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Probability Theory and Related Fields

On the Equivalence of Three Potential Principles for Right Markov Processes

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Summary. We examine three of the principles of probabilistic potential theory in a nonclassical setting. These are: (i) the bounded maximum principle, (ii) the positive definiteness of the energy (of measures of bounded potential), and (iii) the condition that each semipolar set is polar. These principles are known to be equivalent in the context of two Markov processes in strong duality, when excessive functions are lower semicontinuous. We show that when the principles are appropriately formulated their equivalence persists in the wider context of a Borel right Markov process X with distinguished excessive measure m. We make no duality hypotheses and m need not be a reference measure. Our main tools are the stationary process (Y, Q_m) associated with X and m, and a correspondence between potentials μU and certain random measures over (Y, Q_m).

1. Introduction

Our object in this paper is to examine three of the "principles" of probabilistic potential theory in a nonclassical setting. Let us recall these principles in the "classical" context of two standard processes X and $\check{X}$ in duality with respect to a σ-finite reference measure m. Assuming that X and $\check{X}$ are transient, their respective potential kernels U and $\check{U}$ have a common density $u(x, y)$:

$$U(x, dy) = u(x, y)\, m(dy), \qquad \check{U}(dx, y) = m(dx)\, u(x, y).$$

If μ is a finite measure on E (the common state space of X and $\check{X}$), then μU is σ-finite, hence an excessive measure (for X). Moreover $\mu U \ll m$ and a version of $d(\mu U)/dm$ that is excessive for $\check{X}$ is given by

$$\check{u}(y) = \int_E \mu(dx)\, u(x, y). \tag{1.1}$$

Consider now the *bounded maximum principle.*

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(M^*) For each finite measure μ with compact support supp (μ), if $\check{u}$ is bounded, then $\sup\{\check{u}(y): y\in E\} = \sup\{\check{u}(y): y\in \operatorname{supp}(\mu)\}$.

(The "*" is to distinguish this principle from the *maximum principle*, in which $\check{u}$ is not assumed bounded.)

Related to (M^*) is Hunt's hypothesis (H) (the *polarity principle*):

(H) Every set semipolar for X is polar for X.

(Of course, X and $\check{X}$ have the same polar sets, and also the same semipolar sets; see [3, VI].) Blumenthal and Getoor [4] have shown that (M^*) and (H) are equivalent, at least when the excessive functions of $\check{X}$ are lower semicontinuous. Smythe [29] has proved a variant of this result under weaker hypotheses.

The third principle that we consider involves the energy of a signed measure μ relative to the kernel u:

$$I(\mu) = \iint \mu(dx)\, u(x,y)\, \mu(dy),$$

defined whenever the integral exists. The *bounded positivity principle* is this:

(P^*) For each bounded signed measure μ of compact support, if $\int |\mu|(dx)\, u(x,\cdot)$ is bounded, then $I(\mu) \geqq 0$ with equality if and only if $\mu = 0$.

Rao [25] has shown that (P^*) implies (M^*). The converse is true if the excessive functions of $\check{X}$ are lower semicontinuous; we have no reference for this assertion, but see Theorem (5.2) and the attendant remarks. For further background the reader can consult [3–5, 9, 16, 17, 24]; a good reference for classical potential theory is [19].

We shall study analogues of the above principles, and the extent to which they are equivalent, in the context of a Borel right process X coupled with a distinguished excessive measure m. We make no duality hypotheses and m need not be a reference measure. Consequently our three principles depend on the choice of m. Precise statements of these principles are postponed until the next section. Roughly speaking, three modifications are required: (i) "polar" is replaced by the appropriate notion "m-polar"; (ii) the suprema in (M^*) are replaced by "essential suprema" allowing an exceptional m-polar set – this since $d(\mu U)/dm$ will be determined modulo an m-polar set; (iii) only measures μ that charge no m-polar set are admitted in (M^*) and (P^*) (for the reason cited in (ii)). The modified principles are labeled (M_m^*), (H_m) and (P_m^*) respectively. Assuming that m is dissipative, we show that (H_m) and (M_m^*) are equivalent (Sec. 4), that (H_m) implies (P_m^*), and that (P_m^*) implies (H_m) under an extra hypothesis (Sect. 5). In the same sections we briefly discuss how these principles are related to the classical principles.

Our main tools are the Kuznetsov process (Y, Q_m) associated with X and m, and a correspondence between potentials μU and certain random measures over (Y, Q_m). These tools are developed in Sects. 2 and 3.

We use the balance of this section to set up our notation and basic hypotheses. We use mostly standard notation (see, e.g., [3]), but remark here on some specifics: a stochastic interval $[\![S, T[\![$ (for example) is understood to be a subset of $\mathbb{R} \times$ (path space); the Borel σ-field on $\mathbb{R}$ is denoted $\mathscr{R}$; if $(A, \mathscr{A})$ is a measurable space then $\mathscr{A}^+$ denotes the class of positive $\mathscr{A}$-measurable functions on A.

We fix once and for all a right Markov process $X=(X_t, P^x)$ with Borel semigroup (P_t) and Lusin state space $(E, \mathscr{E})$. Let $\Delta \notin E$ be the cemetery for X: we assume only that $P_t 1 \leqq 1$. It is convenient to realize X and (Y, Q_m) as coordinate processes on canonical path spaces Ω and W respectively. Let W denote the space of paths w: $\mathbb{R} \to E \cup \{\Delta\}$ that are E-valued and right continuous on some open interval $]\alpha(w), \beta(w)[\subset \mathbb{R}$, taking the value Δ outside of this interval. (The "dead" path $[\Delta]$: $t \to \Delta$ satisfies $\alpha([\Delta])=+\infty$, $\beta([\Delta])=-\infty$.) Let $(Y_t$: $t \in \mathbb{R})$ denote the coordinate process on W, with associated σ-fields

$$\mathscr{G}_t^0=\sigma\{Y_s: s \leqq t\}, \qquad \mathscr{G}^0=\sigma\{Y_s: s \in \mathbb{R}\}.$$

A family of shift operators is defined on W by

$$(\theta_t w)(s)=\begin{cases} w(t+s), & s>0, t \in \mathbb{R}, \\ \Delta, & s<0, t \in \mathbb{R}. \end{cases}$$

Let $\Omega=\{w \in W$: $\alpha(w)=0$, $Y_{\alpha+}(w)$ exists in $E\} \cup \{[\Delta]\}$ and for $t \geqq 0$ let X_t, $\mathscr{F}^0$, $\mathscr{F}_t^0$ denote the restrictions to Ω of Y_{t+}, $\mathscr{G}^0$, $\mathscr{G}_t^0$ respectively. Since (P_t) is a Borel right semigroup, there is a Borel measurable family $\{P^x, x \in E\}$ of probability measures on $(\Omega, \mathscr{F}^0)$ such that $X=(\Omega, \mathscr{F}^0, \mathscr{F}_{t+}^0, X_t, \theta_t, P^x)$ is a strong Markov realization of (P_t).

Now let Exc denote the class of excessive measures for X: $m \in \mathrm{Exc}$ if and only if m is a σ-finite measure on E such that $m P_t \leqq m$ for all $t>0$. By our right hypotheses on (P_t), and a theorem of Kuznetsov [18], given $m \in \mathrm{Exc}$ there is a unique measure Q_m on $(W, \mathscr{G}^0)$ such that $Q_m([\Delta])=0$,

$$(1.2) \qquad Q_m(Y_t \in A)=m(A), \qquad \forall t \in \mathbb{R}, A \in \mathscr{E},$$

and

$$(1.3) \qquad Q_m(F \circ \theta_T | \mathscr{G}_{T+}^0)=P^{Y_T}(F), \qquad Q_m\text{-a.s.} \quad \text{on } \{\alpha<T<\beta\},$$

for each $F \in (\mathscr{F}^0)^+$ and $(\mathscr{G}_{t+}^0)$-stopping time T. (It is implicit in (1.3) that Q_m restricted to $\mathscr{G}_{T+}^0$ is σ-finite on $\{\alpha<T<\beta\}$; see Mitro [22].) It follows that Q_m is a σ-finite measure on $\mathscr{G}^0$. (Y_t, Q_m) is the *Kuznetsov process* associated with (P_t) and m. Evidently Q_m is *invariant* with respect to the shift operators σ_t, $t \in \mathbb{R}$, defined by

$$(\sigma_t w)(s)=w(t+s), \qquad s \in \mathbb{R}.$$

We close this section by recalling from [10] the balayage operator L_B: Exc $\to$ Exc. Given a nearly Borel set $B \subset E$ define the hitting time

$$\tau_B=\inf\{t>\alpha: Y_t \in B\}.$$

If $m \in \mathrm{Exc}$ then τ_B is a stopping time of the Q_m-completion of $(\mathscr{G}^0_{t+})$. For such m and B we define a measure $L_B m$ on E by

(1.4) $$L_B m(f) = Q_m(f \circ Y_t; \tau_B < t), \quad f \in \mathscr{E}^+.$$

Since $t + \tau_B \circ \sigma_t = \tau_B$ for all $t \in \mathbb{R}$, it is clear that the R.H.S. of (1.4) does not depend on t. Moreover, $L_B m \in \mathrm{Exc}$ and $L_B m \leqq m$. If ξ and m are excessive measures with $\xi \leqq m$, then $L_B \xi \leqq L_B m$; see [10, (5.14)].

Let $U = \int_0^\infty P_t\, dt$ denote the potential kernel of X. From [10, §4] we know that $m \in \mathrm{Exc}$ is dissipative if and only if there is a sequence of potentials $\mu_n U$ with $\mu_n U \uparrow m$. This being the case we have [10, (5.8)]

(1.5) $$L_B m = \uparrow \lim \mu_n P_B U.$$

where P_B is the hitting operator associated with B.

We require one more fact concerning $L_B m$. In what follows "$\wedge$" denotes the infimum in the lattice Exc (endowed with the simple order: $\xi \leqq m$ if and only if $\xi(A) \leqq m(A)$, $\forall A \in \mathscr{E}$). Of course, "$\xi \geqq m$ on G" means $1_G \cdot \xi \geqq 1_G \cdot m$.

(1.6) **Proposition.** [12, (2.7)]. *If G is a finely open, nearly Borel set, then for all $m \in \mathrm{Exc}$,*

$$L_G m = \wedge \{\xi \in \mathrm{Exc} : \xi \geqq m \text{ on } G\}.$$

2. Preliminaries

For the rest of the paper we fix $m \in \mathrm{Exc}$. Various objects defined in the sequel depend on m, but this dependence is seldom acknowledged in our notation.

Let $\mathscr{G}$ denote the Q_m-completion of $\mathscr{G}^0$, and let $\mathscr{G}_t$ (resp. $\hat{\mathscr{G}}_t$) denote $\mathscr{G}^0_{t+}$ (resp. $\hat{\mathscr{G}}^0_t \equiv \sigma\{Y_s\colon s > t\}$) augmented by the Q_m-null sets in $\mathscr{G}$. Let $\mathscr{I}$ denote the class of Q_m-evanescent subsets of $\mathbb{R} \times W$ and put $\mathscr{M} = (\mathscr{R} \otimes \mathscr{G}) \vee \mathscr{I}$. The *optional* and *copredictable* σ-fields are defined on $\mathbb{R} \times W$ by

$$\begin{aligned}\mathscr{O} = \sigma\{Z \in \mathscr{M}^+ &: Z \text{ is } (\mathscr{G}_t)\text{-adapted and right continuous on} \\ &\mathbb{R}, Q_m\text{-a.s.}; Z = 0 \text{ on }]\!]-\infty, \alpha[\![\,\} \\ \hat{\mathscr{P}} = \sigma\{Z \in \mathscr{M}^+ &: Z \text{ is } (\hat{\mathscr{G}}_t)\text{-adapted and right continuous on} \\ &\mathbb{R}, Q_m\text{-a.s.}; Z = 0 \text{ on } [\![\beta, +\infty[\![\,\}.\end{aligned}$$

A $\mathscr{G}$-measurable random time $T\colon W \to \mathbb{R} \cup \{+\infty, -\infty\}$ is optional (resp. copredictable) if $[\![T, +\infty[\![\, \in \mathscr{O}$ (resp. $]\!]-\infty, T]\!] \in \hat{\mathscr{P}}$). The optional (resp. copredictable) projection of a process $Z \in \mathscr{M}^+$ is defined as usual, and is denoted ${}^oZ$ (resp. ${}^{\hat{p}}Z$). See [11] for precise definitions and properties of these projections.

Of central importance is the process (l_t) defined by

(2.1) $$l_t = {}^{\hat{p}}(1_{]\!]\alpha, \beta[\![})_t.$$

A version of l can be chosen such that $l \in \mathcal{O} \cap \hat{\mathcal{P}}$, $0 \leqq l \leqq 1$, and $l_t \circ \sigma_s = l_{t+s}$, $s, t \in \mathbb{R}$. Moreover, setting

$$\Lambda = \{(t, w) \in \mathbb{R} \times W: l_t(w) > 0\},$$

we can (and do) assume that $]\!]\alpha, \beta[\![\subset \Lambda \subset [\![\alpha, \beta[\![$ and that $l_t(w) > 0 \Rightarrow Y_{t+}(w)$ exists in E. See [11, §3] for these facts. We now define an extension $\bar{Y}$ of the basic process Y:

$$\begin{aligned} \bar{Y}_t &= Y_{t+}, \quad -\infty < \alpha = t, \quad l_t > 0, \\ &= Y_t, \quad \text{otherwise.} \end{aligned}$$

Evidently $\bar{Y}_t \circ \sigma_s = \bar{Y}_{t+s}$, $f \circ \bar{Y} \in \mathcal{O} \cap \hat{\mathcal{P}}$ if $f \in \mathcal{E}^+$, and $\Lambda = \{(t, w)$: $\bar{Y}_t(w) \in E\}$. To state an important property of $\bar{Y}$ we need some terminology:

$B \in \mathcal{E}$ is *m-polar* if and only if $Q_m(Y_t \in B$ for some $t \in \mathbb{R}) = 0$;

$B \in \mathcal{E}$ is *m-semipolar* if and only if $Q_m(Y_t \in B$ for uncountably many $t \in \mathbb{R}) = 0$.

An arbitrary $A \subset E$ is m-polar (resp. m-semipolar) if it is contained in a Borel set of the same species. Clearly $B \in \mathcal{E}$ is m-polar if and only if $L_B m = 0$.

(2.2) **Proposition.** [11, (3.22)]. *$B \in \mathcal{E}$ is m-polar if and only if $Q_m(\bar{Y}_t \in B$ for some $t \in \mathbb{R}) = 0$.*

We can regard $(\bar{Y}_t)$ as a process adapted to the reverse filtration $(\hat{\mathcal{G}}_t)$; as such $(\bar{Y}_t)$ is *moderately* Markovian under Q_m. Indeed, following Chung and Walsh [7], and others ([2, 28]), in [11] we constructed a moderate Markov process $\hat{X} = (\hat{X}_t, \hat{P}^x)$ in duality with X relative to m. To state matters precisely we need some notation:

$$\begin{aligned} \hat{\Omega} &= \{w \in W: \beta(w) = 0\} \cup \{[\Delta]\}; \\ \hat{X}_s(w) &= \bar{Y}_{-s}(w), \quad s > 0, \quad w \in \hat{\Omega}; \\ \hat{\theta}_t(w)(s) &= \begin{cases} w(t+s), & \text{if } s < 0, \quad t \in \mathbb{R}, \\ \Delta, & \text{if } s \geqq 0, t \in \mathbb{R}; \end{cases} \\ \hat{\mathcal{F}}_t^0 &= \sigma\{\hat{X}_s: 0 < s \leqq t\}, \quad \hat{\mathcal{F}}^0 = \sigma\{\hat{X}_s: s > 0\}. \end{aligned}$$

Note that $\hat{\theta}_t$: $\{t < \beta\} \to \hat{\Omega}$ and that $\hat{X}_s \circ \hat{\theta}_t = \bar{Y}_{t-s}$ on $\{t < \beta\}$ if $s > 0$. Clearly, $s \to \hat{X}_s(w)$ is left continuous on the random interval $\{s > 0$: $l_{-s}(w) > 0\}$. Theorem (4.6) of [11] asserts the existence of a Borel measurable family $\{\hat{P}^x$: $x \in E\}$ of probability measures on $(\hat{\Omega}, \hat{\mathcal{F}}^0)$ such that $(\hat{X}_t$: $t > 0)$ is moderately Markov under each $\hat{P}^x$, with transition semigroup

$$\hat{P}_t f(x) = \hat{P}^x(f \circ \hat{X}_t), \quad t > 0, \quad f \in \mathcal{E}^+.$$

That is, for each $(\hat{\mathcal{F}}_t^0)$-predictable time S and $t > 0$,

$$\hat{P}^x(f \circ \hat{X}_{S+t} | \hat{\mathcal{F}}_{S-}^0) = \hat{P}_t f \circ \hat{X}_S, \quad \hat{P}^x\text{-a.s.} \quad \text{on } \{0 < S < \infty\}. \tag{2.3}$$

Moreover for each Q_m-copredictable time T and all $F \in (\hat{\mathscr{F}}^0)^+$,

(2.4) $$Q_m(F \circ \hat{\theta}_T \mid \hat{\mathscr{G}}_{T+}) = \hat{P}^{Y_T}(F), \qquad Q_m\text{-a.s.} \quad \text{on } \{l_T > 0\}.$$

Obviously (2.3) and (2.4) imply the duality relation

(2.5) $$m(g \cdot P_t f) = m(\hat{P}_t g \cdot f).$$

It follows easily from (2.5) that if $\xi \in \mathrm{Exc}$ with $\xi \ll m$, then any version $\psi \in \mathscr{E}^+$ of $d\xi/dm$ satisfies (i) $\hat{P}_t \psi \leqq \psi$ m-a.e. ($\forall t > 0$), and (ii) $\hat{P}_t \psi \uparrow \psi$ m-a.e. as $t \downarrow 0$ through any fixed sequence. Thus $d\xi/dm$ is "almost coexcessive". Actually, a truly coexcessive version of $d\xi/dm$ can always be chosen, as the next result indicates. In the sequel a function $f \in \mathscr{E}^*$ (the universal completion of $\mathscr{E}$) is *coexcessive* if it is excessive for the semigroup $(\hat{P}_t)$. A statement $S(x)$ depending on $x \in E$ is true *m-quasi everywhere* (m-q.e.) provided $\{x: S(x) \text{ is false}\}$ is m-polar.

(2.6) **Proposition.** (a) *Given* $\xi \in \mathrm{Exc}$ *with* $\xi \ll m$, *there exists a Borel coexcessive version* $\hat{u}$ *of* $d\xi/dm$; $\hat{u}$ *is uniquely determined* m-q.e.

(b) *If* f *is coexcessive, the there exists a Borel coexcessive function* g *such that* $f = g$ m-q.e. *In particular if* f *is coexcessive, then* $f \circ \bar{Y}_{\cdot} \in \mathcal{O} \cap \hat{\mathscr{P}}$.

Proof. It follows from (2.5) that $m\hat{P}_t \uparrow m$ as $t \downarrow 0$. This fact in hand the existence of $\hat{u}$ follows by the argument of [15, (6.19)]. The uniqueness of $\hat{u}$, and the existence of the modification, g, of f follow as in the proof of [15, (6.11)]. □

We call the density $\hat{u}$ of (2.6b) the *coexessive version* of $d\xi/dm$. (Of course, $\hat{u}$ is really an equivalence class of versions...) It follows from [6, Lemma 2] or [11, §4] that $t \to \hat{u}(\bar{Y}_t)$ has left limits on $]\!]-\infty, \beta[\![$ and right limits on $\mathbb{R}$, Q_m-a.s. By [11, (4.15)] there is a function $\bar{u} \in \mathscr{E}^+$, unique up to an m-polar set, such that

(2.7) $$\bar{u}(\bar{Y}_t) = \lim_{s \downarrow t} \hat{u}(\bar{Y}_s), \qquad \forall t \in \Lambda, \qquad Q_m\text{-a.s.}$$

Since $\bar{u}(\bar{Y}_{\cdot})$ is right continuous on Λ, we refer to $\bar{u}$ as the *fine version* of $d\xi/dm$; see (2.10) below. Of course $\bar{u} = \hat{u}$, m-a.e.; indeed it follows easily from (2.7) that

(2.8) $$\{\hat{u} \neq \bar{u}\} \quad \text{is } m\text{-semipolar.}$$

From (2.7) we also deduce that

(2.9) $$\hat{u} \leqq a \;\; m\text{-q.e.} \Rightarrow \bar{u} \leqq a \;\; m\text{-q.e.}$$

for each $a > 0$. The following result justifies our name for $\bar{u}$.

(2.10) **Proposition.** *Let* $f \in \mathscr{E}^+$ *and suppose that* $t \to f \circ Y_t$ *is right continuous on* $]\alpha, \beta[$, Q_m-a.s. *Then there is a Borel m-polar set* B *such that* $E \backslash B$ *is absorbing for* X *and for* $\hat{X}$, *and* $f|_{E \backslash B}$ *is finely continuous on* $E \backslash B$.

Proof. Define

$$\Omega_0 = \{w \in \Omega: t \to f(X_t(w)) \quad \text{is right continuous on } [0, +\infty[\}.$$

As noted by Meyer [21, p. 236], $A \equiv \{x \in E: P^x(\Omega_0^c) > 0\}$ is a Souslin subset of E. Therefore to show that A is m-polar it suffices to show that each Borel subset of A is m-polar. But this is clear from (1.3) and the optional section theorem [11, (3.16b)] since

$$\{\alpha < T < \beta\} \cap \theta_T^{-1} \Omega_0^c = \{w \in W: \alpha(w) < T(w) < \beta(w), s \to f(Y_s(w)) \text{ is not right continuous on } [T(w), +\infty[\}$$

whenever T is a Q_m-optional time. Thus A is m-polar, and so by [11, (4.14)] there is a Borel m-polar set $B \supset A$ such that $E \setminus B$ is absorbing for X and for $\hat{X}$. By [3, II(4.8)], f is finely continuous at each point of the finely open set $E \setminus B$. □

(2.11) **Definition.** $\mathscr{C}$ denotes the class of finite measures μ on E such that (i) $\mu U \leqq a \cdot m$ for some constant $a = a_\mu > 0$, and (ii) μ charges no m-polar set. Given $\mu \in \mathscr{C}$ we write $\bar{U}(\mu)$ and $\hat{U}(\mu)$ for the fine and coexcessive versions of $d(\mu U)/dm$.

Note that $\bar{U}(\mu) \leqq a$ m-q.e. if $\mu U \leqq a \cdot m$. We shall see in (3.7) that $\hat{U}(\mu) \leqq \bar{U}(\mu)$ m-q.e.

We can now state our analogues of (H), (M^*), and (P^*). The first of these is

(H_m) Every Borel semipolar set is m-polar.

According to [15, (6.13)] every Borel m-semipolar set is the union of a semipolar set and an m-polar set. Thus (H_m) is unchanged if "m-semipolar" is substituted for "semipolar". Next is the bounded maximum principle:

(M_m^*) Given $\mu \in \mathscr{C}$ carried by $A \in \mathscr{E}$, if $\hat{U}(\mu) \leqq 1$ m-q.e. on A, then $\hat{U}(\mu) \leqq 1$ m-q.e.

Finally, define the energy of a signed measure $\mu - \nu$ ($\mu, \nu \in \mathscr{C}$) by

$$I(\mu - \nu) = (\mu - \nu)(\hat{u} - \hat{v}), \tag{2.12}$$

where $\hat{u} = \hat{U}(\mu)$, $\hat{v} = \hat{U}(\nu)$. (Clearly $\hat{u} - \hat{v}$ is uniquely determined off an m-polar set not charged by $\mu + \nu$, and $(\mu + \nu)(\hat{u} + \hat{v}) < \infty$; thus the R.H.S. of (2.12) is well defined and finite.) The bounded positivity principle is this:

(P_m^*) Given $\mu, \nu \in \mathscr{C}$, $I(\mu - \nu) \geqq 0$ with equality if and only if $\mu = \nu$.

As mentioned in section 1, the relationships between these three principles are discussed in Sects. 4 and 5.

We conclude this section with a domination principle involving the *fine* density $\bar{U}(\mu)$. This result extends [11, (7.7)] and is related to work of Azéma [1, 5.8] and Doob [9, 2.IV.13].

Let μ be a measure on E that charges no m-polar set, such that μU is σ-finite. Then $\mu U \in \text{Exc}$ and $\mu U \ll m$. (For if $m(A) = 0$ then $\{U 1_A > 0\}$ is finely open and m-null, hence m-polar; thus $m(A) = 0$ implies $\mu U(A) = \mu(U 1_A) = 0$.) Let

ξ be a second excessive measure such that $\xi \ll m$. Let $\bar{u}$ and $\bar{v}$ denote fine versions of $d(\mu U)/dm$ and $d\xi/dm$ respectively.

(2.13) **Theorem.** If $\bar{u} \leqq \bar{v}$ μ-a.e., *then* $\bar{u} \leqq \bar{v}$ m-q.e.

Proof. Fix $\varepsilon > 0$ and let G denote the fine interior of $\{\bar{u} < (1+\varepsilon)\bar{v}\}$. Then G is nearly Borel, and $\{\bar{u} \leqq \bar{v}\} \setminus G$ is m-polar (see (2.10) and note that $\{\bar{u} = \infty\}$ is m-polar by [11, (4.15)]). On G we have $\mu U = \bar{u} \cdot m \leqq (1+\varepsilon)\ \bar{v} \cdot m = (1+\varepsilon)\xi$, and so by (1.6),

$$L_G(\mu U) \leqq (1+\varepsilon)\xi. \tag{2.14}$$

But $L_G(\mu U) = \mu P_G U$ by (1.5), and $\varepsilon_x P_G = \varepsilon_x$ if $x \in G$. By hypothesis μ is carried by $\{\bar{u} \leqq \bar{v}\}$, hence by G since $\{\bar{u} \leqq \bar{v}\} \setminus G$ is m-polar. Thus $\mu U = \mu P_G U = L_G(\mu U)$ $\leqq (1+\varepsilon)\ \xi$ by (2.14). Since $\varepsilon > 0$ was arbitrary we must have $\mu U \leqq \xi$. Thus $\bar{u} \leqq \bar{v}$ m-q.e. as required. □

(2.15) **Corollary.** *Keeping the notation of* (2.13), *suppose that μ is carried by $A \in \mathscr{E}$. If $\bar{u} \leqq \bar{v}$ m-*q.e. *on A, then $\bar{u} \leqq \bar{v}$ m-*q.e.

Proof. This follows from (2.13) since μ does not charge the m-polar set $\{\bar{u} > \bar{v}\} \cap A$. □

3. Homogeneous Random Measures

In this section we record several facts from [11, §5] to be used in Sects. 4 and 5.

(3.1) **Definition.** A homogeneous random measure (HRM) is a positive kernel $\kappa = \kappa(w, dt)$ from $(W, \mathscr{G})$ to $(\mathbb{R}, \mathscr{R})$ with the following properties:

(i) For each $w \in W$, the measure $\kappa(w, \cdot)$ is carried by $[\alpha(w), \beta(w)[$. There is a sequence (κ_n) of kernels from $(W, \mathscr{G})$ to $(\mathbb{R}, \mathscr{R})$ such that $Q_m(\kappa_n(\mathbb{R}) = +\infty) = 0$ for all $n \in \mathbb{N}$, and $\kappa = \sum_n \kappa_n$.

(ii) (Homogeneity) For each $s \in \mathbb{R}$, the measure $\kappa(\sigma_s w, \cdot - s)$ coincides with $\kappa(w, \cdot)$ for Q_m-a.e. $w \in W$.

Two HRM's κ and γ are identified if $Q_m(\kappa(w, \cdot) \neq \gamma(w, \cdot)) = 0$. A HRM κ is *carried by* a set $\Gamma \in \mathscr{M}$ if $Q_m \int 1_{\Gamma^c}(t, \cdot)\ \kappa(dt) = 0$. We say that a HRM κ is *copredictable* if

$$Q_m \int_{\mathbb{R}} Z_t\, \kappa(dt) = Q_m \int_{\mathbb{R}} {}^{\hat{p}}Z_t\, \kappa(dt), \qquad \forall Z \in \mathscr{M}^+, \tag{3.2}$$

and if the L.H.S. of (3.2) is finite for some strictly positive $Z \in \hat{\mathscr{P}}$. *Optionality* for a HRM carried by Λ is defined analogously. See [11] for details. In view of [11, (5.27)], if κ is an optional, copredictable HRM carried by Λ then the exceptional set implicit in (3.1) (ii) can be taken to be empty.

The characteristic measure ρ_κ of a HRM κ carried by Λ is defined by

$$\rho_\kappa(f)=Q_m\Big(\int_{]0,1]} f\circ \bar{Y}_t\,\kappa(dt)\Big),\qquad f\in\mathscr{E}^+.$$

For $f\in(\mathscr{R}\otimes\mathscr{E})^+$ we have [11, (5.11)],

$$Q_m\Big(\int_{\mathbb{R}} f(t,\bar{Y}_t)\,\kappa(dt)\Big)=\int_{\mathbb{R}} dt\int_E \rho_\kappa(dx)\,f(t,x). \tag{3.3}$$

The following result is one of our basic tools.

(3.4) **Proposition.** [11, (5.22)]. (a) *Let κ be an optional, copredictable HRM carried by Λ. Then ρ_κ charges no m-polar set. Moreover, κ is diffuse (i.e., $t\to\kappa\{t\}$ is Q_m-evanescent) if and only if ρ_κ charges no m-semipolar set.*

(b) *Conversely, if ρ is a σ-finite measure on E that charges no m-polar set, then there exists a unique optional copredictable HRM carried by Λ with characteristic measure ρ.*

In particular, to each $\mu\in\mathscr{C}$ Proposition (3.4)(b) associates a unique HRM $\kappa=\kappa_\mu$ with characteristic measure μ. For the rest of this section we fix such a $\mu\in\mathscr{C}$ and the associated HRM κ.

First note that if we set $\hat{u}(x)=\hat{P}^x(\kappa]-\infty,0[)$, then

$$\mu U=\hat{u}\cdot m. \tag{3.5}$$

To see (3.5) take $f(t,x)=1_{]-\infty,0[}(t)\;P_{-t}\,g(x)$ in (3.3) and use the optionality of κ. Next, by [11, §5] the *discrete* part $\kappa^d\equiv\sum_t \kappa\{t\}\;\varepsilon_t$ of κ is Q_m-indistinguishable from

$$\sum_{t\in\mathbb{R}} k(\bar{Y}_t)\,\varepsilon_t, \tag{3.6}$$

where $k\in\mathscr{E}^+$ is uniquely determined m-q.e., and $\{k>0\}$ is m-semipolar. Recall from (2.11) the fine (resp. coexcessive) version $\bar{U}(\mu)$ (resp. $\hat{U}(\mu)$) of $d(\mu U)/dm$.

(3.7) **Proposition.** *Let $\hat{u}=\hat{P}(\kappa]-\infty,0[)$ and k be as above, and set $\bar{u}=\hat{u}+k$. Then $\bar{u}=\bar{U}(\mu)$, $\hat{u}=\hat{U}(\mu)$ (m-q.e.). In particular, $\bar{U}(\mu)\geqq\hat{U}(\mu)$ m-q.e. Moreover, up to Q_m-evanescence we have*

$$\bar{u}\circ\bar{Y}_\cdot=1_\Lambda\,{}^{\hat{p}}(\kappa]-\infty,\cdot]),\qquad \hat{u}\circ\bar{Y}_\cdot=1_\Lambda\,{}^{\hat{p}}(\kappa]-\infty,\cdot[). \tag{3.8}$$

Proof. It is easy to check that $\hat{u}$ is coexcessive. From (2.6) (a) we see that $\hat{u}=\hat{U}(\mu)$ m-q.e.; in particular $\hat{u}\circ\bar{Y}_\cdot\in\mathscr{O}\cap\hat{\mathscr{P}}$. Since $k\in\mathscr{E}^+$, we also have $\bar{u}\circ\bar{Y}_\cdot\in\mathscr{O}\cap\hat{\mathscr{P}}$. The relations in (3.8) now follow from [11, §6] and the section theorem. Because of (3.8) and the preservation of right continuity by the copredictable projection [11, (3.19)], we see that $t\to\bar{u}\circ\bar{Y}_t$ is right continuous on Λ, Q_m-a.s. This forces $\bar{u}=\bar{U}(\mu)$ m-q.e. □

The following evaluation will be used in section 5. The notation is that of the previous proposition.

(3.9) **Lemma.** *Given* $\psi \in \mathscr{R}^+$ *with* $\int_{\mathbb{R}} \psi(t)\,dt = 1$, *define* $g(s,t) = 1_{\{s<t\}}\,\psi(t) + 1_{\{s \geqq t\}}\,\psi(s)$. *Then*

$$Q_m\left(\iint g(s,t)\,\kappa(ds)\,\kappa(dt)\right) = \mu(\hat{u}) + \mu(\bar{u}). \tag{3.10}$$

Proof. We break the double integral on the L.H.S. of (3.10) into two, according as $s<t$ or $s \geqq t$. The first of these is equal to

$$Q_m \int_{\mathbb{R}} \kappa(dt)\,\psi(t)\,\kappa(]-\infty, t[) = Q_m \int_{\mathbb{R}} \kappa(dt)\,\psi(t)\,\hat{u}(\bar{Y}_t) = \mu(\hat{u}),$$

where the first equality uses (3.8), the second (3.3). In the same way the integral over $s \geqq t$ has Q_m-expectation $\mu(\bar{u})$. □

4. The Bounded Maximum Principle

For the rest of the paper we assume that the distinguished excessive measure m is *dissipative*: for each $f \in \mathscr{E}^+$, if $m(f) < \infty$ then $Uf < \infty$ m-a.e. (See [10].) This transience hypothesis ensures that $\mathscr{C}$ is sufficiently rich, and can always be engineered by passing to the q-subprocess for any $q > 0$.

The main result of this section is the following

(4.1) **Theorem.** (H_m) *and* (M_m^*) *are equivalent.*

Proof that (H_m) *implies* (M_m^*). Given $\mu \in \mathscr{C}$ it follows from (2.8) that $\{\hat{U}(\mu) \neq \bar{U}(\mu)\}$ is m-semipolar, hence m-polar under (H_m). The implication in (M_m^*) is now seen to be a special case of (2.15). □

For the converse we need a lemma. Recall that a Borel set $B \subset E$ is *totally thin* if $\sup\{P^x(e^{-T_B}) : x \in B\} < 1$, where T_B is the hitting time of B by X.

(4.2) **Lemma.** *Let* $A \in \mathscr{E}$ *be a semipolar set that is not m-polar. Then there exist a Borel set* $B \subset A$, *a measure* $\mu \in \mathscr{C}$ *carried by B, a function* $k \in \mathscr{E}^+$, *and a number* $b > 0$ *such that*

(i) *B is totally thin but not m-polar;*
(ii) $\{k > 0\} = \{k \geqq b\} = B$;
(iii) $\mu U \leqq m$, *and* $k = \bar{U}(\mu) - \hat{U}(\mu)$ *m-q.e.*

Proof. We assume without loss of generality that A is totally thin, hence finely closed. Since m is dissipative an appeal to [10, (4.6)] yields a sequence (ν_n) of finite measures on E, each ν_n absolutely continuous with respect to m, such that $\nu_n U \uparrow m$. By (1.5), $\nu_n P_A U \uparrow L_A m$. Since A is not m-polar we can choose n so large that $\nu_n P_A U \neq 0$. Let $\mu_0 = \nu_n P_A$. Then $\mu_0 \neq 0$, $\mu_0 U \leqq m$, $\mu_0(1) \leqq \nu_n(1) < \infty$, and μ_0 charges no m-polar set. Thus $\mu_0 \in \mathscr{C}$, and μ_0 is carried by A. By (3.4)(b), μ_0 is the characteristic measure of a unique optional, copredictable HRM κ_0, and κ_0 is carried by $\{(t,w): \bar{Y}_t(w) \in A\}$ since μ_0 is carried by A. The set A being totally thin, it follows that κ_0 is purely discrete. By the discussion in Sect. 3 there is a function $k_0 \in \mathscr{E}^+$ with $\{k_0 > 0\} \subset A$ such that $\kappa_0 = \sum_{t \in \mathbb{R}} k_0(\bar{Y}_t)\, \varepsilon_t$ up to Q_m-evanescence. Since $\kappa_0 \neq 0$, there exists $b > 0$ such that $\{k_0 \geqq b\}$ is not m-polar. Set $B = \{k_0 \geqq b\}$, $k = 1_B k_0$, and define an optional, copredictable HRM κ, carried by A, by

$$\kappa = \sum_{t \in \mathbb{R}} k(\bar{Y}_t)\, \varepsilon_t.$$

Evidently κ has characteristic measure $\mu = 1_B \mu_0$, and $k = \bar{U}(\mu) - \hat{U}(\mu)$ m-q.e. by (3.7). □

Proof that (M_m^*) *implies* (H_m). If (H_m) fails then there is a semipolar Borel set A that is not m-polar. Let B, μ, k, and b be as promised by Lemma (4.2). Write $\bar{u} = \bar{U}(\mu)$, $\hat{u} = \hat{U}(\mu)$, so that $\bar{u} - \hat{u} = k$ m-q.e. and $\bar{u} \leqq 1$ m-q.e. Moreover,

$$\hat{u} = \bar{u} - k \leqq \bar{u} - b \leqq 1 - b, \qquad m\text{-q.e.} \quad \text{on} \quad B,$$

so by (M_m^*), $\hat{u} \leqq 1 - b$ m-q.e. But (2.9) now forces $\bar{u} \leqq 1 - b$ m-q.e. as well. Iterating this argument we obtain successively $\hat{u} \leqq 1 - 2b$, $\hat{u} \leqq 1 - 3b$, ... (m-q.e.) and eventually $\hat{u} < 0$ m-q.e., which is absurd. Thus (H_m) cannot fail if (M_m^*) holds. □

We close this section by comparing (H_m) and (M_m^*) to (H) and (M^*) when additional hypotheses are imposed. First, assume that m is a reference measure and that (relative to m) X has a dual process $\check{X}$ that is special standard. In this case "m-polar" is the same as "polar", so $(H_m) \equiv (H)$. It can also be shown that each coexcessive function agrees with some $\check{X}$-excessive function off a polar set. In particular, (M_m^*) can be rephrased as

(4.3) Given $\mu \in \mathscr{C}$ carried by $A \in \mathscr{E}$, if $\check{U}(\mu) \leqq 1$ q.e. on A, then $\check{U}(\mu) \leqq 1$.

Here $\check{U}(\mu)$ is the version of $d(\mu U)/dm$ provided by (1.1), and "q.e." means "quasi-everywhere" (i.e. except for a polar set).

Let us assume additionally that $\check{X}$-excessive functions are lower semicontinuous. Aside from the fact that X need not be a standard process, this is the (dual of the) context of [4]. It follows from [4, (5.1)] that a finite measure μ lies in $\mathscr{C}$ provided $\check{U}(\mu)$ is bounded. (See also Revuz [26] on this point.) Now $\{\check{U}(\mu) \leqq 1\}$ is closed, and assuming as we may that the support set A in (4.3) is closed, if $\check{U}(\mu) \leqq 1$ q.e. on A then $\check{U}(\mu) \leqq 1$ everywhere on A. It follows that (M_m^*) and (M^*) are equivalent under the present hypotheses, so our Theorem (4.1) contains [4, (5.3)] as a special case.

5. The Bounded Positivity Principle

As in the previous section, m is assumed to be dissipative. To state our final result we introduce an auxiliary hypothesis:

(5.1) Given a bounded coexcessive function $\hat{u}$, the coexcessive regularization of $\hat{u} \wedge 1$ agrees with $\hat{u} \wedge 1$ m-q.e.

This hypothesis amounts to a weak form of the 0–1 law for the measures $\hat{P}^x$. See Remark (5.11) (c).

(5.2) **Theorem.** *(H_m) implies (P_m^*). If (5.1) holds then (P_m^*) implies (H_m).*

To prove (5.2) we introduce a modified energy I_0; this functional is manifestly positive and coincides with I under (H_m). Given $\mu \in \mathscr{C}$ let κ denote the associated HRM (see sect. 3). Since m is dissipative there exists a $\mathscr{G}$-measurable random time S: $W \to \mathbb{R} \cup \{+\infty\}$ such that

$$\text{(5.3)} \quad \text{(i) } \alpha < S < \beta \quad \text{on} \quad \{S \neq +\infty\};$$
$$\text{(ii) } S = t + S \circ \sigma_t, \qquad \forall t \in \mathbb{R};$$
$$\text{(iii) } Q_m(S = +\infty) = 0.$$

(Such an S exists if and only if m is dissipative; see [13].) Define

$$I_0(\mu) = \tfrac{1}{2} Q_m([\kappa(\mathbb{R})]^2;\ 0 < S \leqq 1), \tag{5.4}$$

and note that $I_0 \geqq 0$ on $\mathscr{C}$. The next result implies that $I_0(\mu) < \infty$ if $\mu \in \mathscr{C}$, and that I_0 does not depend on the choice of S.

(5.5) **Lemma.** *Given $\mu \in \mathscr{C}$ set $\bar{u} = \bar{U}(\mu)$, $\hat{u} = \hat{U}(\mu)$, and $k = \bar{u} - \hat{u}$. Then*

$$I_0(\mu) = \mu(\hat{u}) + \tfrac{1}{2}\mu(k) = I(\mu) + \tfrac{1}{2}\mu(k).$$

We defer the proof of (5.5) to the end of this section.

Note that $I_0(\mu) \leqq \mu(\bar{u}) < \infty$ if $\mu \in \mathscr{C}$, and so

$$\kappa(\mathbb{R}) < \infty, \qquad Q_m\text{-a.s.} \quad \text{on} \quad \{0 < S \leqq 1\},$$

where κ is the HRM associated with μ as before. But $\kappa(\mathbb{R})$ is (σ_t)-invariant, and $\bigcup_{t \in \mathbb{R}} \sigma_t^{-1}\{0 < S \leqq 1\} = \{S \in \mathbb{R}\}$ is Q_m-full. Thus

$$\kappa(\mathbb{R}) < \infty, \qquad Q_m\text{-a.s.} \tag{5.6}$$

This allows us to extend I_0 coherently to $\mathscr{C} - \mathscr{C}$:

$$I_0(\mu - \nu) \equiv \tfrac{1}{2} Q_m([\kappa(\mathbb{R}) - \gamma(\mathbb{R})]^2;\ 0 < S \leqq 1), \tag{5.7}$$

where γ is the HRM associated with ν. Clearly $I_0 \geqq 0$ on $\mathscr{C} - \mathscr{C}$.

Proof of (5.2). (i) Assume (H_m). Let $\mu, \nu \in \mathscr{C}$ with associated HRM's κ and γ. By (H_m) and (3.4) (a), κ and γ are diffuse. In particular $\bar{u} = \hat{u}$, $\bar{v} = \hat{v}$ m-q.e. (Here $\bar{u} = \bar{U}(\mu)$, $\bar{v} = \bar{U}(\nu)$, etc.) Thus $I(\mu - \nu) = I_0(\mu - \nu) \geqq 0$. Suppose now that $I(\mu - \nu) = 0$. Then by (5.7),

$$\kappa(\mathbb{R}) = \gamma(\mathbb{R}) < \infty, \qquad Q_m \quad \text{a.s.} \tag{5.8}$$

Applying the copredictable projection to both sides of (5.8), and using (3.8), we obtain

$$\kappa(]t, +\infty[) - \gamma(]t, +\infty[) = (\bar{v} - \bar{u}) \circ \bar{Y}_t, \tag{5.9}$$

up to Q_m-evanescence. The L.H.S. of (5.9) is finite continuous in t and of bounded variation for t in compact subintervals of $]\alpha, \beta[$. The same is therefore true for the R.H.S. But if $t \to f(t)$ is a (deterministic) continuous function of locally bounded variation, then for Lebesgue a.e. $y \in \mathbb{R}$, $\{t\colon f(t)=y\}$ is countable. See [27, p. 279]. Fubini's theorem now allows us to conclude that for Lebesgue a.e. $y \in \mathbb{R}$,

(5.10) $$\{t \in]\alpha, \beta[: (\bar{v}-\bar{u}) \circ \bar{Y}_t(w)=y\} \quad \text{is countable for } Q_m\text{-a.e. } w.$$

Upon reflection we see that (H_m) and the continuity of $t \to (\bar{v}-\bar{u}) \circ \bar{Y}_t$ force the "level set" in (5.10) to be empty except possibly for *one* value of y, Q_m-a.s. (y may depend on w). In short, $t \to (\bar{v}-\bar{u}) \circ \bar{Y}_t$ is constant on $]\alpha, \beta[$, Q_m-a.s. Thus $\kappa=\gamma$ by (5.9), hence $\mu=v$ as desired.

(ii) Assume (P_m^*) and (5.1). Fix $\mu \in \mathscr{C}$ and suppose that μ is carried by $A \in \mathscr{E}$ and that $\hat{u} \equiv \hat{U}(\mu) \leqq 1$ m-q.e. on A. Define $\xi=\mu U \wedge m$ so that $\xi \in \mathrm{Exc}$ and $\xi \leqq \mu U$. Then by [14, (4.2)] there is a measure v on E such that $\xi=vU$. Clearly $v \in \mathscr{C}$. Let $\hat{v}=\hat{U}(v)$. Since $\hat{u} \wedge 1$ is also a version of $d\xi/dm$, (5.1) implies that $\hat{v}=\hat{u} \wedge 1$ m-q.e. In particular $\hat{v} \leqq \hat{u}$ m-q.e., and $\hat{v}=\hat{u}$ m-q.e. on A, hence μ-a.e. Therefore

$$0 \leqq I(\mu-v)=(\mu-v)(\hat{u}-\hat{v})=-v(\hat{u}-\hat{v}) \leqq 0.$$

Consequently $I(\mu-v)=0$, and so $\mu=v$ by (P_m^*). That is, $\mu U=vU=\mu U \wedge m \leqq m$ which forces $\hat{u} \leqq \bar{u} \leqq 1$ m-q.e. Thus (M_m^*) holds and (5.2) is proved, since (M_m^*) implies (H_m) by (4.1). □

(5.11) *Remarks.* (a) The proof of the implication $(P_m^*) \Rightarrow (M_m^*)$ is adapted from Rao [25].

(b) The energy I_0 of (5.4) is related to the energy of an additive functional, introduced by Meyer [20] (see also Weil [30]). Let (A_t) be a finite additive functional of X; the energy of (A_t) is the quantity

(5.12) $$\uparrow \lim_n \frac{1}{2} \int \mu_n(dx)\, P^x(A_\infty^2)$$

where (μ_n) is any sequence of measures on E such that $\mu_n U \uparrow m$. Now associated with (A_t) is an optional HRM κ carried by $]\!]\alpha, \beta[\![$ and determined by

$$\kappa(dt+s)=dA_t \circ \theta_s \quad \text{on} \quad \{\alpha<s\}, \qquad t>0.$$

Although κ need not be copredictable, it is not hard to check that the R.H.S. of (5.4) coincides with (5.12). See [13, §4].

(c) Note that if $\hat{u}$ is bounded and coexcessive, then $\hat{u} \wedge 1$ differs from its coexcessive regularization only on an m-semipolar set (cf. [6]). Thus (H_m) implies (5.1), and it is not unreasonable to conjecture that (P_m^*) implies (5.1). If X has a Borel right dual process $\check{X}$ relative to m (m need not be a reference measure), and if $\check{X}$ is m-special standard [15, §16], then modulo m-polar sets, $\hat{X}$ and $\check{X}$ have identical classes of excessive functions. Since the cone of $\check{X}$-excessive functions is $\wedge$-stable and contains 1, (5.1) holds in this case. The reader can check that in general (5.1) is equivalent to the following statement: for each bounded coexcessive function $\hat{u}$, the random variable $\lim_{t \downarrow 0} \hat{u}(\hat{X}_t)$ is $\hat{P}^x$ degenerate for m-q.e. $x \in E$.

It remains to prove (5.5). Fix $\mu \in \mathscr{C}$ with associated HRM κ. Let S be as in the definition (5.4) of $I_0(\mu)$. Define a measure M on $\mathbb{R}$ by

$$M(f) = Q_m\left(\iint f(t-s)\,\kappa(ds)\,\kappa(dt); 0 < S \leqq 1\right), \quad f \in \mathscr{R}^+. \tag{5.13}$$

It is easy to check that for $g \in (\mathscr{R} \otimes \mathscr{R})^+$,

$$Q_m \iint g(s,t)\,\kappa(ds)\,\kappa(dt) = \int_{\mathbb{R}} M(db) \int_{\mathbb{R}} g(a, a+b)\,da. \tag{5.14}$$

Now on the one hand, by (5.4) we have $I_0(\mu) = M(1)/2$. On the other hand, if $\psi \in \mathscr{R}^+$ with $\int_{\mathbb{R}} \psi(t)\,dt = 1$, then defining $g(s,t) = 1_{\{s<t\}}\psi(t) + 1_{\{s \geqq t\}}\psi(s)$ we have $\int_{\mathbb{R}} g(a, a+b)\,da = 1$, so by (5.14) and (3.9),

$$I_0(\mu) = M(1)/2 = [\mu(\hat{u}) + \mu(\bar{u})]/2,$$

which agrees with (5.5) since $\bar{u} = \hat{u} + k$ m-q.e. □

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