Probab. Th. Rel. Fields 86, 501–516 (1990)

Probability Theory and Related Fields

On quantum stochastic differential equations with unbounded coefficients

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Summary. We prove an existence, uniqueness and unitarity theorem for quantum stochastic differential equations with unbounded coefficients which satisfy an analyticity condition on a common dense invariant domain. This result, applied to the quantum harmonic oscillator, gives a rigorous meaning to a large class of stochastic differential equations that have been considered formally in quantum probability.

1. Introduction

We consider quantum stochastic differential equations of the form

$$(1.1)\qquad \begin{cases} dU = U((W-I)d\Lambda + L\,dA^+ - L^+ W dA + (iK - \tfrac{1}{2}L^+L)\,dt) \\ U(0) = I \end{cases}$$

in which Λ, A^+, A are the number, creation, annihilation processes in the boson Fock space over $L^2(\mathbb{R}_+)$ and the coefficients W, K and L are unitary, selfadjoint and closed operators respectively in the initial space. These equations naturally arise in the study of quantum evolutions ([1, 2, 4, 9, 10, 14–16]). In [15] an existence, uniquenes and unitarity theorem is proved under the assumption that W, K and L are also bounded. This result has many applications: the dilation of norm continuous quantum dynamical semigroups [15], the construction of quantum diffusions in the sense of [9], modelling physical systems [4] and so on.

In many interesting cases ([1, 2, 4, 10, 14, 16]), however, the coefficients are not bounded (typically L, L^+ are annihilation or creation operators of a quantum harmonic oscillator) and therefore the problem arises to extend this theorem. Some results in this direction can be found in [12, 13, 16] and in [6, 7] in the classical case. In [13] Frigerio gives the outline of a construction using quantum Poisson processes under the assumptions that W is the identity operator, K is the zero operator and L has a spectral property implying an analyticity condition.

In this paper, which can be considered a generalization of [13], we prove the existence, uniqueness and unitarity theorem (cf. Theor. 3.1) for Eq. (1.1) when the unitary operator W and the unbounded operators K, L satisfy an analyticity condition on a common invariant dense domain. This assumption is easily verifiable (cf. Prop. 4.1) when K is a symmetric polynomial of second degree in a, a^+ (the creation and annihilation operators of a quantum harmonic oscillator), L a linear combination of a^+, a and W is the unitary operator $\exp(i\omega a^+ a)$.

The proof is based on a new estimate of some iterated stochastic integrals (cf. (3.6)) refining that of [15]. In spite of the long computations needed to establish this estimate the idea of the proof is very simple. First we consider the adjoint equation to (1.1) and prove the existence of a unique local solution consisting of isometries by the iteration method. The adjoint process is a local solution of (1.1) that can be easily extended using the cocycle property; then we prove uniqueness and unitarity extending a method of Hudson and Parthasarathy [15] to the unbounded coefficient case. The analyticity condition is crucial in the proof both of existence and unitarity.

In Sect. 4 we apply theorem 3.1 to the case of the quantum harmonic oscillator as initial space and we prove (cf. Prop. 4.1, Prop. 4.2) that a certain class of unbounded operators satisfying the formal unitarity conditions generates a markovian cocycle in Fock space whose associated semigroup is (in general) not invertible as in [17]. A counterexample, suggested by Frigerio, shows that the same theorem is no longer true if we consider a weaker analyticity condition. We give also an existence result (cf. Prop. 4.4) for the adjoint equation to (1.1).

After writing this work we knew that quantum stochastic differential equation (1.1) has been studied at the same time by Applebaum [3] (in the multidimensional case), Chebotarev [5] and Vincent-Smith [19]. Our result is stronger than those of [3, 19] because we use a better estimate of iterated stochastic integrals.

We wish to thank L. Accardi and A. Frigerio for useful comments and suggestions.

2. Fock space notations

Let h be a complex separable Hilbert space and $\Gamma(h)$ be the boson Fock space over h. For each $g \in h$ let $\psi(g)$ be the corresponding exponential vector, in particular $\psi(0)$ is the Fock vacuum. It is a well known fact that the family $\{\psi(f) \mid f \in h\}$ is linearly independent and total in $\Gamma(h)$. For all $f \in h$ let $W(f)$ be the unitary Weyl operator on $\Gamma(h)$ defined by its action on exponential vectors $\psi(g)$, with $g \in h$

$$W(f)\psi(g) = \exp(-\tfrac{1}{2}\|f\|^2 - \langle f, g\rangle)\psi(f+g)$$

Let $\mathscr{H}$, $\mathscr{H}_t$, $\mathscr{H}^t$ be the boson Fock space over $L^2(\mathbb{R}_+)$, $L^2(0, t)$, $L^2(t, \infty)$. We have the tensor product decomposition $\mathscr{H} = \mathscr{H}_t \otimes \mathscr{H}^t$. The Hilbert spaces $\mathscr{H}_t$ and $\mathscr{H}^t$ will be identified with the subspaces $\mathscr{H}_t \otimes \psi^t(0)$ and $\psi_t(0) \otimes \mathscr{H}^t$ of $\mathscr{H}$ where $\psi^t(0)$ and $\psi_t(0)$ denote respectively the Fock vacuum in $\mathscr{H}^t$ and $\mathscr{H}_t$. We will denote by $\mathscr{E}$ the set of exponential vectors $\psi(g)$ with $\|g\| \leqq 1$ and $\operatorname{esssup}_{t \in \mathbb{R}_+} |g(t)| \leqq 1$; $\mathscr{E}$ is total in $\mathscr{H}$.

Let $\mathscr{K}$ be a complex separable Hilbert space called the *initial space* and $\mathscr{D}$ a dense linear manifold in $\mathscr{K}$. For $u \in \mathscr{D}$ and $\psi(g) \in \mathscr{E}$ we abbreviate $u \otimes \psi(g)$ to $u\psi(g)$. Moreover all the operators defined on a factor of $\mathscr{K} \otimes \mathscr{H}$ will be identified with their canonical extension to the whole space.

A *stochastic process* is a family $X = (X(t))_{t \geqq 0}$ of operators on $\mathscr{K} \otimes \mathscr{H}$ with domain containing $\mathscr{D} \otimes \mathscr{E}$ strongly measurable on $\mathscr{D} \otimes \mathscr{E}$. Two processes are equivalent if they agree on this domain for almost all t. A stochastic process is *bounded* or *contraction* or *unitary* if the operators $X(t)$ are bounded or contraction or unitary for all $t\in\mathbb{R}_+$. A stochastic process is *adapted* if, for all $u\in\mathcal{D}$, $\psi(g)\in\mathcal{E}$, $t\in\mathbb{R}_+$

(a) $X(t)u\psi(g\chi_{(0,t)})\in\mathcal{K}\otimes\mathcal{H}_t$

(b) $X(t)u\psi(g)=(X(t)u\psi(g\chi_{(0,t)}))\otimes\psi(g\chi_{(t,\infty)})$

We will consider also processes indexed by an interval of $\mathbb{R}_+$; in this case the above definitions admit an obvious translation.

The basic martingales of Fock space stochastic calculus are the adapted processes given by

$$M^{10}(t)u\psi(f)=\frac{d}{d\varepsilon}u\psi(f+\varepsilon\chi_{(0,t)})|_{\varepsilon=0}$$

$$M^{11}(t)u\psi(f)=-i\frac{d}{d\varepsilon}u\psi(e^{i\varepsilon\chi_{(0,t)}}f)|_{\varepsilon=0}$$

$$M^{01}(t)u\psi(f)=\int_0^t f(s)\,ds\,u\psi(f)$$

They are called the creation, number and annihilation process. M^{00} denotes the identity function on $\mathbb{R}_+$. For all adapted process $F=(F(t))_{t\geqq 0}$ such that

$$\int_0^t \|F(s)u\psi(f)\|^2\,ds<\infty \tag{2.1}$$

for all $t\in\mathbb{R}_+$, $u\in\mathcal{D}$, $\psi(f)\in\mathcal{E}$ one can form the stochastic integral of F with respect to the fundamental martingales. Let $F_{\alpha\beta}$ $\alpha,\beta\in\{0,1\}$ be four processes satisfying (2.1), then, for all $u,v\in\mathcal{D}$ and all $f,g\in L^2(\mathbb{R}_+)$ (cf. [15, 18])

$$\left\langle v\psi(g),\int_0^t F_{\alpha\beta}(s)\,dM^{\alpha\beta}(s)u\psi(f)\right\rangle=\int_0^t k^{\alpha\beta}(s)\langle v\psi(g),F_{\alpha\beta}(s)u\psi(f)\rangle\,ds \tag{2.2}$$

where $k^{\alpha\beta}=f_\beta\bar{g}_\alpha$ with the understanding $f_1=f$, $g_1=g$, $f_0=g_0=1$. The Einstein summation convention is used. The Itô table (cf. [15, 18]) can be written in the form

$$dM^{\alpha\beta}\,dM^{\gamma\delta}=\begin{cases}dM^{\alpha\delta} & \text{if } \beta=\gamma=1\\ 0 & \text{elsewhere}\end{cases} \tag{2.3}$$

For all $u\in\mathcal{D}$, $\psi(f)\in\mathcal{E}$ and $t\in\mathbb{R}_+$ we have the inequalities (cf. [18])

$$\left\|\int_0^t F(s)\,dM^{\alpha\beta}(s)u\psi(f)\right\|\leqq\int_0^t \|F(s)u\psi(f)\|\,ds \qquad \text{if } \alpha=0 \tag{2.4}$$

$$\left\|\int_0^t F(s)\,dM^{\alpha\beta}(s)u\psi(f)\right\|\leqq 3\left(\int_0^t \|F(s)u\psi(f)\|^2\,ds\right)^{1/2} \qquad \text{if } \alpha=1 \tag{2.5}$$

Let X_0, Y_0, $L_{\alpha\beta}(\alpha, \beta \in \{0, 1\})$ be operators on $\mathscr{H}$ with domain containing $\mathscr{D}$. An adapted process X (resp. Y) is a *solution* of the *right* (resp. *left*) stochastic differential equation

$$dX(t) = X(t) L_{\alpha\beta} \, dM^{\alpha\beta}(t) \tag{2.6}$$

resp.

$$dY(t) = L_{\alpha\beta} Y(t) \, dM^{\alpha\beta}(t) \tag{2.7}$$

with initial condition X_0 (resp. Y_0) if, for all $t \in \mathbb{R}_+$, $\alpha, \beta \in \{0, 1\}$:

(i) $L_{\alpha\beta}(\mathscr{D} \otimes \mathscr{E}) \subseteq D(X(t))$ (resp. $Y(t)(\mathscr{D} \otimes \mathscr{E}) \subseteq D(L_{\alpha\beta})$), the map $s \to X(s) L_{\alpha\beta}$ (resp. $s \to L_{\alpha\beta} Y(s)$) is strongly measurable on $\mathscr{D} \otimes \mathscr{E}$ and, for all $u \in \mathscr{D}$, $\psi(f) \in \mathscr{E}$

$$\int_0^t \|X(s) L_{\alpha\beta} u \psi(f)\|^2 \, ds < \infty$$

$$\left(\text{resp. } \int_0^t \|L_{\alpha\beta} Y(s) u \psi(f)\|^2 \, ds < \infty\right)$$

(ii)

$$X(t) = X_0 + \int_0^t X(s) L_{\alpha\beta} \, dM^{\alpha\beta}(s)$$

$$\left(\text{resp. } Y(t) = Y_0 + \int_0^t L_{\alpha\beta} Y(s) \, dM^{\alpha\beta}(s)\right).$$

A process X (resp. Y) is a *local* solution of the right (resp. left) stochastic differential equation (2.6) (resp. (2.7)) if there exists a $T > 0$ such that these conditions hold for all $t \in [0, T]$ and all $\alpha, \beta \in \{0, 1\}$.

3. The existence, uniqueness and unitarity theorem

In this section we will prove our theorem on stochastic differential equations with unbounded coefficients. Since we are interested in unitary solutions we will suppose that the conditions that are necessary and sufficient in the bounded case are satisfied on the domain $\mathscr{D}$.

Let W, K, L, L^+ be four operators in $\mathscr{H}$ satisfying the following conditions:

(i) W is unitary, K is selfadjoint, L is closed and L^+ is the adjoint of L,

(ii) $\mathscr{D}$ is an invariant domain for the operators W, K, L, L^+.

Let us consider the operators (defined on $\mathscr{D}$) $L_{11} = W - I$, $L_{10} = L$, $L_{01} = -L^+ W$, $L_{00} = iK - \frac{1}{2} L^+ L$ and denote $L^+_{\alpha\beta}$ the adjoint of $L_{\alpha\beta}(\alpha, \beta \in \{0, 1\})$. The operators $L^+_{\alpha\beta}$ are closed ([20] Theor. 2, p. 196). It is easy to see that $L^+_{01} = -W^+ L$.

Theorem 3.1. *Suppose that, for all $u \in \mathscr{D}$, $n \in \mathbb{N}$, there exists a positive constant $c(n, u)$ such that:*

(a) *for some* $\rho>0$ (*independent of* u)

$$\sum_{n=0}^{\infty} c(n,u)\rho^n<\infty$$

(b) *for all* $n\geqq 1$ *and all* $\alpha_j, \beta_j\in\{0,1\}$, $1\leqq j\leqq n$, *with* $\alpha_j=0$ *for h indices j*

$$\|L_{\alpha_n\beta_n}\dots L_{\alpha_1\beta_1}u\|\leqq c(n,u)\sqrt{(n+h)!} \tag{3.1}$$

$$\|L^+_{\beta_n\alpha_n}\dots L^+_{\beta_1\alpha_1}u\|\leqq c(n,u)\sqrt{(n+h)!}$$

and the second inequality, with $L^+_{\beta_n\alpha_n}$ *replaced by K, holds for all* α_j, $\beta_j\in\{0,1\}$, $1\leqq j\leqq n-1$, *with* $\alpha_j=0$ *for* $h-1$ *indices j.*

Then there exists a unique unitary process $(U(t))_{t\geqq 0}$ *that is a solution of the right stochastic differential equation* (2.6) *with initial condition I.*

In Sect. 4 we show that conditions (a), (b) are satisfied by the unbounded operators W, K, L appearing in many interesting applications ([1, 2, 4, 10, 14]). Here we remark that, as a consequence of condition (b), $\mathscr{D}$ is a set of analytic vectors for the operators $\pm iK-\frac{1}{2}L^+L$, L^+W and L^+. Moreover, for all $n\in\mathbb{N}$, we have $\|L^nu\|\leqq c(n,u)\sqrt{n!}$ so that the assumption on L is stronger than analyticity on $\mathscr{D}$. When $W=I$, $K=0$ then L^+L is a selfadjoint operator ([20] Theor. 2, p. 200) and condition (3.1) is also closely related to the spectral properties of L^+L as shown in [13] (Theor. 4.1).

Remark. We can suppose also that, for all $u\in\mathscr{D}$, the sequence $(c(n,u))_{n\geqq 0}$ is increasing. In fact, for all sequence $(a_n)_{n\geqq 0}$ of nonnegative real numbers with $\limsup_{n\to\infty}\sqrt[n]{a_n}=L>0$ denoting $a'_n=\max_{0\leqq h\leqq n} a_h$ it is easy to show that $\limsup_{n\to\infty}\sqrt[n]{a'_n}=\max\{1,L\}$. Therefore, if the convergence radius of the series $\sum_{n=0}^{\infty} a_n\rho^n$ is nonzero, then is nonzero also the convergence radius of the series $\sum_{n=0}^{\infty} a'_n\rho^n$. We will use this fact in the proof of the isometricity of the solution (Lemma 3.4).

We will divide the proof of Theorem (3.1) into several steps.

Proposition 3.2. *Let* ρ *be as in Theorem* (3.1) *and* $T=\min\{1,\frac{1}{2}(\rho/48)^2\}$. *There exists a unique local solution* $V=(V(t))_{t\in[0,T]}$ *of the left equation*

$$dV(t)=L^+_{\beta\alpha}V(t)\,dM^{\alpha\beta}(t) \tag{3.2}$$

with initial condition I on the interval $[0,T]$ *such that* $V(t)(\mathscr{D}\otimes\mathscr{E})\subseteq D(K)\cap D(L^+L)$ *for all* $t\in[0,T]$. *Moreover,* $V(t)$ *is an isometry for all* $t\in[0,T]$.

Proof. Using the Itô formula (2.3) it is easy to check that, for all local solution $(Y(t))_{t\in[0,x]}$ of (3.2) on the interval $[0,x]$ with $Y(t)(\mathscr{D}\otimes E)\subseteq D(K)\cap D(L^+L)$, we have

$$\langle Y(t)v\psi(g), Y(t)u\psi(f)\rangle=\langle Y_0v\psi(g), Y_0u\psi(f)\rangle$$

for all $u, v \in \mathscr{D}$, $\psi(g)$, $\psi(f) \in \mathscr{E}$. Then, if Y_0 is zero (resp. an isometry), $Y(t)$ will be zero (resp. an isometry) for all $t \in [0, x]$.

We construct now a local solution by iteration. Let

$$V^{(0)}(t) = I, \qquad V^{(n+1)}(t) = \int_0^t L^+_{\beta\alpha} V^{(n)}(s)\, dM^{\alpha\beta}(s).$$

It is easy to see that the sequence is well defined and, for all $n \in \mathbb{N}$

$$V^{(n)}(t) = L^+_{\beta_n\alpha_n} \ldots L^+_{\beta_1\alpha_1} \int_0^t dM^{\alpha_n\beta_n}(t_n) \ldots \int_0^{t_2} dM^{\alpha_1\beta_1}(t_1) \tag{3.3}$$

on the domain $\mathscr{D} \otimes \mathscr{E}$. For all $\alpha, \beta \in \{0, 1\}$, $V^{(n)}(\mathscr{D} \otimes \mathscr{E}) \subseteq D(L^+_{\beta\alpha}) \cap D(K)$ and, denoting by $I^{\alpha_n\beta_n, \ldots, \alpha_1\beta_1}(t)$ the integral in the right-hand side of (3.2) we can write, for all $u \in \mathscr{D}$ and $\psi(f) \in \mathscr{E}$,

$$V^{(n)}(t) u\, \psi(f) = (L^+_{\beta_n\alpha_n} \ldots L^+_{\beta_1\alpha_1} u) \otimes (I^{\alpha_n\beta_n, \ldots, \alpha_1\beta_1}(t)\, \psi(f)). \tag{3.4}$$

We will prove that the series

$$\sum_{n=0}^{\infty} V^{(n)}(t) u\, \psi(f), \qquad \sum_{n=0}^{\infty} L^+_{\beta\alpha} V^{(n)}(t) u\, \psi(f) \tag{3.5}$$

$$\sum_{n=0}^{\infty} K V^{(n)}(t) u\, \psi(f), \qquad \sum_{n=0}^{\infty} L^+ L V^{(n)}(t) u\, \psi(f)$$

converge on $\mathscr{D} \otimes \mathscr{E}$ uniformly in the interval $[0, T]$ and the conclusion will follow by a standard argument. To this end we have to prove an estimate of (3.4). For all integer h with $0 \leqq h \leqq n$ let $\mathscr{A}_{n,h} = \{(\alpha_n\beta_n, \ldots, \alpha_1\beta_1) \mid \alpha_j = 0$ for h indices $j\}$, using (3.1) we have then

$$\begin{aligned} &\| V^{(n)}(t) u\, \psi(f) \| \\ &\leqq \sum_{h=0}^{n} c(n, u) \sqrt{(n+h)!} \sum_{\alpha_n\beta_n, \ldots, \alpha_1\beta_1 \in \mathscr{A}_{n,h}} \| I^{\alpha_n\beta_n, \ldots, \alpha_1\beta_1}(t)\, \psi(f) \| \end{aligned} \tag{3.6}$$

The following estimate is crucial, however we will defer the proof since it is rather technical. For all $t \in [0, 1]$

$$\sum_{\alpha_n\beta_n, \ldots, \alpha_1\beta_1 \in \mathscr{A}_{n,h}} \| I^{\alpha_n\beta_n, \ldots, \alpha_1\beta_1}(t)\, \psi(f) \| \leqq \frac{(24\sqrt{t})^n}{\sqrt{n!}\sqrt{h!}} \| \psi(f) \| \tag{3.7}$$

Using (3.7) we majorize the left-hand side of (3.6) by

$$\| \psi(f) \| \sum_{h=0}^{n} c(n, u) \left(\binom{n+h}{h} \right)^{1/2} (24)^n t^{n/2}$$

and then, due to the elementary inequalities

$$\binom{n+h}{h} \leqq \binom{2n}{n} \leqq 2^{2n}$$

we have

$$\|V^{(n)}(t)u\psi(f)\| \leqq \|\psi(f)\|(n+1)\,c(n,u)(48)^n t^{n/2}$$

In the same way we obtain

$$\|L^+_{\beta\alpha} V^{(n)}(t)u\psi(f)\| \leqq 2\,\|\psi(f)\|(n+1)^2 c(n+1,u)(48)^n t^{n/2}$$

$$\|K V^{(n)}(t)u\psi(f)\| \leqq 2\,\|\psi(f)\|(n+1)^2 c(n+1,u)(48)^n t^{n/2}$$

$$\|L^+ L V^{(n)}(t)u\psi(f)\| \leqq 2\,\|\psi(f)\|(n+1)^2 c(n+1,u)(48)^n t^{n/2}$$

It follows then from assumption (a) of (3.1) that the series (3.5) are convergent on $\mathscr{D}\otimes\mathscr{E}$ uniformly in the interval $[0, T]$. For all $t\in[0, T]$, let $V(t)$ be the sum of the first one. Since the operators $L^+_{\beta\alpha}$, K, L^+L are closed, $V(t)(\mathscr{D}\otimes\mathscr{E})\subseteq D(L^+_{\beta\alpha})\cap D(K)\cap D(L^+L)$ for all $t\in[0, T]$. The process $V=(V(t))_{t\in[0,T]}$ is a local solution of the (3.2) with the initial condition I. □

Proof of (3.7). For all α, $\beta\in\{0, 1\}$ and all positive continuous function x let

$$J^\alpha x(t) = \begin{cases} \int_0^t |x(s)|\,ds & \text{if } \alpha=0 \\ \left(\int_0^t |x(s)|^2\,ds\right)^{1/2} & \text{if } \alpha=1 \end{cases}$$

To prove (3.7) we majorize the stochastic integrals with respect to dM^{11} and dM^{10} (resp. dM^{01} and dM^{00}) using (2.5) (resp. (2.4)) and we obtain

$$\|I^{\alpha_n\beta_n,\ldots,\alpha_1\beta_1}(t)\,\psi(f)\| \leqq 3\,J^{\alpha_n}\,\|I^{\alpha_{n-1}\beta_{n-1},\ldots,\alpha_1\beta_1}(\cdot)\,\psi(f)\|\,(t).$$

From this it follows, iterating the estimates and considering 1 as a constant function

$$\|I^{\alpha_n\beta_n,\ldots,\alpha_1\beta_1}(t)\,\psi(f)\| \leqq \|\psi(f)\|\,3^n J^{\alpha_n} J^{\alpha_{n-1}}\ldots J^{\alpha_1} 1(t) \tag{3.8}$$

When x is the function $s\to s^p (p>0)$ it is easy to see that we have

$$J^\alpha x(t) \leqq \frac{2\,t^{p+\frac{2-\alpha}{2}}}{(2p+2)^{\frac{2-\alpha}{2}}},$$

and then, by induction, since we suppose $t\leqq 1$,

$$J^{\alpha_n}\ldots J^{\alpha_1} 1(t) \leqq 2^n t^{n-\frac{1}{2}(\alpha_1+\ldots+\alpha_n)}(2^{2-\alpha_2} 3^{2-\alpha_3}\ldots n^{2-\alpha_n})^{-1/2}$$

$$\leqq \frac{2^n t^{n/2}}{\sqrt{n!}}(2^{1-\alpha_2} 3^{1-\alpha_3}\ldots n^{1-\alpha_n})^{-1/2}.$$

From this and (3.8) we obtain

$$\sum_{\alpha_n\beta_n,\dots,\alpha_1\beta_1\in\mathscr{A}_{n,h}} \|I^{\alpha_n\beta_n,\dots,\alpha_1\beta_1}(t)\,\psi(f)\|$$
$$\leqq \|\psi(f)\|\,\frac{6^n t^{n/2}}{\sqrt{n!}} \sum_{\alpha_n\beta_n,\dots,\alpha_1\beta_1\in\mathscr{A}_{n,h}} (2^{1-\alpha_2}3^{1-\alpha_3}\dots n^{1-\alpha_n})^{-1/2}$$
$$= \|\psi(f)\|\,\frac{6^n 2^n t^{n/2}}{\sqrt{n!}} \sum_{0<j_1<\dots<j_h\leqq n} (j_1\dots j_h)^{-1/2}.$$

The right-hand side sum has $\binom{n}{h}$ addends and the greatest is $1/\sqrt{h!}$; then it will be less than $2^n/\sqrt{h!}$. This completes the proof of the estimate (3.7). □

Unfortunately we can not obtain the global existence by successive steps of length T because we have no estimates like (3.1) of

$$\|L^+_{\beta_n\alpha_n}\dots L^+_{\beta_1\alpha_1} V_T u\,\psi(f\chi_{(0,T)})\| \tag{3.9}$$

We will prove the global existence for the left equation (3.2) in Sect. 4 under some more assumptions. Here we prove this result for the right equation (2.6) with initial condition I. In fact the adjoint of a bounded local solution of (3.2) is a local solution of (2.6) that can be extended using the *cocycle property* as in [12]. The same can not be done for (3.2) since, due to the unboundedness of $L_{\alpha\beta}$, domain problems arise.

Let Γ_t be the second quantization of the time shift on $L^2(\mathbb{R}_+)$ given by

$$(\sigma_t f)(x)=\begin{cases} f(x-t) & \text{if } x\geqq t\\ 0 & \text{otherwise}\end{cases}$$

For all bounded operator A on $\mathscr{K}\otimes\mathscr{H}$ and all $t\in\mathbb{R}_+$, $\Gamma_t A\Gamma_t^*$ is an operator on $\mathscr{K}\otimes\mathscr{H}^t$ that will be identified with its canonical extension to $\mathscr{K}\otimes\mathscr{H}$. If A is an isometry then $\Gamma_t A\Gamma_t^*$ is an isometry. A contraction valued adapted stochastic process $U=(U(t))_{t\geqq 0}$ is a *cocycle* if, for all $s, t>0$

$$U(t+s)=U(t)(\Gamma_t U(s)\Gamma_t^*).$$

From the uniqueness part of theorem (3.1) it will follow that the solution of the right stochastic differential (2.6) equation with a contraction as initial condition is a cocycle.

Lemma 3.3. *There exists a contraction solution $U=(U(t))_{t\geqq 0}$ of the right stochastic differential equation* (2.6) *with the initial condition I.*

Proof. Let T, V be as in Proposition 3.2 and $U(t)=V^*(t)$ for all $t\in[0, T]$. Using (2.2) it is easy to see that the process $U=(U(t))_{t\in[0,T]}$ is a local solution of equation of (2.6). For all $t\in[0, T]$ let us define

$$U(t+T)=U(T)(\Gamma_T U(t)\Gamma_T^*).$$

The process $(U(t))_{t\in[0,2T]}$ is a local solution of (2.6) on $[0, 2T]$. We have in fact, for all u, $v\in\mathscr{D}$ and $\psi(g)$, $\psi(f)\in\mathscr{E}$

$$\begin{aligned}&\langle v\psi(g), U(t+T)u\psi(f)\rangle\\ &\quad=\langle U^*(T)v\psi(g), \Gamma_T U(t)\Gamma_T^* u\psi(f)\rangle=\langle U^*(T)v\psi(g), u\psi(f)\rangle\\ &\quad+\left\langle U^*(T)v\psi(g), \int_0^t \Gamma_T U(s)\Gamma_T^* L_{\alpha\beta}\,dM^{\alpha\beta}(s+T)u\psi(f)\right\rangle\\ &\quad=\langle v\psi(g), u\psi(f)\rangle+\left\langle v\psi(g), \int_0^{t+T} U(s)L_{\alpha\beta}\,dM^{\alpha\beta}(s)u\psi(f)\right\rangle\end{aligned}$$

By successive steps of length T we can extend the local solution to $\mathbb{R}_+$. □

Lemma 3.4. *The contraction $U=(U(t))_{t\geqq 0}$ constructed in lemma* (3.3) *is the unique contraction that is a solution of the right stochastic differential equation* (2.6) *with initial condition I. Moreover U is unitary.*

Proof. We prove that U is the unique contraction solution of (2.6) and that it is an isometry extending the method of Hudson and Parthasarathy [15] to the unbounded coefficient case.

Let $X=(X(t))_{t\geqq 0}$ be a contraction that is a solution of (2.6) and $\psi(g)$, $\psi(f)\in\mathscr{E}$ fixed. There exists bounded operators $K(t)$, $t\geqq 0$, on $\mathscr{K}$ such that, for all v, $u\in\mathscr{D}$

$$\langle v, K(t)u\rangle=\langle X(t)v\psi(g), X(t)u\psi(f)\rangle.$$

Using the fundamental formulas of quantum stochastic calculus on $\Gamma(L^2(\mathbb{R}_+))$ (2.2) and (2.3) we obtain the linear equation for K

$$\begin{aligned}\text{(3.10)}\qquad \langle v, K(t)u\rangle&=\langle v, K(0)u\rangle\\ &+\int_0^t \{\langle v, K(s)L_{\alpha\beta}u\rangle+\langle L_{\beta\alpha}v, K(s)u\rangle\\ &+\langle L_{1\alpha}v, K(s)L_{1\beta}u\rangle\}\,k^{\alpha\beta}(s)\,ds\end{aligned}$$

We show now that, for all bounded operator $K(0)$ on $\mathscr{K}$, if (3.10) has a solution $(K(t))_{t\geqq 0}$ with $\sup_{t\geqq 0}\|K(t)\|=S<\infty$, then it must be unique. As a consequence, when $X(0)=0$, we have $K(t)=0$ for all $t\in\mathbb{R}_+$ then $X(t)=0$ for all $t\in\mathbb{R}_+$, and the uniqueness part of lemma (3.4) follows. Moreover, when $X(0)=I$, due to the expression of the coefficients $L_{\alpha\beta}$ with respect to the operators W, K, L, $K(t)=I\langle\psi(f), \psi(g)\rangle$ for all $t\in\mathbb{R}_+$ is a solution of (3.10), then it is the unique solution and so $X(0)$ is an isometry.

For all α, $\beta\in\{0, 1\}$, $\gamma\in\{0, 1, 2\}$ let

$$\text{(3.11)}\qquad L_{\alpha\beta\gamma}=\begin{cases} I & \text{if } \gamma=0\\ L_{\alpha\beta} & \text{if } \gamma=1\\ L_{1\beta} & \text{if } \gamma=2\end{cases}\qquad \tilde{L}_{\alpha\beta\gamma}=\begin{cases} L_{\beta\alpha} & \text{if } \gamma=0\\ I & \text{if } \gamma=1\\ L_{1\alpha} & \text{if } \gamma=2\end{cases}$$

and denote, for all integers h_0, h_1, h_2 with $h_0+h_1+h_2=n$

$$\Gamma^n_{h_0,h_1,h_2}=\{(\alpha_n\beta_n\gamma_n,\dots,\alpha_1\beta_1\gamma_1)\,|\,\gamma_j=\gamma \text{ for } h_\gamma \text{ indices } j, \text{ for all } \gamma=0,1,2\}.$$

Iterating n times the equation (3.10) with the initial condition $K(0)=0$ we can show that $|\langle v, K(t)u\rangle|$ is bounded from above by $St^n/n!$ times

$$\sum_{h_0+h_1+h_2=n}\ \sum_{(\alpha_n\beta_n\gamma_n,\dots)\in\Gamma^n_{h_0,h_1,h_2}} \|\tilde{L}_{\alpha_n\beta_n\gamma_n}\dots\tilde{L}_{\alpha_1\beta_1\gamma_1}v\|\cdot\|L_{\alpha_n\beta_n\gamma_n}\dots L_{\alpha_1\beta_1\gamma_1}u\|$$

For all $\alpha_n\beta_n\gamma_n\dots\alpha_1\beta_1\gamma_1\in\Gamma^n_{h_0,h_1,h_2}$, due to (3.11), (3.1) and the remark on the assumption a) of Theorem 3.1, we have

$$\|L_{\alpha_n\beta_n\gamma_n}\dots L_{\alpha_1\beta_1\gamma_1}u\|\leqq c(n,u)\sqrt{(2h_1+h_2)!}$$
$$\|\tilde{L}_{\alpha_n\beta_n\gamma_n}\dots\tilde{L}_{\alpha_1\beta_1\gamma_1}v\|\leqq c(n,v)\sqrt{(2h_0+h_2)!}$$

From the elementary inequalities $\sqrt{(2h_1+h_2)!}\sqrt{(2h_0+h_2)!}\leqq\sqrt{(2n)!}\leqq 2^n n!$, since $\sum_{h_0+h_1+h_2=n}|\Gamma^n_{h_0,h_1h_2}|=(12)^n$, for all $v, u\in\mathcal{D}$ we obtain the inequality

$$|\langle v, K(t)u\rangle|\leqq S(24)^n c(n,u)\,c(n,v)\,t^n$$
$$\leqq S(24t)^n(c(n,u)^2+c(n,v)^2)$$

Therefore, if $\sqrt{24t}<\rho^2/2$, the convergence of the series $\sum_{n=0}^{\infty} c(n,u)^2\rho^{2n}$ (assumption (a) of Theorem 3.1) implies that the right-hand side tends to zero as n tend to infinity. Hence $\langle v, K(t)u\rangle=0$ for $t\in[0,\rho^4/96]$ and we can prove global uniqueness by successive evolution steps of length $\rho^4/96$.

This shows that U is an isometry; since U^* is an isometry by construction it follows that U is unitary. □

4. Applications to the quantum harmonic oscillator

Throughout this section the initial space will be the quantum harmonic oscillator i.e. the Fock space $\Gamma(\mathbb{C})$ with creation and annihilation operators a^+, a and number operator $N=a^+a$; a^+, a are closed, N is selfadjoint and, for all $\omega\in\mathbb{R}$, we can consider the unitary operator $\exp(i\omega N)$.

We have the commutation relations

$$aN=(N+1)a, \qquad Na^+=a^+(N+1), \tag{4.1}$$
$$a\exp(i\omega N)=e^{i\omega}\exp(i\omega N)a, \qquad \exp(i\omega N)a^+=e^{i\omega}a^+\exp(i\omega N)$$

the first two being meant on $D(N^{2/3})$ and the other on $D(N^{1/2})$. For all $x\in D(N^{1/2})$, the following equalities hold

$$\|ax\|=\|N^{1/2}x\|, \qquad \|a^+x\|=\|(N+1)^{1/2}x\| \tag{4.2}$$

Let us consider the operators

$$(4.3)\qquad W=\exp(i\omega N),\qquad L=c_+a^+ +c_-a+c_0$$
$$K=d_{++}a^{+2}+d_{+-}a^+a+d_{--}a^2+d_+a^+ +d_-a+d_0$$

where d with indices $++$, $+-$, ..., are complex constants and d_{+-}, $d_0\in\mathbb{R}$, $\bar{d}_{++}=d_{--}$, $\bar{d}_+=d_-$. It is a well known fact that W, K, L verify conditions (i), (ii) of Sect. 3. Let

$$\rho=15\max\{|c_+|,|c_-|,|c_0|,|d_{++}|,|d_{+-}|,|d_{--}|,|d_+|,|d_-|,|d_0|\}.$$

Using (4.1), (4.2) it is easy to see that, for all $m\in\mathbb{N}$ and all $\alpha_n\beta_n$, ..., $\alpha_1\beta_1$ we have

$$(4.4)\qquad \|L_{\alpha_n\beta_n}\dots L_{\alpha_1\beta_1}a^{+m}\psi(0)\|\leqq\rho^n\|(N+1)^{1/2}\dots(N+n+h)^{1/2}a^{+m}\psi(0)\|$$

where h is the number of indices $\alpha_k\beta_k$ equal to 00. The right-hand side can be computed explicitly and is equal to $\rho^n\sqrt{(m+n+h)!}$. Using the elementary inequality $(m+p)!\leqq 2^{m+p}m!\,p!$ (for all m, $p\in\mathbb{N}$) we can majorize (4.4) by

$$(2^{m/2}\sqrt{m!})\,2^n\sqrt{(n+h)!}.$$

Therefore conditions (a), (b) of Theorem (3.1) are satisfied with $\mathcal{D}$ equal to the linear manifold generated by the set $\{a^{+m}\psi(0)\,|\,m\in\mathbb{N}\}$.

Thus we obtain the proposition

Proposition 4.1. *Let W, L, K be as in* (4.3). *Then there exists a unique unitary solution of the quantum stochastic differential equation*

$$\begin{cases} dU(t)=U(t)((W-I)\,dA(t)+L\,dA^+(t)-L^+W\,dA(t)+(-\frac{1}{2}L^+L+iK)\,dt)\\ U(0)=I\end{cases}$$

In particular $(iK-\frac{1}{2}L^+L,\,L,\,-L^+W,\,W)$ *are the generators, in the sense of* [1, 17], *of a unitary markovian cocycle in the Fock space* $\Gamma(L^2(\mathbb{R}^+))$.

As a corollary we obtain a proof of existence, uniqueness and unitarity for the stochastic differential equations

$$dU(t)=U(t)\left(\rho a^+\,dA(t)-\rho a\,dA^+(t)-\frac{\rho^2}{2}a^+a\,dt\right)$$

$$dU(t)=U(t)\left(\rho a\,dA(t)-\rho a^+\,dA^+(t)-\frac{\rho^2}{2}aa^+\,dt\right),$$

with ρ positive constant and initial condition I, defining respectively the quantum Ornstein-Uhlenbeck and anti-Ornstein-Uhlenbeck process [2, 16].

The existence result for these equations can not be obtained by the iteration method using the estimates of iterated integrals in [3, 15, 19]. In fact, let us consider, for example, the former. For all $n \in \mathbb{N}$ we have $\|(a^+ a)^n a^{+n} \psi(0)\| = n^n (n!)^{1/2}$ so that, with the same notation of [15], $M_{2n}(\psi(0)) \geqq 2^{-n} \rho^{3n} (n!)^{1/2} n^n$ and the series

$$\sum_{n=0}^{\infty} \frac{n^n (n!)^{1/2}}{((2n)!)^{1/2}} 2^{-n} \rho^{3n} x^n$$

is not convergent for all $x > 0$.

Here is another example where Theorem (3.1) applies. It shows that the estimates (3.1) might hold when the coefficients of the stochastic differential equation are polynomials in a^+, a of arbitrary degree, satisfying the formal unitarity conditions in Sect. 3. However, because of the commutation and selfadjointness of the coefficients, it can be considered as a classical one.

Proposition 4.2. *Let p, q be two positive real numbers. There exists a unique unitary solution of the quantum stochastic differential equation*

$$\begin{cases} dU(t) = U(t)(N^p dA(t) - N^p dA^+(t) + (-\frac{1}{2} N^{2p} + i N^q) dt) \\ U(0) = I \end{cases}$$

Proof. Suppose $q \leqq 2p$ for simplicity; the other case can be treated in the same way as it will be clear from our discussion.

Let $\mathscr{D}$ be the linear manifold generated by $\{a^{+m} \psi(0) \mid m \in N\}$. For all $n \in N$ and all indices $\alpha_n \beta_n, \ldots, \alpha_1 \beta_1$, we have

$$\| L_{\alpha_n \beta_n} \ldots L_{\alpha_1 \beta_1} a^{+m} \psi(0) \| \leqq \sqrt{m!}\, 2^n m^{2pn}.$$

The conditions (a), (b) of theorem (3.1) are fulfilled. Moreover we can consider constants $c(n, u)$ such that the series $\sum_{n=0}^{\infty} c(n, u) \rho^n$ converges for all $\rho > 0$. Hence, from proposition 3.2, we can deduce also the global existence for the left equation. □

Other examples of quantum stochastic differential equations with unbounded coefficients satisfying the assumption (3.1) (and the formal unitarity conditions) can be found in [10].

Theorem (3.1) can not hold, in general, when L is a polynomial in the operators a, a^+ of degree greater than 1 as shows the following counterexample suggested by Frigerio and inspired by example (3.3) in the paper by Davies [8].

Suppose that the quantum stochastic differential equation

$$dU(t) = U(t)(a^2 dA(t) - a^{+2} dA^+(t) - \tfrac{1}{2} a^2 a^{+2} dt) \tag{4.5}$$

with initial condition I has a solution. We will show that $V = U^*$ can not be an isometry. Let $(e_n)_{n \geqq 0}$ be the canonical of $\Gamma(\mathbb{C})$ ($e_n = (n!)^{-1/2} a^{+n} \psi(0)$ for all $n \in \mathbb{N}$) and Φ be the Fock vacuum in $\Gamma(L^2(\mathbb{R}_+))$. For all $n \in \mathbb{N}$ let Π_n be the orthogonal projection onto the subspace $e_n \otimes \Gamma(L^2(\mathbb{R}_+))$. Due to the commutation relations $\Pi_n a^+ = a^+ \Pi_{n-1}$, $\Pi_n a = a \Pi_{n+1}$ (with the convention $\Pi_m = 0$ if $m < 0$), the processes $\Pi_n V$ satisfy the recursion relations

$$\Pi_n V(t)=\int_0^t a^{+2}\Pi_{n-2}V(s)\,dA^+(s)$$
$$-\int_0^t \left(a^2\Pi_{n+2}V(s)\,dA(s)+\tfrac{1}{2}a^{+2}a^2\Pi_n V(s)\,ds\right)$$

Hence, for all $t\in\mathbb{R}_+$ and $n\in\mathbb{N}$, we have

$$\|\Pi_n V(t)e_0\Phi\|^2=\delta_{n,0}+\int_0^t \langle a^{+2}\Pi_{n-2}V(s)e_0\Phi, a^{+2}\Pi_{n-2}V(s)e_0\Phi\rangle\,ds$$
$$-\Re e\int_0^t \langle \Pi_n V(s)e_0\Phi, a^2a^{+2}\Pi_n V(s)e_0\Phi\rangle\,ds$$

Using the canonical commutation relations we can write a^2a^{+2} as $(aa^+ +1)(aa^+ +2)$ and, since the e_m are eigenvectors of the number operator aa^+, we obtain

$$\|\Pi_n V(t)e_0\Phi\|^2=\delta_{n,0}+n(n-1)\int_0^t \|\Pi_{n-2}V(s)e_0\Phi\|^2\,ds$$
$$-(n+1)(n+2)\int_0^t \|\Pi_n V(s)e_0\Phi\|^2\,ds$$

From this we deduce that, if n is odd, $\|\Pi_n V(t)e_0\Phi\|=0$ for all $t\in\mathbb{R}_+$. Denote $p_n(t)=\|\Pi_{2n}V(t)e_0\Phi\|^2$. Clearly, if (4.5) has a unitary solution U, we have

$$\sum_{n=0}^{\infty} p_n(t)=\sum_{n=0}^{\infty}\|\Pi_n V(t)e_0\Phi\|^2=1 \tag{4.6}$$

However, the sequence $(p_n(t))_{n\in\mathbb{N}}$ satisfies the system of differential equations

$$\begin{cases} p_n'(t)=-(2n+2)(2n+1)\,p_n(t)+2n(2n-1)\,p_{n-1}(t) & \text{if } n>0\\ p_0'(t)=-2\,p_0(t) \end{cases}$$

with the initial conditions $p_n(0)=\delta_{n,0}$ for all $n\in\mathbb{N}$. This is a basic system of differential equations of a divergent pure birth process in classical probability (cf. [11] Chap. XVII 3, 4 pp. 448–453) and then one has $\sum_{n=0}^{\infty} p_n(t)<1$ for all $t>0$. This contradicts (4.6) and shows that (4.5) can not have isometric solutions.

In the literature quantum stochastic differential equations are usually written as right equations and up to now we have not studied the problem of global existence for the left equation (2.7), however sometimes it is useful to consider the stochastic differential of the adjoint of U. We will show that the global existence for a left equation holds when the coefficients are as in Prop. 4.1 with the restriction on $K: d_{++}=d_{--}=0$. The proof is based on some estimates of the commutators of the coefficients with polynomials in the number operator

N. This method can be used in more general cases provided that the same kind of estimates hold (for example when $L=N^{1/2}-a$ and W, K are as above).

For all $k\in\mathbb{N}$ let N_k denote the operator $(N+1)\ldots(N+k)$; we state first some easy estimates.

Proposition 4.3. *For all $k\in\mathbb{N}$ the following inequalities hold*

$$|\langle x,[N_k,a]x\rangle|\leqq k\|N_k^{1/2}x\|^2 \qquad x\in D(N^{k+\frac{1}{2}})$$
$$|\langle Wx,[N_k,a]x\rangle|\leqq k\|N_k^{1/2}x\|^2 \qquad x\in D(N^{k+\frac{1}{2}})$$
$$|\langle Wx,[N_k,a^+]x\rangle|\leqq k\|N_k^{1/2}x\|^2 \qquad x\in D(N^{k+\frac{1}{2}})$$
$$|\langle x,a[N_k,a^+]x\rangle|\leqq k\|N_k^{1/2}x\|^2 \qquad x\in D(N^{k+1})$$
$$|\langle x,a^+[N_k,a]x\rangle|\leqq k\|N_k^{1/2}x\|^2 \qquad x\in D(N^{k+1})$$

Proof. The proof is a simple computation using the chaos decomposition of $\Gamma(\mathbb{C})$. Let us check, for example, the third inequality. For all $x=(x_n)_{n\geqq 0}\in D(N^{k+\frac{1}{2}})$ we have $\sum_{n=0}^{\infty} n^{2k+1}|x_n|^2<\infty$ and

$$\begin{aligned}
&|\langle Wx,[N_k,a^+]x\rangle|\\
&=k\left|\sum_{n=0}^{\infty}\bar{x}_{n+1}x_n\exp(-i\omega(n+1))\sqrt{n+1}(n+2)\ldots(n+k)\right|\\
&\leqq\frac{k}{2}\sum_{n=0}^{\infty}(n+2)\ldots(n+k)|x_{n+1}|^2+\frac{k}{2}\sum_{n=0}^{\infty}(n+1)\ldots(n+k)|x_n|^2\\
&\leqq k\|N_k^{1/2}x\|^2. \quad \square
\end{aligned}$$

Proposition 4.4. *Let W, K, L be as in Prop. 4.1 and suppose, moreover, that $d_{++}=d_{--}=0$. Then the quantum stochastic differential equation*

$$\text{(4.7)}\quad \begin{cases} dV(t)=((W-I)\,dA(t)+L\,dA^+(t)-L^+W\,dA(t)+(-\frac{1}{2}L^+L+iK)\,dt)\,V(t)\\ V(t)(0)=I\end{cases}$$

has a global solution V which is the adjoint of the solution U of the corresponding right equation. The solution is unique, unitary and has the following property

$$\text{(4.8)}\qquad V(t)u\psi(f)\in\bigcap_{p>0}D(N^p)\qquad \textit{for all } u\in\mathscr{D},\ \psi(f)\in\mathscr{E},\ t\in\mathbb{R}_+$$

Proof. The adjoint of each solution of (4.7) satisfies the corresponding right equation, therefore uniqueness and unitarity follow from Prop. 4.1. From Prop. 3.2 we know that there exists a local solution $V=(V(t))_{t\in[0,T]}$ of Eq. (4.7). We can prove also the following "regularity" property in $[0, T]$

$$\text{(4.9)}\qquad V(t)u\psi(f)\in\bigcap_{p>0}D(N^p)\qquad \text{for all } u\in\mathscr{D},\ \psi(f)\in\mathscr{E}$$

As a matter of fact we can show that the series $\sum_{n=0}^{\infty} N^p V^{(n)}(t)$ converge strongly on $\mathscr{D}\otimes\mathscr{E}$ uniformly in $[0, T]$ for all $p>0$. Let us consider the set

$$B=\{x\in\mathbb{R}_+ \mid \exists V \text{ solution of (4.7) with the property (4.9) on } [0, x)\}$$

We prove that $B=\mathbb{R}_+$. $B\supset[0, T]$ and so it is nonempty. Suppose that B is bounded and let b denote its supremum. For all $u\in\mathscr{D}$, $\psi(f)\in\mathscr{E}$, $k\in\mathbb{N}$ and $t\in[0, b)$ a simple computation gives the equality

$$\begin{aligned}\|N_k^{1/2} V(t)u\psi(f)\|^2 = \|N_k^{1/2} u\psi(f)\|^2 & \\ +2\,\Re e \int_0^t \langle WV(s)u\psi(f), [N_k, L]\, V(s)u\psi(f)\rangle\, \bar{f}(s)\, ds & \\ +\Re e \int_0^t \langle V(s)u\psi(f), L^+ [N_k, L]\, V(s)u\psi(f)\rangle\, ds & \\ +i \int_0^t \langle V(s)u\psi(f), [N_k, K]\, V(s)u\psi(f)\rangle\, ds &\end{aligned}$$

From Prop. 4.3 we obtain the inequality

$$\|N_k^{1/2} V(t)u\psi(f)\|^2 \leqq \|N_k^{1/2} u\psi(f)\|^2 + ck \int_0^t \|N_k^{1/2} V(s)u\psi(f)\|^2\, ds$$

where $c>0$ is an easily computable constant depending only on W, K, L. Then, using Gronwall's lemma

$$\|N_k^{1/2} V(t)u\psi(f)\|^2 \leqq \exp(ckt)\, \|N_k^{1/2} u\psi(f)\|^2.$$

Thus we get estimates of expressions like (3.8) that can be used to construct a solution of (4.7) satisfying (4.9) on intervals larger than $[0, b]$, say $[0, b+\frac{1}{2}\exp(-cb)T]$ starting from $b-\frac{1}{2}\exp(-cb)T$ by an evolution step of length $\exp(-cb)T$. This shows that B can not be bounded and then $B=\mathbb{R}_+$. □

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