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Probability Theory and Related Fields

Completely Analytic Interactions with Infinite Values

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Summary. In this note we extend the notion of completely analytic interactions of Gibbs random fields that is known for finite interactions with finite range to interactions that can have infinite values, too. We formulate a set of ten conditions on such interactions in terms of analyticity properties of the partition functions, or correlation decay. The main theorem states that all these conditions are equivalent. Therefore, an interaction is called a completely analytic interaction, if it satisfies one of these conditions.

1. Introduction

In their recent papers [5, 7] Dobrushin and Shlosman introduced a natural class of interactions of Gibbs random fields. It is called the class of completely analytic interactions and it consists of interactions that possess all usual properties of interactions in the high temperature region. This class is natural because of the fact that it can be defined by very different conditions (by the description of the area, where the partition functions have no zeros, by estimates of the semiinvariants of the finite volume Gibbs distributions or by estimates of the variation distance of the finite volume Gibbs distributions with different boundary conditions). However, the main theorem states that all these conditions are equivalent.

The main point to call these interactions completely analytic is the fact that the free energy and the correlation functions of the corresponding Gibbs random field are analytic functions. The examination of analyticity properties and of decay properties of the correlation functions is one of the classical problems in statistical mechanics. Therefore, there is a wide spread literature about such properties of Gibbs random fields. The main acquisition of the papers [5–7] is the general, unifying approach to them. In these papers the reader can find a detailed survey on the literature and a discussion of the history of the problem. We will not repeat it here.

In the papers [5, 7], where completely analytic interactions were introduced, the authors restricted themselves to the following cases: 1. All interactions are of finite range. 2. The interactions take finite values, only. It is the aim of the present paper to cancel the second of these restrictions. This enables us for example to include into the theory of completely analytic interactions hard core lattice gas models (see Remark 2.5). First results concerning general state spaces and infinite range interactions are published in [4].

The interactions under consideration in this paper are defined on the lattice $\mathbb{Z}^\nu$, $\nu \geqq 1$. The state spaces S is supposed to be finite and to contain a special element 0 in the sense that the value of the interaction for some configurations must be finite, if there is one component in the configuration that equals $0 \in S$. In hard core lattice gas models the role of $0 \in S$ plays the vacuum element, that represents the absence of a particle in the corresponding lattice point. The above condition can obviously be weakened, but only for the price of essentially more complicated constructions and further other conditions. As examples we mention the properties (D^*) and (D) of configuration spaces in [10, pp. 22, 60]. Verifying these conditions, one has to check whether or not it is possible to compose a local configuration on a finite part of the lattice with any other configuration outside some possibly larger finite part in such a way that there do not occur infinite interactions for the resulting configuration. Such compositions are used in the present paper several times. They are not complicated in the case of the presence of a special element $0 \in S$ as above. On the other hand one cannot include into the class of completely analytic interactions for example in dimension one those, for which the state space of the corresponding Markov chain breaks up into several classes or subclasses (see [10, Chap. 5]).

Let us remark that for the enlargement of the theory of completely analytic interactions to infinite interactions it was necessary to change the point of view of the examination of the partition function as an analytic function of the interaction. In the previous papers [5–7] the partition function was understood as a function of the values $U(A, \omega)$ of the interaction U. In the present paper we carry over to the generalized activities $\Gamma(A, \omega) = \exp(-U(A, \omega))$ which are finite in the case when $U(A, \omega) = \infty$, too, and thus allow continuous variations. This led to a reformulation of the second group of conditions. In the present paper we use in the second group of conditions a new class of quantities that take the place of the semiinvariants in the second group of conditions in [5–7]. The new quantities are the coefficients of the Taylor expansion of the logarithm of the partition functions, where the derivatives are calculated with respect to the generalized activities. In some special cases (see [11, Chapt. 4]) such functions are called Ursell functions. By this approach we have found a new group of conditions for complete analyticity in the case of finite interactions.

In section 2 we state our main theorem together with the set of conditions, defining the class of completely analytic interactions. Then we show that hard core lattice gas models will enter this class, if the chemical potential of the particles is large enough. At the end of this section we give a scheme of the proof of the theorem, that is divided into three steps, each of them is contained in one of the following sections.

2. The Main Result

2.1. *Basic Definitions and Notations.* Let $\mathbb{Z}^\nu$, $\nu \geqq 1$ be the ν-dimensional integer lattice with points $t = (t^1, \ldots, t^\nu)$, where t^i are integers. We denote by $\mathscr{P}_f(\mathbb{Z}^\nu)$ the set of all finite subsets of $\mathbb{Z}^\nu$. One-point subsets we denote for simplicity by t, where $t \in \mathbb{Z}^\nu$ is the point contained in this set. For $V, \Lambda \subset \mathbb{Z}^\nu$ dist(V, Λ) denotes the distance between the subsets V and Λ with respect to the norm $\|t\| = \max\{|t^i| \,|\, i = 1, \ldots, \nu\}$ on $\mathbb{Z}^\nu$. For natural numbers r we denote by $\partial V = \partial_r V = \{t \in \mathbb{Z}^\nu \setminus V \,|\, \mathrm{dist}(t, V) \leq r\}$ the r-boundary of $V \subset \mathbb{Z}^\nu$. If $V \subset \mathbb{Z}^\nu$ is some subset of the lattice, then $V^c = \mathbb{Z}^\nu \setminus V$.

Let $S = \{0, 1, \ldots, |S| - 1\}$ be the finite set of states. The set of all maps $\sigma : \mathbb{Z}^\nu \to S$ is denoted by Ω. It is called the space of configurations. For $V \subset \mathbb{Z}^\nu$ the set Ω_V is defined analogously. For $V \subset W \subseteq \mathbb{Z}^\nu$ and $\sigma \in \Omega_W$ we denote by $\sigma_V = \sigma|_V$ the restriction of the configuration σ to the subset V. For $V, W \subset \mathbb{Z}^\nu$, $V \cap W = \emptyset$, $\sigma_V \in \Omega_V$, $\omega_W \in \Omega_W$ we denote by $\sigma_V \omega_W \in \Omega_{V \cup W}$ the configuration for which $(\sigma_V \omega_W)|_V = \sigma_V$ and $(\sigma_V \omega_W)|_W = \omega_W$.

Let r be a fixed natural number and $\mathfrak{N}$ be the set of all nonempty finite subsets of $\mathbb{Z}^\nu$ with diameter smaller or equal than r. Each map $U: \mathscr{P}_f(\mathbb{Z}^\nu) \times \Omega \to \mathbb{C} \cup \{\infty\}$ that fulfills the conditions (i)–(v) below is called an *interaction*.

(i) The value $U(A, \sigma)$ of U for $A \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $\sigma \in \Omega$ depends on $\sigma|_A$, only.

(ii) $U(A, \cdot) < \infty$, if $|A| = 1$, where $|A|$ denotes the cardinality of the set $A \in \mathscr{P}_f(\mathbb{Z}^\nu)$. (2.1)

(iii) $U(A, \cdot) \equiv 0$, if $A \notin \mathfrak{N}$. (2.2)

(iv) The map U is invariant with respect to simultaneous translations of both arguments.

(v) $U(A, \sigma) < \infty$, if there exists a $t \in A$ with $\sigma|_t = 0$. (2.3)

The set of all interactions is denoted by $\mathfrak{A}_r^{\mathbb{C}}$. It contains in a natural sense the set $\mathfrak{A}_r$ of interactions that take values in the set $\mathbb{R} \cup \{\infty\}$, only. Sometimes we use non-translation invariant interactions, i.e. maps that fulfill (i)–(iii) and (v), but not (iv). The set of these maps is denoted by $\tilde{\mathfrak{A}}_r^{\mathbb{C}}$. We introduce on $\tilde{\mathfrak{A}}_r^{\mathbb{C}}$ a metric R:

$$R(U_1, U_2) = \max\left(\sup_{\substack{A \in \mathfrak{N} \\ \sigma \in \Omega}} |\exp(-\mathrm{Re}\, U_1(A, \sigma)) - \exp(-\mathrm{Re}\, U_2(A, \sigma))|, \right.$$
$$\left. \sup_{\substack{A \in \mathfrak{N} \\ \sigma \in \Omega}} |\mathrm{Im}\, U_1(A, \sigma) - \mathrm{Im}\, U_2(A, \sigma)| \right), \tag{2.4}$$

where we suppose that $e^{-\infty} = 0$ and $\mathrm{Im}\, \infty = 0$. This metric R on $\tilde{\mathfrak{A}}_r^{\mathbb{C}}$ induces metrics on $\mathfrak{A}_r^{\mathbb{C}}$ and $\mathfrak{A}_r$ which are denoted by R, too.

In some situations we identify $U \in \tilde{\mathfrak{A}}_r^{\mathbb{C}}$ with the map $\Gamma : \mathscr{P}_f(\mathbb{Z}^\nu) \times \Omega \to \mathbb{C}$, with

$$\Gamma(A, \sigma) = \exp(-U(A, \sigma)) . \tag{2.5}$$

The values are called *generalized activities*.

For each subset of real interactions $\mathfrak{a} \subset \mathfrak{A}_r$, that contains the zero interaction $U^0(U^0(\cdot, \cdot) \equiv 0)$, we call its main component $\mathscr{M}(\mathfrak{a})$ the maximal open connected subset of $\mathfrak{a}$ that contains U^0. By $\mathfrak{a}_\alpha$ we denote the set of real interactions that fulfill the condition α formulated below, where α runs through the set $\{I_a, I_b, II_a, II_b, II_c, III_a, \ldots, III_e\}$.

2.2 **Main Theorem.** *The main components $\mathcal{M}(\mathcal{O}_\alpha)$ are identical for all $\alpha \in \{I_a, I_b, II_a, II_b, II_c, III_a, \ldots, III_e\}$. This general component is called the class of completely analytic interactions.*

2.3 *The Conditions.* For $U \in \tilde{\mathfrak{A}}_r^{\mathbb{C}}$, $V \in \mathcal{P}_f(\mathbb{Z}^\nu)$, $\bar{\sigma} \in \Omega$ the partition function is defined by

$$Z_V(U|\bar{\sigma}) = \sum_{\sigma_V \in \Omega_V} \exp(-H_V^U(\sigma_V|\bar{\sigma})), \tag{2.6}$$

where

$$H_V^U(\sigma_V|\bar{\sigma}) = \sum_{A \cap V \neq \emptyset} U(A, \sigma_V \bar{\sigma}_{V^c}). \tag{2.7}$$

Throughout this paper we suppose that $Z_\phi(U|\bar{\sigma}) = 1$ for $U \in \tilde{\mathfrak{A}}_r^{\mathbb{C}}$, $\bar{\sigma} \in \Omega$. Let us mention that in the case $U \in \mathfrak{A}_r$, i.e. in the case of real interactions, the condition (2.3) ensures that $Z_V(U|\bar{\sigma}) \neq 0$ for all choices of $V \in \mathcal{P}_f(\mathbb{Z}^\nu)$, $\bar{\sigma} \in \Omega$. The property of being non-zero of the partition functions for complex interactions, too, will be of importance in the first group of conditions.

Condition I_a. $U \in \mathcal{O}_{I_a}$ iff there exists $\varepsilon > 0$ such that for all $V \in \mathcal{P}_f(\mathbb{Z}^\nu)$, $\bar{\sigma} \in \Omega$ the partition functions $Z_V(\tilde{U}|\bar{\sigma})$ are nonvanishing provided

$$\tilde{U} \in \mathcal{O}_\varepsilon^T(U) = \{\tilde{U} \in \mathfrak{A}_r^{\mathbb{C}} | R(U, \tilde{U}) < \varepsilon\}. \tag{2.8}$$

Condition I_b. $U \in \mathcal{O}_{I_b}$ iff there exist $C < \infty$ and $\varepsilon > 0$ such that for all $V \in \mathcal{P}_f(\mathbb{Z}^\nu)$, $\bar{\sigma} \in \Omega$ the partition functions $Z_V(\tilde{U}|\bar{\sigma})$ are nonvanishing, provided

$$\tilde{U} \in \mathcal{O}_\varepsilon(U) = \{\tilde{U} \in \tilde{\mathfrak{A}}_r^{\mathbb{C}} | R(U, \tilde{U}) < \varepsilon\} \tag{2.9}$$

and, moreover,

$$|\ln[Z_V(\tilde{U}_1|\bar{\sigma})/Z_V(\tilde{U}_2|\bar{\sigma})]| \leqq C|(V \cup \partial V) \cap \operatorname{supp}(\tilde{U}_1 - \tilde{U}_2)| \tag{2.10}$$

for all $\tilde{U}_1, \tilde{U}_2 \in \mathcal{O}_\varepsilon(U)$, where for $\tilde{U} \in \tilde{\mathfrak{A}}_r^{\mathbb{C}}$

$$\operatorname{supp} \tilde{U} = \bigcup_{A: U(A, \cdot) \not\equiv 0} A.$$

Here and in the sequel we suppose that $\infty + a = \infty$ for $a \in \mathbb{C}$ and $\infty - \infty = 0$.

We define now the functions, called generalized Ursell functions, that replace the semiinvariants in the theory of completely analytic interactions without infinite values. With the help of them we formulate the conditions of the second group. For the definition of the generalized Ursell functions we use the representation of interactions $U \in \mathfrak{A}_r$ by the generalized activities (see (2.5)). Let V be a finite subset of $\mathbb{Z}^\nu$ and $\{A_1, \ldots, A_m\}$ be a finite collection of subsets $A_i \in \mathcal{P}_f(\mathbb{Z}^\nu)$ such that $A_i \cap V \neq \emptyset$, $i = 1, \ldots, m$. For each set $\{\psi_1, \ldots, \psi_m\}$ of functions $\psi_i: \Omega \to \mathbb{R}$ such that $\psi_i(\omega)$ depends on $\omega|_{A_i}$, only, and all $\Gamma \in \mathfrak{A}_r$ we define

$$\hat{\Gamma}_{(z_1, \ldots, z_m)}(A, \cdot) = \Gamma(A, \cdot) + \sum_{i: A_i = A} z_i \psi_i(\cdot), \tag{2.11}$$

where $z_i \in \mathbb{C}$, $i = 1, \ldots, m$. For each multiindex $K = (k_1, \ldots, k_m)$ and each $\bar{\sigma} \in \Omega$

we define *the generalized Ursell functions* by

$$[\psi_1'^{k_1}, \ldots, \psi_m'^{k_m} | \Gamma, V, \bar{\sigma}] = \frac{\partial^{|K|}}{\partial z_1^{k_1} \ldots \partial z_m^{k_m}} (\ln Z_V(\hat{\Gamma}_{(z_1, \ldots, z_m)} | \bar{\sigma})) \Big|_{\substack{z_1 = 0 \\ \vdots \\ z_m = 0}} \tag{2.12}$$

where $|K| = k_1 + \ldots + k_m$.

Condition II_a. $U \in \mathfrak{A}_{II_a}$ iff there exist $C < \infty$ and $\varepsilon > 0$ such that for all $V, \bar{\sigma}, m$, $A_1, \ldots, A_m$, $\psi_1, \ldots, \psi_m$, $K = (k_1, \ldots, k_m)$ with $|\psi_i| \leqq 1$ and $A_i \in \mathfrak{N}$, $i = 1, \ldots, m$ the generalized Ursell functions $[\psi_1'^{k_1}, \ldots, \psi_m'^{k_m} | \tilde{\Gamma}, V, \bar{\sigma}]$, that are defined by (2.12) for real $\tilde{\Gamma}$, can be extended to analytic functions on $\mathcal{O}_\varepsilon(\Gamma)$ and it holds the following estimate:

$$|[\psi_1'^{k_1}, \ldots, \psi_m'^{k_m} | \tilde{\Gamma}, V, \bar{\sigma}]| \leqq k_1! \ldots k_m! \, C^{|K|} \tag{2.13}$$

for all $\tilde{\Gamma} \in \mathcal{O}_\varepsilon(\Gamma)$.

Let us remark that the partition functions $Z_V(\Gamma | \bar{\sigma})$ depend on the values $\Gamma(A, \omega|_A)$ for $A \cap V \neq \emptyset$, only, i.e. they depend on a finite set of values, only. By being analytic we mean the usual property of functions of several variables.

Condition II_b. $U \in \mathfrak{A}_{II_b}$ iff there exist $C < \infty$ and a function $\varphi: \mathbb{Z}_+ \to \mathbb{R}_+$ with the property

$$\sum_{t \in \mathbb{Z}^\nu} \varphi(\|t\|) \|t\|^{\nu - 1} < \infty \tag{2.14}$$

such that for all $V, \bar{\sigma}, m, A_1, \ldots, A_m, \psi_1, \ldots, \psi_m, K = (k_1, \ldots, k_m)$ with $|\psi_i| \leqq 1$ and $A_i \in \mathfrak{N}$, $i = 1, \ldots, m$

$$|[\psi_1'^{k_1}, \ldots, \psi_m'^{k_m} | \Gamma, V, \bar{\sigma}]| \leqq k_1! \ldots k_m! \, C^{|K|} \sum_{G \in \mathscr{G}(A_1, \ldots, A_m)} \prod_{g \in \mathscr{R}(G)} \varphi(|g|) , \tag{2.15}$$

where $\mathscr{G}(A_1, \ldots, A_m)$ is the set of all trees G with m vertices identified with the sets $A_1, \ldots, A_m$, $\mathscr{R}(G)$ is the set of all edges $g = (A_{i_g}, A_{j_g})$ of the tree G and $|g| = \operatorname{dist}(A_{i_g}, A_{j_g})$.

Condition II_c. $U \in \mathfrak{A}_{II_c}$ iff there exist $C < \infty$ and $\alpha > 0$ such that for all $V, \bar{\sigma}, m$, $A_1, \ldots, A_m$, $\psi_1, \ldots, \psi_m$, $K = (k_1, \ldots, k_m)$ with $|\psi_i| \leqq 1$ and $A_i \in \mathfrak{N}$, $i = 1, \ldots, m$

$$|[\psi_1'^{k_1}, \ldots, \psi_m'^{k_m} | \Gamma, V, \bar{\sigma}]| \leqq k_1! \ldots k_m! \, C^{|K|} \exp(-\alpha d(A_1, \ldots, A_m)) , \tag{2.16}$$

where $d(A_1, \ldots, A_m) = \min\{|B| \, | B \subset \mathbb{Z}^\nu, \; B \cup (A_1 \cup \ldots \cup A_m) \text{ connected}\}$ and connectivity is meant in the sense of the graph $\mathbb{Z}^\nu$ with edges joining nearest neighbours.

In the next group of conditions the notion of Gibbs distributions $Q_V^U(\sigma_V | \bar{\sigma})$ in finite volumes $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$ with boundary conditions $\bar{\sigma} \in \Omega$ is used. It is defined for $U \in \mathfrak{A}_r$ and $\sigma_V \in \Omega_V$ by

$$Q_V^U(\sigma_V | \bar{\sigma}) = Z_V(U | \bar{\sigma})^{-1} \exp(-H_V^U(\sigma_V | \bar{\sigma})) , \tag{2.17}$$

where $Z_V(U | \bar{\sigma})$ and $H_V^U(\sigma_V | \bar{\sigma})$ are defined by (2.6) and (2.7) respectively. Let us remark that the Gibbs distributions in the finite volumes with boundary conditions are well-defined because for real U the condition (2.3) ensures that the partition functions do not vanish. For $\Lambda \subset V \in \mathscr{P}_f(\mathbb{Z}^\nu)$ and $\omega_\Lambda \in \Omega_\Lambda$ let

$$Q^U_{V,\Lambda}(\omega_\Lambda|\bar\sigma) = \sum_{\sigma_V \in \Omega_V : \sigma_V|_\Lambda = \omega_\Lambda} Q^U_V(\sigma_V|\bar\sigma) . \tag{2.18}$$

Condition III_a. $U \in \mathcal{O}_{III_a}$ iff for some $\delta < 1$, $\rho > 0$ and all $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $t \in \partial V$, $\bar\sigma^1, \bar\sigma^2 \in \Omega$ with $\bar\sigma^1|_s = \bar\sigma^2|_s$ if $s \neq t$

$$\mathrm{Var}(Q^U_{V,B(t,\rho,V)}(\cdot|\bar\sigma^1), Q^U_{V,B(t,\rho,V)}(\cdot|\bar\sigma^2)) \leqq \tfrac{1}{2}\delta|B(t,\rho,V)|^{-1} , \tag{2.19}$$

where $B(t,\rho,V) = \{s \in V | \rho < \|s-t\| \leqq \rho + r\}$ and by $\mathrm{Var}(\cdot,\cdot)$ is denoted the variation distance of two probability measures on the same finite measurable space, i.e. if Q_1, Q_2 are two probability measures on the finite measurable space X, then

$$\mathrm{Var}(Q_1, Q_2) = \frac{1}{2}\sum_{x \in X} |Q_1(x) - Q_2(x)| .$$

Condition III_b. $U \in \mathcal{O}_{III_b}$ iff for some decreasing function $\varphi : \mathbb{Z}_+ \to \mathbb{R}_+$ with

$$\lim_{d \to \infty} \varphi(d)\, d^{2(\nu-1)} = 0 \tag{2.20}$$

and all $V, t, \bar\sigma^1, \bar\sigma^2$ as specified in Condition III_a and all $\Lambda \subset V$

$$\mathrm{Var}(Q^U_{V,\Lambda}(\cdot|\bar\sigma^1), Q^U_{V,\Lambda}(\cdot|\bar\sigma^2)) \leqq \sum_{s \in \Lambda} \varphi(\|s-t\|) . \tag{2.21}$$

Condition III_c. $U \in \mathcal{O}_{III_c}$ iff for some $\bar K < \infty$, $\kappa > 0$ and all $V, \Lambda, t, \bar\sigma^1, \bar\sigma^2$ as specified in Condition III_b

$$\mathrm{Var}(Q^U_{V,\Lambda}(\cdot|\bar\sigma^1), Q^U_{V,\Lambda}(\cdot|\bar\sigma^2)) \leqq \bar K \exp(-\kappa\, \mathrm{dist}(t,\Lambda)) . \tag{2.22}$$

Condition III_d. $U \in \mathcal{O}_{III_d}$ iff for some $K < \infty$, $\kappa > 0$, all $V, \Lambda, t, \bar\sigma^1, \bar\sigma^2$ as specified in Condition III_b and all $\sigma_\Lambda \in \Omega_\Lambda$ such that $Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^2) \neq 0$

$$\left|\frac{Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^1)}{Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^2)} - 1\right| \leqq K \exp(-\kappa\, \mathrm{dist}(t,\Lambda)) . \tag{2.23}$$

Condition III_e. $U \in \mathcal{O}_{III_e}$ iff for some $\bar K < \infty$, $\kappa > 0$ and some large enough $d = d(\bar K, \kappa)$ the bound (2.22) holds for all $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$ with diam $V < d$ and all $\Lambda, t, \bar\sigma^1, \bar\sigma^2$ as specified in Condition III_b.

The Condition III_e is called a constructive condition. It is of the same type as Condition III_c with the main difference that the corresponding bounds have to be checked only for volumes with a diameter smaller than a given constant $d(\bar K, \kappa)$, which can be evaluated explicitly. In [6] it is proved that for each condition α, $\alpha \in \{I_a, \ldots, III_d\}$, there is a corresponding constructive condition. The same can be done in our situation, too, without new ideas. That is why we do not go into details here.

2.4. *Remark.* The Conditions I_a, I_b, $III_a, \ldots, III_e$ almost coincide with the conditions introduced in [5–7] for completely analytic interactions that take finite values, only. Therefore we refer the reader to [5–7] for a discussion of the significance of these conditions and for the history of the problem, together with the corresponding references. Moreover, the reader can find there several examples of classes of completely analytic interactions with finite values.

2.5. *Remark.* We consider the interactions corresponding to hard core lattice gas models with pair interactions as an example of a class of completely analytic interactions that take infinite values, too. Let $S = \{0, 1, \ldots, |S| - 1\}$ be the state space and $U \in \mathfrak{A}_r$ such that

$$\text{(i)} \qquad U(V, \cdot) \equiv 0 \quad \text{if} \quad V \in \mathfrak{N} \quad \text{and} \quad |V| > 2\,, \tag{2.24}$$

$$\text{(ii)} \qquad U(\{s, t\}, \sigma) = \infty \quad \text{iff} \quad \|s - t\| = 1 \quad \text{and} \quad \sigma|_s \neq 0\,, \sigma|_t \neq 0\,, \tag{2.25}$$

$$\text{(iii)} \qquad U(t, \sigma) = \mu^i \quad \text{iff} \quad \sigma|_t = i \in S\,, \quad \text{where } \mu^0 = 0\,. \tag{2.26}$$

For μ^i, $i = 1, \ldots, |S| - 1$ large enough the following estimate holds obviously:

$$\sum_{s \in \mathbb{Z}^\nu, s \neq 0} \quad \sup_{\substack{\bar\sigma^1, \bar\sigma^2 \in \Omega \\ \bar\sigma^1|_t = \bar\sigma^2|_{t'}\, t \neq s}} \operatorname{Var}(Q_0^U(\cdot|\bar\sigma^1), Q_0^U(\cdot|\bar\sigma^2)) < 1\,. \tag{2.27}$$

In [2, Theorem 5] estimates of the variation distance of the Gibbs distributions in finite volumes were derived from (2.27). They show in the case of interactions with finite range that the corresponding interaction fulfills Condition III_c. It follows from this result that the class of interactions defined by (2.24)–(2.26) is a class of completely analytic interactions. Because of the implication $U \in \mathfrak{A}_{III_c} \Rightarrow U \in \mathfrak{A}_{I_a}$, we find examples of completely analytic interactions not only in the case when U is a pair interaction, but when (2.24) is not fulfilled, too. For this we only have to suppose $\sup_{\sigma \in \Omega} |\Gamma(A, \sigma) - 1| < \varepsilon$ for $A \in \mathfrak{N}$, $|A| > 2$ and ε sufficiently small. For this situation the analyticity of the free energy was shown with the help of cluster expansions in [8]. Let us remark that interactions for hard core lattice gas models are interesting for the examination of the Ising antiferromagnet in the neighbourhood of the critical point, too (see [3, 9]). The condition μ^i, $i = 1, \ldots, |S| - 1$, to be large is essential, because of the well known result in [1], that shows already in the case $S = \{0, 1\}$, $\nu \geqq 2$ an example of an interaction, fulfilling (2.24)–(2.26), but having more than one limit Gibbs state. The nonuniqueness of the limit Gibbs state contradicts the condition of complete analyticity.

2.6. *Remark.* Using inequality (2.27) and the results in [2], it is easy to see that the following assertion is true. Let $U \in \mathfrak{A}_r$ be a real interaction of finite range, fulfilling (2.3) and let $\mu^i = \mu^i(U) \in \mathbb{R}$, $i = 1, \ldots, |S| - 1$ be sufficiently large numbers. Define $U_\mu \in \mathfrak{A}_r$ by

$$U_\mu(t, \bar\sigma) = U(t, \bar\sigma) + \sum_{i=1}^{|S|-1} \mu^i \chi_i(\bar\sigma)\,,$$

where $\chi_i(\bar\sigma) = 1$ if $\bar\sigma|_t = i \in S$ and $\chi_i(\bar\sigma) = 0$ otherwise.

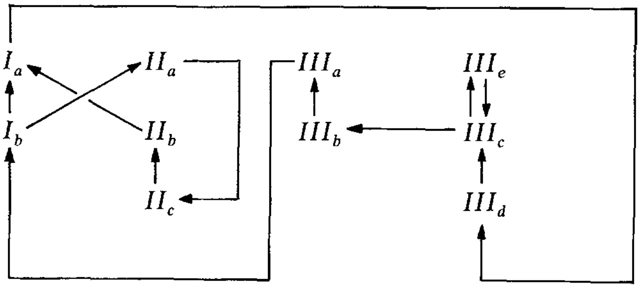
$$U_\mu(V, \sigma) = U(V, \sigma)\,,$$

if $V \in \mathfrak{N}$ and $|V| > 1$.

Then U_μ is a completely analytic interaction.

2.7. *Remark.* We want to point out that the assertion of the main theorem is not restricted to the case of hard spheres that do not allow simultaneous occupations of neighbouring lattice points. It comprises the case of more complicated hard cores, too. Thus it gives new insight into general hard core systems such as examinated in [12]. In [12] there are given conditions on the generalized activities that ensure that the partition functions do not vanish, i.e. they ensure complete analyticity by Condition I_a. Now, the main theorem states a series of other properties of such systems.

2.8. *Remark.* The proof of the Theorem 2.2 follows the scheme



An arrow $X \to Y$ means that the condition Y is fulfilled for the elements of the main component of the set of interactions fulfilling condition X. In Proposition 3.4 the implication $III_a \to I_b$ is proved, in Proposition 4.6 the implication $I_a \to III_d$, in Proposition 4.7 the implication $III_d \to III_c$ and in Proposition 5.1 the implication $II_a \to II_c$. The proofs of the implications $I_b \to I_a$, $III_c \to III_b \to III_a$, $III_c \to III_e$ are obvious. The implications $I_b \to II_a$, $II_c \to II_b \to I_a$ can be proved in the same way as in [5]. We only want to give a comment on the implication $III_e \to III_c$ that is proved in [6, Proposition 4.1]. The main point is that the Conditions III_e and III_c are not conditions directly expressed by the interactions. They are conditions on the variation distance of the Gibbs distributions in finite volumes with different boundary conditions, i.e. they are conditions on specifications. In the proof of Proposition 4.1 in [6] only measurability properties of specifications are used, but not the property that the specification is a positive one.

3. Condition III_a Implies Condition I_b

During the proof of the proposition that Condition III_a implies Condition I_b we use the following lemma.

3.1. **Lemma.** *Let $U \in \mathfrak{A}_r$ and $\rho > 0$, $\delta < 1$, $\nu \geqq 1$ be given and fixed in all the following. Define for $V \in \mathcal{P}_f(\mathbb{Z}^\nu)$ and $t \in V^c$ the sets $B(t, \rho, V)$ and $\tilde{V}$ by*

$$B(t, \rho, V) = \{s \in V \mid \rho < \|s - t\| \leqq \rho + r\} \tag{3.1}$$

$$\tilde{V} = \{s \in V \mid \|s - t\| > \rho + r\} . \tag{3.2}$$

Suppose that for all $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $t \in V^c$, $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ *such that*

$$\bar{\sigma}^1|_s = \bar{\sigma}^2|_s \quad \text{if} \quad s \neq t \tag{3.3}$$

the following estimate holds:

$$\operatorname{Var}(Q^U_{V, B(t,\rho,V)}(\cdot|\bar{\sigma}^1), Q^U_{V, B(t,\rho,V)}(\cdot|\bar{\sigma}^2)) \leqq \tfrac{1}{2}\delta |B(t,\rho,V)|^{-1}. \tag{3.4}$$

Then there exists an $\varepsilon_0 = \varepsilon_0(U, \rho, \delta) > 0$ *such that the partition functions* $Z_V(\tilde{U}|\bar{\sigma})$ *are nonvanishing for all* $\tilde{U} \in \mathcal{O}_{\varepsilon_0}(U)$, $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $\bar{\sigma} \in \Omega$. *Moreover, one can choose* $\varepsilon_0 > 0$ *so small that for all* $0 < \varepsilon < \varepsilon_0$ *and all* $\tilde{U} \in \mathcal{O}_\varepsilon(U)$ *and* $t \in V^c$

I) *there exist functions* $C_1 = C_1(\varepsilon; U)$, $\vartheta_1 = \vartheta_1(\bar{\sigma}; U, \tilde{U}, V, t)$ *and* $\kappa = \kappa(U)$ *such that* $\lim_{\varepsilon \to 0} C_1(\varepsilon) = 0$ *for the fixed* U, $|\vartheta_1(\bar{\sigma})| \leqq 1$ *for all* $\bar{\sigma} \in \Omega$ *and all values of the other variables and*

$$\frac{Z_V(\tilde{U}|\bar{\sigma}) Z_{\tilde{V}}(U|\bar{\sigma})}{Z_V(U|\bar{\sigma}) Z_{\tilde{V}}(\tilde{U}|\bar{\sigma})} = 1 + \vartheta_1 C_1 \exp(-\kappa \operatorname{dist}(t, V \cap \operatorname{supp}(U - \tilde{U}))) \tag{3.5}$$

for all $\bar{\sigma} \in \Omega$,

II) *there exist functions* $C_2 = C_2(\varepsilon; U)$, $\vartheta_2 = \vartheta_2(\bar{\sigma}^1, \bar{\sigma}^2; U, \tilde{U}, V, t)$ *and* $\kappa = \kappa(U)$ *such that* $\lim_{\varepsilon \to 0} C_2(\varepsilon) = 0$, $|\vartheta_2(\bar{\sigma}^1, \bar{\sigma}^2)| \leqq 1$ *for all* $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ *that fulfill* (3.3) *and all fixed values of the other variables and*

$$\frac{Z_V(\tilde{U}|\bar{\sigma}^1) Z_V(U|\bar{\sigma}^2)}{Z_V(U|\bar{\sigma}^1) Z_V(\tilde{U}|\bar{\sigma}^2)} = 1 + \vartheta_2 C_2 \exp(-\kappa \operatorname{dist}(t, V \cap \operatorname{supp}(U - \tilde{U}))) \tag{3.6}$$

for all $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ *that fulfill* (3.3).

For the sake of simplicity we omitted in the formulation of the lemma the specification of the dependence of C_1, C_2, ϑ_1, ϑ_2 and κ of the fixed parameters ν, r, ρ, δ.

Proof. Let $U \in \mathfrak{A}_r$ be an interaction that meets the conditions of the lemma with respect to the numbers ρ and δ. If U takes finite values only, then one can find ε_0 so small that all interactions $\tilde{U} \in \mathcal{O}_{\varepsilon_0}(U)$ have this property, too. Moreover, one can find $0 < \varepsilon_0' < \varepsilon_0$ such that for all $\tilde{U} \in \mathcal{O}_{\varepsilon_0}(U)$

$$\sup_{\substack{A \in \mathfrak{A} \\ \bar{\sigma} \in \Omega}} |U(A, \bar{\sigma}) - \tilde{U}(A, \bar{\sigma})| < \varepsilon_0 .$$

Now, one can prove the lemma in the same way as it was done in [5, Theorem 3.1], where the authors investigated finite interactions with respect to a distance defined by the left hand side of the above inequality. We want to emphasize that in this case ε_0 does not depend on U!

It remains the case, when U takes infinite values. In each ε-neighbourhood of U one can find interactions that take infinite values, too. But one can choose ε_0 so small that for all $\tilde{U} \in \mathcal{O}_{\varepsilon_0}(U)$ and all $A \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $\sigma \in \Omega$ such that $U(A, \sigma) < \infty$ $\tilde{U}(A, \sigma) < \infty$ holds, too. Hence there are two possible cases:

(i) $\tilde{U} \in \mathcal{O}_{\varepsilon_0}(U)$ and $\tilde{U}(A, \sigma) = \infty$ iff $U(A, \sigma) = \infty$.
(ii) $\tilde{U} \in \mathcal{O}_{\varepsilon_0}(U)$ and there exist $A_0 \in \mathfrak{N}$, $\sigma_0 \in \Omega$ such that $U(A_0, \sigma_0) = \infty$ but $\tilde{U}(A_0, \sigma_0) < \infty$.

In the first case the proof of our assertion follows the scheme of the proof in the case, when U takes finite values, only, if we suppose (and we do it throughout this paper) that $\infty - \infty = 0$. For the proof of the second case we introduce a real-valued intermediate interaction U^h. Let $\mathfrak{S} \subset \mathfrak{N} \times \Omega$ be the set of all pairs (A, σ) for which $U(A, \sigma) = \infty$ but $\tilde{U}(A, \sigma) < \infty$. We define

$$U^h(A, \sigma) = \operatorname{Re} \tilde{U}(A, \sigma), \quad \text{if} \quad (A, \sigma) \in \mathfrak{S} \tag{3.7}$$

and

$$U^h(A, \sigma) = U(A, \sigma) \text{ in all other cases}. \tag{3.8}$$

Obviously $U^h \in \mathcal{O}_{\varepsilon_0}(U)$. We assert that under the conditions of the lemma one can choose ε_0 so small that for all $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $t \in V^c$, all $0 < \varepsilon < \varepsilon_0$ and all $\tilde{U} \in \mathcal{O}_\varepsilon(U)$ there exist functions $\hat{C}_1 = \hat{C}_1(\varepsilon; U)$, $K_1 = K_1(\bar{\sigma}; U, U^h, V, t)$, $\hat{C}_2 = \sqrt{\varepsilon}$, $K_2 = K_2(\bar{\sigma}^1, \bar{\sigma}^2; U, U^h, V, t)$, a number $\tilde{\delta} = \tilde{\delta}(\delta, \nu, r, \rho)$ and a function $\kappa = \kappa(U)$ such that $\lim_{\varepsilon \to 0} \hat{C}_1(\varepsilon) = 0$ for fixed values of the other variables and $|K_1(\bar{\sigma})| \leqq 1$ for all $\bar{\sigma} \in \Omega$, $|K_2(\bar{\sigma}^1, \bar{\sigma}^2)| \leqq 1$ for all $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ and for all $\bar{\sigma} \in \Omega$

$$\frac{Z_V(U|\bar{\sigma}) Z_{\tilde{V}}(U^h|\bar{\sigma})}{Z_V(U^h|\bar{\sigma}) Z_{\tilde{V}}(U|\bar{\sigma})} = 1 + K_1 \hat{C}_1 \exp(-\kappa \operatorname{dist}(t, V \cap \operatorname{supp}(U - U^h))), \tag{3.9}$$

and for all $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ that fulfill (3.3)

$$\operatorname{Var}(Q^{U^h}_{V, B(t, \rho, V)}(\cdot|\bar{\sigma}^1), Q^{U^h}_{V, B(t, \rho, V)}(\cdot|\bar{\sigma}^2)) \leqq \tfrac{1}{2} \tilde{\delta} |B(t, \rho, V)|^{-1} \tag{3.10}$$

and

$$\frac{Z_V(U|\bar{\sigma}^1) Z_V(U^h|\bar{\sigma}^2)}{Z_V(U^h|\bar{\sigma}^1) Z_V(U|\bar{\sigma}^2)} = 1 + K_2 \hat{C}_2 \exp(-\kappa \operatorname{dist}(t, V \cap \operatorname{supp}(U - U^h))). \tag{3.11}$$

As before we omitted the specification of the dependence of the functions $\hat{C}_1$, K_1, K_2 and κ of ν, r, ρ, δ. Let us mention two facts that will be used in the proof several times. At first we draw the reader's attention to the fact that, if (3.11) is true, it follows by subsequent application of this formula that for $\bar{\sigma}^1, \bar{\sigma}^2 \in \Omega$ that differ at any finite set of points of the set ∂V for the given $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$

$$\frac{Z_V(U|\bar{\sigma}^1) Z_V(U^h|\bar{\sigma}^2)}{Z_V(U^h|\bar{\sigma}^1) Z_V(U|\bar{\sigma}^2)} = \prod_i (1 + K_{2,i} \hat{C}_2 \exp(-\kappa D_i)), \tag{3.12}$$

where i indicates the points $t_i \in \partial V$, where $\bar{\sigma}^1$ and $\bar{\sigma}^2$ differ from each other and $D_i = \operatorname{dist}(t_i, V \cap \operatorname{supp}(U - U^h))$. Furthermore it will be necessary to distinguish between "good" and "bad" configurations. For $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$ and $\sigma_{V^c} \in \Omega_{V^c}$ we call the configuration $\omega_V \in \Omega_V$ a good configuration and write $\omega_V \in \Omega_V^{\text{good}}(\sigma_{V^c})$, if for all $A \in \mathfrak{N}$, $A \subset V \cup \partial V$ the value $U(A, \omega_V \sigma_{V^c})$ is smaller than infinity. Otherwise the configuration $\omega_V \in \Omega_V$ is called a bad configuration and we write $\omega_V \in \Omega_V^{\text{bad}}(\sigma_{V^c})$. In the following we denote for $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$ by 0_V the configuration of Ω_V that is identical $0 \in S$.

Let $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $t \in V^c$ and $\bar{\sigma} \in \Omega$ be fixed. Then

$$\begin{aligned}
&\sum_{\omega_{V\setminus\tilde{V}} \in \Omega^{\mathrm{bad}}_{V\setminus\tilde{V}}(0_{\tilde{V}}\bar{\sigma}_{V^c})} Q^{U^h}_{V, V\setminus\tilde{V}}(\omega_{V\setminus\tilde{V}}|\bar{\sigma}) \\
&= \sum_{\omega_{V\setminus\tilde{V}} \in \Omega^{\mathrm{bad}}_{V\setminus\tilde{V}}(0_{\tilde{V}}\bar{\sigma}_{V^c})} \sum_{\omega_{\tilde{V}} \in \Omega_{\tilde{V}}} Q^{U^h}_{V,\tilde{V}}(\omega_{\tilde{V}}|\bar{\sigma}) Q^{U^h}_{V\setminus\tilde{V}}(\omega_{V\setminus\tilde{V}}|\bar{\sigma}_{\tilde{V}^c}\omega_{\tilde{V}}) \\
&\leqq \sum_{\omega_{V\setminus\tilde{V}} \in \Omega^{\mathrm{bad}}_{V\setminus\tilde{V}}(0_{\tilde{V}}\bar{\sigma}_{V^c})} \max_{\omega_{\tilde{V}} \in \Omega_{\tilde{V}}} Q^{U^h}_{V\setminus\tilde{V}}(\omega_{V\setminus\tilde{V}}|\bar{\sigma}_{\tilde{V}^c}\omega_{\tilde{V}}) \\
&= K(U, U^h, V, t, \bar{\sigma})
\end{aligned} \tag{3.13}$$

and

$$\lim_{R(U, U^h)\to 0} K(U, U^h, V, t, \bar{\sigma}) = 0 \tag{3.14}$$

uniformly in V, t, $\bar{\sigma}$. Indeed, the set $V\setminus\tilde{V}$ is contained in the bounded sphere $B(t, \rho + r) = \{s \in \mathbb{Z}^\nu \mid \|s - t\| \leqq \rho + r\}$. Hence, the values $Q^{U^h}_{V\setminus\tilde{V}}(\cdot|\bar{\sigma})$ do not depend on the restriction of $\bar{\sigma}$ to the complement of $B(t, \rho + r) \cup \partial B(t, \rho + r)$. It follows from $\omega_{V\setminus\tilde{V}} \in \Omega^{\mathrm{bad}}_{V\setminus\tilde{V}}(0_{\tilde{V}}\sigma_{V^c})$ and (2.4) that at least one factor of the product

$$\prod_{A \cap (V\setminus\tilde{V}) \neq \varnothing} \exp(-U^h(A, \omega_{\tilde{V}}\,\omega_{V\setminus\tilde{V}}\,\bar{\sigma}_{V^c})) Z^{-1}_{V\setminus\tilde{V}}(U^h|\bar{\sigma}) = Q^{U^h}_{V\setminus\tilde{V}}(\omega_{V\setminus\tilde{V}}|\omega_{\tilde{V}}\bar{\sigma}_{\tilde{V}^c})$$

is smaller or equal than $R(U, U^h)$. Furthermore, we conclude from

$$\lim_{R(U, U^h)\to 0} Z_{V\setminus\tilde{V}}(U^h|\bar{\sigma}) = Z_{V\setminus\tilde{V}}(U|\bar{\sigma})$$

and again $V\setminus\tilde{V} \subset B(t, \rho + r)$ that

$$K(U, U^h, V, t, \bar{\sigma})/R(U, U^h) < C < \infty \tag{3.15}$$

for some constant C that does not depend on V, t, $\bar{\sigma}$ and U^h, if $R(U, U^h)$ is small enough.

Now we can start with the proof of (3.9)–(3.11). We do it by induction on the cardinality of $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$. The first step is easy because $Z_\phi(\tilde{U}|\bar{\sigma}) = 1$ by definition. Suppose that (3.9)–(3.11) are valid for all $V \in P_f(\mathbb{Z}^\nu)$ with $|V| \leqq n - 1$. Let $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$ be a set with $|V| = n$ and $t \in \partial V$. A short calculation shows that

$$\frac{Z_V(U|\bar{\sigma})}{Z_V(U^h|\bar{\sigma})} = \left\langle \frac{Z_{\tilde{V}}(U|\cdot\bar{\sigma}_{V^c})}{Z_{\tilde{V}}(U^h|\cdot\bar{\sigma}_{V^c})}[1 + \varphi(\cdot\bar{\sigma}_{V^c})] \right\rangle^{U^h}_{V,\bar{\sigma}} \tag{3.16}$$

where $\langle\cdot\rangle^{U^h}_{V,\bar{\sigma}}$ denotes the expectation value with respect to the Gibbs distribution $Q^{U^h}_V(\cdot|\bar{\sigma})$ (see (2.17)) and the function φ is defined by

$$\varphi(\sigma) = \exp\left(-\sum_{\substack{A \cap V \neq \varnothing \\ A \cap \tilde{V} = \varnothing}} (U(A, \sigma) - U^h(A, \sigma))\right) - 1 \,. \tag{3.17}$$

Let us emphasize that U^h is a real-valued interaction and thus both sides of (3.16) are well defined.

Let $D = \mathrm{dist}(t, V \cap \mathrm{supp}(U - U^h))$. Then, if $D > \rho + r$, the function φ is identical zero. Hence, we get from (3.11) and (3.12)

$$\frac{Z_V(U|\bar\sigma)Z_{\tilde V}(U^h|\bar\sigma)}{Z_V(U^h|\bar\sigma)Z_{\tilde V}(U|\bar\sigma)} = \left\langle \frac{Z_{\tilde V}(U|\cdot\bar\sigma_{V^c})Z_{\tilde V}(U^h|\bar\sigma)}{Z_{\tilde V}(U^h|\cdot\bar\sigma_{V^c})Z_{\tilde V}(U|\bar\sigma)} \right\rangle_{V,\bar\sigma}^{U^h} \tag{3.18}$$

$$= \left\langle \prod_{i=1}^{|V\cap\partial\tilde V|} (1 + K_{2,i}\hat C_2 \exp(-\kappa D_i)) \right\rangle_{V,\bar\sigma}^{U^h}.$$

Using the facts that $|V \cap \partial\tilde V| \leqq |B(t,\rho,V)|$ and $D \leqq D_i + \rho + r$ for all i, it is not hard to see from (3.18) the existence of functions K_1^1 and $\hat C_1^1(\hat C_2)$ such that $|K_1^1| \leqq 1$, $\lim_{\hat C_2 \to 0} \hat C_1^1(\hat C_2) = 0$ and

$$\frac{Z_V(U|\bar\sigma)Z_{\tilde V}(U^h|\bar\sigma)}{Z_V(U^h|\bar\sigma)Z_{\tilde V}(U|\bar\sigma)} = 1 + K_1^1 \hat C_1^1 \exp(-\kappa D), \tag{3.19}$$

where we omitted to write down the corresponding dependence of K_1^1 and $\hat C_1^1(\hat C_2)$ on the parameters U, U^h, V, t, v, r, ρ, δ.

Let now $D \leqq \rho + r$. Then

$$\varphi(\omega_V \bar\sigma_{V^c}) = \begin{cases} -1, & \text{if } \omega_V|_{V\setminus\tilde V} \in \Omega^{\mathrm{bad}}_{V\setminus\tilde V}(0_{\tilde V}\bar\sigma_{V^c}) \\ 0, & \text{if } \omega_V|_{V\setminus\tilde V} \in \Omega^{\mathrm{good}}_{V\setminus\tilde V}(0_{\tilde V}\bar\sigma_{V^c}) \end{cases} \tag{3.20}$$

and, hence, we get from (3.16)

$$\frac{Z_V(U|\bar\sigma)Z_{\tilde V}(U^h|\bar\sigma)}{Z_V(U^h|\bar\sigma)Z_{\tilde V}(U|\bar\sigma)} = \left\langle \frac{Z_{\tilde V}(U|\cdot\bar\sigma_{V^c})Z_{\tilde V}(U^h|\bar\sigma)}{Z_{\tilde V}(U^h|\cdot\bar\sigma_{V^c})Z_{\tilde V}(U|\bar\sigma)} \right\rangle_{V,\bar\sigma}^{U^h} \tag{3.21}$$

$$- \sum_{\omega_{V\setminus\tilde V} \in \Omega^{\mathrm{bad}}_{V\setminus\tilde V}(0_{\tilde V}\bar\sigma_{V^c})} \frac{Z_{\tilde V}(U|\omega_{V\setminus\tilde V}0_{\tilde V}\bar\sigma_{V^c})Z_{\tilde V}(U^h|\bar\sigma)}{Z_{\tilde V}(U^h|\omega_{V\setminus\tilde V}0_{\tilde V}\bar\sigma_{V^c})Z_{\tilde V}(U|\bar\sigma)} Q^{U^h}_{V,V\setminus\tilde V}(\omega_{V\setminus\tilde V}|\bar\sigma).$$

For the first term on the right hand side of (3.21) we obtain a representation in the form of equation (3.19) in the same way as it was shown in the case $D > \rho + r$. The second term on the right hand side of (3.21) can be estimated with the help of the induction hypothesis (3.11) and (3.12)

$$\left| \sum_{\omega_{V\setminus\tilde V} \in \Omega^{\mathrm{bad}}_{V\setminus\tilde V}(0_{\tilde V}\bar\sigma_{V^c})} \frac{Z_{\tilde V}(U|\omega_{V\setminus\tilde V}0_{\tilde V}\bar\sigma_{V^c})Z_{\tilde V}(U^h|\bar\sigma)}{Z_{\tilde V}(U^h|\omega_{V\setminus\tilde V}0_{\tilde V}\bar\sigma_{V^c})Z_{\tilde V}(U|\bar\sigma)} Q^{U^h}_{V,V\setminus\tilde V}(\omega_{V\setminus\tilde V}|\bar\sigma) \right|$$

$$\leqq \prod_{i=1}^{|V\cap\partial\tilde V|} |1 + \hat C_2 \exp(-\kappa D_i)| \sum_{\omega_{V\setminus\tilde V} \in \Omega^{\mathrm{bad}}_{V\setminus\tilde V}(0_{\tilde V}\bar\sigma_{V^c})} Q^{U^h}_{V,V\setminus\tilde V}(\omega_{V\setminus\tilde V}|\bar\sigma). \tag{3.22}$$

Because of (3.13), (3.14) and $|V \cap \partial\tilde V| \leqq |B(t,\rho + r)|$ the right hand side of (3.22) tends to zero, if $R(U, U^h)$ tends to zero. Hence, we have in the case $D \leqq \rho + r$ for some $\hat C_1^2(\hat C_2)$, K_1^2, C^3

$$\frac{Z_V(U|\bar\sigma)Z_{\tilde V}(U^h|\bar\sigma)}{Z_V(U^h|\bar\sigma)Z_{\tilde V}(U|\bar\sigma)} = 1 + K_1^2 \hat C_1^2 \exp(-\kappa D) + C^3, \tag{3.23}$$

where $\lim_{R(U,U^h)\to 0} C^3 = 0$, uniformly in V and $\bar\sigma$, $|K_1^2| \leqq 1$ and $\lim_{\hat C_2 \to 0} \hat C_1^2(\hat C_2) = 0$. We get from (3.19) and (3.23) the needed equation (3.9) with new functions K_1 and $\hat C_1$ that possess the asserted properties.

We want to point out that the function $\hat C_1$ is constructed from $\hat C_2 = \sqrt{\varepsilon}$ and further functions that do not depend on V, if $R(U, U^h)$ is small enough. Hence, $\hat C_1$ does not depend on V.

In the next step we prove (3.10) for $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$ with $|V| = n$, assuming that (3.9) is true for the same V. We have to show that for any small enough $\hat\delta > 0$ one can choose $\varepsilon_0 > 0$ independently of V, $t \in \partial V$ and $\bar\sigma \in \Omega$ such that for all $t \in \partial V$, $\bar\sigma \in \Omega$ and all $\tilde U \in \mathfrak{A}_r^{\mathbb{C}}$ with $R(U, \tilde U) < \varepsilon_0$

$$\mathrm{Var}(Q^{U^h}_{V,B(t,\rho,V)}(\cdot|\bar\sigma),\ Q^U_{V,B(t,\rho,V)}(\cdot|\bar\sigma)) < \hat\delta\ . \tag{3.24}$$

For the sake of simplicity we write B instead of $B(t, \rho, V)$. Then

$$\mathrm{Var}(Q^{U^h}_{V,B}(\cdot|\bar\sigma),\ Q^U_{V,B}(\cdot|\bar\sigma)) = \frac{1}{2} \sum_{\omega_B \in \Omega_B} \left(\sum_{\omega_{V\setminus B} \in \Omega_{V\setminus B}} Q^U_V(\omega_B \omega_{V\setminus B}|\bar\sigma) \right) \times |1 - \Psi(\omega_B)|\,, \tag{3.25}$$

where

$$\Psi(\omega_B) = \frac{Z_V(U|\bar\sigma)}{Z_V(U^h|\bar\sigma)} \frac{\sum_{\omega_{V\setminus B} \in \Omega_{V\setminus B}} \exp\left(- \sum_{A \cap V \neq \varnothing} U^h(A, \omega_B \omega_{V\setminus B} \bar\sigma_{V^c}) \right)}{\sum_{\omega_{V\setminus B} \in \Omega_{V\setminus B}} \exp\left(- \sum_{A \cap V \neq \varnothing} U(A, \omega_B \omega_{V\setminus B} \bar\sigma_{V^c}) \right)}\,.$$

Because of $Q^U_V(\omega_B \omega_{V\setminus B}|\bar\sigma) = 0$, if $\omega_B \in \Omega_B^{\mathrm{bad}}(0_{V\setminus B}\bar\sigma_{V^c})$, we have to examine Ψ for $\omega_B \in \Omega_B^{\mathrm{good}}(0_{V\setminus B}\bar\sigma_{V^c})$, only. Since B has the thickness r we get with $W = V\setminus(B \cup \tilde V)$

$$\Psi(\omega_B) = \frac{Z_V(U|\bar\sigma)}{Z_V(U^h|\bar\sigma)} \frac{\sum_{\omega_W \in \Omega_W} \sum_{\omega_{\tilde V} \in \Omega_{\tilde V}} \exp\left(- \sum_{A \cap \tilde V \neq \varnothing} U^h(A, \omega_{\tilde V} \omega_W \omega_B \bar\sigma_{V^c}) \right)}{\sum_{\omega_W \in \Omega_W} \sum_{\omega_{\tilde V} \in \Omega_{\tilde V}} \exp\left(- \sum_{A \cap \tilde V \neq \varnothing} U(A, \omega_{\tilde V} \omega_W \omega_B \bar\sigma_{V^c}) \right)} \times$$

$$\times \frac{\exp\left(- \sum_{A \cap V \neq \varnothing, A \cap \tilde V = \varnothing} U^h(A, \omega_B \omega_W \bar\sigma_{(B\cup W)^c}) \right)}{\exp\left(- \sum_{A \cap V \neq \varnothing, A \cap \tilde V = \varnothing} U(A, \omega_B \omega_W \bar\sigma_{(B\cup W)^c}) \right)}$$

$$= \frac{Z_V(U|\bar\sigma) Z_{\tilde V}(U^h|\omega_B \bar\sigma_{B^c})}{Z_V(U^h|\bar\sigma) Z_{\tilde V}(U|\omega_B \sigma_{B^c})} \times$$

$$\times \left(1 + \frac{\sum_{\omega_W \in \Omega_W^{\mathrm{bad}}(\omega_B \bar\sigma_{(B\cup W)^c})} \exp\left(- \sum_{\substack{A \cap V \neq \varnothing \\ A \cap \tilde V = \varnothing}} U^h(A, \omega_B \omega_W \bar\sigma_{(B\cup W)^c}) \right)}{\sum_{\omega_W \in \Omega_W^{\mathrm{good}}(\omega_B \bar\sigma_{(B\cup W)^c})} \exp\left(- \sum_{\substack{A \cap V \neq \varnothing \\ A \cap \tilde V = \varnothing}} U(A, \omega_B \omega_W \bar\sigma_{(B\cup W)^c}) \right)} \right).$$

It follows from the induction hypothesis (3.9) that, if the distance $R(U, U^h)$ tends to zero, the first factor tends to one. The same is true for the second factor, because of $\omega_W \in \Omega_W^{\mathrm{bad}}(\omega_B \bar\sigma_{(B\cup W)^c})$ in the numerator. Since the function $\hat C_1$ in (3.9) does not depend on V and $W \subset \{s \in \mathbb{Z}^\nu \mid \|t-s\| \leq \rho\}$ the convergence is uniform in V and $\bar\sigma$. This proves (3.24) that leads together with (3.3) and (3.4) to

$$\begin{aligned}\mathrm{Var}(Q_{V,B}^{U^h}(\cdot|\bar\sigma^1), Q_{V,B}^{U^h}(\cdot|\bar\sigma^2)) &\leqq \mathrm{Var}(Q_{V,B}^{U^h}(\cdot|\bar\sigma^1), Q_{V,B}^{U}(\cdot|\bar\sigma^1)) + \\ &\quad + \mathrm{Var}(Q_{V,B}^{U}(\cdot|\bar\sigma^1), Q_{V,B}^{U}(\cdot|\bar\sigma^2)) + \\ &\quad + \mathrm{Var}(Q_{V,B}^{U}(\cdot|\bar\sigma^2), Q_{V,B}^{U^h}(\cdot|\bar\sigma^2)) \\ &\leqq \frac{1}{2}\delta|B|^{-1} + 2\hat\delta ,\end{aligned}$$

if ε_0 is sufficiently small. Now, we choose $\hat\delta$ so small that

$$\tfrac{1}{2}\delta|B|^{-1} + 2\hat\delta < \tfrac{1}{2}\tilde\delta|B|^{-1}$$

for some $\tilde\delta < 1$. Inequality (3.10) is proved.

In the last step of the proof we suppose that (3.9) and (3.10) are true for all $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $|V| \leqq n$ and that (3.11) is true for all $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $|V| \leqq n-1$. We prove that (3.11) is true for all $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $|V| = n$, too. We write for the corresponding parameters V, $\bar\sigma^1$, $\bar\sigma^2$

$$\frac{Z_V(U|\bar\sigma^1)\, Z_V(U^h|\bar\sigma^2)}{Z_V(U^h|\bar\sigma^1)\, Z_V(U|\bar\sigma^2)} =$$

$$1 + \frac{(Z_V(U|\bar\sigma^1)/Z_V(U^h|\bar\sigma^1) - Z_V(U|\bar\sigma^2)/Z_V(U^h|\bar\sigma^2))(Z_{\tilde V}(U^h|\bar\sigma^2)/Z_{\tilde V}(U|\bar\sigma^2))}{(Z_V(U|\bar\sigma^2)/Z_V(U^h|\bar\sigma^2))(Z_{\tilde V}(U^h|\bar\sigma^2)/Z_{\tilde V}(U|\bar\sigma^2))} . \quad (3.26)$$

Using (3.16) and (3.17) the numerator of the right hand side of (3.26) can be written in the form

$$\begin{aligned}&\left\langle \frac{Z_{\tilde V}(U|\cdot\bar\sigma^1_{V^c})Z_{\tilde V}(U^h|\bar\sigma^2)}{Z_{\tilde V}(U^h|\cdot\bar\sigma^1_{V^c})Z_{\tilde V}(U|\bar\sigma^2)} - 1\right\rangle_{V,\bar\sigma^1}^{U^h} - \left\langle \frac{Z_{\tilde V}(U|\cdot\bar\sigma^2_{V^c})Z_{\tilde V}(U^h|\bar\sigma^2)}{Z_{\tilde V}(U^h|\cdot\bar\sigma^2_{V^c})Z_{\tilde V}(U|\bar\sigma^2)} - 1\right\rangle_{V,\bar\sigma^2}^{U^h} \\ &+ \left\langle \frac{Z_{\tilde V}(U|\cdot\bar\sigma^1_{V^c})Z_{\tilde V}(U^h|\bar\sigma^2)}{Z_{\tilde V}(U^h|\cdot\bar\sigma^1_{V^c})Z_{\tilde V}(U|\bar\sigma^2)}\,\varphi(\cdot\,\bar\sigma^1_{V^c})\right\rangle_{V,\bar\sigma^1}^{U^h} \\ &- \left\langle \frac{Z_{\tilde V}(U|\cdot\bar\sigma^2_{V^c})Z_{\tilde V}(U^h|\bar\sigma^2)}{Z_{\tilde V}(U^h|\cdot\bar\sigma^2_{V^c})Z_{\tilde V}(U|\bar\sigma^2)}\,\varphi(\cdot\,\bar\sigma^2_{V^c})\right\rangle_{V,\bar\sigma^2}^{U^h} .\end{aligned} \quad (3.27)$$

Notice that within the $\langle\cdot\rangle$-brackets of the first term one can replace $\bar\sigma^1_{V^c}$ by $\bar\sigma^2_{V^c}$. We can use the induction hypothesis (3.11) in the form of (3.12) and get

$$\left| \left\langle \frac{Z_{\tilde V}(U|\cdot\bar\sigma^2_{V^c})Z_{\tilde V}(U^h|\bar\sigma^2)}{Z_{\tilde V}(U^h|\cdot\bar\sigma^2_{V^c})Z_{\tilde V}(U|\bar\sigma^2)} - 1\right\rangle_{V,\bar\sigma^1}^{U^h} - \left\langle \frac{Z_{\tilde V}(U|\cdot\bar\sigma^2_{V^c})Z_{\tilde V}(U^h|\bar\sigma^2)}{Z_{\tilde V}(U^h|\cdot\bar\sigma^2_{V^c})Z_{\tilde V}(U|\bar\sigma^2)} - 1\right\rangle_{V,\bar\sigma^2}^{U^h} \right|$$

$$\leqq 2\left(\prod_i (1 + \hat C_2 \exp(-\kappa \tilde D_i)) - 1\right) \mathrm{Var}(Q_{V,B}^{U^h}(\cdot|\bar\sigma^1), Q_{V,B}^{U^h}(\cdot|\bar\sigma^2)) , \quad (3.28)$$

where all $\tilde D_i \geqq \tilde D = \mathrm{dist}(V \cap \partial\tilde V,\ V \cap \mathrm{supp}(U - U^h))$. The number of points t_i

where $(\cdot \bar{\sigma}^1_{\tilde{V}^c})|_{V \cap \partial \tilde{V}}$ and $\bar{\sigma}^1|_{V \cap \partial \tilde{V}}$ differ from each other is bounded from above by $|B(t, \rho, V)|$. Hence, one can choose $\kappa(\tilde{\delta})$ and ε_0 so small that for $D < \tilde{D} + \rho + r$ and $R(U, U^h) < \varepsilon_0$

$$\prod_{i=1}^{|B|} (1 + \hat{C}_2 \exp(-\kappa \tilde{D}_i)) - 1 \leqq |B| \cdot \hat{C}_2 \exp(-\kappa D) \tilde{\delta}^{-1/2}$$

Then it follows from (3.10) that the right hand side of (3.28) does not exceed

$$\hat{C}_2 \exp(-\kappa D) \tilde{\delta}^{1/2} . \tag{3.29}$$

In the case $D > \rho + r$ the function φ is identical zero and we still have to examine the denominator of (3.26). But it follows straightforward from (3.9) that for sufficiently small ε_0

$$\left| \frac{Z_V(U|\bar{\sigma}^2) Z_{\tilde{V}}(U^h|\bar{\sigma}^2)}{Z_V(U^h|\bar{\sigma}^2) Z_{\tilde{V}}(U|\bar{\sigma}^2)} \right| \geqq \tilde{\delta}^{1/4} . \tag{3.30}$$

Now we get (3.11) from (3.30), (3.29), (3.27) and (3.26). In the case $D \leqq \rho + r$ we use (3.11) in the form (3.12) and (3.20) for the set $\tilde{V} \subset V$ and for $\bar{\sigma}^j, j = 1, 2$, to estimate the last two terms in (3.27).

$$\begin{aligned}
&\left| \frac{Z_{\tilde{V}}(U^h|\bar{\sigma}^2)}{Z_{\tilde{V}}(U|\bar{\sigma}^2)} \sum_{\omega_V \in \Omega_V} \frac{Z_{\tilde{V}}(U|\omega_V \bar{\sigma}^j_{V^c})}{Z_{\tilde{V}}(U^h|\omega_V \bar{\sigma}^j_{V^c})} \varphi(\omega_V \bar{\sigma}^j_{V^c}) Q_V^{U^h}(\omega_V|\bar{\sigma}^j) \right| \\
&\leqq \sum_{\omega_{V \setminus \tilde{V}} \in \Omega^{\mathrm{bad}}_{V \setminus \tilde{V}}(0_{\tilde{V}} \bar{\sigma}^j_{V^c})} \sum_{\omega_{\tilde{V}} \in \Omega_{\tilde{V}}} \left| \frac{Z_{\tilde{V}}(U^h|\bar{\sigma}^2) Z_{\tilde{V}}(U|\omega_{\tilde{V}} \omega_{V \setminus \tilde{V}} \bar{\sigma}^j_{V^c})}{Z_{\tilde{V}}(U|\bar{\sigma}^2) Z_{\tilde{V}}(U^h|\omega_{\tilde{V}} \omega_{V \setminus \tilde{V}} \bar{\sigma}^j_{V^c})} \right| Q_V^{U^h}(\omega_{\tilde{V}} \omega_{V \setminus \tilde{V}}|\bar{\sigma}^j) \\
&\leqq (1 + \hat{C}_2)^{|B|} \sum_{\omega_{V \setminus \tilde{V}} \in \Omega^{\mathrm{bad}}_{V \setminus \tilde{V}}(0_{\tilde{V}} \bar{\sigma}^j_{V^c})} Q^{U^h}_{V, V \setminus \tilde{V}}(\omega_{V \setminus \tilde{V}}|\bar{\sigma}^j) \qquad (3.31) \\
&\leqq (1 + \hat{C}_2)^{|B|} K(U, U^h, V, t, \bar{\sigma}^j) ,
\end{aligned}$$

where $K(U, U^h, V, t, \bar{\sigma}^j)$ is the function defined in (3.13) with the properties expressed by (3.14) and (3.15). Now one can choose $\kappa(\tilde{\delta})$ and ε_0 so small that (3.28) and (3.30) are satisfied and that the bound (3.29) holds for (3.28). In this case the absolute value of (3.27) does not exceed

$$\hat{C}_2 \exp(-\kappa D) \tilde{\delta}^{1/2} + 2K(U, U^h, V, t, \bar{\sigma}^j) \cdot (1 + \hat{C}_2)^{|B|} .$$

Due to the special choice of $\hat{C}_2$ as $\hat{C}_2 = \sqrt{\varepsilon}$ the above term can be estimated by $\hat{C}_2 \exp(-\kappa D) \tilde{\delta}^{1/4}$, if ε_0 is small enough. Now (3.11) follows straightforward from (3.30) and (3.26).

We want to point out that during the proof of (3.9)–(3.11) we always ensured that the construction of the functions $\hat{C}_1$ and κ did not depend on the actual volume V in our induction procedure. Moreover, in each step we had to choose a suitable small ε_0. We did this choice independently of V, too. Thus we can use as the needed ε_0 the minimal one of all steps. The rest of the proof of the lemma is easy. Indeed, we find that the following condition is fulfilled by definition of U^h: For $A \in \mathfrak{N}$ and $\omega \in \Omega$

$$\tilde{U}(A, \omega) = \infty \quad \text{if and only if} \quad U^h(A, \omega) = \infty .$$

This corresponds to the case (i) of our proof and, hence, (3.5) and (3.6) are valid for U^h instead of U. The assertions of the lemma are now consequences of the following calculations.

$$\frac{Z_V(\tilde{U}\,|\bar\sigma)Z_{\tilde V}(U|\bar\sigma)}{Z_V(U|\bar\sigma)Z_{\tilde V}(\tilde U|\bar\sigma)} = \frac{Z_V(\tilde U\,|\bar\sigma)}{Z_V(U^h|\bar\sigma)}\,\frac{Z_{\tilde V}(U^h|\bar\sigma)}{Z_{\tilde V}(\tilde U\,|\bar\sigma)}\,\frac{Z_V(U^h|\bar\sigma)}{Z_V(U|\bar\sigma)}\,\frac{Z_{\tilde V}(U|\bar\sigma)}{Z_{\tilde V}(U^h|\bar\sigma)}$$
$$= (1+\vartheta_1 C_1\exp(-\kappa d))(1+K_1\hat C_1\exp(-\hat\kappa D))^{-1}$$
$$= 1+\tilde\vartheta_1\tilde C_1\exp(-\tilde\kappa d)\,,$$

where $\hat\kappa$ is the constant κ used in (3.9) and $\tilde\vartheta_1$, $\tilde C_1$, $\tilde\kappa$ are some new constants, that possess the needed properties. This proves (3.5). Analogously we get

$$\frac{Z_V(\tilde U\,|\bar\sigma^1)Z_V(U|\bar\sigma^2)}{Z_V(U|\bar\sigma^1)\,Z_V(\tilde U\,|\bar\sigma^2)} = (1+\vartheta_2 C_2\exp(-\kappa d))(1+K_2\hat C_2\exp(-\hat\kappa D))^{-1}$$
$$= 1+\tilde\vartheta_2\tilde C_2\exp(-\tilde\kappa d)\,,$$

for some new constants that possess the needed properties. The equation (3.6) is proved.

In the course of the proof of the proposition that $U\in\mathcal{O}\!\!\mathcal{C}_{III_a}$ implies $U\in\mathcal{O}\!\!\mathcal{C}_{I_b}$ we make use of the following lemma.

3.2. **Lemma.** *Then, if* $U\in\mathcal{O}\!\!\mathcal{C}_{III_a}$, *there exists* $\varepsilon>0$ *such that for all* $W\subset V\in\mathcal{P}_f(\mathbb{Z}^\nu)$, $\bar\sigma\in\Omega$, $\tilde U\in\mathcal{O}_\varepsilon(U)$ *the following estimate holds for some constant* $C=C(U,|W|)<\infty$:

$$\max_{\sigma_W\in\Omega_W}|Z_{V\setminus W}(\tilde U\,|\sigma_W\bar\sigma_{W^c})/Z_V(\tilde U\,|\bar\sigma)|<C\,. \tag{3.32}$$

Proof. In a first step we prove an estimate for the ratio of partition functions for $U\in\mathfrak{A}_r$. Namely, we show that for $V\in\mathcal{P}_f(\mathbb{Z}^\nu)$, $t\in V$, $\sigma_t\in S$, $\bar\sigma\in\Omega$ there exists a constant $C_1(U)>0$ that does not depend on V, t, σ_t, $\bar\sigma$ such that

$$Z_V(U|\bar\sigma)/Z_{V\setminus t}(U|\sigma_t\bar\sigma_{t^c})>C_1(U)\,. \tag{3.33}$$

Using the abbreviation

$$q(U,V,t,\sigma_t,\bar\sigma)=\exp\Bigg(-\sum_{A\cap V=t}U(A,\sigma_t\,\bar\sigma_{t^c})\Bigg)$$

we get

$$Z_V(U|\bar\sigma)/Z_{V\setminus t}(U|\sigma_t\bar\sigma_{t^c})=\sum_{\tau_t\in S}\frac{Z_{V\setminus t}(U|\tau_t\bar\sigma_{t^c})}{Z_{V\setminus t}(U|\sigma_t\bar\sigma_{t^c})}q(U,V,t,\tau_t,\bar\sigma)$$
$$\geqq\frac{Z_{V\setminus t}(U|0\,\bar\sigma_{t^c})}{Z_{V\setminus t}(U|\sigma_t\bar\sigma_{t^c})}q(U,V,t,0,\bar\sigma)\,. \tag{3.34}$$

If follows from (2.3) and $U\in\mathfrak{A}_r$ that

$$q(U,V,t,0,\bar\sigma)\geqq C_2(U)>0 \tag{3.35}$$

for some constant $C_2(U)$ that does not depend on $V, t, \bar{\sigma}$. For the examination of the ratio on the right hand side of (3.34) we look at the terms $\exp(-H^U_{V\setminus t}(\omega|\sigma_t\bar{\sigma}_{t^c}))$ (see (2.7)) the sum of which gives the partition function $Z_{V\setminus t}(U|\sigma_t\bar{\sigma}_{t^c})$. In case of $\exp(-H^U_{V\setminus t}(\omega|\sigma_t\bar{\sigma}_{t^c})) \neq 0$ there is a number $C_3 = C_3(U, V, t, \omega, \sigma_t, \bar{\sigma}) > 0$ such that $\exp(-H^U_{V\setminus t}(\omega|\sigma_t\bar{\sigma}_{t^c}))\cdot C_3 = \exp(-H^U_{V\setminus t}(\omega|0\bar{\sigma}_{t^c}))$. Since $U\in\mathfrak{A}_r$, the constant C_3 depends on a finite number of variables, only, and, hence, there exists

$$\min C_3(U, V, t, \omega, \sigma_t, \bar{\sigma}) = C_4(U) > 0\,. \tag{3.36}$$

Now, we get from (3.36)–(3.38)

$$Z_V(U|\bar{\sigma})/Z_{V\setminus t}(U|\sigma_t\bar{\sigma}_{t^c}) \geqq C_2(U)\cdot C_4(U)\left(1+\frac{\sum\limits_{\omega\in\Omega^{\text{bad}}_{V\setminus t}(\sigma_t\bar{\sigma}_{V^c})}\exp(-H^U_{V\setminus t}(\omega|\sigma_t\bar{\sigma}_{t^c}))}{C_4(U)Z_{V\setminus t}(U|\sigma_t\bar{\sigma}_{t^c})}\right)$$

$$\geqq C_2(U)\cdot C_4(U) = C_1(U) > 0\,.$$

The estimate (3.33) is proved. After these preliminary examinations we go about the proof of (3.32). Let $\varepsilon > 0$ be so small, that for all $\tilde{U}\in\mathcal{O}_\varepsilon(U)$ the conclusions of Lemma 3.1 can be used. To show (3.32), it is enough to handle the case $W = t$ for some $t\in V\in\mathcal{P}_f(\mathbb{Z}^\nu)$ and $\bar{\sigma}\in\Omega$. If V, t and $\bar{\sigma}$ are fixed, then S splits into two parts S_1 and S_2:

$$\tau_t\in S_1\,,\quad \text{iff}\quad U(A, \tau_t\bar{\sigma}_{t^c}) < \infty \quad \text{for all}\quad A\in\mathfrak{R}\,,\ A\cap V = t\,.$$

We get for all $\sigma_t\in S$

$$\left|\frac{Z_V(\tilde{U}|\bar{\sigma})}{Z_{V\setminus t}(\tilde{U}|\sigma_t\bar{\sigma}_{t^c})}\right| = \left|\sum_{\tau_t\in S_1}\frac{Z_{V\setminus t}(\tilde{U}|\tau_t\bar{\sigma}_{t^c})}{Z_{V\setminus t}(\tilde{U}|\sigma_t\bar{\sigma}_{t^c})}q(\tilde{U}, V, t, \tau_t, \bar{\sigma})\right.$$
$$\left.+\sum_{\tau_t\in S_2}\frac{Z_{V\setminus t}(\tilde{U}|\tau_t\bar{\sigma}_{t^c})}{Z_{V\setminus t}(\tilde{U}|\sigma_t\bar{\sigma}_{t^c})}q(\tilde{U}, V, t, \tau_t, \bar{\sigma})\right|\,. \tag{3.37}$$

Using Lemma 3.1, equation (3.6), we can write

$$\frac{Z_{V\setminus t}(\tilde{U}|\tau_t\bar{\sigma}_{t^c})}{Z_{V\setminus t}(\tilde{U}|\sigma_t\bar{\sigma}_{t^c})} = (1+\vartheta_2 C_5)\frac{Z_{V\setminus t}(U|\tau_t\bar{\sigma}_{t^c})}{Z_{V\setminus t}(U|\sigma_t\bar{\sigma}_{t^c})}\,, \tag{3.38}$$

where we included the term $\exp(-\kappa d)$ from (3.6) into ϑ_2.

For $\tau_t\in S_1$ we get

$$q(\tilde{U}, V, t, \tau_t, \bar{\sigma})/q(U, V, t, \tau_t, \bar{\sigma}) = 1+\vartheta_3 C_6\,, \tag{3.39}$$

where $|\vartheta_3| \leqq 1$ and C_6 tends to zero, if ε tends to zero, and the convergence is uniformly in V, t, τ_t and $\bar{\sigma}$.

For $\tau_t\in S_2$ we get

$$\lim_{\varepsilon\to 0} q(\tilde{U}, V, t, \tau_t, \bar{\sigma}) = 0 \tag{3.40}$$

uniformly in V, t, τ_t, $\bar{\sigma}$.

It follows from (3.37)–(3.40) and (3.33) that

$$\left|\frac{Z_{V\setminus t}(\tilde U|\sigma_t\bar\sigma_{t^c})}{Z_V(\tilde U|\bar\sigma)}\right| = \frac{Z_{V\setminus t}(U|\sigma_t\bar\sigma_{t^c})}{Z_V(U|\bar\sigma)}\left|\left\{(1+\vartheta_2C_5)(1+\vartheta_3C_6)+\right.\right.$$

$$\left.\left.+\sum_{\tau_t\in S_2}(1+\vartheta_2C_5)\frac{Z_{V\setminus t}(U|\tau_t\bar\sigma_{t^c})}{Z_V(U|\bar\sigma)}q(\tilde U,V,t,\tau_t,\bar\sigma)\right\}^{-1}\right|$$

$$\leqq C_7(U) < \infty\ .$$

The bound C in (3.32) can be chosen as $C = |W|\cdot C_7(U)$. The lemma is proved.

3.3. **Corollary.** *Then, if $U\in\mathfrak{A}_{III_a}$, there exists $\varepsilon > 0$ such that for all $V\in\mathscr{P}_f(\mathbb{Z}^\nu)$, $\bar\sigma\in\Omega$, $\tilde U\in\mathcal{O}_\varepsilon(U)$ and each complex-valued function φ on Ω that depends for some $W\subset V$ on $\omega|_W$, only, the following estimate holds:*

$$|\langle\varphi\rangle^{\tilde U}_{V,\bar\sigma}| \leqq C\|\varphi\|\ , \tag{3.41}$$

where $C = C(|W|, U, r, \nu, \varepsilon)$ is independent of V and $\bar\sigma$ and

$$\|\varphi\| = \sup_{\omega\in\Omega}|\varphi(\omega)|\ .$$

Before proving this assertion we want to remark that in (3.41) the left hand side is defined by

$$\langle\varphi\rangle^{\tilde U}_{V,\bar\sigma} = \sum_{\omega_V\in\Omega_V}\varphi(\omega_V\bar\sigma_{V^c})Q^{\tilde U}_{V,\bar\sigma}(\omega_V|\bar\sigma)\ , \tag{3.42}$$

where $Q^{\tilde U}_V(\omega_V|\bar\sigma)$ is defined by (2.17) with $\tilde U$ instead of U. In Lemma 3.1 it was shown that for $U\in\mathfrak{A}_{III_a}$ and $\varepsilon>0$ small enough the partition functions $Z_V(\tilde U|\bar\sigma)$ are nonvanishing for $\tilde U\in\mathcal{O}_\varepsilon(U)$. Hence, for small enough ε $\langle\varphi\rangle^{U}_{V,\bar\sigma}$ is well-defined.

Proof. Let $U\in\mathfrak{A}_{III_a}$, $\varepsilon>0$ be small enough, $\tilde U\in\mathcal{O}_\varepsilon(U)$ and $\varphi:\Omega\to\mathbb{C}$ a function that for some $W\subset V\in\mathscr{P}_f(\mathbb{Z}^\nu)$ depends on $\omega|_W$, only. Then it follows from (2.6), (3.42) and (3.32) that

$$|\langle\varphi\rangle^{\tilde U}_{V,\bar\sigma}| \leqq |S|^{|W|}\|\varphi\|\max_{\sigma_W\in\Omega_W}\left|\frac{Z_{V\setminus W}(\tilde U|\sigma_W\bar\sigma_{W^c})}{Z_V(\tilde U|\bar\sigma)}\right|\times$$

$$\times\max_{\sigma_W\in\Omega_W}\left|\exp\left(\sum_{\substack{A\cap W\neq\varnothing\\ A\cap(V\setminus W)=\varnothing}}\tilde U(A,\sigma_W\bar\sigma_{W^c})\right)\right|$$

$$\leqq |S|^{|W|}\|\varphi\|C_1\max_{\sigma_W\in\Omega_W}\left|\exp\left(-\sum_{\substack{A\cap W\neq\varnothing\\ A\cap(V\setminus W)=\varnothing}}\tilde U(A,\sigma_W\bar\sigma_{W^c})\right)\right|, \tag{3.43}$$

where C_1 is the bound in (3.32). Let $\hat\sigma_W\in\Omega_W$ be the configuration for which the maximum is reached on the right hand side of (3.43). It is easy to see that $\hat\sigma_W\in\Omega^{\text{good}}_W(\bar\sigma_{W^c})$, if ε is small enough. Hence, $\operatorname{Re}\tilde U(A,\hat\sigma_W\bar\sigma_{W^c}) < C_2(U,\varepsilon)$ for all $A\in\mathfrak{N}$, $A\cap W\neq\varnothing$, $A\cap(V\setminus W)=\varnothing$. As a consequence we get that the right hand side of (3.43) is bounded by a constant

$$C = |S|^{|W|} \|\varphi\| C_1 \cdot C_3(U, \varepsilon, r, |W|),$$

Q.E.D.

3.3. **Proposition.** *The Condition* $U \in \mathcal{A}_{III_a}$ *implies* $U \in \mathcal{A}_{I_b}$.

Proof. Let $U \in \mathcal{A}_{III_a}$ and $\varepsilon_0 > 0$ be the number for which the assertions of Lemma 3.1 and Lemma 3.2 are valid. For $\tilde{U}_1, \tilde{U}_2 \in \mathcal{O}_{\varepsilon_0}(U)$ we choose a sequence of interactions $\hat{U}^i \in \mathcal{O}_{\varepsilon_0}(U)$, $i = 1, \ldots, k$ such that

(i) $\hat{U}^1(A, \omega) = \tilde{U}_1(A, \omega)$ and $\hat{U}^k(A, \omega) = \tilde{U}_2(A, \omega)$ for all $A \subset V \cup \partial V$, $\omega \in \Omega$.
(ii) Then, if $(\hat{U}^{i+1} - \hat{U}^i)(A, .) \not\equiv 0$, the interactions differ from each other on this set A, only, and for all $\bar{\sigma} \in \Omega$ with $\bar{\sigma}|_A \neq \omega|_A$ the values $\hat{U}^{i+1}(A, \bar{\sigma})$ and $\hat{U}^i(A, \bar{\sigma})$ are equal.
(iii) k is the smallest number such that (i) and (ii) are satisfied.

It is clear that

$$k \leqq C_1(r, v, |S|)|(V \cup \partial V) \cap \operatorname{supp}(\tilde{U}_1 - \tilde{U}_2)|. \tag{3.44}$$

Our assertion will follow from (3.44) and the estimate

$$|\ln[Z_V(\hat{U}^i|\bar{\sigma})/Z_V(\hat{U}^{i+1}|\bar{\sigma})]| \leqq C_2, \tag{3.45}$$

where $C_2 = C_2(U, r, \varepsilon, v)$.

Let A_0 be the set for which $\hat{U}^i$ and $\hat{U}^{i+1}$ differ from each other. Using (ii) and (3.32) we get for all $V \in \mathcal{P}_f(\mathbb{Z}^v)$, $\bar{\sigma} \in \Omega$

$$\left|\frac{Z_V(\hat{U}^i|\bar{\sigma})}{Z_V(\hat{U}^{i+1}|\bar{\sigma})} - 1\right| = \left|\sum_{\omega \in \Omega_{A_0}} \frac{Z_{V \setminus A_0}(\hat{U}^i|\omega\bar{\sigma}_{A_0^c})}{Z_V(\hat{U}^{i+1}|\bar{\sigma})}\{\exp(-\hat{H}^i(\omega)) - \exp(-\hat{H}^{i+1}(\omega))\}\right|$$
$$\leqq C \sum_{\omega \in \Omega_{A_0}} |\{\exp(-\hat{H}^i(\omega)) - \exp(-\hat{H}^{i+1}(\omega))\}|, \tag{3.46}$$

where $\hat{H}^j(\omega) = \sum_{W \subset A_0 \cup \partial V,\, W \cap A_0 \neq \emptyset} \hat{U}^j(W, \omega\bar{\sigma}_{A_0^c})$ for $j = i, i+1$.

But the last sum in (3.46) can obviously be estimated by a constant $\tilde{C} \cdot \varepsilon_0$, where $\tilde{C} = \tilde{C}(U, r, v, \varepsilon_0, |S|)$. This already proves (3.45). Hence, the proposition is proved.

4. Condition I_a Implies Condition III_b

For the proof of the assertion that $U \in \mathcal{M}(\mathcal{A}_{I_a})$ implies $U \in \mathcal{A}_{III_a}$ we need the following lemmata. One of them is already published, but we repeat it here without proof for the sake of completeness.

4.1. **Lemma** [6, Lemma 3.1]. *Suppose the function* $\varphi(z)$ *is analytic in the disc* $\{z \in \mathbb{C} \,|\, |z| < 1 + \delta\}$, $\delta > 0$, *with* $|\varphi(z)| \leqq C_1$ *for* $|z| \leqq 1$ *and* $\varphi(0)$ *is real,* $\varphi(0) > \alpha > 0$. *Let* $E = 1 - \exp(-\alpha/2C_1)$ *and* $C_2 = 1 + \max\{|\ln \alpha|, |\ln C_1|\}$. *Then* $\varphi(z) \neq 0$ *for* $|z| \leqq E$ *and* $|\ln \varphi(z)| \leqq C_2$ *for* $|z| \leqq E$, *where we choose the branch of the logarithm in such a way that* $\ln \varphi(0)$ *is real.*

4.2. **Lemma.** *Let $U \in \mathfrak{A}_{I_a}$. Then there exist $\varepsilon > 0$ and $C < \infty$ such that for all $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $\bar{\sigma} \in \Omega$, $\tilde{U} \in \mathcal{O}_\varepsilon^T(U)$*

$$|\ln Z_V(\tilde{U}\,|\bar{\sigma})| \leqq C|V|\,. \tag{4.1}$$

Proof. Because of $U \in \mathfrak{A}_{I_a}$ there exists $\varepsilon_1 > 0$ such that for all $\tilde{U} \in \mathcal{O}_{\varepsilon_1}^T(U)$ the partition functions are nonvanishing. We define

$$\bar{u} = \sup\{|U(A,\omega)|\,|A \in \mathfrak{N}\,,\ \omega \in \Omega\,,\ U(A,\omega) < \infty\}$$

and
$$\kappa = \operatorname{card}\{V \subset \mathbb{Z}^\nu|\,0 \in V,\ \operatorname{diam} V \leqq r\}\,.$$

We show that the numbers ε and C looked for are

$$\varepsilon = [1 - \exp(-\exp(-\kappa(2\bar{u} + \varepsilon_1) + \ln|S|))]\varepsilon_1 \tag{4.2}$$

$$C = [1 + \kappa(\bar{u} + \varepsilon_1) + \ln|S|]\,. \tag{4.3}$$

According to (2.5) we use instead of U respectively $\tilde{U}$ the corresponding generalized activities Γ respectively $\tilde{\Gamma}$. It is not hard to check that for $\tilde{\Gamma} \in \mathcal{O}_\varepsilon^T(\Gamma)$

$$|Z_V(\tilde{\Gamma}\,|\bar{\sigma})^{1/|V|}| \leqq \exp(\kappa(\bar{u} + \varepsilon_1) + \ln|S|)\,. \tag{4.4}$$

For $|z| \leqq 1$ we introduce the functions

$$\Gamma^z(A,\omega) = \Gamma(A,\omega)\exp(z \cdot \ln(\tilde{\Gamma}(A,\omega)/\Gamma(A,\omega))) \quad \text{if} \quad \Gamma(A,\omega) \neq 0$$

and $\Gamma^z(A,\omega) = z\tilde{\Gamma}(A,\omega)$ if $\Gamma(A,\omega) = 0$.

Obviously $\Gamma^z \in \mathcal{O}_{\varepsilon_1}^T(\Gamma)$ for all $|z| \leqq 1$. Thus, the partition functions $Z_V(\Gamma^z|\bar{\sigma})$ are analytic functions of the parameter z in the disc $\{z \in \mathbb{C}\,|\,|z| \leqq 1\}$. Moreover, it is easy to see that

$$|Z_V(\Gamma^z|\bar{\sigma})^{1/|V|}| \leqq \exp(\kappa(\bar{u} + \varepsilon_1) + \ln|S|)$$

and

$$Z_V(\Gamma|\bar{\sigma})^{1/|V|} \geqq \exp(-\kappa\bar{u})\,.$$

Hence, we can apply Lemma 4.1 to $\varphi(z) = Z_V(\Gamma^z|\bar{\sigma})^{1/|V|}$, which gives by definition of ε and C (see (4.2), (4.3)) the needed result.

4.3. **Lemma.** *Let $\mathscr{W} \subset \mathfrak{U}_r$ be an open and connected subset that contains the interaction $\Gamma^0 = \{\Gamma(A,\cdot) \equiv 1,\ A \in \mathfrak{N}\}$ corresponding to U^0. Define $\mathcal{O}_\varepsilon(\mathscr{W}) = \bigcup_{\Gamma \in \mathscr{W}} \mathcal{O}_\varepsilon(\Gamma)$ and $\mathcal{O}_\varepsilon^T(\mathscr{W}) = \bigcup_{\Gamma \in \mathscr{W}} \mathcal{O}_\varepsilon^T(\Gamma)$. Then there exist $\varepsilon_0 > 0$ and a function $\alpha(\Gamma,\varepsilon) > 0$, $\Gamma \in \mathscr{W}$, $\varepsilon < \varepsilon_0$, such that for all functions $g(\tilde{\Gamma})$, $\tilde{\Gamma} \in \mathcal{O}_\varepsilon(\mathscr{W})$, that are analytic in $\mathcal{O}_\varepsilon^T(\mathscr{W})$ the conditions*

(i) $|g(\tilde{\Gamma})| \leqq M$ for all $\tilde{\Gamma} \in \mathcal{O}_\varepsilon^T(\mathscr{W})$ (4.5)

and

(ii) *for some $n \geqq 0$*

$$\left.\frac{\partial^m g(\tilde{\Gamma})}{\partial^{l_1}\tilde{\Gamma}(B_1)\ldots\partial^{l_q}\tilde{\Gamma}(B_q)}\right|_{\tilde{\Gamma}=\Gamma^0} \equiv 0 \tag{4.6}$$

$\sum_{i=1}^{q} l_i = m \leqq n$, $B_1,\ldots,B_q \in \mathscr{P}_f(\mathbb{Z}^\nu)$ imply

$$|g(\Gamma)| \leqq M\exp(-n\alpha(\Gamma,\varepsilon)) \quad \textit{for all} \quad \Gamma \in \mathscr{W}\,. \tag{4.7}$$

By the partial derivatives (4.6) of the function $g(\tilde{\Gamma})$, $\tilde{\Gamma} \in \mathcal{O}_\varepsilon(\mathscr{W})$, we mean the family of functions

$$\left\{ \frac{\partial^m g(\tilde{\Gamma})}{\partial^{l_{11}} \tilde{\Gamma}(B_1, \sigma^{1,1}) \partial^{l_{12}} \tilde{\Gamma}(B_1, \sigma^{1,2}) \dots \partial^{l_{q u_q}} \tilde{\Gamma}(B_q, \sigma^{q, u_q})} \right\},$$

where the parameters are chosen as follows: $l_{ij} \geqq 0$, $l_{i1} + \dots + l_{iu_i} = l_i$, $\sigma^{i,j} \in \Omega_{B_i}$, $j = 1, \dots, u_i > 0$, $i = 1, \dots, q$. The proof of the lemma proceeds in the same way as the proof of Theorem 4.2 in [5].

A family $\{V_i \in \mathscr{P}_f(\mathbb{Z}^\nu) | i = 1, \dots, I\}$ is said to be connected, if for all $\emptyset \neq J \subset \{1, \dots, I\}$

$$\left(\bigcup_{i \in J} V_i \right) \cap \left(\bigcup_{i \notin J} V_i \right) \neq \emptyset .$$

In the next two lemmata we examine the generalized Ursell functions, that are defined by (2.12), to find the crucial properties of these functions.

4.4. **Lemma.** *Let* $V \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $A_i \in \mathfrak{N}$, $A_i \cap V \neq \emptyset$, ψ_i: $\Omega \to \mathbb{R}$ *such that* $\psi_i(\omega)$ *depends on* $\omega|_{A_i}$ *only,* $i = 1, \dots, m$ *and* $K = (k_1, \dots, k_m)$ *be given. Then, if the set* $\{A_1, \dots, A_m\}$ *is nonconnected,*

$$[\psi_1'^{k_1}, \dots, \psi_m'^{k_m} | \Gamma^0, V, \bar{\sigma}] = 0 . \tag{4.8}$$

Proof. By definition (see (2.11), (2.5)–(2.7)) we have

$$Z_V(\Gamma^0_{(z_1, \dots, z_m)} | \bar{\sigma}) = \sum_{\omega_V \in \Omega_V} \prod_{A \cap V \neq \emptyset} \left(1 + \sum_{i:\, A_i = A} z_i \psi_i(\omega_V \bar{\sigma}_{V^c}) \right).$$

From the nonconnectivity of $\{A_1, \dots, A_m\}$ it follows the existence of a nonempty set $J \subset \{1, \dots, m\}$ such that $\left(\bigcup_{i \in J} A_i \right) \cap \left(\bigcup_{i \notin J} A_i \right) = \emptyset$. Now it is not hard to see that

$$Z_V(\Gamma^0_{(z_1, \dots, z_m)} | \bar{\sigma}) = Z_{V \cap \left(\bigcup_{i \in J} A_i \right)}(\Gamma^0_{(z_i,\, i \in J)} | \bar{\sigma}) \cdot Z_{V \cap \left(\bigcup_{i \notin J} A_i \right)}(\Gamma^0_{(z_i,\, i \notin J)} | \bar{\sigma}) .$$

Thus the logarithm of $Z_V(\Gamma^0_{(z_1, \dots, z_m)} | \bar{\sigma})$ breaks up into a sum of two logarithms and, consequently, the derivatives defining the generalized Ursell functions for Γ^0 are vanishing.

4.5. **Lemma.** *Let* $\{A_1, \dots, A_m\}$, $\{B_1, \dots, B_q\}$ *be two families of elements of* $\mathfrak{N}$ *and* $\{\psi_1, \dots, \psi_m\}$ *be a family of maps* ψ_i: $\Omega \to \mathbb{R}$ *such that* $\psi_i(\omega)$ *depends on* $\omega|_{A_i}$, *only.*
I) *Then, if* $\{A_1, \dots, A_m, B_1, \dots, B_q\}$ *is nonconnected,*

$$\left. \frac{\partial^{l_1 + \dots + l_q}}{\partial^{l_1} \tilde{\Gamma}(B_1) \dots \partial^{l_q} \tilde{\Gamma}(B_q)} [\psi_1'^{k_1}, \dots, \psi_m'^{k_m} | \tilde{\Gamma}, V, \bar{\sigma}] \right|_{\tilde{\Gamma} = \Gamma^0} = 0 , \tag{4.9}$$

for all choices of the parameters V, $\bar{\sigma}$, $l_i = (l_{i1}, \dots, l_{iu_i})$, $i = 1, \dots, q$, $K = (k_1, \dots, k_m)$.

II) *Let* $V_i \in \mathscr{P}_f(\mathbb{Z}^\nu)$, $\bar{\sigma}^i \in \Omega$, $i = 1, 2$ *be such that for* $D = \left(\bigcup_{i=1}^m A_i \right) \cup \left(\bigcup_{i=1}^q B_i \right)$ *the following conditions hold:*

$$D \cap V_1 = D \cap V_2, \quad D \cap \partial V_1 = D \cap \partial V_2 \quad \textit{and} \quad \bar{\sigma}^1|_s = \bar{\sigma}^2|_s \quad \textit{for all} \quad s \in \partial V_1 \cap D .$$

Then, for all choices of the other parameters

$$\frac{\partial^{l_1+\dots+l_q}}{\partial^{l_1}\tilde{\Gamma}(B_1)\dots\partial^{l_q}\tilde{\Gamma}(B_q)}[\psi_1'^{k_1},\dots,\psi_m'^{k_m}|\tilde{\Gamma},V_1,\bar{\sigma}^1]\Big|_{\tilde{\Gamma}=\Gamma^0}$$
$$=\frac{\partial^{l_1+\dots+l_q}}{\partial^{l_1}\tilde{\Gamma}(B_1)\dots\partial^{l_q}\tilde{\Gamma}(B_q)}[\psi_1'^{k_1},\dots,\psi_m'^{k_m}|\tilde{\Gamma},V_2,\bar{\sigma}^2]\Big|_{\tilde{\Gamma}=\Gamma^0} \tag{4.10}$$

Proof. The left hand side of (4.9) is equal to

$$[\psi_1'^{k_1},\dots,\psi_m'^{k_m},\chi_{1,1}'^{l_{11}},\dots,\chi_{q,u_q}'^{l_{qu_q}}|\Gamma^0,V,\bar{\sigma}]| , \tag{4.11}$$

where

$$\chi_{i,j}(\omega_V\bar{\sigma}_{V^c})=\begin{cases}1 & \text{for } (\omega_V\bar{\sigma}_{V^c})|_{B_i}=\sigma^{ij}\\ 0 & \text{otherwise}\end{cases} \tag{4.12}$$

if $\sigma^{ij}|_{\partial V\cap B_i}=\bar{\sigma}|_{\partial V\cap B_i}$ for all $i=1,\dots,q, j=1,\dots,u_i$.

Now the first assertion follows immediately from Lemma 4.4. The second assertion is a direct consequence of the representation (4.11) for the derivatives and of the properties of D.

4.6. **Proposition.** *The Condition* $U\in\mathcal{M}(\mathcal{O}\!\mathcal{C}_{I_a})$ *implies* $U\in\mathcal{O}\!\mathcal{C}_{III_a}$.

Proof. Let $U\in\mathcal{M}(\mathcal{O}\!\mathcal{C}_{I_a})$ and consider the function

$$\begin{aligned} g(U) &= \ln[Q^U_{V,\Lambda}(\sigma_\Lambda|\bar{\sigma}^1)/Q^U_{V,\Lambda}(\sigma_\Lambda|\bar{\sigma}^2)]\\ &= \ln Z_V(U|\bar{\sigma}^2)-\ln Z_V(U|\bar{\sigma}^1)+\ln Z_{V\setminus\Lambda}(U|\sigma_\Lambda\bar{\sigma}^1_{\Lambda^c})\\ &\quad -\ln Z_{V\setminus\Lambda}(U|\sigma_\Lambda\bar{\sigma}^2_{\Lambda^c}) , \end{aligned} \tag{4.13}$$

where the parameters are chosen as follows: $V\in\mathcal{P}_f(\mathbb{Z}^\nu)$, $\Lambda\subset V$, $t\in\partial V$, $\bar{\sigma}^1,\bar{\sigma}^2\in\Omega$ such that $\bar{\sigma}^1|_s=\bar{\sigma}^2|_s$ for $s\neq t$, $\sigma_\Lambda\in\Omega^{\text{good}}_\Lambda(\bar{\sigma}^1_{\Lambda^c})\cap\Omega^{\text{good}}_\Lambda(\bar{\sigma}^2_{\Lambda^c})$. Define $\mathcal{W}=\mathcal{W}_{C,\varepsilon}$ to be the main component of the set of interactions that fulfill (4.1) with the constants C and 2ε. It follows from Lemma 4.2 that the sets $\mathcal{W}_{C,\varepsilon}$ exhaust $\mathcal{M}(\mathcal{O}\!\mathcal{C}_{I_a})$. Moreover, we get

$$|g(\tilde{U})|\leqq 4C|V| \tag{4.14}$$

for all $\tilde{U}\in\mathcal{O}^T_\varepsilon(\mathcal{W})$. In the next step of the proof we want to show an estimate similar to (4.7). For this we need for some $n\geqq 0$

$$\frac{\partial^{l_1+\dots+l_q}g(\tilde{\Gamma})}{\partial^{l_1}\tilde{\Gamma}(B_1)\dots\partial^{l_q}\tilde{\Gamma}(B_q)}\Big|_{\tilde{\Gamma}=\Gamma^0}=0 . \tag{4.15}$$

To prove (4.15) we use Lemma 4.5 for the case $m=0$. By this lemma the derivative on the left hand side of (4.15) can be nonvanishing only if the family $\{B_1,\dots,B_q\}$ is connected with $t\in B_i$ for some i and $\Lambda\cap B_i\neq\emptyset$ for some i. Indeed, if connectivity does not hold, the derivative vanishes according to I) of Lemma 4.5. If on the other hand the family is connected, but $t\in B_i$ for all i or all intersections $\Lambda\cap B_i$ are empty, then according to II) of the same lemma we have a complete cancellation of equal terms with different signs in (4.13). Hence, if the derivative is nonvanishing, then $q\geqq[r^{-1}\text{dist}(t,\Lambda)]$, where $[x]$ denotes the integer part of the number x. Now, it is a consequence of Lemma 4.3 that

$$|g(U)| \leqq 4C|V|\exp(-\alpha(U)\operatorname{dist}(t, \Lambda)) . \tag{4.16}$$

Let us recall that in (4.10) the boundary conditions $\bar\sigma^1, \bar\sigma^2 \in \Omega$ differ from each other at t, only. For general $\bar\sigma^1, \bar\sigma^2 \in \Omega$ such that for the fixed $\sigma_\Lambda \in \Omega_\Lambda$ holds $\sigma_\Lambda \in \Omega_\Lambda^{\text{good}}(\bar\sigma^1_{\Lambda^c}) \cap \Omega_\Lambda^{\text{good}}(\bar\sigma^2_{\Lambda^c})$ we get by subsequent application of (4.15)

$$\left|\ln\frac{Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^1)}{Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^2)}\right| \leqq 4C|V|2|\Delta(\bar\sigma^1,\bar\sigma^2)|\exp(-\alpha(U)\operatorname{dist}(\Delta(\bar\sigma^1,\bar\sigma^2),\Lambda)) , \tag{4.17}$$

where $\Delta(\bar\sigma^1,\bar\sigma^2) = \{s \in \partial V | \bar\sigma^1|_s \neq \bar\sigma^2|_s\}$.

Now we can attack the original problem. Remark that without loss of generality we can restrict ourselves to the situation, where $V = \Lambda \cup V_1$ with $\Lambda \cap V_1 = \emptyset$ and

$$V_1 = \{s \in V | \operatorname{dist}(s,t) \leqq \operatorname{dist}(t,\Lambda)\} . \tag{4.18}$$

Indeed, assume that there exists $s \in V$, $s \notin \Lambda$, but $\operatorname{dist}(s,t) > \operatorname{dist}(t,\Lambda)$. Then

$$\frac{Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^1)}{Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^2)} = \frac{\sum\limits_{\tau_s \in S} Q^U_{V,\tilde\Lambda}(\sigma_\Lambda \tau_s|\bar\sigma^1)}{\sum\limits_{\tau_s \in S} Q^U_{V,\tilde\Lambda}(\sigma_\Lambda \tau_s|\bar\sigma^2)} ,$$

where $\tilde\Lambda = \Lambda \cup \{s\}$ and Λ is situated in V such that $V \backslash \tilde\Lambda$ fulfills (4.17). Assuming that (2.23) is true for V and $\tilde\Lambda$ and using the implication

$$a_i/b_i < K, i = 0, \ldots, |S| \Rightarrow \sum_{i=0}^{|S|} a_i \Big/ \sum_{i=0}^{S} b_i < K ,$$

which is true for positive real numbers a_i and b_i, we get that (2.23) is true for V and Λ, too.

In the following we use the notations

$$\Lambda' = \{s \in \Lambda | \operatorname{dist}(s,t) \leqq \operatorname{dist}(t,\Lambda) + r\}$$

$$\Lambda'' = \Lambda \backslash \Lambda' \quad \text{and}$$

$$\tilde V = \{s \in \Lambda'' | \operatorname{dist}(s,\Lambda') > \operatorname{dist}(t,\Lambda)\} .$$

Taking into account the finiteness of the range of the interaction a long but simple calculation shows that

$$Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^1)/Q^U_{V,\Lambda}(\sigma_\Lambda|\bar\sigma^2) = Q^U_{V,\Lambda'}(\sigma_{\Lambda'}|\bar\sigma^1)/Q^U_{V,\Lambda'}(\sigma_{\Lambda'}|\bar\sigma^2) . \tag{4.19}$$

This enables us to restrict ourselves to the set Λ'. For this set and $j = 1, 2$ we get by the formula of the total probability

$$Q^U_{V,\Lambda'}(\sigma_{\Lambda'}|\bar\sigma^j) = \sum_{\omega_{\tilde V} \in \Omega_{\tilde V}} Q^U_{V\backslash\tilde V,\Lambda'}(\sigma_{\Lambda'}|\omega_{\tilde V}\bar\sigma^j_{\tilde V^c})Q^U_{V,\tilde V}(\omega_{\tilde V}|\bar\sigma^j) . \tag{4.20}$$

Before doing the next step let us remark that we can restrict ourselves to the case $\operatorname{dist}(t,\Lambda) > r$. This assertion is true because of the possibility to change the needed constant K in (2.23) for the values of $\mathrm{dist}(t, \Lambda)$ up to the finite number r. It follows from this remark that $\mathrm{dist}(\tilde{V}, \Lambda') > r$. We use (4.17) for the volume $V \setminus \tilde{V}$ and get for all $\omega'_{\tilde{V}}, \omega''_{\tilde{V}} \in \Omega_{\tilde{V}}$

$$\left| \ln \frac{Q^U_{V\setminus\tilde{V},\Lambda'}(\sigma_{\Lambda'} | \omega'_{\tilde{V}} \bar{\sigma}^1_{\tilde{V}^c})}{Q^U_{V\setminus\tilde{V},\Lambda'}(\sigma_{\Lambda'} | \omega''_{\tilde{V}} \bar{\sigma}^2_{\tilde{V}^c})} \right| \leqq 4C|V\setminus\tilde{V}|(|\partial(V\setminus\tilde{V}) \cap V| + 1)\exp(-\alpha(U)\mathrm{dist}(t, \Lambda)) . \tag{4.21}$$

It is easy to see that $|V\setminus\tilde{V}| \leqq C_1|\Lambda'|(\mathrm{dist}(t, \Lambda))^\nu$, $|\Lambda'| \leqq C_2(\mathrm{dist}(t, \Lambda))^\nu$ and $|\partial(V\setminus\tilde{V})| \leqq C_3(\mathrm{dist}(t, \Lambda))^\nu$, where $C_1 = C_1(\nu)$, $C_2 = C_2(r, \nu)$ and $C_3 = C_3(r, \nu)$ are constants. Hence, it follows from (4.20) and (4.21) that

$$\left| \frac{Q^U_{V,\Lambda'}(\sigma_{\Lambda'} | \bar{\sigma}^1)}{Q^U_{V,\Lambda'}(\sigma_{\Lambda'} | \bar{\sigma}^2)} - 1 \right| \leqq K\exp(-\kappa \,\mathrm{dist}(t, \Lambda)) \tag{4.22}$$

for some $\kappa > 0$ and some $K < \infty$. Now the estimate (2.23) follows immediately from (4.22) and (4.19). The proposition is proved.

4.7. **Proposition.** *The Condition $U \in \mathfrak{A}_{III_d}$ implies $U \in \mathfrak{A}_{III_c}$.*

Proof. Let $U \in \mathfrak{A}_{III_d}$ and the parameters $V, \Lambda, t, \bar{\sigma}^1, \bar{\sigma}^2$ be chosen as necessary for the Conditions III_c and III_d. We denote by $\Omega'_\Lambda = \Omega^{\mathrm{good}}_\Lambda(\bar{\sigma}^1_{\Lambda^c}) \cap \Omega^{\mathrm{good}}_\Lambda(\bar{\sigma}^2_{\Lambda^c})$. Now the proof proceeds in the following way: Then, if $\mathrm{dist}(t, \Lambda) > r$, it follows from (2.23) that

$$\begin{aligned}\mathrm{Var}(Q^U_{V,\Lambda}(\cdot | \bar{\sigma}^1), Q^U_{V,\Lambda}(\cdot | \bar{\sigma}^2)) &\leqq \tfrac{1}{2} \max_{\sigma_\Lambda \in \Omega'_\Lambda} \left| \frac{Q^U_{V,\Lambda}(\sigma_\Lambda | \bar{\sigma}^1)}{Q^U_{V,\Lambda}(\sigma_\Lambda | \bar{\sigma}^2)} - 1 \right| \\ &\leqq \tfrac{1}{2} K\exp(-\kappa \mathrm{dist}(t, \Lambda)) .\end{aligned}$$

The case $\mathrm{dist}(t, \Lambda) \leqq r$ is trivial because it is possible to increase the needed constant $\bar{K}$ in (2.22) for the values of $\mathrm{dist}(t, \Lambda)$ up to the finite number r.

5. The Second Group of Conditions

The basis to include the second group of conditions into the cycle of the theorem are the properties of the generalized Ursell functions for the zero-interaction U^0 that are proved in the Lemmata 4.4 and 4.5 We use them to prove the implication $U \in \mathcal{M}(\mathfrak{A}_{II_a}) \Rightarrow U \in \mathfrak{A}_{II_c}$. Before doing this we remark that the implication $U \in \mathfrak{A}_{I_b} \Rightarrow U \in \mathfrak{A}_{II_a}$ follows from the estimate

$$|\mathrm{supp}(\tilde{\Gamma}_{(z_1, \ldots, z_m)} - \Gamma)| = \left| \bigcup_{i=1}^m A_i \right| \leqq C \cdot m$$

for some constant $C = C(r, \nu)$ and the Cauchy formula for derivatives of analytic functions. The Condition II_b follows from Condition II_c with $\varphi(d) = \exp(-\kappa d)$.

5.1. **Proposition.** *The Condition $U \in \mathcal{M}(\mathfrak{A}_{II_a})$ implies $U \in \mathfrak{A}_{II_c}$.*

Proof. We apply Lemma 4.3 to the function

$$g(\tilde{\Gamma}) = [\psi'^{k_1}_1, \ldots, \psi'^{k_m}_m | \tilde{\Gamma}, V, \bar{\sigma}] ,$$

where $\tilde{\Gamma} \in \mathcal{O}_\varepsilon^T(\Gamma)$. The a priori estimate is (2.13), i.e. $M = C^{|K|}k_1! \ldots k_m!$. It follows from assertion I) of Lemma 4.5 that the derivative (4.6) can be nonvanishing only if $\{A_1, \ldots, A_m, B_1, \ldots, B_q\}$ is connected. Since $\operatorname{diam} B_i \leqq r$ and by definition of $d(A_1, \ldots, A_m)$ it follows that $q \geqq c \cdot d(A_1, \ldots, A_m)$ for some constant $c = c(r, v)$. This immediately gives the asserted estimate (2.16).

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