$

$$\eta(x_{kl}^*) = \xi_{lk} \tag{5.2}$$

$$\eta(x_{kl}) = -\sum_{n=1}^{d} B_{kn} \xi_{nl}, \tag{5.3}$$

$k, l = 1, \ldots, d$.

Proof. Let π be a *-representation of $\mathscr{K}_d$ on H. By Proposition 5.1 the operator $\underline{B}$ on $\mathbb{C}^d \otimes H$ given by (5.1) is unitary. Let η be a 1-cocycle with respect to π. Then we have, using the relations in $\mathscr{K}_d$ and the fact that $\eta(\mathbf{1})=0$,

$$0=\sum_{n=1}^{d}(\pi(x_{kn})\eta(x_{ln}^*)-\eta(x_{kn}x_{ln}^*)+\eta(x_{kn})\delta(x_{ln}^*))$$

$$=\sum_{n=1}^{d} B_{kn}\eta(x_{ln}^*)+\eta(x_{kl}),$$

so $(\underline{B}, \underline{\xi})$ with $\xi_{kl}=\eta(x_{lk}^*)$ satisfies (5.1) to (5.3). Conversely, let $\underline{\xi}\in M_d\otimes H$ and $\underline{B}\in M_d\otimes B(H)$ unitary be given. Equation (5.1) defines a *-representation of $\mathscr{K}_d$. Moreover, there is a uniquely determined 1-cocycle η with respect to π satisfying (5.2) and (5.3). □

Theorem 5.1. *Let H be a Hilbert space, $d\in\mathbb{N}$, $\underline{B}\in M_d\otimes B(H)$ unitary and $\underline{\xi}\in M_d\otimes H$. Let $(\mathscr{H}^{\mathscr{V}}, \mathscr{U}^{\mathscr{V}}, \Phi_0^{\mathscr{V}})$ be the d-dimensional unitary process with independent and stationary increments given by the solution $\mathscr{V}$ of the quantum stochastic differential equation*

$$\mathrm{d}V_t=V_t(\mathrm{d}A_t(\underline{\xi})-\mathrm{d}A_t^\dagger(\underline{B}\,\underline{\xi})+\mathrm{d}\Lambda_t(\underline{B}-\mathbf{1})-\tfrac{1}{2}\langle\underline{\xi},\underline{\xi}\rangle\,\mathrm{d}t) \tag{5.4}$$

with the initial condition $V_0=\mathbf{1}$, and let (π, η) be the pair associated to $(\underline{B}, \underline{\xi})$ by Proposition 5.3. Then the bilinear form $\mathscr{L}$ on $\mathscr{K}_d$ given by

$$\mathscr{L}(b, c)=-\langle\eta(b^*), \eta(c)\rangle,$$

$b, c\in\mathscr{K}_d$, is the coboundary of the generator of $(\mathscr{H}^{\mathscr{V}}, \mathscr{U}^{\mathscr{V}}, \Phi_0^{\mathscr{V}})$.

Proof. We must proof that for the generator ψ of $(\mathscr{H}^{\mathscr{V}}, \mathscr{U}^{\mathscr{V}}, \Phi_0^{\mathscr{V}})$

$$\psi(b^*c)=\overline{\psi(b)}\,\delta(c)+\overline{\delta(b)}\,\psi(c)+\langle\eta(b), \eta(c)\rangle \tag{5.5}$$

for all $b, c\in\mathscr{K}_d$. We have

$$\psi(b^*c)=\frac{\mathrm{d}}{\mathrm{d}t}\langle\Phi_0^{\mathscr{V}}, j(U_t)(a)^\dagger j(U_t)(b)\,\Phi_0^{\mathscr{V}}\rangle|_{t=0}. \tag{5.6}$$

For $k, l=1, \ldots, d$ the operators $(U_t)_{kl}$ satisfy the quantum stochastic differential equations

$$\mathrm{d}(U_t)_{kl}=\sum_{n=1}^{d}(U_t)_{kn}(\mathrm{d}A_t(\eta(x_{nl}^*))+\mathrm{d}A_t^\dagger(\eta(x_{nl}))+\mathrm{d}\Lambda_t(B_{nl}-\delta_{nl}\mathbf{1})$$

$$-\tfrac{1}{2}\sum_{m=1}^{d}\langle\eta(x_{nm}^*), \eta(x_{lm}^*)\rangle\,\mathrm{d}t) \tag{5.7}$$

with the initial condition $(U_0)_{kl}=\delta_{kl}\mathbf{1}$. It follows that $j(U_t)(b)$ for arbitrary b in $\mathscr{K}_d$ satisfies a quantum stochastic differential equation and by quantum Ito's formula

$$\begin{aligned} \mathrm{d}(j(U_t)(b)^\dagger j(U_t)(c)) = {} & \mathrm{d}(j(U_t)(b)^\dagger) j(U_t)(c) + j(U_t)(b)^\dagger \,\mathrm{d}(j(U_t)(c)) \\ & + \mathrm{d}(j(U_t)(b)^\dagger)\,\mathrm{d}(j(U_t)(c)) \end{aligned} \tag{5.8}$$

where the third term is calculated according to the quantum Ito table. But in Eq. (5.6) we are just dealing with expectations in the vacuum state $\Phi_0^{\mathscr{V}}$, so we only need to compute the coefficient of the $\mathrm{d}t$-part of (5.8). This coefficient will be a complex-valued function in $t\in\mathbb{R}_+$, and its value at $t=0$ gives the right hand side of (5.6). Using the initial condition, one checks that the first two terms on the right hand side of (5.8) give rise to the first two terms on the right hand side of (5.5). We are left with the computation of the coefficient of the $\mathrm{d}t$-part of

$$\mathrm{d}(j(U_t)(b)^\dagger)\,\mathrm{d}(j(U_t)(c))$$

at $t=0$. To this goal we compute the $\mathrm{d}A_t^\dagger$-parts of $\mathrm{d}(j(U_t)(b))$ and $\mathrm{d}(j(U_t)(c))$ and use

$$\mathrm{d}A_t(\xi)\,\mathrm{d}A_t^\dagger(\zeta)=\langle\xi,\zeta\rangle\,\mathrm{d}t,$$

$\xi,\zeta\in H$. Of course, we can assume that both b and c are monomials in x_{kl} and x_{kl}^*. Assuming that

$$b=x_{k_1l_1}^{\varepsilon_1}\dots x_{k_rl_r}^{\varepsilon_r}$$

for some $n\in\mathbb{N}$, $k_1,\dots,k_r$, $l_1,\dots,l_r\in\{1,\dots,d\}$ and $\varepsilon_1,\dots,\varepsilon_r\in\mathbb{Z}_2$, we obtain from (5.7) and quantum Ito's formula for the $\mathrm{d}A_t^\dagger$-part of $\mathrm{d}(j(U_t)(b))$ the expression

$$\begin{aligned} & \sum_{n=1}^{r}\sum_{m_1,\dots,m_r=1}^{d} U_{k_1m_1}^{\varepsilon_1}\dots U_{k_rm_r}^{\varepsilon_r}\,\mathrm{d}A(B_{m_1l_1}^{\varepsilon_1}) \\ & \dots\mathrm{d}A(B_{m_{n-1},l_{n-1}}^{\varepsilon_{n-1}})\,\mathrm{d}A^\dagger(\eta(x_{m_nl_n}^{\varepsilon_n}))\,\delta_{m_{n+1},l_{n+1}}\dots\delta_{m_rl_r} \\ & =\sum_{n=1}^{r}\sum_{m_1,\dots,m_r=1}^{d} U_{k_1m_1}^{\varepsilon_1}\dots U_{k_rm_r}^{\varepsilon_r}\,\mathrm{d}A^\dagger(B_{m_1l_1}^{\varepsilon_1} \\ & \dots B_{m_{n-1},l_{n-1}}^{\varepsilon_{n-1}}\,\eta(x_{m_nl_n}^{\varepsilon_n}))\,\delta_{m_{n+1},l_{n+1}}\dots\delta_{m_rl_r}. \end{aligned} \tag{5.9}$$

Finally, the fact that η is a 1-cocycle with respect to π gives

$$\sum_{n=1}^{r}(B_{k_1l_1}^{\varepsilon_1}\dots B_{k_{n-1},l_{n-1}}^{\varepsilon_{n-1}})\,\eta(x_{k_nl_n}^{\varepsilon_n})\,\delta_{k_{n+1},l_{n+1}}\dots\delta_{k_rl_r}=\eta(x_{k_1l_1}^{\varepsilon_1}\dots x_{k_rl_r}^{\varepsilon_r})$$

which together with (5.9) yields (5.5). □

Proof of Theorem 3.2. Let ψ be the generator of $(\mathscr{H},\mathscr{U},\Phi_0)$. As ψ is a hermitian, conditionally positive linear functional on $\mathscr{K}_d$, it gives rise to a *-representation $\pi=\pi^{(\psi)}$ of $\mathscr{K}_d$ on a Hilbert space H and a 1-cocycle $\eta=\eta_{L_\psi}$ with respect to π by Proposition 4.1. Let $(\underline{B},\underline{\xi})$ be the pair associated to (π,η) by Proposition 5.3.

By Theorem 5.1 the generator $\tilde{\psi}$ of the process $(\mathscr{H}^{\tilde{\mathscr{V}}}, \mathscr{U}^{\tilde{\mathscr{V}}}, \Phi_0^{\tilde{\mathscr{V}}})$ which is given by the solution $\tilde{\mathscr{V}}$ of the quantum stochastic differential equation (5.4) has the coboundary $\mathscr{L}$ with

$$\mathscr{L}(b, c) = -\langle \eta(b^*), \eta(c) \rangle,$$

$b, c \in \mathscr{K}_d$. But $\mathscr{L}$ is also the coboundary of ψ. It follows that $\tilde{\psi}$ and ψ differ only by a *-derivation on $\mathscr{K}_d$, that is by a hermitian linear functional D on $\mathscr{K}_d$ satisfying

$$D(bc) = D(b)\,\delta(c) - \delta(b)\,D(c),$$

$b, c \in \mathscr{K}_d$. The *-derivation D is determined by the selfadjoint matrix $\underline{h} \in M_d$ with

$$h_{kl} = -\mathrm{i} D(x_{kl});$$

see [3]. So ψ must be the generator of the process $(\mathscr{H}^{\mathscr{V}}, \mathscr{U}^{\mathscr{V}}, \Phi_0^{\mathscr{V}})$ where $\mathscr{V}$ is the solution of the quantum stochastic differential equation

$$\mathrm{d} V_t = V_t(\mathrm{d}A_t(\underline{\xi}) - \mathrm{d}A_t^\dagger(\underline{B}\,\underline{\xi}) + \mathrm{d}A_t(\underline{B} - \mathbf{1}) + (\mathrm{i}\underline{h} - \tfrac{1}{2}\langle \underline{\xi}, \underline{\xi} \rangle)\,\mathrm{d}t)$$

with the initial condition $V_0 = \mathbf{1}$. Moreover, we have

$$\mathrm{i}h_{kl} - \tfrac{1}{2}\langle \underline{\xi}, \underline{\xi} \rangle_{kl} = \psi(x_{kl})$$

which gives

$$h_{kl} = \tfrac{1}{2}(\overline{\psi(x_{lk})} - \psi(x_{kl})). \quad \square$$

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