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Probability Theory and Related Fields

Excursions of a $\mathrm{BES}_0(d)$ and its Drift Term $(0<d<1)$

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Summary. Let X be a $\mathrm{BES}_0(d)$ $(0<d<1)$ with canonical decomposition $X=B+(d-1)H$, where B is a brownian motion and H locally of zero energy. The process $(X;H)$ is shown to have a local time at $(0;0)$, and the characteristic measure of its excursions (in Itô's sense) is described. This study leads us to new determinations of the – space variable – process defined by the occupation densities of H taken at some optional times.

1. Introduction

The theory of the excursions of a Markov process out of a regular point, initiated by Itô [7], is a powerful tool to study diffusions on the line such as brownian motion and Bessel processes. It both provides global information on the paths and allows explicit computation of distributions (see Williams [12]; Rogers [9]; Pitman and Yor [8]; Biane and Yor [4]; Barlow et al. [1]).

In this paper, we consider the Markovian couple formed by a Bessel process X of dimension $d\in(0;1)$ and its drift term H which appears in the canonical decomposition of X

(1.1) $$X(t)=B(t)+(d-1)\,H(t)$$

as the sum of a real brownian motion B and a locally of zero energy process $(d-1)H$. We firstly note that $(0;0)$ is a regular point for $(X;H)$ and that its associated local time satisfies an analogy of Lévy's downcrossing theorem. Following Itô [7], the process of the excursions of $(X;H)$ out of $(0;0)$ is a Poisson point process, and we describe its characteristic measure.

The initial motivation of this work was an attempt to explain the Ray-Knight theorems obtained in [2] for the occupation densities of H. We will see in Sect. 4 that the excursion theory not only gives a direct proof of those theorems, but also yields to new results. There are now two natural excursion processes related to X (the classical one, as it is described for instance in Pitman and Yor [8], and the present one); and a comparizon of the two would be interesting since each is likely to produce new information about the other.

In this paper, $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqq 0}, \mathbf{P})$ will stand for a complete probability space, endowed with a right-continuous family of σ-fields. We will denote by X a Bessel process of dimension $d\in(0;1)$, starting from 0 and with an instantaneous reflecting barrier at 0 (in short, X is a $\mathrm{BES}_0(d)$); and by $\{L_t^a: a\in\mathbb{R}_+, t\geqq 0\}$ a jointly continuous version of its local times, i.e. $\mathbf{P}$ a.s., for every positive t and bounded Borel φ, we have

$$\int_0^t \varphi(X(s))\,ds = \int_{\mathbb{R}_+} \varphi(a)\,L_t^a\,a^{d-1}\,da.$$

We saw in [2] that the drift term H of X [see (1.1)] admits a representation as a "partie finie" (p.f.) in Hadamard's sense associated to the local times of X:

$$(1.2)\qquad H(t)=\tfrac{1}{2}\,\text{p.f.}\int_0^t X^{-1}(s)\,ds=\tfrac{1}{2}\int_{\mathbb{R}_+}(L_t^a-L_t^0)\,a^{d-2}\,da$$

$\left(\text{remember that } d\in(0;1), \text{ so } \int_0^t X^{-1}(s)\,ds=\infty \text{ p.s. for every positive } t\right)$. This formula implies that

(1.3) p.s., H increases on every interval on which X is never zero.

Note however that, according to (1.1) and since $d-1$ is negative, H is negative when B is positive. More precisely, if we introduce

$$T(b)=\inf\{t\geqq 0: H(t)=b\} \qquad (b\in\mathbb{R}),$$

then $T(b)$ is finite a.s. (cf. Lemma 3.7. in [2]), and it follows from (1.3) that $X(T(b))=0$ provided that b is negative.

2. Local Time and Down-Crossings Number

In this paragraph, we introduce a local time at $(0;0)$ for $(X;H)$. For every $n\in\mathbb{N}$ and $t\in\mathbb{R}_+$, let us denote by $d_n(t)$ the number of down-crossings of H from 0 to -2^{-n} during the interval of time $[0;t]$. Our main result is

Theorem 2.1. $\mathbf{P}$ a.s., *for all* t, $2^{n(d-1)}d_n(t)$ *converges as* n *goes to* $+\infty$ *to a continuous non-decreasing process* $\delta(t)$. *Furthermore, the set of times at which* δ *increases is* $\{t: X(t)=H(t)=0\}$ *and* $\lim_{+\infty}\delta=+\infty$.

Remark. $(0;0)$ is a regular point for the Markov process $(X;H)$ (indeed, if g_ε denotes the last time before $T(-\varepsilon)$ when H is zero, then there is no neighbourhood of g_ε on which H increases, so according to (1.3), $X(g_\varepsilon)=H(g_\varepsilon)=0$, $0<g_\varepsilon$

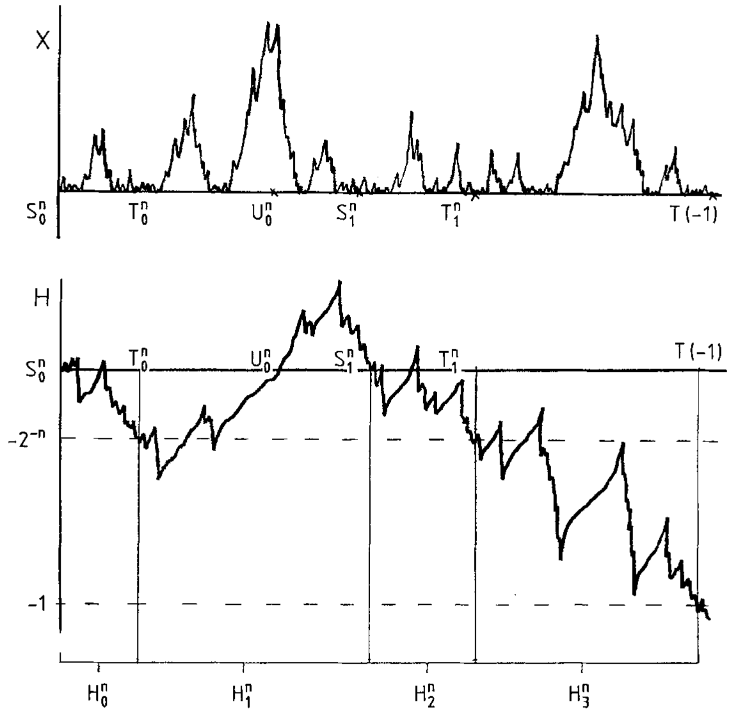

Fig. 1. $(X; H)$ graph on $[0; T(-1)]$

$<T(-\varepsilon)$ and $T(-\varepsilon)\downarrow 0$ p.s.), and since δ is clearly a positive continuous homogeneous additive functional of (X, H) which increases only when $(X; H)=(0; 0)$, δ is the (unique up to a multiplicative constant) local time at $(0; 0)$ of $(X; H)$.

Proof. Two lemmas are interspersed through the proof; the first is

Lemma 2.2. $\{2^{n(d-1)}d_n(T(-1)): n\in\mathbb{N}\}$ *is a positive martingale (with regards to its natural filtration). Hence it converges* **P** a.s. *and in* $L^1(\mathbf{P})$, *and its limit,* $\delta(T(-1))$ *has an exponential distribution with parameter* 1.

Proof of Lemma 2.2. Let us set, for all $(k, n)\in\mathbb{N}^2$: $S_0^n\equiv 0$,

$$T_k^n=\inf\{t\geqq S_k^n: H(t)=-2^{-n}\}\wedge T(-1),$$

$$U_k^n=\inf\{t\geqq T_k^n: H(t)=0\}\wedge T(-1),$$

and

$$S_{k+1}^n=\inf\{t>U_k^n: H(t)=0\}\wedge T(-1).$$

To be at ease, it is important to understand what is related to those definitions (see Fig. 1):

T_0^n is the first hitting time of -2^{-n} by H, and we saw in the introduction that $X(T_0^n)=0$. If $U_0^n<T(-1)$, then U_0^n is the first time after T_0^n when H hits 0 again. Since $X(U_0^n)\neq 0$ a.s. on $\{U_0^n<T(-1)\}$ (see Proposition 5.4. in [2]), H increases on a neighbourhood of U_0^n; and S_1^n is the second time after T_0^n when

H hits 0 again. Since H is positive immediately after U_0^n, the same arguments as before imply $X(S_1^n)\equiv 0$.

We split H paths as in Fig. 1: let us denote

$$H_{2k}^n(t)=H(S_k^n+t) \quad \text{for } t\in[0;T_k^n-S_k^n]$$

$$H_{2k+1}^n(t)=H(T_k^n+t) \quad \text{for } t\in[0;S_{k+1}^n-T_k^n].$$

Since $X(S_k^n)=X(T_k^n)=0$, the process $t\mapsto X(S_k^n+t)$ (respectively $t\mapsto X(T_k^n+t)$) is a $\mathrm{BES}_0(d)$, independent of $\mathscr{F}_{S_k^n}$ (respectively of $\mathscr{F}_{T_k^n}$); hence:

(i) Conditionally on $H(S_k^n)$, H_{2k}^n is independent of $\mathscr{F}_{S_k^n}$ and of $\check{\mathscr{F}}_{T_k^n}=\sigma\{X(t): t\geqq T_k^n\}$, is identically 0 when $H(S_k^n)=-1$, and is distributed as H_0^n when $H(S_k^n)=0$.

(ii) Conditionally on $(H(T_k^n), H(S_{k+1}^n))$, H_{2k+1}^n is independent of $\mathscr{F}_{T_k^n}$ and of $\check{\mathscr{F}}_{S_{k+1}^n}=\sigma\{X(t): t>S_{k+1}^n\}$, is identically 0 if $H(T_k^n)=-1$, and is distributed as $(H_1^n|H(S_1^n)=0)$ (respectively $(H_1^n|H(S_1^n)=-1)$) when $H(S_{k+1}^n)=0$ (respectively $H(S_{k+1}^n)=-1$ and $H(T_k^n)=-2^{-n}$).

Now, notice that, for every $p\in\mathbb{N}^*$, $\{d_n(T(-1))=p\}=\{H(S_k^n)=0$ and $H(T_k^n)=-2^{-n}$: $k\leqq p-1\}\cap\{H(S_p^n)=-1\}$. Since for every non negative integer r, down-crossings of H from 0 to -2^{-n-r} only occur on intervals $[S_k^n; T_k^n]$, and since for every positive integer $q<n$, $d_q(T(-1))$ depends only on $\{H_{2k+1}^n: 0\leqq k\leqq p-1\}$; we deduce from above that, conditionally on $d_n(T(-1))$, $\{d_{n+r}(T(-1)): r\in\mathbb{N}\}$ and $\{d_q(T(-1)): q<n\}$ are independent (i.e. $\{d_n(T(-1)): n\in\mathbb{N}\}$ is a Markov chain).

On the one hand, we saw in §5 of [2] that, if f is a continuous function and F is the primitive of f which is nul at 0, then

$$\mathscr{E}_t^f=\exp\left\{X(t)\cdot f(H(t))+(1-d)\,F(H(t))-\tfrac{1}{2}\int_0^t (f'+f^2)(H(s))\,ds\right\}$$

is a continuous local martingale. If we take $f(x)=(x-1/2)^{-1}$, then $f'+f^2\equiv 0$ on $(-\infty;1/2)$, and the optional sampling theorem applied to $T(-1/2)\wedge T(1/2)$ gives

$$\mathbf{P}(\{T(-1/2)<T(1/2)\})=2^{d-1}=\mathbf{P}(\{S_1^1=T(-1)\}).$$

Since $X(S_1^1)\equiv 0$, the strong Markov property implies

$$\mathbf{P}(\{d_1(T(-1))>q\})=(1-2^{d-1})^q \quad (q\in\mathbb{N}); \tag{2.1}$$

and particularly, $\mathbf{E}[d_1(T(-1))]=2^{1-d}$.

On the other hand, since conditionally on $\{d_n(T(-1))=p\}$, $\{H_{2k}^n: k\leqq p-1\}$ is a family of p independent processes, all having the same distribution as $\{2^{-n}H(2^{n^2}t): t\leqq 2^{-n^2}\ T(-1)\}$ (use the scaling property), we deduce from (2.1) that

$$\mathbf{E}(d_{n+1}(T(-1))|d_n(T(-1)))=2^{1-d}\,d_n(T(-1)),$$

and the first part of Lemma 2.2. is proved (because $\{d_n(T(-1)): n\in\mathbb{N}\}$ is a Markov chain).

Thus, (2.1) implies that $\mathbf{P}(\{d_n(T(-1))>q\})=(1-2^{n(d-1)})^q$. Hence, taking the limit as $n\uparrow+\infty$, we obtain that

$$\mathbf{P}(\{\delta(T(-1))>x\})=e^{-x}. \quad \square$$

Proof of theorem 2.1 (*continuation*). Let us first show that if T is an optional time such that $T\leqq T(-1)$ $\mathbf{P}$ a.s., then $2^{n(d-1)}d_n(T)$ converges $\mathbf{P}$ a.s. as $n\uparrow+\infty$ (according to Theorem 21, Chap. 1 in Dellacherie and Meyer [6], this shall imply the convergence in $L^1(\mathbf{P})$, since, $2^{n(d-1)}d_n(T)\leqq 2^{n(d-1)}d_n(T(-1))$, and $\{2^{n(d-1)}d_n(T(-1)): n\in\mathbb{N}\}$ is uniformly integrable).

Let us set $S=\inf\{s\geqq T: H(s)=X(s)=0\}\wedge T(-1)$. As in the proof of Lemma 2.2., we see that on $\{H(T)>0\}\cup\{H(T)=0$ and $X(T)\neq 0\}$, $S=\inf\{s>T: H(s)=0\}$, and on $\{X(T)=H(T)=0\}$, $S=T$. We have

(i) On $\{S=T(-1)\}$, $H(T)$ is negative (since $H(T)\geqq 0$ implies $S=\inf\{s>T: H(s)=0\}$; and the last quantity is clearly less than $T(-1)$); and provided that $H(T)<-2^{-p}$, $d_n(T)=d_n(S)$ for every $n>p$. Consequently, $2^{n(d-1)}d_n(T)$ converges.

(ii) On $\{S<T(-1)\}$, let us define $\tilde{X}(t)=X(S+t)$. Since $X(S)=0$, $\tilde{X}$ is a $\mathrm{BES}_0(d)$: $\tilde{X}(t)=\tilde{B}(t)+(d-1)\tilde{H}(t)$, with $\tilde{H}(t)=\frac{1}{2}\,\text{p.f.}\int_0^t \tilde{X}^{-1}(s)\,ds$. Let $\tilde{T}(-1)$ denote the first hitting time of -1 by $\tilde{H}$; and $\tilde{d}_n(\tilde{T}(-1))$ the number of downcrossings from 0 to -2^{-n} of $\tilde{H}$ during $[0;\tilde{T}(-1)]$. We have

$$d_n(T(-1))=d_n(S)+\tilde{d}_n(\tilde{T}(-1)).$$

So, according to Lemma 2.2., $2^{n(d-1)}d_n(S)$ converges as $n\uparrow+\infty$. Now, notice that $d_n(S)=d_n(T)$ in each of the following cases:

(a) For every n if $(H(T)>0)$ or if $(H(T)=0$ and $X(T)\neq 0)$ (since then $S=\inf\{s>T: H(s)=0\}$).

(b) For every n if $X(T)=H(T)=0$ (since then $S=T$).

(c) For every $n>p$ if $H(T)<-2^{-p}$ (since then, if $U=\inf\{s\geqq T: H(s)=0\}$, then $S=\inf\{s>U: H(s)=0\}$, and H is negative on $(T;U)$ and positive on $(U;S)$).

Hence, in any case, $2^{n(d-1)}d_n(T)$ converges.

Now, we have to prove that $\mathbf{P}$ a.s., $2^{n(d-1)}d_n(t)$ converges for all positive t. For every positive ε, let us introduce

$$A_\varepsilon=\{t<T(-1): \liminf_{n\uparrow+\infty} 2^{n(d-1)}d_n(t)+\varepsilon<\limsup_{n\uparrow+\infty} 2^{n(d-1)}d_n(t)\},$$

and

$$S_\varepsilon=\inf\{t\geqq 0: t\in A_\varepsilon\}\wedge T(-1).$$

Since A_ε is a progressive set, S_ε is a stopping time; and we deduce from above that $2^{n(d-1)}d_n(S_\varepsilon)$ converges, so $S_\varepsilon\notin A_\varepsilon$. If we suppose that $\mathbf{P}(\{A_\varepsilon\neq\emptyset\})$ is positive, then, conditionally on $\{A_\varepsilon\neq\emptyset\}$, $H(S_\varepsilon)=X(S_\varepsilon)=0$ (if $H(S_\varepsilon)\neq 0$, then, for every s close enough to S_ε, $d_n(S_\varepsilon)=d_n(s)$ provided that n being sufficiently large; so

$s \notin \Lambda_\varepsilon$; and if $H(S_\varepsilon)=0$ and $X(S_\varepsilon)>0$, then H increases on a neighbourhood of S_ε, and the same arguments apply). Once again, $\tilde{X}(t)=X(S_\varepsilon+t)$ is a $\mathrm{BES}_0(d)$; if we define the corresponding $\tilde{d}_n(t)$, $\tilde{T}(-1)$ and $\tilde{\delta}(\tilde{T}(-1))$, then, for all positive s, the definition of S_ε implies the existence of a positive s' such that $s'<s$ and $\limsup_{n\uparrow+\infty} 2^{n(d-1)}\tilde{d}_n(s')>\varepsilon$. Since $2^{n(d-1)}\tilde{d}_n(s')\leqq 2^{n(d-1)}\tilde{d}_n(\tilde{T}(-1))$, we obtain $\tilde{\delta}(\tilde{T}(-1))>\varepsilon$; and this is false because $\tilde{\delta}(\tilde{T}(-1))$ has an exponential distribution. Hence $\mathbf{P}(\{\Lambda_\varepsilon \neq \emptyset\})=0$ and after usual scaling arguments, we have proven that $\mathbf{P}$ a.s., $2^{n(d-1)}d_n(t)$ converges for all positive t.

The continuity of δ is a consequence of the following

Lemma 2.3. *For each positive ε, there exists an integer N such that, for every $n>N$ and every stopping time T, $T\leqq T(-1)$,*

$$|\mathbf{E}(2^{n(d-1)}d_n(T)-\delta(T))|<\varepsilon.$$

Proof of Lemma 2.3. Since the martingale $\{2^{n(d-1)}d_n(T(-1)): n\in\mathbb{N}\}$ is uniformly integrable, for every positive ε, there exists an integer N such that, if $n>N$, then

$$\mathbf{E}(|2^{n(d-1)}d_n(T(-1))-\delta(T(-1))|)\leqq\varepsilon.$$

Let S denote $\inf\{t\geqq T: H(t)=X(t)=0\}\wedge T(-1)$. In particular, we have $\mathbf{E}(|2^{n(d-1)}d_n(S)-\delta(S)|\,\mathbf{1}_{\{S=T(-1)\}})\leqq\varepsilon$. On the other hand, conditionally on $\{S<T(-1)\}$, $\tilde{X}(t)=X(S+t)$ is a $\mathrm{BES}_0(d)$, and with the corresponding notations,

$$d_n(T(-1))=d_n(S)+\tilde{d}_n(\tilde{T}(-1)), \qquad \delta(T(-1))=\delta(S)+\tilde{\delta}(\tilde{T}(-1)).$$

So

$$\begin{aligned}&\mathbf{E}(|2^{n(d-1)}d_n(S)-\delta(S)|\,\mathbf{1}_{\{S<T(-1)\}})\\ &\leqq\mathbf{E}(|2^{n(d-1)}d_n(T(-1))-\delta(T(-1))|\\ &\quad+|2^{n(d-1)}\tilde{d}_n(\tilde{T}(-1))-\tilde{\delta}(\tilde{T}(-1))|)\leqq 2\varepsilon.\end{aligned}$$

The lemma follows from the obvious inequality: $d_n(T)\leqq d_n(S)\leqq d_n(T)+1$. □

Proof of theorem 2.1 (end). Since δ is a previsible non-decreasing process, according to Dellacherie and Meyer ([6]: Theorem 48, Chap. VI), in order to prove the continuity of δ, it is sufficient to show that for any stopping time $T\leqq T(-1)$ $\mathbf{P}$ a.s., and for any sequence $\{T_n: n\in\mathbb{N}\}$ of stopping times, $T_n\leqq T(-1)$ and $\lim T_n=T$ $\mathbf{P}$ a.s., we have $\lim \mathbf{E}(\delta(T_n))=\mathbf{E}(\delta(T))$, which is an easy consequence of the former lemma.

Since we already know that δ increases only when X and H are zero, it remains to prove the converse: let a and b be two real numbers, $0\leqq a\leqq b$, and let us define $T=\inf\{t>a: X(t)=H(t)=0\}$. $\tilde{X}(t)=X(T+t)$ is a $\mathrm{BES}_0(d)$; and according to Lemma 2.2. and the scaling invariance property, for every positive ε, $\tilde{\delta}(\tilde{T}(-\varepsilon))$ has an exponential distribution. Since $\delta(a)=\delta(T)$ and $\lim_{\varepsilon\downarrow 0}\tilde{T}(-\varepsilon)=0$, we have $\mathbf{P}(\{\delta(T)=\delta(b),\ T<b\})=0$, and so

$$\mathbf{P}\Big[\bigcup_{\substack{0\leqq a\leqq b\\ a,b\in\mathbb{Q}}}\{\exists t\in]a;b[: X(t)=H(t)=0 \text{ and } \delta(a)=\delta(b)\}\Big]=0.$$

Finally, Lemma 2.2. and the scaling invariance imply that $\lim_{+\infty} \delta = +\infty$. □

The local time at $(0;0)$ is a natural measure on the zero set of $(X;H)$. In order to make a more rigourous statement, let us introduce the right-continuous inverse of δ:

$$\sigma(t) = \inf\{s: \delta(s) > t\}.$$

We easily deduce from Theorem 2.1.

Proposition 2.4. *σ is a stable subordinator of index $(1-d)/2$. More precisely, for all positive k,* $\mathbf{E}[\exp - k\,\sigma(t)] = \exp - t(8\,k)^{(1-d)/2}$.

Proof. For all positive k, $\{H(kt): t \geqq 0\} \overset{(d)}{=} \{k^{1/2} H(t): t \geqq 0\}$. Hence $\{\delta(kt): t \geqq 0\} \overset{(d)}{=} \{k^{(1-d)/2}\,\delta(t): t \geqq 0\}$ and $\{\sigma(kt): t \geqq 0\} \overset{(d)}{=} \{k^{2/(1-d)}\,\sigma(t): t \geqq 0\}$. Furthermore, since δ only increases when X and H are both zero, the strong Markov property implies that $\{X(\sigma(t)+r): r \geqq 0\}$ is a $\mathrm{BES}_0(d)$ independent of $\mathscr{F}_{\sigma(t)}$; and σ is a non-decreasing process with homogeneous independent increments. Eventually, the Laplace transform of σ is obtained by the computation of the Laplace transform of $\inf\{t: d_n(t) = x\}$ which is done by the same techniques as in [2]. □

Corollary 2.5. *There is a finite positive constant C such that,* $\mathbf{P}$ a.s., *for all positive t, $\varphi - m(\{s \leqq t: X(s) = H(s) = 0\}) = C\,\delta(t)$; where $\varphi - m$ stands for the Hausdorff φ-measure, and*

$$\varphi(h) = h^{(1-d)/2} (\log|\log h|)^{(1+d)/2}.$$

Proof. σ has the same distribution as the right-continuous inverse of the local time at 0 of a stable process of index $2/(1+d)$; and so δ has the same distribution as the local time at 0 of this stable process. According to Taylor and Wendel [11], there is a positive finite constant C such that

$$\varphi - m(\{s \leqq t: H(s) = X(s) = 0\}) = C\,\delta(t). \quad \square$$

Remark. It is easy to prove that

$$\{t: H(t)=0\} = \{t: H(t) = X(t) = 0\} \cup \{t: H(t) = 0; X(t) \neq 0\}$$

is the canonical decomposition of the closed set $\{t: H(t)=0\}$ as the union of a perfect closed set and the set of its isolated points. Furthermore, between two isolated points (respectively two accumulation points), there exist infinitely many accumulation points (respectively at least one isolated point). In particular, we also have $\varphi - m(\{s \leqq t: H(s) = 0\}) = C\,\delta(t)$.

3. Excursions of $(X;H)$

We saw in the former paragraph that $(0;0)$ is a regular and recurrent point for $(X;H)$. Following Itô [7], we introduce the excursion process $e = (e^1; e^2)$, where

$$e^1(t) = \{X(\sigma(t-)+r)\,\mathbf{1}_{\{r \leqq \sigma(t) - \sigma(t-)\}}: r \in \mathbb{R}_+\},$$
$$e^2(t) = \{H(\sigma(t-)+r)\,\mathbf{1}_{\{r \leqq \sigma(t) - \sigma(t-)\}}: r \in \mathbb{R}_+\}.$$

Theorem 3.1. (Itô). *e is a Poisson point process.*

We will denote by m its characteristic measure on Ω_0^{abs}, the set of continuous $\omega: \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}$, $\omega(0)=(0;0)$ and ω absorbed at $(0;0)$ after the first return to the origine.

The purpose of this paragraph is to present a decomposition of the generic excursion (see Theorem 3.4.). In comparizon with Williams' decomposition at the maximum of the brownian excursion (Williams [12]; Rogers [9]), it is intersesting to note that here, the splitting time is a stopping time in the natural filtration on Ω_0^{abs}. Several applications of this result are discussed in Sect. 4. The key point for the description of m is the following

Lemma 3.2. *$\{X(s): s \leqq \sigma(t)\}$ and $\{X(\sigma(t)-s): s \leqq \sigma(t)\}$ are identical in law. Hence, so are the processes $\{H(s): s \leqq \sigma(t)\}$ and $\{-H(\sigma(t)-s): s \leqq \sigma(t)\}$.*

Proof. Let τ stand for the right-continuous inverse of L^0 ($\tau_t = \inf\{s: L_t^0 > t\}$). $H(\tau.)$ is a stable process of index $2-d$ (the strong Markov property implies that $H(\tau.)$ has homogeneous independent increments, and it remains to use the scaling invariance property, see Biane and Yor [4]). According to Boylan [5], there exists a jointly continuous version $\{\Lambda_t^a: a \in \mathbb{R}, t \geqq 0\}$ of the occupation densities of $H(\tau.)$ i.e. a.s., for every bounded Borel φ and positive t,

$$\int_0^t \varphi(H(\tau_s))\, ds = \int_{\mathbb{R}} \varphi(a) \Lambda_t^a\, da.$$

Since $d_n(\tau_t)$ is the number of down-crossings from 0 to -2^{-n} of $H(\tau.)$ during $[0; t]$ (because H increases on every excursion interval of X), $\delta(\tau.)$ is a continuous positive homogeneous additive functional of $H(\tau.)$ which increases only when $H(\tau.)$ is nul, and so there is a positive finite constant c such that $\delta(\tau_t) = c\Lambda_t^0$. Since δ increases only when $X=0$, $\delta(t) = c\Lambda_{L_t^0}^0$. Now set

$$l_t^a = \Lambda_{L_t^0}^a \quad \text{and} \quad l_t^* = \operatorname{Sup}\{l_t^a: a \in \mathbb{R}\}.$$

(note that a.s., $(a, t) \mapsto l_t^a$ is continuous). We have

$$\int_0^t \varphi(H(s))\, dL_s^0 = \int_{\mathbb{R}} \varphi(a) l_t^a\, da.$$

Consider the process

$$Z(t) = \mathbf{1}_{\{l_t^* < x\}} \phi(\delta(t)) \exp\left\{-\sum_{i=1}^n \alpha_i X(t_i \wedge t)\right\},$$

where n and x are two positive integers, $0 \leqq t_1 < \ldots < t_n$, $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_+)^n$, and ϕ is a non negative with compact support C^∞ function. Since a.s. for every a, the measure dl_t^a does not charge $\inf\{t: l_t^* = x\}$, the only discontinuous time of Z, $a \mapsto \int_0^\infty Z(s)\, dl_s^a$ is continuous.

Now, let $\{f_k: k \in \mathbb{N}\}$ be an approximation of the Dirac mass at 0. On the one hand,

$$\mathbf{E}\left[\int_0^\infty c f_k(H(t))\, Z(t)\, dL_t^0\right] = \mathbf{E}\left[\int_{\mathbb{R}} da\, f_k(a) \int_0^\infty Z(s)\, d(c\, l_s^a)\right],$$

and the last term converges as $k \uparrow +\infty$ to

$$\mathbf{E}\left[\int_0^\infty Z(s)\, d\delta(s)\right].$$

On the other hand,

$$\mathbf{E}\left[\int_0^\infty c f_k(H(t))\, Z(t)\, dL_t^0\right]$$

$$= \int_0^\infty \mathbf{E}[\mathbf{1}_{\{l_{\tau_t}^* < x\}}\, c f_k(H(\tau_t))\, \phi(c\, l_{\tau_t}^0) \exp\{-\sum \alpha_i X(t_i \wedge \tau_t)\}]\, dt.$$

Since $\{X(s): s \leqq \tau_t\}$ has the same distribution as $\{\check{X}(s) = X(\tau_t - s): s \leqq \tau_t\}$ (because, according to Pitman and Yor [8], the processes of excursions of those two processes are two Poisson point processes stopped at the instant t, with the same characteristic measure), if we set $\check{H}(s) = \frac{1}{2}\,\text{p.f.} \int_{v^0}^{s} du/\check{X}(u)$ and define the corresponding $\{\check{l}_t^a: a \in \mathbb{R}, t \geqq 0\}$, then $\check{H}(s) = H(\tau_t) - H(\tau_t - s)$; hence $\check{H}(\tau_t) = H(\tau_t)$ and $\check{l}_{\tau_t}^a = l_{\tau_t}^{H(\tau_t) - a}$. So,

$$\begin{aligned}&\mathbf{E}[\mathbf{1}_{\{l_{\tau_t}^* < x\}}\, c f_k(H(\tau_t))\, \phi(c\, l_{\tau_t}^0) \exp\{-\sum \alpha_i X(t_i \wedge \tau_t)\}]\\ &\quad = \mathbf{E}[\mathbf{1}_{\{l_{\tau_t}^* < x\}}\, c f_k(H(\tau_t))\, \phi(c\, l_{\tau_t}^{H(\tau_t)}) \exp\{-\sum \alpha_i X(\tau_t - t_i \wedge \tau_t)\}],\end{aligned}$$

consequently

$$\mathbf{E}\left[\int_0^\infty c f_k(H(t))\, Z(t)\, dL_t^0\right]$$

$$= \mathbf{E}\left[\int_{\mathbb{R}} da\, f_k(a) \int_0^\infty \mathbf{1}_{\{l_s^* < x\}}\, \phi(c\, l_s^{H(s)}) \exp\{-\sum \alpha_i X(s - t_i \wedge s)\}\, d(c\, l_s^a)\right];$$

and this last quantity converges as $k \uparrow +\infty$ to

$$\mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{l_s^* < x\}}\, \phi(c\, l_s^{H(s)}) \exp\{-\sum \alpha_i X(s - t_i \wedge s)\}\, d\delta(s)\right].$$

Furthermore, since δ increases only when $H = 0$, the last expression is equal to

$$\mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{l_s^* < x\}}\, \phi(\delta(s)) \exp\{-\sum \alpha_i X(s - t_i \wedge s)\}\, d\delta(s)\right].$$

So, eventually, we have

$$\mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{l_s^* < x\}}\, \phi(\delta(s)) \exp\{-\sum \alpha_i X(t_i \wedge s)\}\, d\delta(s)\right]$$

$$= \mathbf{E}\left[\int_0^\infty \mathbf{1}_{\{l_s^* < x\}}\, \phi(\delta(s)) \exp\{-\sum \alpha_i X(s - t_i \wedge s)\}\, d\delta(s)\right].$$

Taking the limit as $x \uparrow +\infty$, we obtain

$$\int_0^{+\infty} ds\, \phi(s)\, \mathbf{E}[\exp\{-\sum \alpha_i X(\sigma(s) - t_i \wedge \sigma(s))\}]$$

$$= \int_0^{+\infty} ds\, \phi(s)\, \mathbf{E}[\exp\{-\sum \alpha_i X(t_i \wedge \sigma(s))\}].$$

Hence, for almost every s,

$$\mathbf{E}[\exp\{-\sum \alpha_i X(\sigma(s) - t_i \wedge \sigma(s))\}] = \mathbf{E}[\exp\{-\sum \alpha_i X(t_i \wedge \sigma(s))\}].$$

We deduce that, for all t,

$$\{X(s) : s \leqq \sigma(t)\} \overset{(d)}{=} \{X(\sigma(t) - s) : s \leqq \sigma(t)\}.$$

In particular,

$$\left\{\text{p.f.} \int_0^s \frac{du}{X(u)} : s \leqq \tau_t\right\} \overset{(d)}{=} \left\{\text{p.f.} \int_0^s \frac{du}{X(\sigma(t) - u)} : s \leqq \tau_t\right\}$$

$$= \left\{\text{p.f.} \int_{\sigma(t)-s}^{\sigma(t)} \frac{du}{X(u)} : s \leqq \tau_t\right\}$$

$$= \left\{\text{p.f.} \int_0^{\sigma(t)} \frac{du}{X(u)} - \text{p.f.} \int_0^{\sigma(t)-s} \frac{du}{X(u)} : s \leqq \tau_t\right\},$$

and since p.f. $\int_0^{\sigma(t)} \frac{du}{X(u)} = 0$, we have the second part of the Lemma. □

We start the description of m introducing the following

Notations. For all $\omega = (\omega^1; \omega^2) \in \Omega_0^{\text{abs}}$, we set

$$u = \inf\{r > 0 : \omega^2(r) = 0\}, \qquad v = \inf\{r > 0 : \omega(r) = (0, 0)\},$$

$$i = \inf\{\omega^2(r) : r \geqq 0\} \quad \text{and} \quad s = \sup\{\omega^2(r) : r \geqq 0\}.$$

– For all positive t, we set

$$g_t = \operatorname{Sup}\{s<t\colon H(s)=X(s)=0\}$$

and

$$d_t = \inf\{s>t\colon H(s)=X(s)=0\}.$$

We have

Lemma 3.3. (α) $m(i=0)=m(s=0)=m(\omega^1(u)=0)=0$.

(β) *For all positive* x, $m(i<-x)=x^{d-1}$.

(γ) $m(\omega^1(u)\in dx)=\dfrac{1-d}{\Gamma(d)}\,x^{d-2}\,\mathbf{1}_{\{x>0\}}\,dx$.

Proof. (α) Let ε be a positive real number. Since $m(i<-\varepsilon)$ is finite and positive, we can introduce the probability $m_\varepsilon(\cdot)=m(\cdot\,|\,i<-\varepsilon)$. We know that m_ε is the law of the process

$$\{(X(g_{T(-\varepsilon)}+s);\, H(g_{T(-\varepsilon)}+s))\,\mathbf{1}_{\{s\leqq d_{T(-\varepsilon)}-g_{T(-\varepsilon)}\}}\colon 0\leqq s\}$$

As in Sect. 2, we notice that if $\tilde g_{T(-\varepsilon)}=\sup\{s<T(-\varepsilon)\colon H(s)=0\}$, then $X(\tilde g_{T(-\varepsilon)})=0$ (because if it were positive, then H would increase on a neighbourhood of $\tilde g_{T(-\varepsilon)}$, and $\tilde g_{T(-\varepsilon)}$ would not be the last time before $T(-\varepsilon)$ at which H is zero); and so $\tilde g_{T(-\varepsilon)}=g_{T(-\varepsilon)}$. In particular, H is negative on $(g_{T(-\varepsilon)};\, T(-\varepsilon))$.

According to Proposition 5.4 in [2], we have

$$m_\varepsilon(s>0)\geqq m_\varepsilon(\omega^1(u)>0)=1,$$

so, taking the limit as $\varepsilon\downarrow 0$,

$$m(s=0;\, i\neq 0)=m(\omega^1(u)=0;\, i\neq 0)=0. \tag{3.1}$$

On the other hand, according to Lemma 3.2,

$$m(s\neq 0;\, i=0)=m(s=0;\, i\neq 0),$$

thus

$$m(i=0)=m(s\neq 0;\, i=0)+m(s=0;\, i=0)=m(\omega^2\equiv 0).$$

On $\{\omega^2\equiv 0\}$, $v=\inf\{r>0\colon \omega^1(r)=0\}$, ω^1 is positive on $(0;v)$, and so

$$0=\omega^2(v)=\tfrac12\,\text{p.f.}\int_0^v \frac{dr}{\omega^1(r)}=\tfrac12\int_0^v \frac{dr}{\omega^1(r)}.$$

Hence, necessarily $v=0$, i.e. $\omega\equiv 0$. So $m(\omega^2\equiv 0)=m(\omega\equiv 0)=0$; and consequently, $m(i=0)=0$. According to (3.1), we deduce that $m(s=0)=m(\omega^1(u)=0)=0$.

(β) According to Lemma 2.2. and to the scaling invariance property,

$$\mathbf{E}[\delta(T(-x))]=x^{1-d},\qquad \text{so } m(i<-x)=x^{d-1}.$$

(γ) According to Proposition 5.4. in [2] and to the scaling invariance property, for every positive θ and ε, we have

$$\mathbf{E}[\exp - \theta X(T(\varepsilon))] = (1+\varepsilon\theta)^{1-d} - (\varepsilon\theta)^{1-d}.$$

Hence,

$$\int m(d\omega)\, \mathbf{1}_{\{i<-\varepsilon\}}[\exp(-\theta\omega^1(u))-1] = \varepsilon^{d-1}[(1+\varepsilon\theta)^{1-d} - (\varepsilon\theta)^{1-d} - 1];$$

and the last quantity converges as $\varepsilon \downarrow 0$ to $-\theta^{1-d}$. This implies that $m(i+0; \omega^1(u) \in dx) = \frac{1-d}{\Gamma(d)} x^{d-2} \mathbf{1}_{\{x>0\}}\, dx$; and we achieve the proof with the help of (α). □

We are now able to describe m:

Theorem 3.4. *For all positive x, let X^x denote a* $\mathrm{BES}_x(d)$*: $X^x(t) = x + B(t) + (d-1) H^x(t)$, where B is a standard brownian motion and $H^x(t) = \frac{1}{2}\,\mathrm{p.f.} \int_0^t \frac{ds}{X^x(s)}$; and let $S^x(0)$ stand for* $\inf\{t>0: H^x(t)=0\}$. *Then (remember that v denotes the life time of the excursion) for every positive x, conditionally on $\omega^1(u)=x$, the processes*

$$\{(\omega^1(r+u); \omega^2(r+u)): 0 \leqq r \leqq v-u\}$$

and

$$\{(\omega^1(u-r); -\omega^2(u-r)): 0 \leqq r \leqq u\}$$

are independent and have both the same distribution as

$$\{(X^x(r); H^x(r)): 0 \leqq r \leqq S^x(0)\}.$$

Proof. According to Lemma 3.3., $\omega^1(u)$ is positive $m(d\omega)$ a.s., so ω^2 increases on a neighbourhood of u, and hence, $v = \inf\{r>u: \omega^2(r)=0\}$ and $0 = \sup\{r<u: \omega^2(r)=0\}$. We deduce that u is the only zero of ω^2 on $]0; v[$. Since Lemma 3.2. implies that, under m, the processes $\{\omega^1(r): 0 \leqq r \leqq v\}$ and $\{\omega^1(v-r): 0 \leqq r \leqq v\}$ have the same distribution, we deduce that $\{\omega^1(u+r): 0 \leqq r \leqq v-u\}$ and $\{\omega^1(u-r): 0 \leqq r \leqq u\}$ are equally distributed too.

For every positive ε, with the notations of the proof Lemma 3.3., let us consider

$$\Phi = \Phi(\{\omega^1(u+r): 0 \leqq r \leqq v-u\})$$

and

$$\Psi = \Psi(\{\omega^1(u-r): 0 \leqq r \leqq u\})$$

two non-negative Borel functionals on Ω_0^{abs}; and let us denote by $\varphi_\varepsilon(\omega^1(u))$ and $\psi_\varepsilon(\omega^1(u))$ their respective m_ε-mean conditionally on $\omega^1(u)$: φ_ε and ψ_ε are both non-negative Borel functions and, for all Borel non-negative functions f,

$$\int m(d\omega)\, \mathbf{1}_{\{i<-\varepsilon\}}\, \Phi f(\omega^1(u)) = \int m(d\omega)\, \mathbf{1}_{\{i<-\varepsilon\}}\, \varphi_\varepsilon(\omega^1(u)) f(\omega^1(u))$$

$$\int m(d\omega)\, \mathbf{1}_{\{i<-\varepsilon\}}\, \Psi f(\omega^1(u)) = \int m(d\omega)\, \mathbf{1}_{\{i<-\varepsilon\}}\, \psi_\varepsilon(\omega^1(u)) f(\omega^1(u)).$$

According to Lemma 3.3.i), the left-sides of the former equalities converge as $\varepsilon \downarrow 0$ respectively to $\int m(d\omega)\, \Phi f(\omega^1(u))$ and $\int m(d\omega)\, \Psi f(\omega^1(u))$. So $\mathbf{1}_{\{i<-\varepsilon}\, \varphi_\varepsilon(\omega^1(u))$ and $\mathbf{1}_{\{i<-\varepsilon\}}\, \psi_\varepsilon(\omega^1(u))$ increase as $\varepsilon \downarrow 0$ respectively to two $\omega^1(u)$-measurable random variables denoted by $\varphi(\omega^1(u))$ and $\psi(\omega^1(u))$.

On the other hand, we know that the distribution of $\{\omega(r)\colon 0 \leqq r \leqq v\}$ under m_ε is equal to the distribution of $\{(X(g_{T(-\varepsilon)}+r);\ H(g_{T(-\varepsilon)}+r))\colon 0 \leqq r \leqq d_{T(-\varepsilon)} - g_{T(-\varepsilon)}\}$ under $\mathbf{P}$. Let us introduce

$$U_\varepsilon = \inf\{r > T(-\varepsilon)\colon H(r)=0\}\ (=\inf\{r > g_{T(-\varepsilon)}\colon H(r)=0\}).$$

Then $d_{T(-\varepsilon)} = \inf\{r > U_\varepsilon\colon H(r)=0\}$ (because, since $X(U_\varepsilon)$ is positive, H increases on a neighbourhood of U_ε), and, according to the strong Markov property, conditionally on $X(U_\varepsilon)=x$, $\{X(U_\varepsilon+r)\colon 0 \leqq r \leqq d_{T(-\varepsilon)} - U_\varepsilon\}$ is independent of $\mathscr{F}_{U_\varepsilon}$ and has the same law as $\{X^x(r)\colon 0 \leqq r \leqq S^x(0)\}$. We deduce firstly that $\varphi_\varepsilon(x) = \mathbf{E}[\Phi(\{X^x(r)\colon 0 \leqq r \leqq S^x(0)\})] = \varphi(x)$; and secondly that, for every non-negative Borel function f,

$$\begin{aligned}&\int m(d\omega)\, \mathbf{1}_{\{i<-\varepsilon\}}\, \Phi\Psi f(\omega^1(u))\\ &\quad = \int m(d\omega)\, \mathbf{1}_{\{i<-\varepsilon\}}\, \varphi(\omega^1(u))\, \psi_\varepsilon(\omega^1(u))\, f(\omega^1(u)).\end{aligned}$$

Taking the limit as $\varepsilon \downarrow 0$, we finally get

$$\int m(d\omega)\, \Phi\Psi f(\omega^1(u)) = \int m(d\omega)\, \varphi(\omega^1(u))\, \psi(\omega^1(u))\, f(\omega^1(u)),$$

so, conditionally on $\omega^1(u)=x$, $\{\omega^1(u+r)\colon 0 \leqq r \leqq v-u\}$ and $\{\omega^1(u-r)\colon 0 \leqq r \leqq u\}$ are two independent processes, the first one having the same distribution as $\{X^x(r)\colon 0 \leqq r \leqq S^x(0)\}$.

Finally, since

$$\omega^2(u+t) = \tfrac{1}{2}\,\text{p.f.} \int_0^{u+t} dr/\omega^1(r)$$

$$= \omega^2(u) + \tfrac{1}{2}\,\text{p.f.} \int_u^{u+t} dr/\omega^1(r) = \tfrac{1}{2}\,\text{p.f.} \int_0^t dr/\omega^1(u+r) \qquad (0 \leqq t \leqq v-u)$$

and

$$\omega^2(u-t) = \tfrac{1}{2}\,\text{p.f.} \int_0^{u-t} dr/\omega^1(r)$$

$$= \omega^2(u) - \tfrac{1}{2}\,\text{p.f.} \int_{u-t}^{u} dr/\omega^1(r) = -\tfrac{1}{2}\,\text{p.f.} \int_0^t dr/\omega^1(u-r) \quad (0 \leqq t \leqq u),$$

Theorem 3.4. is proven. □

Finally, let us see an easy application

Proposition 3.5. *Consider* Z^x *a* $\mathrm{BES}_x(d)$ *and* $K^x(t) = \frac{1}{2}\,\text{p.f.} \int_0^t \frac{dr}{Z^x(r)}$ *conditioned on* $\operatorname{Sup}\{K^x(r)\colon r \leqq S^x(0)\} \geqq 1$ *(where* $S^x(0) = \inf\{t > 0\colon K^x(r)=0\}$*), and let* $D^x(1) = \operatorname{Sup}\{t \leqq S^x(0)\colon\ K^x(t)=1\}$. *Then, conditionally on* $X(T(1))=x$, *the process* $\{X(T(1)-t)\colon t \leqq T(1)\}$ *is distributed as* $\{Z^x(t)\colon t \leqq D^x(t)\}$.

Proof. When $\mathbf{Q}$ is a probability measure on a set of paths and W a process, let us denote by $\mathfrak{L}(W, \mathbf{Q})$ the law of W under $\mathbf{Q}$. We also denote by Θ_r the translation operator of r. We have

$$\begin{aligned}\mathfrak{L}(\{X(r): r \leqq T(1)\}, \mathbf{P}) &= \mathfrak{L}(\{X(r) \circ \Theta_{T(-1)}: r \leqq T(0) \circ \Theta_{T(-1)}\}, \mathbf{P}) \\ &= \mathfrak{L}(\{\omega^1(r + \ell(-1)): 0 \leqq r \leqq u - \ell(-1)\}, m_1),\end{aligned}$$

where $\ell(-1) = \inf\{r: \omega^2(r) = -1\}$. Hence

$$\begin{aligned}&\mathfrak{L}(\{X(T(1)-r): r \leqq T(1)\}, \mathbf{P}(\cdot \mid X(T(1)) = x)) \\ &\quad = \mathfrak{L}(\{\omega^1(u-r): 0 \leqq r \leqq u - \ell(-1)\}, m_1(\cdot \mid \omega^1(u) = x)) \\ &\quad = \mathfrak{L}(\{\omega^1(u+r): 0 \leqq r \leqq d(1) - u\}, m(\cdot \mid \omega^1(u) = x \text{ and } s \geqq 1)),\end{aligned}$$

where $d(1) = \sup\{r > 0: \omega^2(r) = 1\}$; it just remains to apply Theorem 3.4. □

4. Some Applications

4.1. Ray-Knight's Type Results

Let us now recall the main results of [2]: $\mathbf{P}$ a.s., the occupation measure of H is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, with densities $\{\lambda_t^a: a \in \mathbb{R}, t \geqq 0\}$, and we have the following analogies of Ray-Knight theorems

(R.K.–1) Conditionally on $\lambda^0_{T(-1)} = x$, $\{\lambda^a_{T(-1)}: a \geqq 0\}$ is the square of a $\text{BES}_{\sqrt{x}}(0)$ (in short $\text{BES}\,\mathrm{Q}_x(0)$).

(R.K. 1) Conditionally on $\lambda^0_{T(1)} = x$, $\{\lambda^{-a}_{T(1)}: a \geqq 0\}$ is a $\text{BES}\,\mathrm{Q}_x(0)$.

In order to explain these results via the excursion theory, let us first give a Ray-Knight theorem for the generic excursion of H: for m-a.e. ω, there exists a family $\{\lambda^a: a \in \mathbb{R}\}$ of r.v. such that for every Borel bounded φ,

$$\int_0^v \varphi(\omega^2(r))\, dr = \int_{\mathbb{R}} \varphi(a) \lambda^a\, da.$$

We have

Lemma 4.1. *Conditionally on* $\omega^1(u) = x$, $\{\lambda^a: a \in \mathbb{R}_+\}$ *and* $\{\lambda^{-a}: a \in \mathbb{R}_+\}$ *are two independent* $\text{BES}\,\mathrm{Q}_{2x}(0)$.

Proof. ω^2 is negative on $(0; u)$ and positive on $(u; v)$. According to Theorem 3.4., conditionally on $\omega^1 = x$, $\{\lambda^a: a \in \mathbb{R}_+\}$ and $\{\lambda^{-a}: a \in \mathbb{R}_+\}$ are independent and have both the same distribution as the occupation densities process of $\{H^x(t): t \leqq S^x(0)\}$. Using generalized stochastic calculus introduced in [3], these occupation densities are easily shown to be a $\text{BES}\,\mathrm{Q}_{2x}(0)$ (see proof of Theorem 4.1. in [2]). □

Remember that σ is the right-continuous inverse of the local time at $(0; 0)$ of $(X; H)$. We are now able to claim the following third Ray-Knight theorem:

Theorem 4.2. (i) $\{\lambda^0_{\sigma(t)}: t \geqq 0\}$ *is a unilateral stable process of index* $1-d$. *More precisely, for all positive k,*

$$\mathbf{E}\left[\exp -\frac{k}{2}\lambda^0_{\sigma(t)}\right] = \exp - t\, k^{1-d}.$$

(ii) *Conditionally on* $\lambda^0_{\sigma(t)} = x$ $(x>0)$, *the processes* $\{\lambda^a_{\sigma(t)}: a \geqq 0\}$ *and* $\{\lambda^{-a}_{\sigma(t)}: a \geqq 0\}$ *are two independent* BES $Q_x(0)$.

Proof. Let f_1 and f_2 be two continuous non-negative functions with compact support; and let Φ_1 and Φ_2 be the non-negative, non-increasing solutions of $\Phi_i'' = f_i \Phi_i$, with $\Phi_i(0)=1$ $(i=1$ or $2)$. According to the exponential formula (see Itô [7]),

$$\mathbf{E}\left[\exp\left\{-\tfrac{1}{2}\int_0^{\sigma(t)} (f_1(H(s))\,\mathbf{1}_{\{H(s)\geqq 0\}} + f_2(-H(s))\,\mathbf{1}_{\{H(s)\leqq 0\}})\,ds\right\}\right]$$

$$= \exp\left\{-t\int m(d\omega)\left[1-\exp\left\{-\tfrac{1}{2}\int_0^v (f_1(\omega^2(r))\,\mathbf{1}_{\omega^2(r)\geqq 0}\right.\right.\right.$$

$$\left.\left.\left. + f_2(-\omega^2(r))\,\mathbf{1}_{\omega^2(r)\leqq 0})\,dr\right\}\right]\right\}$$

$$= \exp\left\{-t\int m(d\omega)\left[1-\exp\left\{-\tfrac{1}{2}\left(\int_0^u f_2(-\omega^2(r))\,dr + \int_u^v f_1(\omega^2(r))\,dr\right)\right\}\right]\right\}$$

$$= \exp\left\{-t\int_{\mathbb{R}_+} \frac{1-d}{\Gamma(d)}\, x^{-2+d}(1-\exp\{x(\Phi_1'(0)+\Phi_2'(0))\})\right\}$$

(use Lemmas 3.3. and 4.1. and Pitman and Yor's description [8] of the Laplace transform of a BES $Q_x(0)$); and the last quantity is equal to $\exp\{t(\Phi_1'(0) + \Phi_2'(0))^{1-d}\}$. On the other hand, according to Corollary 3.10. in [2], $\lambda^0_{\sigma(t)} = \sum_{s\leqq \sigma(t)} 2X(s)\,\mathbf{1}_{\{H(s)=0\}}$, and the exponential formula implies easily that, for all positive k,

$$\mathbf{E}\left[\exp -\frac{k}{2}\lambda^0_{\sigma(t)}\right] = \exp - t\, k^{1-d}.$$

Hence,

$$\mathbf{E}\left[\exp\left\{-\tfrac{1}{2}\int_0^{+\infty} f_1(a)\,\lambda^a_{\sigma(t)}\,da - \tfrac{1}{2}\int_0^{+\infty} f_2(a)\,\lambda^{-a}_{\sigma(t)}\,da\right\}\right]$$

$$= \mathbf{E}\left[\exp\left\{\frac{1}{2}(\Phi_1'(0)+\Phi_2'(0))\,\lambda^0_{\sigma(t)}\right\}\right],$$

that is, conditionally on $\lambda^0_{\sigma(t)} = x$, $\{\lambda^a_{\sigma(t)}: a \geqq 0\}$ and $\{\lambda^{-a}_{\sigma(t)}: a \geqq 0\}$ are two independent BES $Q_x(0)$ (see Pitman and Yor [8]). □

Let us give now an alternative proof of the descriptions (R.K.–1) and (R.K. 1):

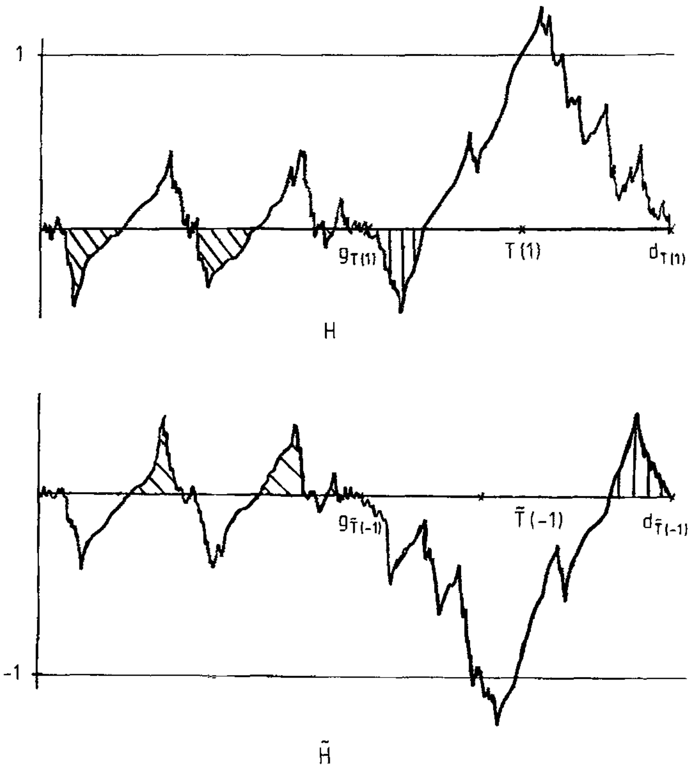

Fig. 2. The transformation $H \mapsto \tilde{H}$ send the negative part of H-graph before $g_{T(1)}$ on the positive part of $\tilde{H}$-graph before $\tilde{T}(-1)$ (stripped areas), and the negative part of H-graph between $g_{T(1)}$ and $T(1)$ on the positive part of $\tilde{H}$-graph between $\tilde{T}(1)$ and $d_{\tilde{T}(-1)}$ (hatched areas)

Proof of (R.K.–1). According to Theorem 3.1., the processes

$$t \mapsto e(t)\,\mathbf{1}_{\{\inf e^2(t) > -1\}} \quad \text{and} \quad t \mapsto e(t)\,\mathbf{1}_{\{\inf e^2(t) \leqq -1\}}$$

are two independent Poisson point processes with respective characteristic measure $m\mathbf{1}_{i>-1}$ and $m\mathbf{1}_{i\leqq -1}$. Thus, Lemma 2.2. implies that the process of the excursions out of (0; 0) of $(X(\cdot \wedge g_{T(-1)}); H(\cdot \wedge g_{T(-1)}))$ is a Poisson point process with characteristic measure $m\mathbf{1}_{i\geqq -1}$ and killed at an independent exponential time with parameter 1. Since Theorem 3.4. allows us to claim that $\{\omega(u+r): 0 \leqq r \leqq v-u\}$ has the same distribution under $(m\mathbf{1}_{i\geqq -1} | \omega^1(u)=x)$ as under $(m|\omega^1(u)=x)$, applying Theorem 4.2., we obtain that, conditionally on $\delta(T(-1))=t$ and $\lambda^0_{T(-1)}=x$, $\{\lambda^a_{T(-1)}: a \geqq 0\}$ is a BES $\mathrm{Q}_x(0)$, and so (R.K.–1) is proven. □

Proof of (R.K. 1). Let us see now that (R.K. 1) is a consequence of (R.K.–1) and of the invariance under time reversal property (Lemma 3.2):

Let us introduce the process $\tilde{H}$: $\tilde{H}(t) = -H(\sigma(s) - t + \sigma(s-))$ for $t \in [\sigma(s-); \sigma(s)]$ (see Fig. 2). According to Lemma 3.2., H and $\tilde{H}$ are equally distributed (since their respective excursions processes are two Poisson point processes with the same characteristic measure); and we denote by $\tilde{\lambda}$, $\tilde{T}(-1)$... the corresponding occupation densities, first hitting time of -1... for $\tilde{H}$. We clearly have for every positive a, (see Fig. 2)

$$\lambda^{-a}_{gT(1)} = \tilde{\lambda}^{a}_{g\tilde{T}(-1)} = \tilde{\lambda}^{a}_{\tilde{T}(-1)},$$

and

$$\lambda^{-a}_{dT(1)} = \lambda^{-a}_{T(1)} = \tilde{\lambda}^{a}_{d\tilde{T}(-1)}.$$

We know that, conditionally on $\tilde{\lambda}^{0}_{\tilde{T}(-1)} = x$, $\{\tilde{\lambda}^{a}_{\tilde{T}(-1)}: a \geqq 0\}$ is a BES $Q_x(0)$. Furthermore $\{\tilde{\lambda}^{a}_{d\tilde{T}(-1)} - \tilde{\lambda}^{a}_{\tilde{T}(-1)}: a \geqq 0\}$ is independent of $\tilde{\mathscr{F}}_{\tilde{T}(-1)}$; and is, conditionally on $\tilde{\lambda}^{0}_{d\tilde{T}(-1)} - \tilde{\lambda}^{0}_{\tilde{T}(-1)} = y$, a BES $Q_y(0)$ (the first part is a consequence of the strong Markov property, and Lemma 4.1. implies the second). The additive property of squares of Bessel processes (Shiga and Watanabe [10]) allows us to claim that, conditionally on $\tilde{\lambda}^{0}_{d\tilde{T}(-1)} = z$, $\{\tilde{\lambda}^{a}_{d\tilde{T}(-1)}: a \geqq 0\}$ is a BES $Q_x(0)$; so (R.K. 1) is proven. □

Remark. Applying the same methods as in the proof of (R.K.–1), we easily obtain other Ray-Knight's representations of the occupation densities of H taken at optional times such as $\inf\{t: H(t)=0 \text{ and } X(t)>1\}$ or $\inf\{t: t-g_t>1 \text{ and } H(t)<0\}$.

4.2. *Computation of Some Distributions*

Section 3 also allows us to compute the distribution of several r.v.'s such as (for instance) $H(1)$, g_1 and $(\delta(T_\theta), \lambda^0_{T_\theta})$ where T_θ denotes and exponential time with parameter $\theta^2/2$ independent of X.

Proposition 4.3. *For every negative x,*

$$\mathbf{P}(H(1) \in dx) = 2^d(1-d)(2\Pi)^{-1/2}\exp(-x^2(1-d)^2/2)\,dx.$$

Proof. According to the scaling invariance property, there is a positive constant c such that, for every positive t, $\mathbf{P}(\{H(t)<0\}) = c$. We have

$$c = \mathbf{E}\left[\int_0^{+\infty} e^{-t}\mathbf{1}_{H(t)<0}\,dt\right] = \mathbf{E}\left[\sum_{s>0}\int_{\sigma(s-)}^{\sigma(s)} e^{-t}\mathbf{1}_{H(t)<0}\,dt\right]$$

$$= \mathbf{E}\left[\sum_{s>0} e^{-\sigma(s-)}\left(\int_0^{v} e^{-r}\mathbf{1}_{\omega^2(r)<0}\,dr\right)\circ\Theta_{\sigma(s-)}\right]$$

(where Θ denotes the translation operator)

$$= \mathbf{E}\left[\int_0^{+\infty} e^{-\sigma(s-)}\,ds\right]\int m(d\omega)\left(\int_0^{u} e^{-r}\,dr\right)$$

(using Maisonneuve's formula)

$$= \int_0^{+\infty} \exp(-8^{(1-d)/2}s)\,ds\int m(d\omega)(1-e^{-u})$$

$$= 8^{(d-1)/2}\int_0^{+\infty} dx(1-e^{-\sqrt{2x}})\frac{1-d}{\Gamma(d)}x^{d-2} = 2^{d-1}.$$

On the one hand, for any negative x,

$$\mathbf{P}(H(1)<x)=\mathbf{P}(T(x)<1;\ \tilde{H}(1-T(x))<0)$$

$\left(\text{where } \tilde{X}(t)=X(t+T(x)) \text{ and } \tilde{H}(t)=H(t+T(x))-x=\frac{1}{2}\,\text{p.f.}\int_0^t \frac{ds}{\tilde{X}(s)}\right)$. Since $\tilde{X}$ is independent of $\mathscr{F}_{T(x)}$, we have

$$\mathbf{P}(H(1)<x)=2^{d-1}\ \mathbf{P}(T(x)<1).$$

On the other hand, we easily deduce from (1.1) that

$$T(x)=\inf\{t>0\colon B(t)=x(1-d)\}. \tag{4.1}$$

Indeed, since X is non-negative, $B(t)\geqq x(1-d)$ for all $t\leqq T(x)$; and since $X(T(x))=0$, $B(T(x))=x(1-d)$. So (4.1) is proven, and

$$\begin{aligned}\mathbf{P}(T(x)<1)&=\mathbf{P}(\inf\{B(s)\colon s\leqq 1\}\leqq x(1-d))\\&=2\,\mathbf{P}(B((1-d)^{-2})<x);\end{aligned}$$

and finally, $\mathbf{P}(H(1)<x)=2^d\,\mathbf{P}(B((1-d)^{-2})<x)$. □

The result for the positive part of $H(1)$ is less simple:

Proposition 4.4. *The law of $H^+(1)$ is given by: for every positive θ,*

$$\mathbf{E}\left(\exp\left\{-\frac{\theta^2}{2}(H^+(1))^{-2}\right\}\right)=(\operatorname{ch}\theta)^{1-d}-(e^\theta/2)^{1-d}.$$

Proof. Let T_θ be an exponential time with parameter $\theta^2/2$ independent of X. We have, for any positive x,

$$\begin{aligned}\mathbf{P}(H(T_\theta)>x)&=\mathbf{E}\left[\frac{\theta^2}{2}\int_0^{+\infty} dt\exp\left(-\frac{\theta^2}{2}t\right)\mathbf{1}_{H(t)>x}\right]\\&=\mathbf{E}\left[\frac{\theta^2}{2}\int_{T(x)}^{S(x)} dt\exp\left(-\frac{\theta^2}{2}t\right)\right.\\&\qquad\left.+\frac{\theta^2}{2}\int_{S(x)}^{+\infty} dt\exp\left(-\frac{\theta^2}{2}t\right)\mathbf{1}_{\tilde{H}(t-S(x))>0}\right],\end{aligned}$$

where $S(x)=\inf\{t>T(x)\colon H(t)=x\}$ and $\tilde{H}(t)=H(t+S(x))-x$. Since $\tilde{H}$ has the same distribution as H and is independent of $\mathscr{F}_{S(x)}$, we obtain

$$\mathbf{P}(H(T_\theta)>x)=\mathbf{E}\left(\exp\left\{-\frac{\theta^2}{2}T(x)\right\}-2^{d-1}\exp\left\{-\frac{\theta^2}{2}S(x)\right\}\right).$$

The computation of $\mathbf{E}\left(\exp-\frac{\theta^2}{2}T(x)\right)$ and $\mathbf{E}\left(\exp-\frac{\theta^2}{2}S(x)\right)$ is done by the same methods as in [2], and we obtain

$$\mathbf{E}\left(\exp-\frac{\theta^2}{2}T(x)\right)=(\operatorname{ch}\theta x)^{1-d}-(\operatorname{sh}\theta x)^{1-d}$$

and

$$\mathbf{E}\left(\exp-\frac{\theta^2}{2}S(x)\right)=\mathrm{e}^{\theta x(1-d)}-(2\operatorname{sh}\theta x)^{1-d}.$$

Finally, we get

$$\mathbf{P}(H(T_\theta)>x)=(\operatorname{ch}\theta x)^{1-d}-(\mathrm{e}^{\theta x}/2)^{1-d}. \tag{4.2}$$

We finish the proof applying (4.2) to $x=1$ and using the scaling invariance property. □

Proposition 4.5. g_1 *follows a* $\beta\left(\frac{1-d}{2},\frac{1+d}{2}\right)$ *distribution.*

Proof. By the same computations as in the proof of Proposition 4.3, we easily obtain for every positive ζ,

$$\mathbf{E}\left[\int_0^{+\infty}\exp-(t+\zeta g_t)\,dt\right]=(1+\zeta)^{(d-1)/2}.$$

Then the scaling invariance property and Barlow, Pitman and Yor's methods [1] establish the proposition. □

Eventually we have

Proposition 4.6. *For every positive* θ, *the law of* $(\delta(T_\theta),\lambda^0_{T_\theta})$ *is given by: for all positive* a *and* b,

$$\begin{aligned}&\mathbf{E}(\exp\{-a\delta(T_\theta)-b\lambda^0_{T_\theta}\})\\&\quad=[(2(\theta+b))^{1-d}-(\theta+2b)^{1-d}+\theta^{1-d}]/[a+(2(\theta+b))^{1-d}].\end{aligned}$$

In particular, $\delta(T_\theta)$ *has an exponential distribution with parameter* $(2\theta)^{1-d}$.

Proof.

$$\begin{aligned}&\mathbf{E}(\exp\{-a\delta(T_\theta)-b\lambda^0_{T_\theta}\})\\&=\mathbf{E}\left[\frac{\theta^2}{2}\int_0^{+\infty}\exp-\left(\frac{\theta^2}{2}t+a\delta(t)+b\lambda^0_t\right)dt\right]\\&=\mathbf{E}\left[\int_0^{+\infty}\exp-\left(\frac{\theta^2}{2}\sigma(t)+at+b\lambda^0_{\sigma(t)}\right)dt\right]\\&\qquad\cdot\left(\int m(d\omega)\frac{\theta^2}{2}\int_0^v\exp-\left(\frac{\theta^2}{2}r+b\lambda^0_r\right)dr\right)\\&=\left(\int_0^{+\infty}e^{-at}\,\mathbf{E}\left(\exp-\left(\frac{\theta^2}{2}\sigma(t)+b\lambda^0_{\sigma(t)}\right)\right)dt\right)\\&\qquad\cdot\left(\int m(d\omega)[(1-\mathrm{e}^{-\theta^2u/2})+\mathrm{e}^{-2b\omega^1(u)}(\mathrm{e}^{-\theta^2u/2}-\mathrm{e}^{-\theta^2v/2})]\right).\end{aligned}$$

After usual computations, we obtain

$$\mathbf{E}\left(\exp-\left(\frac{\theta^2}{2}\sigma(t)+b\lambda^0_{\sigma(t)}\right)\right)=\exp\{-t(2(\theta+b))^{1-d}\},$$

$$\int m(d\omega)(1-e^{-\theta^2 u/2})=\theta^{1-d},$$

and

$$\int m(d\omega)\,e^{-2b\omega^1(u)}(e^{-\theta^2 u/2}-e^{-\theta^2 v/2})=(2(\theta+b))^{1+d}-(\theta+2b)^{1-d};$$

which proves the proposition. □

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