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Selfdecomposable Distributions for Maxima of Independent Random Vectors

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Summary. In the present paper the limit laws for conveniently normalized multivariate sample extremes are characterized by means of the decomposability of probability distributions. Continuous automorphisms of $\bar{\mathbf{R}}^d = [-\infty, \infty)^d$ with respect to the operation "$\vee$" defined by $x \vee y = (\max(x_i, y_i), i = 1 \ldots d)$ are treated as norming mappings. An integral representation of the limit distributions is found using their log-concavity and a decomposition of $\bar{\mathbf{R}}^d$ in orbits of the norming family. Finally an example is given as an illustration.

1. Introduction

In the classical theory selfdecomposable laws appear to be limit laws for normalized sums of independent random variables. Linear mappings are used as norming ones since they (and only they) preserve the summing operation. This paper is aimed at studying the corresponding class of limit laws on $\mathbf{R}^d$ in a stochastic model with a $\vee$-operation (instead of "+") between the random vectors defined by $\mathbf{X} \vee \mathbf{Y} = (\max(X^{(i)}, Y^{(i)}), i = 1 \ldots d)$. Denote the set of all df's on $\mathbf{R}^d$ by $\mathscr{P}$. For $\mathbf{Z} = \mathbf{X} \vee \mathbf{Y}$, where $\mathbf{X}, \mathbf{Y}$ are independent with df's $F_{\mathbf{X}}, F_{\mathbf{Y}}$, it holds $F_{\mathbf{Z}} = F_{\mathbf{X}} \cdot F_{\mathbf{Y}}$. The unit (neutral) element of $\mathscr{P}$ with respect to the multiplication is the distribution $\delta_{-\infty}$ degenerated at the point $-\infty$, and $\delta_{-\infty} \notin \mathscr{P}$. Even if $F_{\mathbf{Z}}$ belongs to $\mathscr{P}$, we cannot say the same for its components $F_{\mathbf{X}}$ and $F_{\mathbf{Y}}$, because one of them may possess "mass" at $-\infty$. Roughly speaking, this is why a number of problems concerning decomposition in components can not be solved satisfactory in the structure $(\mathscr{P}, \cdot)$ endowed with the topology of the weak convergence. Here we shall consider random vectors in $\bar{\mathbf{R}}^d = [-\infty, \infty)^d$ and expand $\mathscr{P}$ to the set $\mathscr{F} = \{$df's F on $\bar{\mathbf{R}}^d$: $\lim F(x_1, \ldots, x_d) \geqq 0$, $x_i \to -\infty\}$, $i \in \{1, \ldots, d\}$. Unfortunately, the topology of the weak convergence does not remain relevant in the structure $(\mathscr{F}, \cdot)$, thus we have constructed in [6] a more successful metric $\mathscr{L}$. However, in the present paper we shall try to avoid using the unconventional $\mathscr{L}$-convergence and in the same time to stay in the framework of $(\mathscr{F}, \cdot)$. This

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is possible under the restriction that all considered limit df's F belong to $\mathscr{P}$ and $\{x: 0<F(x)<1\}=\mathbf{R}^d$. In this case the weak convergence can be used instead of $\mathscr{L}$. So we suppose this restriction to be in force further on.

Let $\mathbf{X}_k=(X_k^{(i)}, i=1 \dots d)$, $k=1, \dots, n$, be independent random vectors in $\bar{\mathbf{R}}^d=[-\infty, \infty)^d$ from which we form the random sequence

$$\mathbf{Z}_n=\mathbf{X}_1 \vee \dots \vee \mathbf{X}_n=(\max_{1 \leq k \leq n} X_k^{(i)}, i=1 \dots d). \tag{1}$$

We denote by GMA$(\bar{\mathbf{R}}^d)$ the set of all continuous $\vee$-automorphisms L of $\bar{\mathbf{R}}^d$. Hence L preserves $\vee$-operation, i.e.

$$L(x \vee y)=L(x) \vee L(y) \tag{2}$$

and there exists the inverse mapping L^{-1}. Let the sequence F_n of df's of normalized maxima (1) converges weakly to a nondegenerate df F, i.e.

$$F_n(x):=\Pr(L_n^{-1}\mathbf{Z}_n<x) \xrightarrow{w} F(x), \quad n \to \infty, \ L_n \in \mathrm{GMA}(\bar{\mathbf{R}}^d). \tag{3}$$

Assume the uniformity assumption

$$\max_{1 \leq k \leq n} (1-\Pr(L_n^{-1}\mathbf{X}_k<x)) \to 0, \quad n \to \infty, \tag{u.a.}$$

holds whenever x is a continuity point of F and $F(x)>0$. Then the possible limit distributions in (3) are said to belong to the class max-L.

It is known [2] that the class MID of max-infinite divisible laws consists of all limit df's for random sequences of the form $\mathbf{Z}_n=\mathbf{X}_{n1} \vee \dots \vee \mathbf{X}_{nk(n)}$ under (u.a.). In our model $\mathbf{X}_{nj}=L_n^{-1}\mathbf{X}_j$, so max-$L \subset$ MID. Thus, two more problems occur here, namely

(A) the form and the properties of the possible norming mappings L_n satisfying (2), (u.a.) and (3); (B) the characterization of the limit class max-L.

The present paper gives the solution of these problems. In Sect. 2 we analyse the properties of the norming sequence L_n by solving the functional equation (2) and by proving a theorem on preserving the type. The latter establishes the existence of a family of mappings $T_\beta \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$, $\beta \in (0, 1]$, with respect to which the limit df $F(x)$ is max-selfdecomposable, i.e. it can be decomposed into a product of $F(T_\beta x)$ and a MID-df. We denote the class of all max-selfdecomposable df's by MSD and state that max-L coincides with MSD. The main result of the paper is given in Sect. 3 where we concentrate on the integral representation of a MSD-df F. The basic idea consists of decomposing $\mathbf{R}^d$ in orbits of the norming family and finding the corresponding decomposition of the exponential measure of F. Then the log-concavity of F leads to the representation formula.

It is worth mentioning two papers on the multivariate sample extremes which appeared in 1977. Balkema and Resnick [2] dealt with a characterization of the MID-df's by their exponential measures, whereas de Haan and Resnick [4] contained a description of the class MS of max-stable laws. An year later the monograph [3] was published where necessary and sufficient conditions were found for weak convergence to a MS-df (Theorem 5.3.1) under linear normalization. The present investigation is based on the above mentioned works as well as on [5] and [6].

2. Class max-L Under Nonlinear Normalization

Let us start with the problem (A). We assume that (2), (u.a.) and (3) hold throughout this section. Since the random sequence (1) is nondecreasing it is reasonable to assume at hoc that the norming sequence L_n is also nondecreasing in n.

First we discuss (2) as a functional equation for the $\vee$-endomorphisms of $\bar{\mathbf{R}}^d$. Recently, the problem of describing the latter was posed in [9]. In order to answer it, let us consider a mapping L: $\bar{\mathbf{R}}^d \to \bar{\mathbf{R}}^d$, $L(x) = (l_1(x), \ldots, l_d(x))$, preserving the $\vee$-operation. Its coordinate functions l_i: $\bar{\mathbf{R}}^d \to \bar{\mathbf{R}}^1$, $i = 1, \ldots, d$, also preserve " $\vee$ " and the following lemma holds.

Lemma 1. *The functions*

$$l(x_1, \ldots, x_d) = \bigvee_{k=1}^{d} l^{(k)}(x_k), \qquad l^{(k)}: \bar{\mathbf{R}}^1 \to \bar{\mathbf{R}}^1, \tag{4}$$

monotonous and nondecreasing in each coordinate, are the unique nontrivial solutions of the functional equation

$$l(x \vee y) = l(x) \vee l(y), \qquad l: \bar{\mathbf{R}}^d \to \bar{\mathbf{R}}^1.$$

Since the proof is entirely analogous to that of the corresponding result for the linear operators we will omit it.

Note that the class of the mappings preserving " $\vee$ " on $\bar{\mathbf{R}}^d$ is closed under the composition $(L \cdot T)(x) = L(Tx)$, passing to the inverse mapping (if it exists) and to a limit.

Thus, according to Lemma 1, the coordinate functions of the norming mapping L_n have the form (4). Additionally L_n is assumed to be continuous and invertible. The latter means that the coordinate functions (4) are strictly increasing in each coordinate. These properties lead to a further specifying of (4), namely

Corollary of Lemma 1. *Let* $L \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$. *Then there exists a permutation* σ *of the coordinats such that*

$$L(x_1, \ldots, x_d) = (l_1^{(\sigma 1)}(x_{\sigma 1}), \ldots, l_d^{(\sigma d)}(x_{\sigma d})). \tag{4a}$$

Let us consider the restriction imposed on the sequence L_n by (u.a.). The limit df F in (3) belongs to MID, hence there exists $q = \inf\{x: F(x) > 0\} \in \bar{\mathbf{R}}^d$ [2]. The set $\{x: F(x) > 0\}$ forms a semigroup with respect to the $\vee$-operation whose unit is q. So (u.a.) implies the asymptotic closeness of $L_n^{-1}\mathbf{X}_k$ to the semigroup unit. In the introduction we assumed that $\{0 < F < 1\} = \mathbf{R}^d$, so $q = -\infty$.

Therefore we concentrate on the weak convergence (3). For all $n \geqq 1$ the df $F_n(x) = \Pr(\mathbf{Z}_n < L_n x)$ decomposes in a product $F_n = F_n^{(1)} \cdot F_n^{(2)}$ with

$$F_n^{(1)}(x) := \Pr(\mathbf{Z}_{[n\beta]} < L_n x), \qquad \beta \in (0, 1),$$
$$F_n^{(2)}(x) := \Pr\left(\bigvee_{k=[n\beta]+1}^{n} \mathbf{X}_k < L_n x \right). \tag{5}$$

Here $[x]$ means the integer part of x. Now we make use of the following lemma which is proved in [6] for the onedimensional case. Taking into account the multivariate Helly theorem we may easily extend the proof to the case $\mathbf{R}^d$. The notion of max-compactness used further is also studied in [6].

Definition. A set $\mathscr{A} \subseteq \mathscr{F}$ is called max-*compact* if for $\varepsilon > 0$ there exists a $x_\varepsilon \in \mathbf{R}^d$ such that $\sup\{1 - F(x_\varepsilon): F \in \mathscr{A}\} \leqq \varepsilon$.

For $\mathscr{A} = \{F_n\}$ max-compactness means that each subsequence $F_{n'}$ contains a convergent in $\mathscr{F}$ subsubsequence.

Lemma 2. *Suppose that for all $n \geqq 1$ one has $G_n = G_n^{(1)} \cdot G_n^{(2)}$ and $G_n \xrightarrow{w} G$, $G \in \mathscr{P}$. Then the sequences $G_n^{(1)}$ and $G_n^{(2)}$ are* max-*compact and if for $\{n'\} \subseteq \{n\}$, $n' \to \infty$,*

$G_{n'}^{(i)} \xrightarrow{w} G^{(i)}$, $i = 1, 2$, then $G^{(i)} \in \mathscr{F}$ and $G = G^{(1)} \cdot G^{(2)}$.

In our case Lemma 2 implies that for $\beta \in (0, 1)$ a sequence $\{n'\} \subseteq \{n\}$ and distributions S_β, $F_\beta \in \mathscr{F}$ exist, such that for $n' \to \infty$ the sequences $F_{n'}^{(1)}$ and $F_{n'}^{(2)}$ defined in (5) converge weakly to S_β and F_β, respectively. Furthermore $F = S_\beta \cdot F_\beta$. It can be directly seen that S_β and F_β belong to MID. Our next aim is to prove an explicit description of the component S_β. For that reason we need an analogue of the known Khinchin theorem on preserving the type. We recall, two df's G and S are said to belong to the same type if there is such a mapping $T \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$ that $S(x) = G(Tx)$. Further, a d-dimensional df G is called nondegenerate if all its onedimensional marginals $G^{(i)}$, $i = 1, \ldots, d$, are nondegenerate.

Theorem 1 (on preserving the type). *Suppose a sequence G_n of df's converges weakly to a nondegenerate df $G \in \mathscr{P}$, $\{0 < G < 1\} = \mathbb{R}^d$, and for all $n \geqq 1$ the mappings $T_n \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$ satisfy $T_n(x) \geqq x$. If the sequence $G_n(T_n x)$ converges weakly to a nondegenerate distribution $S \in \mathscr{F}$, $\{0 < S < 1\} = \mathbb{R}^d$, then both G and S belong to the same type.*

In particular it means that S is also a df from $\mathscr{P}$.

Proof. By assumption the sequence $T_n(x) = (t_{nj}(x),\ j = 1 \ldots d)$ is bounded from below. We will show that it has an upper bound, too. To obtain a contradiction we assume that $t_{nj}(x) \to \infty$, $n \to \infty$, for some j. Let $j = 1$. According to Lemma 1 there is an index $i \in \{1, \ldots, d\}$ and a subsequence $\{n'\} \subseteq \{n\}$ such that for $n = n' \to \infty$ $t_{n1}^{(i)}(x_i) \to \infty$.

The sequence $\{G_n \cdot T_n\}$ is max-compact, hence there exists such a x_ε that for all $n \geqq 1$ and $\varepsilon > 0$ $G_n(T_n x_\varepsilon) \geqq 1 - \varepsilon$. The latter can be rewritten as $P_n(A_{n\varepsilon}) \geqq 1 - \varepsilon$ where P_n is the probability measure corresponding to G_n and $A_{n\varepsilon} := \{x \in \bar{\mathbf{R}}^d: x \leqq T_n(x_\varepsilon)\}$. Define a $(d-1)$-dimensional hyperplane $H := \{x \in \bar{\mathbf{R}}^d: x_i = c\}$, $c = $ constant, and continuous functions $f_k: \bar{\mathbf{R}}^d \to \bar{\mathbf{R}}^1$, $k \geqq 1$, $f_k(x) := 1 - 1/t_{k1}(x)$. For sufficient large k $0<f_k<1$ and $f_k(x)\to 1$, $k\to\infty$. Now using the notation P_G for the probability measure corresponding to G and $I_H(x)$ for the indicator function of H we may conclude that

$$\begin{aligned} P_G(H) &\sim \int_H f_k(x)\,dG(x) \xleftarrow[n]{} \int_H f_k(x)\,dG_n(x) \\ &\geqq \int_{A_{n\varepsilon}} I_H(x)\, f_k(x)\,dG_n(x) \\ &\geqq \inf\{f_k(x)\colon x\in H\cap A_{n\varepsilon}\}\, P_n(A_{n\varepsilon}) \\ &\geqq (1-1/t_{k1}^{(i)}(c))\,(1-\varepsilon). \end{aligned}$$

Letting $k\to\infty$ and $\varepsilon\to 0$ we find that $P_G(H)=1$, hence G must be degenerate.

This contradiction implies the boundedness of the sequence $T_n(x)$. Hence, any subsequence $\{n'\}$ will have a subsubsequence $\{n''\}$ such that $T_{n''}(x)$ converges to a $\vee$-endomorphism $T(x)$ of $\bar{\mathbf{R}}^d$. So for $b=(b_1,\dots,b_d)$, $b_i>0$, (and if we apply the operations " + " and " − " coordinatewise) we get

$$G_{n''}(Tx-b)\leqq G_{n''}(T_{n''}x)\leqq G_{n''}(Tx+b)$$

for sufficiently large n''. Let $n''\to\infty$ and $b\to 0$. We have

$$S(x)=G(Tx) \tag{6}$$

for $x\in\bar{\mathbf{R}}^d$ such that $T(x)$ is a continuity point of G belonging to $\{t\colon G(t)>0\}$. Now, the mapping T defined by (6) does not depend on the choice of $\{n''\}$. Further $T(-\infty)=-\infty$, since T satisfies (2). This implies that S can have no defect at $-\infty$, i.e. $S\in\mathscr{P}$.

It remains to prove that $T\in\mathrm{GMA}(\bar{\mathbf{R}}^d)$. Let us consider the sequence $T_n^{-1}(x)$ for which $T_n^{-1}(x)\leqq x$ holds. We show that it is bounded from below as well. Indeed, let us assume that there is a subsequence $\{n'\}$, $n'\to\infty$, such that $T_{n'}^{-1}(x)\to-\infty$. Then $T_{n'}^{-1}(x)<-N$ becomes true for sufficiently large n' and $N>0$. Thus we have $x=T_{n'}\cdot T_{n'}^{-1}x<T_{n'}(-N)$. According to $T_n(-\infty)=-\infty$ one can choose N so large that $T_{n'}(-N)$ becomes smaller than x. But this is in contradiction to the previous inequality. Now passing to a subsequence, if necessary, it may be assumed that for $n\to\infty$ $T_n^{-1}(x)\to R(x)$ and $T_n(x)\to T(x)$ where R and T are $\vee$-endomorphisms of $\bar{\mathbf{R}}^d$. So $T(x)-b<T_n(x)<T(x)+b$ for sufficiently large n and $b>0$. The strict monotonicity of T_n^{-1} leads to

$$T_n^{-1}(Tx-b)<T_n^{-1}\cdot T_n(x)<T_n^{-1}(Tx+b).$$

If $n\to\infty$ and $b\to 0$ we may obtain $R(Tx-0)\leqq x\leqq R(Tx+0)$ for all x. Analogously $T(Rx-0)\leqq x\leqq T(Rx+0)$ holds as well. Comparing the last two relations we conclude that $R\cdot T(x)=x=T\cdot R(x)$ for all $x\in\mathbf{R}^d$, i.e. $R=T^{-1}$ and the two mappings are continuous. This completes the proof.

Let us return to the random sequence $\mathbf{Z}_{[n\beta]}$ from (5). For it we have (passing to a subsequence if necessary) the convergences

$$\Pr(\mathbf{Z}_{[n\beta]} < L_{[n\beta]}\, x) = F_{[n\beta]}(x) \xrightarrow{w} F(x), \qquad \beta \in (0, 1),$$

and

$$\Pr(\mathbf{Z}_{[n\beta]} < L_n\, x) = F_{[n\beta]}(L_{[n\beta]}^{-1} \cdot L_n\, x) \xrightarrow{w} S_\beta(x)$$

when $n \to \infty$. Since L_n is nondecreasing in n, the inequality $T_{n,\beta}(x) \geqq x$ will hold for the mapping $T_{n,\beta} := L_{[n\beta]}^{-1} \cdot L_n \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$. Further, the (u.a.) ensures a non-empty intersection $\{x\colon\ 0 < F(x) < 1\} \cap \{x\colon\ 0 < S_\beta(x) < 1\}$ for all $\beta \in (0, 1]$. By assumption F is nondegenerate, hence such is S_β. Now we are able to apply Theorem 1. According to it there exists a mapping $T_\beta \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$ such that $S_\beta(x) = F(T_\beta\, x)$ and if $T_{n,\beta}$, $n \to \infty$, is convergent, then $T_\beta = \lim T_{n,\beta}$.

We end discussing the asymptotic properties of the norming sequence L_n by the following

Lemma 3. *Let $F \in$ max-L with respect to the norming sequene L_n. The family $\mathcal{T} = \{T_\beta\colon\ \beta \in (0, 1]\}$ of continuous $\vee$-automorphisms of $\bar{\mathbf{R}}^d$, such that*

$$T_\beta(x) = \lim L_{[n\beta]}^{-1} \cdot L_n(x), \qquad n \to \infty, \tag{7}$$

forms an one-parameter semigroup with following properties:

(i) *$T_\beta(x)$ satisfies the functional equation of Abbel's type*

$$T_s(T_\beta\, x) = T_{s\cdot\beta}(x), \qquad s, \beta \in (0, 1], \tag{8}$$

with boundary conditions $T_1(x) = x$ and

$$\lim T_\beta(x) = \infty, \qquad \beta \to 0; \tag{9}$$

(ii) *$T_\beta(x)$ is increasing in x and decreasing in β. Moreover $T_\beta(x) > x$ for $\beta \in (0, 1)$;*

(iii) *there exists such a reversible continuous mapping $H\colon \mathbf{R}^d \to \mathbf{R}^d$ which generates $\mathcal{T}$ by*

$$T_\beta(x) = H^{-1}(H\,x - e \cdot \log \beta), \qquad e = (1, \ldots, 1) \in \mathbf{R}^d.$$

We would briefly sketch out the proof of Lemma 3:

(i) Let $\beta, s \in (0, 1)$ and $n \to \infty$. Then the equality

$$L_{[n\beta s]}^{-1} \cdot L_n(x) = (L_{[n\beta s]}^{-1} \cdot L_{[n\beta]})\,(L_{[n\beta]}^{-1} \cdot L_n)\,(x)$$

implies (8). From (7) one can see that $T_1(x) = x$ and moreover $T_\beta^{-1} = T_{1/\beta}$. Thus (8) is valid for all positive parameters. In order to show (9) note that

$$-\log \Pr(\mathbf{Z}_{[n\beta]} < L_n\, x) \sim \sum_{i=1}^{[n\beta]} (1 - \Pr(\mathbf{X}_i < L_n\, x))$$

$$\to -\log F(T_\beta\, x), \qquad n \to \infty.$$

Setting here $\beta = 1/n \to 0$ we obtain (9).

(ii) $T_\beta \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$, hence it is increasing in x. Since L_n is nondecreasing in n, we have $T_\beta(x) \geqq T_s(x)$ for $\beta < s$. If we assume that $T_\beta(x) = T_s(x)$, then there is a $t \in (0,1)$, $t=\beta/s$, with $T_t(x)=x$. Therefore, for all $n \geqq 1$ $T_{t^n}(x)=x$. But this leads to a contradiction to (9). Consequently T_β is strictly decreasing in β.

(iii) This statement is a modification of Theorem 20 in [7] which has been adapted to our model.

For the family $\{T_\beta^{-1}: \beta \in (0,1]\}$ one may determine also properties analogous to these of Lemma 3. In particular

$$\lim T_\beta^{-1}(x) = -\infty, \qquad \beta \to 0. \tag{9a}$$

Thus we complete the answer of A) and go on to problem B) concerning the characterization of the class max-L. In fact, we have already shown that, if a nondegenerate df F belongs to max-L (i.e., (3) and (u.a.) are valid), there exists a one-parameter semigroup $\mathscr{T}=\{T_\beta: \beta \in (0,1]\}$, $T_\beta \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$, such that F can be decomposed in the product

$$F(x) = F(T_\beta x)\, F_\beta(x), \qquad \beta \in (0,1], \tag{10}$$

where F_β is a MID-df. The property (10) will be called max-selfdecomposability. Thus, the class MSD of all max-selfdecomposable laws on $\mathbf{R}^d$ contains max-L. The inverse statement is also true and it may be proved as simply as in the onedimensional case [5]. In other words, the following statement holds

Theorem 2. *The class* max-L *coincides with the class MSD.*

Using the characteristic decomposition (10) rewritten as $F \circ H^{-1}(Hx)/F \circ H^{-1}(Hx - e \cdot \log \beta) = F_\beta(x)$ one may observe that the function $f(x) = -\log F \circ H^{-1}(x)$, $f: \mathbf{R}^d \to \mathbf{R}^1_+$, is decreasing and convex. Hence F is continuous and it has the expression

$$F(x) = \exp(-f(Hx)) \tag{11}$$

if and only if F is a MSD-df with respect to the semigroup $\mathscr{T}=\{T_\beta(x) = H^{-1}(Hx - e \cdot \log \beta): \beta \in (0,1]\}$ generated by H.

Evidently, any MSD-df is also a MID-df. Conversely, let F belongs to MID and let $\log(F \cdot H^{-1})$ be non concave. Then F cannot be a MSD-df with respect to $\mathscr{T}$.

We call the mapping H a parameter of the max-selfdecomposability of F. Denote by $\mathscr{H}(F)$ the set of all reversible continuous mappings $H: \mathbf{R}^d \to \mathbf{R}^d$, $H(\pm\infty) = \pm\infty$, such that $\log(F \cdot H^{-1})$ is concave. If H and $G \in \mathscr{H}(F)$, then F is a MSD-df with respect to $\mathscr{T}$ and also to the semigroup $\{G_\beta = G^{-1}(G - e \cdot \log \beta): \beta \in (0,1]\}$.

The question about the uniqueness of (11) could be reasonable posed here. It may be answered by applying the following

Corollary of Theorem 1. *Suppose*

$$\Pr(\mathbf{Z}_n < L_n x) \xrightarrow{w} F_1(x) = F_1(T_\beta x)\, F_{1\beta}(x)$$

and $\qquad n \to \infty,$

$$\Pr(\mathbf{Z}_n < M_n x) \xrightarrow{w} F_2(x) = F_2(G_\beta x)\, F_{2\beta}(x),$$

where F_1 *and* F_2 *are nondegenerate df's. Then there exists a mapping* $R \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$ *such that* $F_1(x) = F_2(Rx)$ *and* $T_\beta = R^{-1} \cdot G_\beta \cdot R$, *i.e.* $H = G \cdot R$.

Thus, we may say that the expression (11) is unique up to a mapping $R \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$ such that P_F is R-invariant. In the next section we return to (11) and proceed with discussing problem B).

3. Integral Representation of a MSD-Exponential Measure

We remind a result from [2]: a df F (in our assumption $\{0<F<1\}=\mathbf{R}^d$) belongs to MID if and only if there is a measure μ on $\bar{\mathbf{R}}^d$, σ-finite everywhere except the point $q=-\infty$, such that $F(x)=\exp(-\mu(A_x^c))$. Here $x=(x_1, \ldots, x_d)$, $A_x = [-\infty, x_1] x \ldots x [-\infty, x_d]$, A_x^c is the complement of A_x. The measure μ is called an exponential measure (exp.m.) of F. Obviously $t\mu$, $t>0$, is also a MID-exp.m.

Let μ and ν_β be exp.m.'s corresponding to the df's F and F_β from (10). This characteristic decomposition is equivalent to

$$\mu(A)-\mu(T_\beta A)=\nu_\beta(A)\geqq 0, \tag{12}$$

for $\beta\in(0, 1]$ and for all subsets A of the Borel σ-algebra $\mathfrak{B}^d$.

Our next goal is to give a convenient decomposition of $\bar{\mathbf{R}}^d$ and find a corresponding decomposition of μ.

First, let us consider the group (under composition) $\mathscr{L}=\{L_t: t>0\}$ of mappings $L_t(x)$ increasing in t and x and generated by $\mathscr{T}$ as follows

$$L_t=\begin{Bmatrix} T_t^{-1}, & t\in(0, 1) \\ T_{1/t}, & t\geqq 1 \end{Bmatrix}=H^{-1}(H+e\cdot\log t), \qquad t>0. \tag{13}$$

Let us define an orbit O_x of $\mathscr{L}$ passing through a point x as $O_x := \{z=L_t(x): t>0\}$. Each orbit begins from $q=-\infty$ and gets up to ∞, according to (9) and (9a). If x_1 and x_2 are distinct points of O_x then there is a $s>0$ such that $x_1=L_s(x_2)$. Hence, each point $x\in\mathbf{R}^d$ lies on one orbit only, i.e. the set of all orbits forms a decomposition of $\mathbf{R}^d$. In this respect there are Borel subsets of $\mathbf{R}^d$ which intersect each orbit of $\mathscr{L}$ at exactly one point. For example, the set $B=\{x^*\in\mathbf{R}^d: \max(x_1^*, \ldots, x_d^*)=1\}$ is of this type. Now, an arbitrary $x\in\mathbf{R}^d$ may be represented by $x=L_s(x^*)$, $x^*\in B$, $s>0$. The set B becomes compact by including the points of ∂B. This results in adding the axes through $q=-\infty$ to the set of the orbits and in this way obtaining a decomposition $\{O_{x^*}: x^*\in B\}$ of $\bar{\mathbf{R}}^d$.

Second, according to the formula for total probability, μ generates a family $\{\mu_{x^*}: x^*\in B\}$ of conditional exp.m.'s μ_{x^*} with $\operatorname{Supp}\mu_{x^*}$ equal to O_{x^*} such that for all $A\subset A_{-N}^c$, $N>0$ fixed,

$$\mu(A)=\int_B \mu_{x^*}(A)\, P(dx^*) \tag{14}$$

where P is a probability measure on B, depending on μ. (In the next section this dependence is explicitly given.) Formula (14) may be interpreted as a decomposition of μ consistent to $\mathscr{L}$. The notion of a "conditional exponential measure" completely corresponds to that of conditional probability measure as for $A \subset A^c_{-N}$ we have $\mu(A) = t\,p(A)$, $t > 0$, p-probability measure on $\mathbf{R}^d$.

Third, the property (12) expressed by $\mathscr{L}$ means: $\mu(A) - \mu(L_t A) = \nu_{1/t}(A) \geqq 0$, $t \geqq 1$, i.e. μ is a MSD-exp.m. with respect to the semigroup $\{L_t\colon t \geqq 1\}$. Each conditional exp.m. inherits this property[1]. Therefore, there exists an one-to-one correspondence between the exp.m. μ_{x^*} and a certain function $f_{x^*}\colon \mathbf{R}^d \to \mathbf{R}^1_+$ which is monotonous, decreasing and convex. Indeed, by virtue of (11)

$$f_{x^*}(H\,y) = \mu_{x^*}(A^c_y), \qquad y \in \mathbf{R}^d,$$

with $f_{x^*}(\infty) = 0$. Hence f_{x^*} has a negative (for instance left-sided) derivative f'_{x^*} satisfying

$$f_{x^*}(a) = -\int_a^\infty f'_{x^*}(y)\,dy.$$

Since $H^{-1}(a) \in \mathbf{R}^d$ there are $s > 0$ and $x_a \in B$ such that $H^{-1}(a) = L_s(x_a)$, i.e. $a = H(x_a) + e \cdot \log s$. Thus

$$f_{x^*}(a) = \mu_{x^*}(A^c_{L_s x_a} \cap O_{x^*}).$$

Let us determine the set in the brackets. Denoting $H(x) = (h_i(x),\ i = 1 \dots d)$ we have

$$\begin{aligned} A_{L_s x_a} \cap O_{x^*} &= \{y \in \mathbf{R}^d\colon y = L_t(x^*) \leqq L_s(x_a)\} \\ &= \{L_t(x^*)\colon h_i(x^*) + \log t \leqq h_i(x_a) + \log s,\ i = 1, \dots, d\} \\ &= \{L_t(x^*)\colon t \leqq s \cdot \min_{1 \leqq i \leqq d} \exp(h_i(x_a) - h_i(x^*))\}. \end{aligned}$$

We abbreviate the expression on the right hand side of the last inequality by $c(x^*, a)$ and obtain

$$A^c_{L_s x_a} \cap O_{x^*} = \{L_t(x^*)\colon t > c(x^*, a)\}.$$

Note that $c(x^*, a) \to 0$ when $a \to -\infty$. Let us set

$$m_{x^*}(s) := \mu_{x^*}\{L_t(x^*)\colon t > s\}. \tag{15}$$

The function $m_{x^*}(s)$ defined on $\mathbf{R}^1_+$ is convex, decreasing and $m_{x^*}(0) = \infty$, $m_{x^*}(\infty) = 0$. For $x^* \in \partial B$ we set $m_{x^*}(s) = \infty$ when $s = 0$ and $m_{x^*}(s) = 0$ when $s > 0$. Then

$$f_{x^*}(a) = m_{x^*}(c(x^*, a)) = -\int_{c(x^*, a)}^\infty dm_{x^*}(t).$$

Finally substituting this expression in (14) and letting $N \to \infty$ we get

Theorem 3. *A df F belongs to MSD with a parameter H if and only if there is a Borel subset B of $\mathbf{R}^d$ which intersects each orbit of $L_t(x) = H^{-1}(H x + e \cdot \log t)$*

---

[1] More precisely, this statement is true for P-almost all $x^* \in B$

at exactly one point, and a probability measure P on B so that for all $x > -\infty$ the corresponding exp.m. μ may be expressed as

$$\mu(A_x^c) = \int_B -\left[\int_{c(x^*, Hx)}^{\infty} dm_{x^*}(t)\right] P(dx^*). \tag{16}$$

Here $c(x^*, Hx)$ is a positive constant and $m_{x^*}(t)$ is a non-negative decreasing, convex function defined on $\mathbf{R}_+^1$ as in (15).

Proof. The integral in (16) is of the form $f(Hx)$ given by (11) which is a sufficient condition for μ to be a MSD-exp.m. We have already shown the necessity of (16).

Analogous integral representation of selfdecomposable (Levy) probability measures in the classical model (with summing operation between the random vectors) is investigated in [8]. The method used there consists of finding the extreme points of a certain convex set of measures. Then the Choquet theorem ([10], § 3) yields the representation formula. The same method is applied to operator-stable distributions in [7]. The decomposition (14), as one may observe, is nothing else but a Choquet representation, because

- the set $\mathfrak{M}$ of all MSD-exp.m's with respect to a given semigroup $\mathscr{L}$ is convex and max-compact;
- the extreme points of $\mathfrak{M}$ are exactly the exp.m's μ_{x^*} concentrated on the orbits of $\mathscr{L}$.

We illustrate (16) in the next section.

4. Example

A df F on $\mathbf{R}^d$ is called max-stable (briefly $F \in \mathrm{MS}$) with respect to a group $\mathscr{L} = \{L_t : t > 0\}$, $L_t \in \mathrm{GMA}(\bar{\mathbf{R}}^d)$, if

$$F(x) = F^t(L_t x) \tag{17}$$

for all $t > 0$. Evidently, $\mathrm{MS} \subset \mathrm{MSD}$. The result corresponding to Theorem 3 for MS-df's has been studied extensively in recent years (for instance [3], Theorem 5.4.3, and [4], Theorem 2). The integral representation given there fits completely to Theorem 3. This fact is illustrated below by (18) and (19).

The exp.m. μ corresponding to a df $F \in \mathrm{MS}$ satisfies $t^{-1}\mu(A) = \mu(L_t A)$, $A \in \mathfrak{B}^d$. This means Supp μ is quasi-invariant under $\mathscr{L}$. The same applies to the conditional exp.m. μ_{x^*}, so we get $m_{x^*}(s) = t\, m_{x^*}(t s)$. Hence $dm_{x^*}(t) = -m_{x^*}(1)\, dt/t^2$. We notice that the max-stability parameter H of F (the generator of $\mathscr{L}$) has the form $H(x_1, \ldots, x_d) = (h_1(x_1), \ldots, h_d(x_d))$ where h_i is a max-stability parameter of the corresponding marginal $F^{(i)}$ of F, $i = 1, \ldots, d$. Now we have by Theorem 3

$$\begin{aligned}\mu(A_x^c) &= \int_B \max_{1 \le i \le d} \exp(h_i(x_i^*) - h_i(x_i))\, dP'(x^*)\\ &= \int_B \max_{1 \le i \le d} \frac{\log F^{(i)}(x_i)}{\log F^{(i)}(x_i^*)}\, dP'(x^*),\end{aligned} \tag{18}$$

where $x=(x_1, \ldots, x_d)$, $x^*=(x_1^*, \ldots, x_d^*)$, and $dP'=m_{x^*}(1)\,dP$ is a positive Borel measure on B, satisfying

$$1=\int_B \exp(h_i(x_i^*))\,dP'(x^*), \qquad i=1, \ldots, d. \tag{19}$$

Formula (18) enables us to construct multivariate max-stable df's with given marginals.

The simplest example is the case of a df F on $\mathbf{R}_+^2$ whose marginals $F^{(i)}$, $i=1,2$, are max-stable with parameters $h_i(x_i)=\log x_i$, i.e. $F^{(i)}(x_i)=\exp(-1/x_i)$, $x_i \geqq 0$. It means that $F \in \mathrm{MS}$ with respect to $L_t(x)=(t x_1, t x_2)$, $t>0$. The corresponding decomposition of $\operatorname{Supp} F$ is a bundle of rays starting from the point $q=\inf\{x\colon F(x)>0\}=0$. Let us give B as a union

$$B_1 \cup B_2=\{x^* \in \mathbf{R}_+^2\colon 0 \leqq x_1^* \leqq 1,\ x_2^*=1\} \cup \{x^* \in \mathbf{R}_+^2\colon 0 \leqq x_2^* \leqq 1,\ x_1^*=1\},$$

and

$$dP'(x_1^*, x_2^*)=\begin{cases} dE(x_2^*)\,dP_1(x_1^*) & \text{on } B_1, \\ dE(x_1^*)\,dP_2(x_2^*) & \text{on } B_2. \end{cases}$$

Here $dP_i=c_i\,dG_i$, $c_i>0$, G_i-df on $[0,1]$, $i=1,2$, and $E(x)$ is the distribution concentrated at x. We denote the mean value of G_i by m_i. The constants c_i could be found by means of (19) which results in the system $c_1 m_1+c_2=1$, $c_1+c_2 m_2=1$. The conditional exp.m's are of the form

$$\mu_{x^*}(A_x^c)=\begin{cases} c_1 \max(x_1^*/x_1, 1/x_2) & \text{if } x^* \in B_1, \\ c_2 \max(1/x_1, x_2^*/x_2) & \text{if } x^* \in B_2. \end{cases}$$

Putting $\beta=x_2/x_1$ we come to the following expression

$$-\log F(x_1, x_2)=\begin{cases} 1/x_1\left[c_1 \int_0^\beta (\beta-u)\,dG_1(u)+1\right], & \beta<1, \\ c_1+c_2, & \beta=1, \\ 1/x_2\left[c_2 \int_0^{1/\beta} (\beta^{-1}-u)\,dG_2(u)+1\right], & \beta>1. \end{cases}$$

Finally, let us treat the limit Theorem 2 as a tool for solving approximation problems. The use of nonlinear normalizations in it enriches the class of limit distributions that can serve for approximation of distributions of sample extremes. This is one practical application of (16) and (18).

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References

1. Aczel, J.: Some general methods in the theory of functional equations with one variable. New applications of functional equations (in Russian). Uspehi Mathem. Nauk **3** (69), 3–67 (1956)
2. Balkema, A.A., Resnick, S.I.: Max-infinite divisibility. J. Appl. Probab. **14**, 309–319 (1977)
3. Galambos, J.: The asymptotic theory for extreme order statistics. New York: Wiley 1978 (Russian translation Nauka, Moskow 1984)
4. de Haan, L.., Resnick, S.I.: Limit theory for multivariate sample extremes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **40**, 317–337 (1977)
5. Pancheva, E.: Limit theorems for extreme order statistics under nonlinear normalization (Lect. Notes Math., vol. 1155, pp. 284–309). Berlin Heidelberg New York: Springer 1984
6. Pancheva, E.: General limit theorems for maxima of independent random variables (in Russian). Probab. Theor. Appl. **4**, 31, 730–744 (1986)
7. Sharp, M.: Operator-stable probability distributions on vector groups. Trans. Am. Math. Soc. **136**, 51–65 (1969)
8. Urbanik, K.: Levy's probability measures on Euclidean spaces. Studia Mathem. XLIV, 119–148 (1972)
9. Zolotarev, V.M.: Two analytic problems in the probability theory (in Russian). Abstracts from XX School-Colloquium on probability theory and Mathem. Statist., Bakuriani 1986, USSR
10. Phelps, R.: Lectures on Choquet's theorem. New Jersey: Princeton 1966

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