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Probability Theory and Related Fields

An Almost Sure Invariance Principle for Stationary Ergodic Sequences of Banach Space Valued Random Variables

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Summary. In this paper we establish an almost sure invariance principle with an error term $o((t \log\log t)^{1/2})$ (as $t \to \infty$) for partial sums of stationary ergodic martingale difference sequences taking values in a real separable Banach space. As partial sums of weakly dependent random variables can often be well approximated by martingales, this result also leads to almost sure invariance principles for a wide class of stationary ergodic sequences such as ϕ-mixing and α-mixing sequences and functionals of such sequences. Compared with previous related work for vector valued random variables (starting with an article by Kuelbs and Philipp [27]), the present approach leads to a unification of the theory (at least for stationary sequences), moment conditions required by earlier authors are relaxed (only second order weak moments are needed), and our proofs are easier in that we do not employ estimates of the rate of convergence in the central limit theorem but merely the central limit theorem itself.

0. Introduction

Throughout this paper, we shall adhere to the following standard notation: $\mathbb{R} := $ set of all real numbers, $\mathbb{R}^+ := \{x \in \mathbb{R}: x > 0\}$, $\mathbb{Z} :=$ set of all integers, $\mathbb{N} := \{x \in \mathbb{Z}: x > 0\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $x \in \mathbb{R}^+$, we use $L_2 x$ to mean $\log\log(\max(e^e, x))$.

The present article is concerned with extensions of the following (by now classical) invariance principle for the law of the iterated logarithm established by Strassen [44] in 1964.

Theorem 0.A. *Let $(X_j)_{j \in \mathbb{N}}$ be a sequence of independent, identically distributed* (i.i.d.) *real-valued random variables* (r.v.'s) *with $EX_1 = 0$ and $EX_1^2 = 1$. Then, without changing its distribution, one can redefine the sequence $(X_j)_{j \in \mathbb{N}}$ on a new probability*

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space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ *on which there exists a Brownian motion* $(W(t))_{t\geq 0}$ *with* $W(0)=0$ *and* $EW(1)^2=1$ *such that*

$$\left|\sum_{k=1}^{[t]} X_k - W(t)\right| = o((t\mathrm{L}_2\, t)^{1/2}) \qquad \bar{P}-\text{a.s.} \qquad (\text{as } t\to\infty). \tag{0.1}$$

Strassen utilized this theorem in order to show that his functional version of the law of the iterated logarithm for a Brownian motion can be carried over to the partial sum process of a sequence of i.i.d. real-valued r.v.'s with mean zero and variance one.

Since the appearance of Strassen's paper, almost sure invariance principles of the form (0.1) have been obtained for a large class of independent and "weakly dependent" sequences (X_k). Strassen himself [45] (Theorem 4.4) also investigated the case when (X_k) is a martingale difference sequence. Partial extensions of this latter result have been published by Heyde and Scott [18] and Hall and Heyde [16]. Weakly dependent sequences such as mixing sequences, functions of mixing sequences, lacunary trigonometric sequences, or moving averages of i.i.d. r.v.'s have been studied by Heyde and Scott [18], Philipp and Stout [34], and Heyde [19]. The key idea exploited in [18], [34], and [19] is that, under a variety of assumptions, partial sums of "weakly dependent" sequences can be well approximated by martingales (see also Gordin [15] and Statulevičius [40]). If the error term of this approximation is small enough, and if the associated martingale admits an almost sure invariance principle, this also leads to an almost sure invariance principle for the original partial sum sequence.

The results quoted till now are confined to real-valued r.v.'s, and the corresponding proofs all depend on the Skorohod embedding for martingales (applied either to the partial sum process itself or to an associated martingale). To prove almost sure invariance principles for r.v.'s taking values in a higher dimensional Euclidean space (or even in a general Banach space), Berkes and Philipp [2] developed a new approximation technique based on the Strassen-Dudley theorem and suitable estimates of the rate of convergence in the central limit theorem with respect to the Prohorov distance. In the meantime this approach has been pursued by several authors. Kuelbs and Philipp [27] investigated sequences of independent, ϕ-mixing, and α-mixing sequences of Banach space valued r.v.'s. In [35], Philipp weakened the assumption of finite absolute $(2+\delta)$-th moments (for some $\delta>0$) still required in the proof of a Banach space analogue of Strassen's invariance principle for i.i.d. r.v.'s given in [27] (see also [7]). Dehling and Philipp [7] are concerned with absolutely regular (or weak Bernoulli) and α-mixing sequences of Banach space valued r.v.'s. Morrow and Philipp [31] derived partial generalizations of Strassen's [45] martingale invariance principle to the case of Hilbert space valued r.v.'s. For further related results, we refer to Philipp's [36] survey and the references given there.

In this paper we shall establish an almost sure invariance principle with an error term $o((t\mathrm{L}_2\, t)^{1/2})$ (as $t\to\infty$) for partial sums of certain stationary ergodic sequences of r.v.'s taking values in a real separable Banach space. More precisely, we consider stationary ergodic sequences having the property that their partial sums can in a sense be approximated by partial sums of stationary ergodic martingale difference sequences. This approach leads to a unification of the theory for stationary vector valued sequences analogous to that described in the book by Hall and Heyde [17], Chapter 5, for real-valued r.v.'s. In the stationary case, our main results (Theorems 3.1 and 3.2 below) contain several results previously derived by the Berkes-Philipp method as special cases (see Corollary 4.1 and Remark 4.3 below). In contrast to earlier work concerning almost sure invariance principles for weakly dependent random vectors, the results of this paper usually hold under minimal moment conditions. (Only second order weak moments are required in Theorems 3.1 and 3.2; previous authors generally needed moment conditions somewhat stronger than finite absolute second moments.) This is achieved by organizing the proofs in such a way that estimates of the rate of convergence in the central limit theorem are completely avoided; only a conditional version of the usual central limit theorem for stationary ergodic martingale difference sequences taking values in a finite-dimensional Euclidean space is employed. Thus our proofs are also easier than earlier proofs of related results in that the tools required are weaker.

We now proceed to describe the organization of this paper. In Sect. 2 we introduce and discuss the dependence structure on which our main results are based. The essential condition (the "M_2-property") ensures that the partial sums of a stationary ergodic sequence can be well approximated by partial sums of a stationary ergodic martingale difference sequence (Proposition 2.1). It turns out (Proposition 2.2) that this condition, though usually much easier to handle, is actually equivalent to a condition occurring in the book by Hall and Heyde [17]. Proposition 2.2 can be regarded as an answer to a question implicitly posed in [17]. The already announced general invariance principles (Theorems 3.1 and 3.2) for stationary ergodic sequences satisfying a suitable variant of the M_2-property are then stated in Sect. 3. Section 4 contains applications of the results of Sect. 3 to sequences of mixing r.v.'s and functions of such sequences (Corollary 4.1). We also relate these results to those obtained in the previous literature (Remarks 4.2 and 4.3). Sections 5–10 are devoted to proving the as yet indicated results. In Sect. 5 we give a new proof of Strassen's invariance principle. This proof illustrates in a simple setting some of the ideas to be used in the proof of Theorems 3.1 and 3.2. The proof of Theorems 3.1 and 3.2 is then carried out in Sects. 6–8. Having the proof in Sect. 5 in mind, the rôle of the auxiliary results stated in Sect. 6 becomes fairly obvious. Section 7 contains an approximation theorem (Proposition 7.1) for random variables taking values in a sequence space. The invariance principle for finite-dimensional Banach spaces (Theorem 3.1) is an immediate consequence of this proposition, while the remaining part of the proof of Theorem 3.2 (where the underlying separable Banach space is admitted to be infinite-dimensional) consists to a considerable extent in simply translating Proposition 7.1 into a Banach space result (see Sect. 8). It seems worthwhile to note that concepts connected with the theory of abstract Wiener spaces or the law of the iterated logarithm in general Banach spaces come into the picture only in Sect. 8. The proof of Proposition 2.2 is given in Sect. 9; Corollary 4.1 and Remark 4.1 are verified in Sect. 10.

1. Notation and Preliminaries

1.1. *General notations.* The topological dual of a Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is denoted by $\mathbb{B}^*$. 1_A is the indicator function of a set A. For $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\|x\|$ will always stand for the Euclidean norm of x, i.e., $\|x\| := \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. I_n is the $n \times n$ identity matrix. For a symmetric, nonnegative definite $n \times n$ matrix C, $N(0, C)$ denotes the normal distribution with mean 0 and covariance matrix C. The convolution of two probability measures μ and ν defined on the Borel σ-field of a separable Banach space $\mathbb{B}$ is written as $\mu * \nu$.

Let $(\Omega_1, \mathscr{F}_1, \mu_1)$ be a measure space, let $(\Omega_2, \mathscr{F}_2)$ be a measurable space, and let $T: \Omega_1 \to \Omega_2$ be a measurable mapping. Then $\mu_1 \circ T^{-1}$ denotes the measure μ_2 defined by $\mu_2(A) := \mu_1(T^{-1}A)$ $(A \in \mathscr{F}_2)$.

1.2. *Further notations and conventions.* For a Polish space Θ, $\mathscr{B}(\Theta)$ denotes the Borel σ-field induced by the topology of Θ.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and let Θ be a Polish space. A mapping $X: \Omega \to \Theta$ will be called a random variable (r.v.) if and only if it is Borel measurable. If X is simply said to be measurable, this will always mean that it is Borel measurable.

For a probability space $(\Omega, \mathscr{F}, P)$, $\mathscr{L}_2(\Omega, \mathscr{F}, P)$ denotes the Banach space of P-equivalence classes of Borel measurable functions $f: \Omega \to \mathbb{R}$ such that $\int f^2 \, dP < \infty$, equipped with the norm $\|\cdot\|_2$ defined by $\|f\|_2 := (\int f^2 \, dP)^{1/2}$. If the underlying probability space is clear from the context, we shall also use the abridged notation $\mathscr{L}_2$ in place of $\mathscr{L}_2(\Omega, \mathscr{F}, P)$. For a sequence $(f_n)_{n \in \mathbb{N} \cup \{\infty\}}$ in $\mathscr{L}_2(\Omega, \mathscr{F}, P)$, the convergence $\|f_n - f_\infty\|_2 \to 0$ (as $n \to \infty$) will usually be indicated by "$f_n \xrightarrow[\mathscr{L}_2]{} f_\infty$ as $n \to \infty$".

Let $a \in \mathbb{R}^+$, and let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable Banach space. Then $C_{\mathbb{B}}[0, a]$ denotes the separable Banach space of all continuous functions $f: [0, a] \to \mathbb{B}$, furnished with the norm $\|\cdot\|_{(a)}$ defined by $\|f\|_{(a)} := \sup\{\|f(x)\|_{\mathbb{B}}: x \in [0, a]\}$.

If $(X_j)_{j \in J}$ (J an arbitrary set) is a family of r.v.'s, $\mathscr{A}(X_j; A(j))$ denotes the σ-field generated by all X_j such that j has the property $A(j)$.

1.3. *Mixing conditions.* Let $(\Omega, \mathscr{F}, P)$ be a probability space, let Θ be a Polish space, and let $(X_k)_{k \in \mathbb{Z}}$ be a stationary sequence of r.v.'s $X_k: \Omega \to \Theta$. For $J \subset \mathbb{Z}$, put $\mathscr{F}_J := \mathscr{A}(X_j; j \in J)$, and let $P|\mathscr{F}_J$ denote the restriction of P to $\mathscr{F}_J$. Write

$$I'_p := \{m \in \mathbb{Z}: m \leqq p\}, \qquad I''_p := \{m \in \mathbb{Z}: m \geqq p\} \qquad (\text{for } p \in \mathbb{Z}),$$

and define (for $n \in \mathbb{N}$)

$$\alpha(n) := \sup\{|P(A \cap B) - P(A)\,P(B)|: \ A \in \mathscr{F}_{I'_0}, B \in \mathscr{F}_{I''_n}\},$$

$$\rho(n) := \sup\{(\int fg \, dP)/((\int f^2 \, dP)^{1/2} (\int g^2 \, dP)^{1/2}): \quad f \in \mathscr{L}_2(\Omega, \mathscr{F}_{I'_0}, P|\mathscr{F}_{I'_0}), g \in \mathscr{L}_2(\Omega, \mathscr{F}_{I''_n}, P|\mathscr{F}_{I''_n}), \ f \neq 0, g \neq 0, \int f \, dP = \int g \, dP = 0\},$$

$$\phi(n) := \sup\{|P(B|A) - P(B)|: A \in \mathscr{F}_{I'_0}, B \in \mathscr{F}_{I''_n}, P(A) > 0\}.$$

The sequence (X_k) is called α-mixing, ρ-mixing, or ϕ-mixing according as $\alpha(n) \to 0$, $\rho(n) \to 0$, or $\phi(n) \to 0$ as $n \to \infty$. It is clear that a ρ-mixing sequence is α-mixing. Moreover, a ϕ-mixing sequence is ρ-mixing (cf. Ibragimov and Linnik [20], Theorem 17.2.3, and Ibragimov [21]).

1.4. *Brownian motion in a Banach space.* Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a real separable Banach space. A probability measure ν on $\mathscr{B}(\mathbb{B})$ is called a mean zero Gaussian measure if for every $f \in \mathbb{B}^*$ $\nu \circ f^{-1}$ is a mean zero Gaussian distribution with variance $\int f(x)^2 \nu(dx)$. Starting from a mean zero Gaussian measure on $\mathscr{B}(\mathbb{B})$, the construction of a corresponding Brownian motion can be carried out analogously to the real case (see, e.g., Kuelbs [23]).

2. Martingale Approximation of Partial Sums of Stationary Sequences

To describe the general setting underlying Theorems 3.1 and 3.2, we begin with some definitions.

Definition 2.1. Let Θ be a Polish space. A quadruple $\mathbb{G} := ((\Omega, \mathscr{F}, P), \tau, \mathscr{F}_0, Y_0)$ consisting of a probability space $(\Omega, \mathscr{F}, P)$, an ergodic automorphism $\tau\colon \Omega \to \Omega$ on $(\Omega, \mathscr{F}, P)$, a sub-σ-field $\mathscr{F}_0 \subset \mathscr{F}$ with $\mathscr{F}_0 \subset \tau^{-1} \mathscr{F}_0$, and a r.v. $Y_0\colon \Omega \to \Theta$ is called the *germ of a Θ-valued stationary ergodic F-sequence.* (Here "F" is used to remind of "filtration"; cf. Definition 2.2 below.)

Definition 2.2. Notation is as in Definition 2.1. The sequence $(Y_k, \mathscr{F}_k)_{k \in \mathbb{Z}}$ defined by

$$Y_k := Y_0 \circ \tau^k \quad \text{and} \quad \mathscr{F}_k := \tau^{-k} \mathscr{F}_0 \tag{2.1}$$

is called the *stationary ergodic F-sequence induced by* $\mathbb{G}$.

Definition 2.3. Let $\mathbb{G} = ((\Omega, \mathscr{F}, P), \tau, \mathscr{F}_0, \xi_0)$ be the germ of an $\mathbb{R}$-valued stationary ergodic F-sequence, and let $(\xi_k, \mathscr{F}_k)_{k \in \mathbb{Z}}$ be the stationary ergodic F-sequence induced by $\mathbb{G}$.

(a) Suppose $E|\xi_0| < \infty$. Then the sequence $(\xi_k, \mathscr{F}_k)_{k \in \mathbb{Z}}$ is said to be a *stationary ergodic martingale difference sequence* if and only if

$$\xi_0 \quad \text{is} \quad \mathscr{F}_0\text{-measurable and } E(\xi_0 | \mathscr{F}_{-1}) = 0 \quad \text{a.s.} \tag{2.2}$$

(b) Suppose $E\xi_0^2 < \infty$. Then the sequence $(\xi_k, \mathscr{F}_k)_{k \in \mathbb{Z}}$ is said to have the *M_2-property* if and only if the sequences

$$\left(\sum_{j=1}^{n} E(\xi_j | \mathscr{F}_0) \right)_{n \in \mathbb{N}} \quad \text{and} \quad \left(\sum_{j=-n}^{0} (\xi_j - E(\xi_j | \mathscr{F}_0)) \right)_{n \in \mathbb{N}} \tag{2.3}$$

are Cauchy in $\mathscr{L}_2$.

Remark 2.1. If $(\xi_k, \mathscr{F}_k)_{k \in \mathbb{Z}}$ is a stationary ergodic martingale difference sequence in the sense of (a), then

$$\xi_k \text{ is } \mathscr{F}_k\text{-measurable and } E(\xi_k | \mathscr{F}_{k-1}) = E(\xi_0 | \mathscr{F}_{-1}) \circ \tau^k = 0 \quad \text{a.s.}$$

for all $k \in \mathbb{Z}$.

The importance of the M_2-property resides in the fact that under this condition the partial sums of the sequence $(\xi_k)_{k\in\mathbb{N}}$ behave in a sense like partial sums of a stationary ergodic martingale difference sequence; this is a consequence of the following

Proposition 2.1. *Let* $\mathbb{G}=((\Omega, \mathscr{F}, P), \tau, \mathscr{F}_0, \xi_0)$ *be the germ of an* $\mathbb{R}$*-valued stationary ergodic F-sequence, and let* $(\xi_j, \mathscr{F}_j)_{j\in\mathbb{Z}}$ *be the stationary ergodic F-sequence induced by* $\mathbb{G}$*. Suppose*

$$E\xi_0^2<\infty. \tag{2.4}$$

If

$$(\xi_j, \mathscr{F}_j)_{j\in\mathbb{Z}} \text{ has the } M_2\text{-property,} \tag{2.5}$$

then the r.v.'s ξ_k $(k\in\mathbb{Z})$ *admit a representation of the form*

$$\xi_k=\tilde{\xi}_k+\eta_{k-1}^{(\infty)}-\eta_k^{(\infty)} \quad \text{a.s.,} \tag{2.6}$$

where

$$(\tilde{\xi}_j, \mathscr{F}_j)_{j\in\mathbb{Z}} \text{ is a stationary ergodic martingale difference sequence with } E\tilde{\xi}_0^2<\infty \tag{2.7}$$

and

$$(\eta_j^{(\infty)})_{j\in\mathbb{Z}} \text{ is a stationary sequence of r.v.'s with } E|\eta_0^{(\infty)}|^2<\infty. \tag{2.8}$$

Proof (see also Gordin [15], and Hall and Heyde [17], Chapter 5). For $k\in\mathbb{Z}$ and $l\in\mathbb{N}$, define

$$\eta_k^{(l)}:=\sum_{j=1}^{l} E(\xi_{j+k}|\mathscr{F}_k)-\sum_{j=-l}^{0}(\xi_{j+k}-E(\xi_{j+k}|\mathscr{F}_k)).$$

Using (2.4) and (2.5), we see that the limits

$$\eta_k^{(\infty)}:=\mathscr{L}_2-\lim_{l\to\infty}\eta_k^{(l)} \tag{2.9}$$

exist and that

$$\xi_{-l}-E(\xi_{-l}|\mathscr{F}_k)\xrightarrow[\mathscr{L}_2]{}0 \quad \text{as} \quad l\to\infty \tag{2.10}$$

and

$$E(\xi_l|\mathscr{F}_k)\xrightarrow[\mathscr{L}_2]{}0 \quad \text{as} \quad l\to\infty. \tag{2.11}$$

(Here we have also made use of the fact that

$$E(\xi_{j+l}|\mathscr{F}_l)=E(\xi_j|\mathscr{F}_0)\circ\tau^l \quad \text{a.s.} \tag{2.12}$$

for all $j, l \in \mathbb{Z}$.) This implies that the right-hand side of the identity

$$\sum_{j=-l}^{l} (E(\xi_{j+k}|\mathscr{F}_k) - E(\xi_{j+k}|\mathscr{F}_{k-1}))$$
$$= \xi_k + \eta_k^{(l)} - \eta_{k-1}^{(l)} - \xi_{k-l-1}$$
$$+ E(\xi_{k-l-1}|\mathscr{F}_{k-1}) - E(\xi_{k+l}|\mathscr{F}_{k-1}) \quad \text{a.s.} \tag{2.13}$$

converges in $\mathscr{L}_2$ (as $l \to \infty$) to

$$\xi_k^* := \xi_k + \eta_k^{(\infty)} - \eta_{k-1}^{(\infty)}. \tag{2.14}$$

Since ξ_k^* can also be obtained as the $\mathscr{L}_2$-limit of the left-hand side of (2.13), it is easy to check that

$$E(\xi_k^* | \mathscr{F}_{k-1}) = 0 \quad \text{a.s.} \tag{2.15}$$

and that

$$\xi_k^* \text{ can be assumed to be } \mathscr{F}_k\text{-measurable.} \tag{2.16}$$

Moreover, (2.12) entails that

$$\xi_k^* = \xi_0^* \circ \tau^k \quad \text{a.s.} \quad \text{and} \quad \eta_k^{(\infty)} = \eta_0^{(\infty)} \circ \tau^k \quad \text{a.s.} \tag{2.17}$$

Taking $\tilde{\xi}_k := \xi_0^* \circ \tau^k$ and using (2.14)–(2.17), we arrive at (2.6).

An alternative characterization of the M_2-property is provided by the following

Proposition 2.2. *Let $\mathbb{G} = ((\Omega, \mathscr{F}, P), \tau, \mathscr{F}_0, \xi_0)$ be the germ of an $\mathbb{R}$-valued stationary ergodic F-sequence, and let $(\xi_j, \mathscr{F}_j)_{j \in \mathbb{Z}}$ be the stationary ergodic F-sequence induced by $\mathbb{G}$. Suppose $E\xi_0^2 < \infty$. Moreover, let*

$$x_l := E(\xi_{-l}|\mathscr{F}_0) - E(\xi_{-l}|\mathscr{F}_{-1}) \quad \text{for } l \in \mathbb{Z},$$

let $\mathscr{F}_{-\infty} := \bigcap_{k \in \mathbb{Z}} \mathscr{F}_k$, and let $\mathscr{F}_\infty$ denote the σ-field generated by $\bigcup_{k \in \mathbb{Z}} \mathscr{F}_k$. Then the following two statements are equivalent:

(i) $\left(\sum_{k=1}^{n} E(\xi_k|\mathscr{F}_0) \right)_{n \in \mathbb{N}}$ *is a Cauchy sequence in $\mathscr{L}_2$;*

(ii) $\sum_{m=1}^{\infty} \limsup_{n \to \infty} E\left(\left(\sum_{l=m}^{n} x_{-l} \right)^2 \right) < \infty$ *and* $E(\xi_0|\mathscr{F}_{-\infty}) = 0$ a.s.

Similarly, the statements (iii) and (iv) are also equivalent:

(iii) $\left(\sum_{k=-n}^{-1} (\xi_k - E(\xi_k|\mathscr{F}_0)) \right)_{n \in \mathbb{N}}$ *is a Cauchy sequence in $\mathscr{L}_2$;*

(iv) $\sum_{m=1}^{\infty} \limsup_{n \to \infty} E\left(\left(\sum_{l=m}^{n} x_l \right)^2 \right) < \infty$ *and* $E(\xi_0|\mathscr{F}_\infty) = \xi_0$ a.s.

Proof. See Sect. 9.

Heyde [19] and Hall and Heyde [17], p. 144, implicitly posed the problem of simplifying conditions (ii) and (iv) in a convenient way. The result stated in Proposition 2.2 can be regarded as an answer to this question.

In [19] and [17], the representation (2.6) is obtained for sequences $(\xi_j, \mathscr{F}_j)_{j\in\mathbb{Z}}$ satisfying (ii) and (iv) instead of (i) and (iii) (as in our Proposition 2.1). Usually, conditions (i) and (iii) are easier to work with; but in some cases (e.g., for stationary linear processes) conditions (ii) and (iv) apply in a natural way. (For further details, see Corollary 4.1 (below) and its proof.)

The next definition extends the M_2-property to the vector space setting.

Definition 2.4. Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a real separable Banach space, let $\mathbb{G} = ((\Omega, \mathscr{F}, P), \tau, \mathscr{F}_0, X_0)$ be the germ of a $\mathbb{B}$-valued stationary ergodic F-sequence, and let $(X_k, \mathscr{F}_k)_{k\in\mathbb{Z}}$ be the stationary ergodic F-sequence induced by $\mathbb{G}$. Suppose that

$$E f(X_0)^2 < \infty \qquad \text{for all } f\in\mathbb{B}^*.$$

Then the sequence $(X_k, \mathscr{F}_k)_{k\in\mathbb{Z}}$ is said to have the *weak M_2-property* if and only if, for all $f\in\mathbb{B}^*$, the sequences $(f(X_k), \mathscr{F}_k)_{k\in\mathbb{Z}}$ have the M_2-property.

3. General Theorems

After these preparations, we are now in a position to state our main results.

Theorem 3.1. *Let $d\in\mathbb{N}$, let $\mathbb{G} = ((\Omega, \mathscr{F}, P), \tau, \mathscr{F}_0, X_0)$ be the germ of an $\mathbb{R}^d$-valued stationary ergodic F-sequence, and let $(X_k, \mathscr{F}_k)_{k\in\mathbb{Z}}$ be the stationary ergodic F-sequence induced by $\mathbb{G}$. Suppose:*

$$E\|X_0\|^2 < \infty; \tag{3.1}$$

$$(X_k, \mathscr{F}_k)_{k\in\mathbb{Z}} \textit{ has the weak } M_2\textit{-property.} \tag{3.2}$$

Write $S_0 := 0$ $(\in\mathbb{R}^d)$ and, for $n\in\mathbb{N}$, $S_n := \sum_{j=1}^{n} X_j$. Then the limit

$$C := \lim_{n\to\infty} \frac{1}{n} \operatorname{Cov}(S_n) \tag{3.3}$$

exists, and, without changing its distribution, one can redefine the sequence $(X_k)_{k\in\mathbb{N}}$ on a new probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ on which there exists an $\mathbb{R}^d$-valued Brownian motion $(W(t))_{t\geq 0}$ with $W(0)=0$ and $\operatorname{Cov}(W(1)) = C$ such that

$$\|S_{[t]} - W(t)\| = o((t \mathrm{L}_2 t)^{1/2}) \quad \bar{P}\text{-a.s.} \qquad (\textit{as } t\to\infty). \tag{3.4}$$

Taking $\kappa(r) = d$ for all $r\in\mathbb{N}$ in Proposition 7.1 (below), the *proof* follows directly by combining this proposition and Lemma 6.4.

Remark 3.1. In the case $d=1$, the assertion of the above theorem coincides with an almost sure invariance principle that can be obtained as a consequence of the proof of a functional law of the iterated logarithm for partial sums of stationary sequences of real-valued r.v.'s given by Heyde and Scott [18] and Heyde [19]. This follows from Proposition 2.2 (above).

Next we consider r.v.'s taking values in an arbitrary real separable Banach space.

Theorem 3.2. *Let* $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ *be a real separable Banach space, let* $\mathbb{G} = ((\Omega, \mathscr{F}, P), \tau, \mathscr{F}_0, X_0)$ *be the germ of a* $\mathbb{B}$*-valued stationary ergodic F-sequence, and let* $(X_k, \mathscr{F}_k)_{k\in\mathbb{Z}}$ *be the stationary ergodic F-sequence induced by* $\mathbb{G}$. *Suppose*:

$$E f(X_0)^2 < \infty \qquad \text{for all } f \in \mathbb{B}^*; \tag{3.5}$$

$$(X_k, \mathscr{F}_k)_{k\in\mathbb{Z}} \text{ has the weak } M_2\text{-property.} \tag{3.6}$$

Write $S_0 := 0$ $(\in \mathbb{B})$ *and, for* $n \in \mathbb{N}$, $S_n := \sum_{j=1}^{n} X_j$. *Then the limits*

$$C(f, g) := \lim_{n\to\infty} \frac{1}{n} E f(S_n)\, g(S_n) \tag{3.7}$$

exist for all $f, g \in \mathbb{B}^*$, *and the following two statements are equivalent*:

(a) *there exists a mean zero Gaussian measure* ν *with covariance function* C *(i.e.,* $C(f, g) = \int f(x)\, g(x)\, \nu(dx)$ *for all* $f, g \in \mathbb{B}^*$*), and the sequence* $((n L_2 n)^{-1/2} S_n)_{n\in\mathbb{N}}$ *is with probability one conditionally* $\|\cdot\|_{\mathbb{B}}$*-compact*;

(b) *without changing its distribution, one can redefine the sequence* $(X_k)_{k\in\mathbb{N}}$ *on a new probability space* $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ *on which there exists a* $\mathbb{B}$*-valued Brownian motion* $(W(t))_{t\geq 0}$ *such that*

$$\|S_{[t]} - W(t)\|_{\mathbb{B}} = o((t L_2 t)^{1/2}) \ \ \bar{P}-\text{a.s.} \qquad (\text{as } t \to \infty) \tag{3.8}$$

(in this case, the covariance function of $W(1)$ *is necessarily equal to* C*).*

The *proof* of Theorem 3.2 will be carried out in Sects. 6–8.

Remark 3.2. Together with the functional law of the iterated logarithm for Brownian motion in a real separable Banach space (established by Kuelbs and LePage [24]), an invariance principle of the form (3.8) implies that the partial sums of the sequence $(X_k)_{k\in\mathbb{N}}$ also satisfy the functional law of the iterated logarithm.

Remark 3.3. The point of view adopted in Theorem 3.2 is similar to that in Kuelbs' [25] Theorem 3.1 in that both results involve the assumption that (with the notation of our Theorem 3.2) the sequence $((n L_2 n)^{-1/2} S_n)_{n\in\mathbb{N}}$ be a.s. conditionally compact in the norm topology.

Remark 3.4. The criterion (a) in Theorem 3.2 is intimately connected with two rather intricate questions, namely:

(i) Under which conditions on C does there exist a mean zero Gaussian measure having C as its covariance function?

(ii) When is the sequence $((n L_2 n)^{-1/2} S_n)_{n\in\mathbb{N}}$ with probability one conditionally norm compact?

As to the first problem, only partial results are known. (For more detailed information, see Vakhania, Tarieladze, and Chobanyan [46], Chapters IV and V.) In the last years, a great deal of research has been focused on the second problem in the special case when the S_n's are partial sums of a sequence $(X_k)_{k\in\mathbb{N}}$ of i.i.d. r.v.'s; an important recent paper on this subject is due to Ledoux and Talagrand [28]. For weakly dependent (nonindependent) r.v.'s much less is known. Some results in this respect can be found in [27] (Theorem 5 in conjunction with conditions (1.7)–(1.9) of Theorem 1), [7] (Theorem 6), and [31]. It is natural to ask which of the results so far established for i.i.d. r.v.'s can be carried over to the class of r.v.'s considered in Theorem 3.2, in particular to stationary ergodic martingale difference sequences.

We emphasize, however, that further progress concerning the questions (i) and (ii) would hardly affect the proofs of the present paper; such results would only be of complementary character.

Remark 3.5. Notation and assumptions are as in Theorem 3.2. The weak M_2-property implies that the limits

$$\gamma_k(f):=\mathscr{L}_2-\lim_{m\to\infty}\sum_{j=1}^{m}E(f(X_{j+k})|\mathscr{F}_k)$$

and

$$\rho_k(f):=\mathscr{L}_2-\lim_{m\to\infty}\sum_{j=-m}^{0}(f(X_{j+k})-E(f(X_{j+k})|\mathscr{F}_k))$$

exist for all $k\in\mathbb{Z}$ and all $f\in\mathbb{B}^*$. Setting

$$\eta_k(f):=\gamma_k(f)-\rho_k(f)\quad\text{and}\quad\xi_k(f):=f(X_k)+\eta_k(f)-\eta_{k-1}(f),$$

it follows from the proof of Proposition 2.1 that

$(\xi_k(f))_{k\in\mathbb{Z}}$ is a stationary martingale difference sequence with $E\xi_0(f)^2<\infty$

and that

$(\eta_k(f))_{k\in\mathbb{Z}}$ is a stationary sequence with $E\eta_0(f)^2<\infty$.

Together with (3.7), this leads to the following alternative definition of C:

$$C(f,g)=E(\xi_0(f)\,\xi_0(g))\qquad(f,g\in\mathbb{B}^*).\tag{3.9}$$

4. Applications

Corollary 4.1 below gives an impression of the variety of applications of Theorems 3.1 and 3.2. To formulate this result, we begin by introducing some notation.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a real separable Banach space, and let $(X_k)_{k\in\mathbb{Z}}$ be a stationary sequence of r.v.'s $X_k\colon \Omega\to\mathbb{B}$. Suppose that

$$E(f(X_0)^2)<\infty \quad \text{and} \quad Ef(X_0)=0 \quad \text{for all } f\in\mathbb{B}^*. \tag{4.1}$$

We consider the following conditions A.1–A.7.

A.1 (stationary ergodic martingale difference sequences). $(X_k)_{k\in\mathbb{Z}}$ is ergodic and, writing $\mathscr{G}_0:=\mathscr{A}(X_{-j}; j\in\mathbb{N}_0)$, one has

$$E(f(X_1)|\mathscr{G}_0)=0 \quad \text{a.s.} \quad \text{for all } f\in\mathbb{B}^*.$$

A.2 (α-mixing sequences). $(X_k)_{k\in\mathbb{Z}}$ is α-mixing, $E(|f(X_1)|^{2+\delta})<\infty$ for some $\delta\in\mathbb{R}^+$ and all $f\in\mathbb{B}^*$, and the mixing coefficients $\alpha(k)$ $(k\in\mathbb{N})$ satisfy one of the following two conditions:

(i) writing $c_l(\delta,f):=\max\limits_{1\leq j\leq 2^l}\left(E\left(\left|f\left(\sum\limits_{i=1}^{j}X_i\right)\right|^{2+\delta}\right)\right)^{1/(2+\delta)}$, the series $\sum\limits_{l=1}^{\infty} c_l(\delta,f)\,\alpha(2^l)^{\delta/(4+2\delta)}$ converges for all $f\in\mathbb{B}^*$;

(ii) $\sum\limits_{k=1}^{\infty}\alpha(k)^{\delta/(4+2\delta)}<\infty$.

A.3 (ρ-mixing sequences). $(X_k)_{k\in\mathbb{Z}}$ is ρ-mixing with $\sum\limits_{k=1}^{\infty}k^{-1/2}\rho(k)<\infty$.

A.4 (ϕ-mixing sequences). $(X_k)_{k\in\mathbb{Z}}$ is ϕ-mixing, $E(|f(X_1)|^{2+\delta})<\infty$ for some $\delta\in[0,\infty)$ and all $f\in\mathbb{B}^*$, and the mixing coefficients $\phi(k)$ $(k\in\mathbb{N})$ satisfy one of the following two conditions:

(i) letting $c_l(\delta,f)$ be defined as in A.2, one has

$$\sum_{l=1}^{\infty} c_l(\delta,f)\,\phi(2^l)^{(1+\delta)/(2+\delta)}<\infty \quad \text{for all } f\in\mathbb{B}^*;$$

(ii) $\sum\limits_{k=1}^{\infty}k^{-1/2}\phi(k)^{(1+\delta^*)/(2+\delta^*)}<\infty$, where $\delta^*:=\min(\delta, 1)$.

A.5 (functions of α-mixing sequences). The X_k's possess a representation of the form $X_k=\Psi((Y_{j+k})_{j\in\mathbb{Z}})$, where

(i) $(Y_j)_{j\in\mathbb{Z}}$ is a stationary α-mixing sequence of r.v.'s on Ω with values in a Polish space Θ,

(ii) $E(|f(X_0)|^{2+\delta})<\infty$ for some $\delta\in\mathbb{R}^+$ and all $f\in\mathbb{B}^*$,

(iii) $\sum\limits_{k=1}^{\infty}\alpha(k)^{\delta/(4+2\delta)}<\infty$ (the $\alpha(k)$'s being the α-mixing coefficients pertaining to the sequence $(Y_j)_{j\in\mathbb{Z}}$),

(iv) $\Psi\colon \Theta^{\mathbb{Z}}\to\mathbb{B}$ is measurable, and, writing $\mathscr{M}_{-k}^{k}:=\mathscr{A}(Y_j;\ -k\leqq j\leqq k)$ for $k\in\mathbb{N}$, one has

(v) $\sum_{k=1}^{\infty}(E|E(f(X_0)|\ \mathscr{M}_{-k}^{k})-f(X_0)|^2)^{1/2}<\infty$ for all $f\in\mathbb{B}^*$.

A.6 (functions of ϕ-mixing sequences). The X_k's possess a representation of the form $X_k=\Psi((Y_{j+k})_{j\in\mathbb{Z}})$, where

(i) $(Y_j)_{j\in\mathbb{Z}}$ is a stationary ϕ-mixing sequence of r.v.'s on Ω with values in a Polish space Θ,

(ii) $E(|f(X_0)|^{2+\delta})<\infty$ for some $\delta\in[0,\infty)$ and all $f\in\mathbb{B}^*$,

(iii) $\sum_{k=1}^{\infty}\phi(k)^{(1+\delta)/(2+\delta)}<\infty$ (the $\phi(k)$'s being the ϕ-mixing coefficients pertaining to the sequence $(Y_j)_{j\in\mathbb{Z}}$),

(iv) $\Psi\colon\Theta^{\mathbb{Z}}\to\mathbb{B}$ is measurable,

and, writing $\mathscr{M}_{-k}^{k}:=\mathscr{A}(Y_j;-k\leqq j\leqq k)$ for $k\in\mathbb{N}$, one has

(v) $\sum_{k=1}^{\infty}(E|E(f(X_0)|\mathscr{M}_{-k}^{k})-f(X_0)|^2)^{1/2}<\infty$ for all $f\in\mathbb{B}^*$.

A.7 (stationary linear processes generated by i.i.d. r.v.'s). The X_k's possess a representation of the form $X_k=\sum_{j=-\infty}^{\infty}c_j\,Y_{k-j}$, where

(i) $(Y_j)_{j\in\mathbb{Z}}$ is a sequence of i.i.d. r.v.'s $Y_j\colon\Omega\to\mathbb{B}$ satisfying $E(f(Y_0)^2)<\infty$ and $Ef(Y_0)=0$ for all $f\in\mathbb{B}^*$,

(ii) $(c_j)_{j\in\mathbb{Z}}$ is a sequence of real numbers with $\sum_{j=-\infty}^{\infty}c_j^2<\infty$ and $\sum_{\varepsilon\in\{-1,1\}}\sum_{m=1}^{\infty}\limsup_{n\to\infty}\left(\sum_{j=m}^{n}c_{\varepsilon j}\right)^2<\infty$.

Corollary 4.1. *Notation is as above. As usual, let* $S_0:=0$ $(\in\mathbb{B})$ *and* $S_n:=\sum_{j=1}^{n}X_j$ *for* $n\in\mathbb{N}$. *Suppose that* (4.1) *and, in addition, one of the conditions* A.1–A.7 *is fulfilled. Then*:

(S1) *In case* $\mathbb{B}=\mathbb{R}^d$ (*for some* $d\in\mathbb{N}$), *the conclusion of Theorem* 3.1 *also holds under the present hypotheses.*

(S2) *The limits* $C(f,g):=\lim_{n\to\infty}\frac{1}{n}E(f(S_n)\,g(S_n))$ *exist for all* $f,g\in\mathbb{B}^*$, *and the equivalence of the statements* (*a*) *and* (*b*) *in Theorem* 3.2 *also holds under the present hypotheses.*

Proof. See Sect. 10.

Remark 4.1. Notation is as in Corollary 4.1. For $n\in\mathbb{N}$ and $f,g\in\mathbb{B}^*$, define

$$U_n(f,g):=E(f(X_0)\,g(X_0))+\sum_{k=1}^{n}E(f(X_0)\,g(X_k))$$
$$+\sum_{k=1}^{n}E(f(X_k)\,g(X_0))$$

if one of the conditions A.1–A.6 is fulfilled, and

$$V_n(f, g) := \left(\sum_{j=-n}^{n} c_j \right)^2 E(f(Y_0)\, g(Y_0))$$

if A.7 is fulfilled. Then the sequences $(U_n(f, g))_{n \in \mathbb{N}}$ ($(V_n(f, g))_{n \in \mathbb{N}}$, respectively) converge to $C(f, g)$.

Proof. See Sect. 10.

Remark 4.2 (related results for real-valued r.v.'s).

(a) If $\mathbb{B} = \mathbb{R}^d$, then Corollary 4.1 is a corollary to Theorem 3.1. As already mentioned, the case $d=1$ of Theorem 3.1 is equivalent to a result of Heyde and Scott [18], [19]. Since Hall and Heyde [17] did not exploit the equivalence given in Proposition 2.2 of the present paper, their assumption on the rate of decay of the ϕ-mixing coefficients in Corollary 5.5 (in [17]) is somewhat stronger than our condition in A.4, namely $\sum_{k=1}^{\infty} \phi(k)^{(1+\delta)/(2+\delta)} < \infty$.

(b) Improving earlier results of Iosifescu [22] and Reznik [38], Oodaira and Yoshihara [32], [33] derived laws of the iterated logarithm for partial sums of stationary sequences satisfying mixing conditions. For ϕ-mixing sequences and functions of ϕ-mixing sequences, their assumptions are more stringent than ours; on the other hand, for α-mixing sequences and functions of α-mixing sequences, they only need that $\sum_{k=1}^{\infty} \alpha(k)^{\delta'/(2+\delta')} < \infty$ for some $\delta' \in (0, \delta)$. (Here δ and $\alpha(k)$ ($k \in \mathbb{N}$) have the same meaning as in A.2 and A.5, respectively.)

(c) In the case $\delta = 0$ (cf. A.4), the best condition on the mixing rate for sequences of ϕ-mixing r.v.'s occurring in the previous literature in connection with results of iterated logarithm type seems to be $\sum_{k=1}^{\infty} \phi(k)^{1/2} < \infty$ (cf. [17], Corollary 5.5). For $\delta \in (0, 1]$, Berkes and Philipp [2] and Dabrowski [6] obtained almost sure invariance principles requiring only a logarithmic rate of decay of the ϕ-mixing coefficients. (For further extensions of these last mentioned results to ρ-mixing and α-mixing sequences, see Bradley [4].)

Remark 4.3 (related results for Banach space valued r.v.'s).

(a) Taking Strassen's converse to the law of the iterated logarithm (see, e.g., [13] or [43]) into account, we see that, under each of the assumptions A.1–A.7 except for A.2 and A.5, the above corollary comprises the independent case (see [7], Theorem 3) in its full generality.

(b) Extending the work of Kuelbs and Philipp [27], Dehling and Philipp [7], Theorem 2, established the equivalence of the statements (a) and (b) in Theorem 3.2 (as to the definition of the covariance function, recall Remark 4.1 above) for weakly stationary, α-mixing sequences $(X_k)_{k \in \mathbb{N}}$ satisfying $\sup_{k \in \mathbb{N}} E\|X_k\|_{\mathbb{B}}^{2+\delta} < \infty$ (for some $\delta \in (0, 1]$), $EX_k = 0$ for all $k \in \mathbb{N}$, and $\alpha(n) = O(n^{-(1+\varepsilon)(2+\delta)/\delta})$ for some $\varepsilon \in \mathbb{R}^+$. In the case of a finite-dimensional Banach space $\mathbb{B}$, our mixing condition in A.2(ii), demanding a rate of decay similar to $\alpha(n)=O(n^{-(1+\varepsilon)(4+2\delta)/\delta})$ for some $\varepsilon\in\mathbb{R}^+$, is somewhat more restrictive.

(c) For some further results related to the present paper, we refer to Morrow and Philipp [31], Eberlein [12], and Philipp [37], where invariance principles for $\mathbb{R}^d$-valued and Hilbert space valued r.v.'s are obtained. These authors require more stringent moment conditions, but, on the other hand, their results also apply to a wide class of nonstationary sequences.

Remark 4.4 (a problem). It would be interesting to know whether the strictly stationary case of Theorem 2 in [7] can be obtained as a consequence of our Theorem 3.2 (cf. Remark 4.3(b)). A possible approach (suggested by A.2(i)) to this problem would be to estimate $E\left|\sum_{k=1}^{n}\xi_k\right|^{2+\delta'}$ $(\delta'\in(0,\delta))$ for stationary α-mixing sequences $(\xi_k)_{k\in\mathbb{Z}}$ of real-valued r.v.'s satisfying $E\xi_1=0$, $E|\xi_1|^{2+\delta}<\infty$ for some $\delta\in\mathbb{R}^+$, and $\sum_{k=1}^{\infty}\alpha(k)^{\delta/(2+\delta)}<\infty$. Some results (unfortunately too weak for our purposes) in this direction have been derived by Yokoyama [47].

5. A New Proof of Strassen's Invariance Principle

Before proceeding to the proof of Theorems 3.1 and 3.2 (in Sects. 6–8), we shall sketch some of the underlying main ideas by giving a new proof of Strassen's invariance principle (Theorem 0.A above) based on the same type of reasoning. Because of its neatness, this proof also seems to be of some interest in its own right.

Compared with previous proofs of Theorem 0.A, the essential advantage of the present proof is that the tools employed are more elementary in the sense that they are easier to extend to more general situations. Indeed, examining this proof and taking into account that Lemma 5.1 is only utilized to simplify the presentation (see Remark 5.1 below), it turns out that the only deeper ingredients are

(a) the easier upper half of the Hartman-Wintner law of the iterated logarithm (a short proof of which can be found in a paper by de Acosta [1])
and

(b) the usual central limit theorem.

Especially, this proof works without estimates of the rate of convergence in the central limit theorem. Apart from the proofs employing the Skorohod embedding method, all previous proofs of Theorem 0.A and extensions thereof required such estimates (see, e.g., Major [29], Kuelbs and Philipp [27], and Philipp [35]). As such estimates are often difficult to prove (note that, e.g., Philipp's [35] proof is based on a rather delicate estimate due to Yurinskii [48] of the rate of convergence in the central limit theorem with respect to the Prohorov distance), our approach appears to be particularly attractive. What is even more important is that estimates of the rate of convergence in the central limit theorem for general stationary ergodic martingale difference sequences (such sequences will play a crucial rôle in the proof of Theorems 3.1 and 3.2) are not available and unlikely to exist.

Though our proof of Theorem 0.A is markedly different from Major's [29] one, it nevertheless has two points in common with his proof in that it also makes use of the quantile transform technique and the argument stated (in a somewhat extended form) in Lemma 6.4 below.

We preface our proof by two lemmas.

Lemma 5.1. (*Skorohod* [39]; *see also Dudley and Philipp* [9], *p.* 521f.) *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space, let* Θ_1 *and* Θ_2 *be Polish spaces, let* $Y\colon \Omega \to \Theta_1$ *and* $U\colon \Omega \to (0, 1)$ *be two r.v.'s, and let* Q *be a probability measure on* $\mathscr{B}(\Theta_1 \times \Theta_2)$. *Suppose:*

(a) *U is uniformly distributed on* (0, 1);

(b) *U and Y are independent;*

(c) *the first marginal of Q is equal to* $P \circ Y^{-1}$.

Then there is a Borel measurable mapping $\Psi\colon \Theta_1 \times (0, 1) \to \Theta_2$ *such that* $P \circ (Y, \Psi(Y, U))^{-1} = Q$.

Lemma 5.2. *Let* $(Q_n)_{n \in \mathbb{N}_0}$ *be a sequence of probability measures on* $\mathscr{B}(\mathbb{R})$ *such that* $\int x^2 Q_n(dx) < \infty$ *for all* $n \in \mathbb{N}_0$. *Suppose:*

(a) $Q_n \to Q_0$ *weakly as* $n \to \infty$;

(b) $\int x^2 Q_n(dx) \to \int x^2 Q_0(dx)$ *as* $n \to \infty$.

Then there is a probability space $(\Omega, \mathscr{F}, P)$ *on which there is defined a sequence* $(\xi_n)_{n \in \mathbb{N}_0}$ *of real-valued r.v.'s* ξ_n *with* $P \circ \xi_n^{-1} = Q_n$ (*for all* $n \in \mathbb{N}_0$) *such that*

$$E(\xi_n - \xi_0)^2 \to 0 \quad \text{as } n \to \infty. \tag{5.1}$$

Proof. Let $(\Omega, \mathscr{F}, P) := ((0, 1), \mathscr{B}((0, 1)), \lambda)$, where λ is the Lebesgue measure on $\mathscr{B}((0, 1))$. For each $k \in \mathbb{N}_0$, let F_k be the distribution function corresponding to Q_k, i.e.,

$$F_k(x) := Q_k((-\infty, x]) \quad \text{for } x \in \mathbb{R}.$$

Moreover, let

$$\xi_k(y) := \inf\{t \in \mathbb{R}\colon F_k(t) \geqq y\} \quad \text{for } y \in (0, 1).$$

It is well-known (see, e.g., Dudley [8], p. 71, or Gänssler and Stute [14], Satz 1.12.6) that

$$P \circ \xi_k^{-1} = Q_k \tag{5.2}$$

and that $\lim_{n \to \infty} \xi_n(y) = \xi_0(y)$ except for at most countably many points $y \in (0, 1)$, i.e.,

$$\lim_{n \to \infty} \xi_n = \xi_0 \quad P\text{-a.s.} \tag{5.3}$$

Because of (b), (5.2) and (5.3), it follows from Scheffé's lemma (see, e.g., Gänssler and Stute [14], Satz 1.6.11) that

$$\text{the sequence } (\xi_n)_{n \in \mathbb{N}_0} \text{ is uniformly square-integrable.} \tag{5.4}$$

Combining (5.3) and (5.4), we arrive at (5.1).

Proof of theorem 0.A. Enlarging the underlying probability space if necessary, we may assume that the probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ on which the sequence $(X_j)_{j\in\mathbb{N}}$ is defined is so rich that there also exists a family $\{U_{\alpha,k}^{(r)}; k, r\in\mathbb{N}, \alpha\in\{1,2\}\}$ (independent of $(X_j)_{j\in\mathbb{N}}$) of i.i.d. r.v.'s $U_{\alpha,k}^{(r)}: \bar{\Omega}\to(0,1)$ with uniform distribution on (0, 1). As usual, we write $S_0:=0$ and $S_n:=\sum_{j=1}^{n} X_j$ for $n\in\mathbb{N}$.

The proof of Theorem 0.A consists of four steps S.1–S.4.

S.1 (approximation of a single partial sum). Combining the central limit theorem and Lemma 5.2, we can find a sequence $(l(r))_{r\in\mathbb{N}}$ in $\mathbb{N}$ with

$$l(r+1)/l(r)\in\mathbb{N} \tag{5.5}$$

and a probability space $(\Omega, \mathscr{F}, P)$ on which there are defined r.v.'s $S_{l(r)}^*, T_{l(r)}^*: \Omega \to\mathbb{R}$ such that

$$P\circ(S_{l(r)}^*)^{-1}=\bar{P}\circ S_{l(r)}^{-1}, \qquad P\circ(l(r)^{-1/2}\, T_{l(r)}^*)^{-1}=N(0,1),$$

and

$$E|S_{l(r)}^*-T_{l(r)}^*|^2\leqq l(r)/(2r^4).$$

By virtue of Lemma 5.1, we may assume that

$$(\Omega, \mathscr{F}, P)=(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P}), \qquad S_{l(r)}^*=S_{l(r)},$$
$$T_{l(r)}^*=\Lambda_r(S_{l(r)}, U_{1,1}^{(r)}),$$

where $\Lambda_r: \mathbb{R}\times(0,1)\to\mathbb{R}$ is a Borel measurable function.

S.2 (approximation of a single block). Let $r\in\mathbb{N}$, and let

$$Z_1^{(r)}:=\Lambda_r(S_{l(r)}, U_{1,1}^{(r)})$$

(with Λ_r being as in S.1). For $a\in\mathbb{R}^+$, let $C[0,a]$ denote the separable Banach space of all continuous functions $f: [0,a]\to\mathbb{R}$, equipped with the supremum norm. According to Lemma 5.1, there is a Borel measurable mapping

$$\Psi_r: \mathbb{R}\times(0,1)\to C[0, l(r)]$$

such that

$$V_1^{(r)}:=\Psi_r(Z_1^{(r)}, U_{2,1}^{(r)})$$

is a Brownian motion with

$$V_1^{(r)}(0)=0 \quad \text{and} \quad E(V_1^{(r)}(1)^2)=1$$

satisfying

$$V_1^{(r)}(l(r))=Z_1^{(r)} \ \ \bar{P}\text{-a.s.}$$

S.3 (a sequence of preliminary approximations of the partial sum process). Let $(l(r))_{r\in\mathbb{N}}$, $(\Lambda_r)_{r\in\mathbb{N}}$, and $(\Psi_r)_{r\in\mathbb{N}}$ be as in S.1 and S.2. Our next aim is to construct a sequence $(W_r)_{r\in\mathbb{N}}$ of Brownian motions $W_r=\{W_r(t);\, t\in[0,\infty)\}$ with

$$W_r(0)=0 \quad \text{and} \quad EW_r(1)^2=1$$

having the following properties:

(a) $\mathscr{A}(W_r((k-1)\,l(r)+t)-W_r((k-1)\,l(r));\ t\in[0,l(r)])$
$\subset\mathscr{A}(S_{kl(r)}-S_{(k-1)l(r)},\,U^{(r)}_{1,k},\,U^{(r)}_{2,k})$ for all $k\in\mathbb{N}$;

(b) $E|(W_r(kl(r))-W_r((k-1)\,l(r)))-(S_{kl(r)}-S_{(k-1)l(r)})|^2$
$\leqq l(r)/(2r^4)$ for all $k\in\mathbb{N}$;

(c) $((W_r(kl(r))-W_r((k-1)\,l(r)))-(S_{kl(r)}-S_{(k-1)l(r)}))_{k\in\mathbb{N}}$
is a sequence of i.i.d. r.v.'s.

To this end, we put

$$Z^{(r)}_k:=\Lambda_r(S_{kl(r)}-S_{(k-1)l(r)},\,U^{(r)}_{1,k})$$

and

$$V^{(r)}_k:=\Psi_r(Z^{(r)}_k,\,U^{(r)}_{2,k})$$

for $k, r\in\mathbb{N}$. The desired Brownian motions W_r ($r\in\mathbb{N}$) are then defined inductively by setting $W_r(0):=0$ and

$$W_r(t):=W_r((k-1)\,l(r))+V^{(r)}_k(t-(k-1)\,l(r))$$

for $t\in((k-1)\,l(r),\,kl(r)]$ and $k\in\mathbb{N}$. It is obvious from S.1 and S.2 that the thus obtained sequence $(W_r)_{r\in\mathbb{N}}$ satisfies (a)–(c). Moreover,

(d) $\limsup_{t\to\infty}(t\,\mathrm{L}_2\,t)^{-1/2}|S_{[t]}-W_r(t)|\leqq r^{-2}$ $\bar{P}$-a.s. for each $r\in\mathbb{N}$.

To prove (d), let $r\in\mathbb{N}$ be arbitrarily fixed, and let the sequences $(M^{(r)}_{\alpha,k})_{k\in\mathbb{N}}$ ($\alpha\in\{1,2\}$) be defined by

$$M^{(r)}_{1,k}:=\max_{(k-1)\,l(r)<m\leqq kl(r)}|S_m-S_{(k-1)l(r)}|$$

and

$$M^{(r)}_{2,k}:=\max_{0\leqq t\leqq 1}|W_r((k-1+t)\,l(r))-W_r((k-1)\,l(r))|.$$

It suffices to show that

$$\limsup_{m\to\infty}(m\,l(r)\,\mathrm{L}_2(m\,l(r)))^{-1/2}|S_{ml(r)}-W_r(m\,l(r))|\leqq r^{-2}\quad \bar{P}\text{-a.s.} \tag{5.6}$$

and that

$$\lim_{k\to\infty}k^{-1/2}M^{(r)}_{\alpha,k}=0\quad \bar{P}\text{-a.s.}\qquad(\alpha\in\{1,2\}). \tag{5.7}$$

In view of (b) and (c), (5.6) is an immediate consequence of the upper class part of the Hartman-Wintner law of the iterated logarithm, (5.7) follows from the fact that the sequences $(M_{\alpha,k}^{(r)})_{k\in\mathbb{N}}$ $(\alpha\in\{1,2\})$ are stationary and satisfy $\int|M_{\alpha,1}^{(r)}|^2\,d\bar{P}<\infty$.

S.4 (the final approximation of the partial sum process). Taking (5.5) and S.3 ((a) and (d)) into account, the conclusion of Theorem 0.A now follows from Lemma 6.4 below, i.e., by gluing increments of the Brownian motions W_r in a suitable way together.

Remark 5.1. From a formal point of view, the use of Lemma 5.1 is not really essential for proving the main results of this paper. Random variables having the desired joint distributions can always be constructed by more elementary techniques based on the existence of regular conditional distributions and the Ionescu-Tulcea theorem. In the case of the above proof, this alternative argument would only lead to minor complications in the presentation; however, a detailed exposition of the proofs of Theorems 3.1 and 3.2 relying on these "more elementary" tools turns out to be extremely cumbersome.

6. Auxiliary Results

6.1. Weak convergence of conditional distributions and approximation of r.v.'s. One key ingredient for proving Theorems 3.1 and 3.2 is the approximation result stated in Proposition 6.1 below. This result will play a similar rôle as Theorems 1 and 2 of Berkes and Philipp [2] do in previous related work. In contrast to the Berkes-Philipp approximation theorems, the proof of Proposition 6.1 does not depend on the Strassen-Dudley theorem (see, e.g., [8], Theorem 1), but on the conditional quantile transform argument described in Lemmas 6.1 and 6.2 below.

Proposition 6.1. *Let $d\in\mathbb{N}$, let $(\Omega,\mathscr{F},P)$ be a probability space, let Θ be a Polish space, and let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of square-integrable r.v.'s $Y_n\colon\Omega\to\mathbb{R}^d$. Moreover, let $U\colon\Omega\to(0,1)$ and $X\colon\Omega\to\Theta$ be two r.v.'s, and let $(Q_n)_{n\in\mathbb{N}_0}$ be a sequence of stochastic kernels $Q_n\colon\Theta\times\mathscr{B}(\mathbb{R}^d)\to[0,1]$. Write $\mu:=P\circ X^{-1}$, and suppose:*

(i) *for each $n\in\mathbb{N}$, $Q_n(x,\cdot)$ is a regular conditional distribution for Y_n given $X=x$ $(x\in\Theta)$;*

(ii) *there is a μ-null set $N\in\mathscr{B}(\Theta)$ such that for each $x\in\Theta\setminus N$, $Q_n(x,\cdot)\to Q_0(x,\cdot)$ weakly as $n\to\infty$;*

(iii) $\iint\|y\|^2\,Q_0(x,dy)\,\mu(dx)<\infty$;

(iv) $E\|Y_n\|^2\to\iint\|y\|^2\,Q_0(x,dy)\,\mu(dx)$ *as $n\to\infty$;*

(v) *U is uniformly distributed on $(0,1)$ and independent of $(X,(Y_n)_{n\in\mathbb{N}})$.*

Then there is a sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of Borel measurable mappings

$$\Lambda_n\colon\mathbb{R}^d\times\Theta\times(0,1)\to\mathbb{R}^d$$

such that the r.v.'s $Y_n^*\colon\Omega\to\mathbb{R}^d$ defined by

$$Y_n^*:=\Lambda_n(Y_n,X,U)$$

have the following properties:

$$P\{X\in A,\ Y_n^*\in B\}=\int_A Q_0(x,B)\,\mu(dx)\qquad \text{for all } A\in\mathscr{B}(\Theta)$$
$$\text{and all } B\in\mathscr{B}(\mathbb{R}^d);\tag{6.1}$$

$$E\|Y_n-Y_n^*\|^2\to 0\qquad \text{as } n\to\infty.\tag{6.2}$$

The proof of Proposition 6.1 depends crucially on Lemma 6.2 below which in turn is basically a corollary to the following

Lemma 6.1. *Let* $d\in\mathbb{N}$, *let* Θ *be a Polish space, let* μ *be a probability measure on* $\mathscr{B}(\Theta)$, *and let* $Q_n\colon\Theta\times\mathscr{B}(\mathbb{R}^d)\to[0,1]$, $Q_n^{(1)}\colon\Theta\times\mathscr{B}(\mathbb{R})\to[0,1]$ *and* $Q_n^{(k)}\colon(\Theta\times\mathbb{R}^{k-1})\times\mathscr{B}(\mathbb{R})\to[0,1]$ $(n\in\mathbb{N}_0,\ k\in\{2,\dots,d\})$ *be stochastic kernels. Suppose*

$$Q_n(x,A)=\int\dots\int 1_A(y_1,\dots,y_d)\,Q_n^{(d)}((x,(y_1,\dots,y_{d-1})),dy_d)\dots Q_n^{(2)}((x,y_1),dy_2)\,Q_n^{(1)}(x,dy_1)$$

for all $n\in\mathbb{N}$, $x\in\Theta$, $A\in\mathscr{B}(\mathbb{R}^d)$. *Also assume that the functions* $F_{n,1}(\cdot\,|\,\cdot)$ *and* $F_{n,k}(\cdot\,|\,\cdot)$ *defined by*

$$F_{n,1}(y_1\,|\,x):=Q_n^{(1)}(x,(-\infty,y_1]),$$
$$F_{n,k}(y_k\,|\,y_1,\dots,y_{k-1},x):=Q_n^{(k)}((x,(y_1,\dots,y_{k-1})),(-\infty,y_k]),$$

$(y_1,\dots,y_d\in\mathbb{R},\ x\in\Theta,\ k\in\{2,\dots,d\})$ *satisfy the following conditions:*

(i) $F_{n,1}(y_1\,|\,x)\to F_{0,1}(y_1\,|\,x)$ *as* $n\to\infty$,
$F_{n,k}(y_k\,|\,y_1,\dots,y_{k-1},x)\to F_{0,k}(y_k\,|\,y_1,\dots,y_{k-1},x)$ *as* $n\to\infty$

for all $k\in\{2,\dots,d\}$, $y_1,\dots,y_d\in\mathbb{R}$, $x\in\Theta$;

(ii) *for all* $k\in\{2,\dots,d\}$, $y_1,\dots,y_{d-1}\in\mathbb{R}$, $x\in\Theta$, *the functions* $F_{0,1}(\cdot\,|\,x)$ *and* $F_{0,k}(\cdot\,|\,y_1,\dots,y_{k-1},x)$ *are continuous and strictly increasing;*

(iii) $$\inf_{\delta>0}\limsup_{n\to\infty}\sup_{z_{(k-1)}\in B_\delta(y_{(k-1)})}|F_{n,k}(y_k\,|\,y_1,\dots,y_{k-1},x)-F_{n,k}(y_k\,|\,z_1,\dots,z_{k-1},x)|=0$$

for all $x\in\Theta$, $k\in\{2,\dots,d\}$, $y_{(k-1)}=(y_1,\dots,y_{k-1})\in\mathbb{R}^{k-1}$, $y_k\in\mathbb{R}$, *where* $z_{(k-1)}=(z_1,\dots,z_{k-1})$ *and* $B_\delta(y_{(k-1)}):=\{z_{(k-1)}\in\mathbb{R}^{k-1}\colon \|z_{(k-1)}-y_{(k-1)}\|<\delta\}$.
Then there is a probability space $(\Omega,\mathscr{F},P)$ *on which there are defined r.v.'s* $X\colon\Omega\to\Theta$ *and* $Y_n\colon\Omega\to\mathbb{R}^d$ $(n\in\mathbb{N}_0)$ *such that*

$$P\{X\in A,\ Y_n\in B\}=\int_A Q_n(x,B)\,\mu(dx)\qquad \text{for all } A\in\mathscr{B}(\Theta)$$
$$\text{and all } B\in\mathscr{B}(\mathbb{R}^d)\tag{6.3}$$

and

$$Y_n(\omega)\to Y_0(\omega)\qquad \text{for all } \omega\in\Omega\quad (\text{as } n\to\infty).\tag{6.4}$$

Proof of Lemma 6.1. Let $(\Omega,\mathscr{F},P)$ be a probability space on which there are defined independent r.v.'s $X, U_1,\dots,U_d$, where $X\colon\Omega\to\Theta$ has distribution μ and

$U_1, \ldots, U_d\colon \Omega \to (0, 1)$ are uniformly distributed on (0, 1). Further, let the r.v.'s $Y_{n,1}, \ldots, Y_{n,d}\colon \Omega \to \mathbb{R}$ ($n \in \mathbb{N}_0$) be defined inductively by

$$Y_{n,1}(\omega) := \inf\{t \in \mathbb{R}\colon F_{n,1}(t \mid X(\omega)) \geqq U_1(\omega)\}$$

and

$$Y_{n,k}(\omega) := \inf\{t \in \mathbb{R}\colon F_{n,k}(t \mid Y_{n,1}(\omega), \ldots, Y_{n,k-1}(\omega), X(\omega)) \geqq U_k(\omega)\}$$

for $k \in \{2, \ldots, d\}$ and $\omega \in \Omega$, and put $Y_n := (Y_{n,1}, \ldots, Y_{n,d})$. It is well-known (and easy to check) that the thus obtained r.v.'s Y_n satisfy (6.3) (see, e.g., Skorohod [39], p. 630). In order to demonstrate that they also satisfy (6.4), we proceed by induction. The case $k = 1$ requires only a minor reinterpretation of the following argument. Suppose $d > 1$, let $1 < k \leqq d$, and assume that the first $k-1$ components of Y_n converge pointwise to those of Y_0. Fix $\omega \in \Omega$, and put $y_k := Y_{0,k}(\omega)$. Then, given $\varepsilon > 0$, we can (by (i)–(iii)) find an $n_0 \in \mathbb{N}$ such that for $n \geqq n_0$

$$\begin{aligned} &F_{n,k}(y_k - \varepsilon \mid Y_{n,1}(\omega), \ldots, Y_{n,k-1}(\omega), X(\omega)) \\ &\quad < F_{0,k}(y_k \mid Y_{0,1}(\omega), \ldots, Y_{0,k-1}(\omega), X(\omega)) \\ &\quad < F_{n,k}(y_k + \varepsilon \mid Y_{n,1}(\omega), \ldots, Y_{n,k-1}(\omega), X(\omega)), \end{aligned}$$

i.e., using (ii) and the definition of the Y_n's,

$$|Y_{n,k}(\omega) - Y_{0,k}(\omega)| \leqq \varepsilon.$$

Lemma 6.2. *Let d, Θ and μ be as in Lemma* 6.1, *let $\delta \in \mathbb{R}^+$, and let $(\tilde{Q}_n)_{n \in \mathbb{N}_0}$ be a sequence of stochastic kernels $\tilde{Q}_n\colon \Theta \times \mathscr{B}(\mathbb{R}^d) \to [0, 1]$. For $x \in \Theta$ and $n \in \mathbb{N}_0$, define*

$$Q_n(x, \cdot) := \tilde{Q}_n(x, \cdot) * N(0, \delta I_d).$$

Suppose that, for each $x \in \Theta$,

$$\tilde{Q}_n(x, \cdot) \text{ converges weakly as } n \to \infty \text{ to } \tilde{Q}_0(x, \cdot).$$

Then there is a probability space $(\Omega, \mathscr{F}, P)$ on which there are defined r.v.'s $X\colon \Omega \to \Theta$ and $Y_n\colon \Omega \to \mathbb{R}^d$ ($n \in \mathbb{N}_0$) satisfying (6.3) *and* (6.4).

Proof of Lemma 6.2. For $x \in \Theta$ and $n \in \mathbb{N}_0$, define

$$\tilde{Q}_{n,k}(x, A) := \tilde{Q}_n(x, A \times \mathbb{R}^{d-k}) \qquad (\text{for } k \in \{1, \ldots, d\},\ A \in \mathscr{B}(\mathbb{R}^k)),$$

$$F_{n,k}(y_k \mid y_1, \ldots, y_{k-1}, x) := \frac{\int_{-\infty}^{y_k} \int_{\mathbb{R}^k} \exp(-(2\delta)^{-1} \|(y_1, \ldots, y_{k-1}, s) - t\|^2)\, \tilde{Q}_{n,k}(x, dt)\, ds}{\int_{\mathbb{R}} \int_{\mathbb{R}^k} \exp(-(2\delta)^{-1} \|(y_1, \ldots, y_{k-1}, s) - t\|^2)\, \tilde{Q}_{n,k}(x, dt)\, ds}$$

(for $k \in \{1, \ldots, d\}$, $y_1, \ldots, y_d \in \mathbb{R}$),

and let $Q_n^{(k)}((x, (y_1, \ldots, y_{k-1})), \cdot)$ be the probability measure (on $\mathscr{B}(\mathbb{R})$) corresponding to the distribution function $F_{n,k}(\cdot \mid y_1, \ldots, y_{k-1}, x)$. Then it follows by routine computations that the thus defined Q_n's, $Q_n^{(k)}$'s, and $F_{n,k}$'s satisfy the hypotheses of Lemma 6.1. (To verify (iii), note that the function

$$u \mapsto \exp(-(2\delta)^{-1}\|u\|^2) \quad (u \in \mathbb{R}^{k-1})$$

is uniformly continuous.) This means that the conclusion of Lemma 6.2 is a consequence of Lemma 6.1.

Proof of Proposition 6.1. Let $Y_0: \Omega \to \mathbb{R}^d$ be a r.v. such that

$$P\{X \in A, Y_0 \in B\} = \int_A Q_0(x, B)\,\mu(dx) \quad \text{for all } A \in \mathscr{B}(\Theta) \text{ and all } B \in \mathscr{B}(\mathbb{R}^d), \tag{6.5}$$

and let $\eta: \Omega \to \mathbb{R}^d$ be a r.v. with $P \circ \eta^{-1} = N(0, I_d)$ which is independent of $(X, (Y_n)_{n \in \mathbb{N}_0})$. (In view of condition (v) and Lemma 5.1, the r.v.'s η and Y_0 can be defined without enlarging the underlying probability space.)

Now let $\delta \in \{j^{-1}: j \in \mathbb{N}\}$ be arbitrarily fixed. Using (i), (ii), and Lemma 6.2, we find that there is a probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ on which there are defined r.v.'s $\bar{Y}'_{n,\delta}: \bar{\Omega} \to \mathbb{R}^d$ $(n \in \mathbb{N}_0)$ and $\bar{X}: \bar{\Omega} \to \Theta$ such that

$$\bar{P} \circ (\bar{Y}'_{n,\delta}, \bar{X})^{-1} = P \circ (Y_n + \delta^{1/2}\eta, X)^{-1} \quad \text{for all } n \in \mathbb{N}_0 \tag{6.6}$$

and

$$\bar{Y}'_{n,\delta} \to \bar{Y}'_{0,\delta} \quad \bar{P}\text{-a.s.} \quad \text{as } n \to \infty. \tag{6.7}$$

Enlarging the probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ if necessary, we may assume that there also exists a r.v. $\bar{U}: \bar{\Omega} \to (0, 1)$ with uniform distribution on (0, 1) which is independent of $(\bar{X}, (\bar{Y}'_{n,\delta})_{n \in \mathbb{N}_0})$. By (6.6) and Lemma 5.1, there is a sequence $(\Psi_{n,\delta})_{n \in \mathbb{N}_0}$ of Borel measurable mappings

$$\Psi_{n,\delta}: \mathbb{R}^d \times \Theta \times (0, 1) \to \mathbb{R}^d \times \mathbb{R}^d$$

such that the r.v.'s $\bar{Y}'_{n,\delta}$ admit a decomposition of the form

$$\bar{Y}'_{n,\delta} = \bar{Y}_{n,\delta} + \delta^{1/2}\bar{\eta}_n, \tag{6.8}$$

where

$$\begin{gathered} (\bar{Y}_{n,\delta}, \bar{\eta}_n) = \Psi_{n,\delta}(\bar{Y}'_{n,\delta}, \bar{X}, \bar{U}), \\ \bar{P} \circ (\bar{Y}_{n,\delta}, \bar{X})^{-1} = P \circ (Y_n, X)^{-1}, \\ \bar{\eta}_n \text{ is independent of } (\bar{Y}_{n,\delta}, \bar{X}), \text{ and } \bar{P} \circ \bar{\eta}_n^{-1} = N(0, I_d). \end{gathered} \tag{6.9}$$

Because of (iii), (iv), and the definition of the r.v.'s $\bar{Y}'_{n,\delta}$, it follows from Scheffé's lemma (see, e.g., Gänssler and Stute [14], Satz 1.6.11) that

$$\text{the sequence } (\bar{Y}'_{n,\delta})_{n \in \mathbb{N}_0} \text{ is uniformly square-integrable.} \tag{6.10}$$

Combining (6.7) and (6.10), we obtain

$$\lim_{n \to \infty} \int \|\bar{Y}'_{n,\delta} - \bar{Y}'_{0,\delta}\|^2 \, d\bar{P} = 0,$$

which entails that

$$\limsup_{n\to\infty} \int \|\bar{Y}_{n,\delta} - \bar{Y}_{0,\delta}\|^2 \, d\bar{P} \leqq 4d\delta \tag{6.11}$$

(by (6.8) and (6.9)).

According to Lemma 5.1, there is a sequence $(A_{n,\delta})_{n\in\mathbb{N}}$ of Borel measurable mappings

$$A_{n,\delta}: \mathbb{R}^d \times \Theta \times (0,1) \to \mathbb{R}^d$$

such that the r.v.'s

$$Y^*_{n,\delta} := A_{n,\delta}(Y_n, X, U) \tag{6.12}$$

satisfy

$$P \circ (Y_n, X, Y^*_{n,\delta})^{-1} = \bar{P} \circ (\bar{Y}_{n,\delta}, \bar{X}, \bar{Y}_{0,\delta})^{-1}. \tag{6.13}$$

Taking into account that $\delta \in \{j^{-1}: j \in \mathbb{N}\}$ in our above considerations was arbitrary, it follows from (6.11) and (6.13) that there is a strictly increasing sequence $(\nu_k)_{k\in\mathbb{N}_0}$ in $\mathbb{N}_0$ such that $\nu_0 = 0$ and

$$\sup_{n > \nu_{l-1}} E\|Y^*_{n,l^{-1}} - Y_n\|^2 \leqq 8dl^{-1} \quad \text{for all } l \in \mathbb{N}\setminus\{1\}. \tag{6.14}$$

We define

$$A_n := A_{n,l^{-1}} \quad \text{for } \nu_{l-1} < n \leqq \nu_l \text{ and } l \in \mathbb{N}.$$

It is obvious from (6.5), (6.9), and (6.12)–(6.14) that the thus obtained mappings A_n meet the requirements of Proposition 6.1.

6.2. A conditional central limit theorem. We begin with a technical lemma. (A similar result appears in a paper of Eagleson [10], but his proof is somewhat unclear.)

Lemma 6.3. *Let Θ be a Polish space, and let P be a probability measure on $\mathscr{B}(\Theta)$. Let $(\Omega_i, \mathscr{F}_i)$ $(i \in \{1,2\})$ be measurable spaces, where the σ-field $\mathscr{F}_2$ is assumed to be countably generated. Let $X: \Theta \to \Omega_1$, $Y: \Theta \to \Omega_2$, and $Z: \Theta \to \mathbb{R}$ be measurable mappings. Suppose $\int |Z| \, dP < \infty$. Put $\mu := P \circ X^{-1}$, and let $Q: \Omega_1 \times \mathscr{B}(\Theta) \to [0,1]$ be a stochastic kernel satisfying*

$$\int_B Q(x, A)\, \mu(dx) = P(A \cap \{X \in B\})$$

(for all $A \in \mathscr{B}(\Theta)$ and all $B \in \mathscr{F}_1$), i.e., $Q(x,\cdot)$ is a regular conditional probability on $\mathscr{B}(\Theta)$ given $X = x$. Then there is a μ-null set $M \in \mathscr{F}_1$ such that for all $x \in \Omega_1 \setminus M$

$$E^x(Z \mid Y = y) = E(Z \mid X = x, Y = y) \quad Q(x,\cdot)\circ Y^{-1}\text{-a.s.} \tag{6.15}$$

Here E^x denotes the conditional expectation on the probability space $(\Theta, \mathscr{B}(\Theta), Q(x,\cdot))$.

Proof. For $x \in \Omega_1$, we define $Q_{Y|x} := Q(x,\cdot)\circ Y^{-1}$. (6.15) will be established by showing that there is a μ-null set $M \in \mathscr{F}_1$ such that for all $x \in \Omega_1 \setminus M$, $B \in \mathscr{F}_2$

$$\int_B E(Z \mid X = x, Y = y)\, Q_{Y|x}(dy) = \int_{Y^{-1}B} Z(\theta)\, Q(x, d\theta). \tag{6.16}$$

For all $A \in \mathscr{F}_1$ and all $B \in \mathscr{F}_2$, we have

$$\int_A \int_B E(Z \mid X=x, Y=y)\, Q_{Y|x}(dy)\, \mu(dx)$$
$$= \int_{X^{-1}A \cap Y^{-1}B} Z\, dP = \int_A \int_{Y^{-1}B} Z(\theta)\, Q(x, d\theta)\, \mu(dx),$$

which entails that for each $B \in \mathscr{F}_2$ there is a μ-null set $N(B) \in \mathscr{F}_1$ such that (6.16) holds for all $x \in \Omega_1 \setminus N(B)$. Since $\mathscr{F}_2$ is countably generated, we can even find a countable field $\mathscr{C}$ generating $\mathscr{F}_2$. We set $M := \bigcup_{B \in \mathscr{C}} N(B)$. Then $\mu(M)=0$. It is obvious that (6.16) holds for all $x \in \Omega_1 \setminus M$ and all $B \in \mathscr{F}_2$.

The following proposition is a conditional version (essentially due to Eagleson [10], [11]) of the central limit theorem for stationary ergodic martingale difference sequences in $\mathbb{R}^d$ $(d \in \mathbb{N})$.

Let Θ be a Polish space, let P be a probability measure on $\mathscr{B}(\Theta)$, and let $\sigma: \Theta \to \Theta$ be an ergodic automorphism on the probability space $(\Theta, \mathscr{B}(\Theta), P)$. Moreover, let $Y_0: \Theta \to \mathbb{R}^d$ be a r.v. with $\int \|Y_0\|^2\, dP < \infty$. For $k, l \in \mathbb{Z}$, put

$$Y_k := Y_0 \circ \sigma^k \quad \text{and} \quad \mathscr{G}_l := \mathscr{A}(Y_j;\, -\infty < j \leqq l).$$

Suppose

$$E(Y_0 \mid \mathscr{G}_{-1}) = 0 \quad \text{a.s.}$$

For each $I \subset \mathbb{Z}$ with $I \neq \emptyset$, let $\Omega_I := \prod_{k \in I} \mathbb{R}^d$. Furnished with the product topology, Ω_I is a Polish space. Let Ω_I be endowed with the corresponding Borel σ-field $\mathscr{B}(\Omega_I)$. The mapping $\hat{Y}_I: \Theta \to \Omega_I$ defined by $\hat{Y}_I(\theta) := (Y_i(\theta))_{i \in I}$ is $\mathscr{B}(\Omega_I)$-measurable. The corresponding distribution $P \circ \hat{Y}_I^{-1}$ will be denoted by μ_I. Write $C := \operatorname{Cov}(Y_0)$, $K := \{j \in \mathbb{Z}: j \leqq 0\}$, and let $Q(\hat{y}, \cdot)$ be a regular conditional probability on $\mathscr{B}(\Theta)$ given $\hat{Y}_K = \hat{y}$ (cf. Lemma 6.3).

Proposition 6.2. *There is a μ_K-null set $M \in \mathscr{B}(\Omega_K)$ such that for all $\hat{y} \in \Omega_K \setminus M$*

$$Q(\hat{y}, \cdot) \circ \left(n^{-1/2} \sum_{k=1}^{n} Y_k\right)^{-1} \to N(0, C) \quad \textit{weakly (as } n \to \infty).$$

Proof. Throughout this proof, the elements $x \in \mathbb{R}^d$ are considered as row vectors, while x^T denotes the corresponding column vector. For $k \in \mathbb{N}$, let $\mathscr{G}_k^+ := \mathscr{A}(Y_j;\, 1 \leqq j \leqq k)$. Together with Lemma 6.3, the assumptions $\int \|Y_0\|^2\, dP < \infty$ and $E(Y_0 \mid \mathscr{G}_{-1}) = 0$ a.s. imply that there exists a μ_K-null set $M_1 \in \mathscr{B}(\Omega_K)$ such that

$$(Y_k, \mathscr{G}_k^+)_{k \in \mathbb{N}} \text{ is a martingale difference sequence on } (\Theta, \mathscr{B}(\Theta), Q(\hat{y}, \cdot)) \tag{6.17}$$

and

$$E^{\hat{y}} \|Y_k\|^2 < \infty \qquad \text{(for all } k \in \mathbb{N}) \tag{6.18}$$

for all $\hat{y} \in \Omega_K \setminus M_1$. (As in Lemma 6.3, $E^{\hat{y}}$ denotes the conditional expectation on the probability space $(\Theta, \mathscr{B}(\Theta), Q(\hat{y}, \cdot))$.) We also have

$$P_1 := P\left\{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E(Y_k^T Y_k \mid \mathscr{G}_{k-1}) = C\right\}$$

$$= \mu_{\mathbb{Z}}\left\{(y_j)_{j\in\mathbb{Z}}: \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E(Y_k^T Y_k \mid \ldots, Y_{-1} = y_{-1}, \ldots, Y_{k-1} = y_{k-1}) = C\right\}$$

$$= \int Q(\hat{y}, \cdot) \circ \hat{Y}_{\mathbb{N}}^{-1}\left\{(y_j)_{j\in\mathbb{N}}: \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E(Y_k^T Y_k \mid \ldots, Y_{-1} = y_{-1}, \ldots, Y_{k-1} = y_{k-1}) = C\right\} \mu_K(d\hat{y})$$

(with $\hat{y} = (y_j)_{j\in K}$)

$$= \int Q(\hat{y}, \cdot) \circ \hat{Y}_{\mathbb{N}}^{-1}\left\{(y_j)_{j\in\mathbb{N}}: \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E^{\hat{y}}(Y_k^T Y_k \mid Y_1 = y_1, \ldots, Y_{k-1} = y_{k-1}) = C\right\} \mu_K(d\hat{y})$$

(by Lemma 6.3)

$$= \int Q\left(\hat{y}, \left\{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E^{\hat{y}}(Y_k^T Y_k \mid \mathscr{G}_{k-1}^{+}) = C\right\}\right) \mu_K(d\hat{y}).$$

Combining this identity and the fact that – in consequence of the Birkhoff ergodic theorem (note that, for all $k \in \mathbb{N}$, $E(Y_k^T Y_k \mid \mathscr{G}_{k-1}) = E(Y_1^T Y_1 \mid \mathscr{G}_0) \circ \sigma^{k-1}$ a.s.) – $P_1 = 1$, we infer that there exists a μ_K-null set $M_2 \in \mathscr{B}(\Omega_K)$ such that for all $\hat{y} \in \Omega_K \setminus M_2$

$$\frac{1}{n} \sum_{k=1}^{n} E^{\hat{y}}(Y_k^T Y_k \mid \mathscr{G}_{k-1}^{+}) \to C \qquad Q(\hat{y}, \cdot)\text{-a.s.} \tag{6.19}$$

(as $n \to \infty$). A similar argument shows that

$$\frac{1}{n} \sum_{k=1}^{n} E^{\hat{y}}(\|Y_k\|^2 1_{\{\|Y_k\| \geq r\}} \mid \mathscr{G}_{k-1}^{+}) \to \int \|Y_1\|^2 1_{\{\|Y_1\| \geq r\}} dP \qquad Q(\hat{y}, \cdot)\text{-a.s.} \tag{6.20}$$

(as $n \to \infty$) for all $r \in \mathbb{N}_0$ and for all $\hat{y}$ in the complement of a suitable μ_K-null set $M_3 \in \mathscr{B}(\Omega_K)$. Setting $M := M_1 \cup M_2 \cup M_3$ and invoking Brown's [5] martingale central limit theorem and the Cramèr-Wold device, the assertion of Proposition 6.2 follows from (6.17)–(6.20).

6.3. A law of the iterated logarithm for stationary martingale difference sequences. The following proposition is a generalization of the upper class part of Stout's [41] martingale analogue of the Hartman-Wintner law of the iterated logarithm to the case of not necessarily ergodic sequences.

Proposition 6.3. *Let $(\Omega, \mathscr{F}, P)$ be a probability space, and let $(\xi_k)_{k\in\mathbb{N}}$ be a stationary martingale difference sequence of r.v.'s $\xi_k\colon \Omega\to\mathbb{R}$. Suppose $E\xi_1^2<\infty$. Then*

$$\limsup_{n\to\infty}(2n\mathrm{L}_2 n)^{-1/2}\sum_{k=1}^{n}\xi_k\leqq E(\xi_1^2\,|\,\mathscr{I})^{1/2}\quad a.s., \tag{6.21}$$

where $\mathscr{I}$ is the σ-field of invariant events pertaining to the sequence (ξ_k).

Remark 6.1. By the very argument given below, "$\leqq$" in (6.21) can be replaced by "$=$"; but we shall not need this refinement of Proposition 6.3.

Proof of proposition 6.3. If $E(\xi_1^2\,|\,\mathscr{I})>0$ a.s., the proof can be carried through (with only minor modifications) by following the lines of Stout's [41] proof in the ergodic case. If $E(\xi_1^2\,|\,\mathscr{I})=0$ a.s., then $\xi_k=0$ a.s. (for all k), so that (6.21) is trivial. Finally, the case $0<P\{E(\xi_1^2\,|\,\mathscr{I})>0\}<1$ can be reduced to these two cases by considering the sequence (ξ_k) separately on $\Omega':=\{E(\xi_1^2\,|\,\mathscr{I})>0\}$ and on $\Omega'':=\Omega\setminus\Omega'$.

Remark 6.2. To establish the upper class part of Stout's [41] theorem (or of its generalized version stated in Proposition 6.3), only the upper half of Stout's [42] martingale analogue of Kolmogorov's law of the iterated logarithm is required. An elegant proof of the latter can be found in Stout's [43] monograph.

6.4. A subsequence argument. The last result of this section generalizes an argument due to Major [29].

Lemma 6.4. *Let $(\Omega, \mathscr{F}, P)$ be a probability space, let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a real separable Banach space (endowed with its Borel σ-field $\mathscr{B}(\mathbb{B})$), and let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of r.v.'s $\xi_k\colon\Omega\to\mathbb{B}$. Suppose that, for each $r\in\mathbb{N}$,* there is a *$\mathbb{B}$-valued Brownian motion $(W_r(t))_{t\geqq 0}$ (defined on the same probability space) with $W_r(0)=0$ and covariance function C of $W_r(1)$ (C being independent of r) such that*

$$P\{\limsup_{t\to\infty}(t\mathrm{L}_2 t)^{-1/2}\|\sum_{k\leqq t}\xi_k-W_r(t)\|\geqq r^{-2}\}\leqq r^{-2}. \tag{6.22}$$

Also assume that there is a system $\mathscr{M}=\{M(s)\colon s\in\mathbb{N}\}$ of infinite subsets of $\mathbb{N}$ satisfying

$$M(s)\subset M(s')\qquad \text{if } s>s' \tag{6.23}$$

such that

$$\mathscr{A}(W_s(t);\, s\leqq r,\, t\leqq n)\quad \text{and}\quad \mathscr{A}(W_r(t)-W_r(n);\, t\geqq n)$$
$$\text{are independent for each } (n,r)\in\mathbb{N}^2 \text{ with } n\in M(r). \tag{6.24}$$

Then one can even find a $\mathbb{B}$-valued Brownian motion $(W(t))_{t\geqq 0}$ with $W(0)=0$ and covariance function C of $W(1)$ such that

$$\|\sum_{k\leqq t}\xi_k - W(t)\| = o((t\,\mathrm{L}_2\,t)^{1/2}) \quad a.s. \tag{6.25}$$

Proof. From (6.22) it follows that there is a sequence $(m_j)_{j\in\mathbb{N}}$ of positive integers with

$$m_{j+1}\geqq 2m_j \qquad \text{for all } j\in\mathbb{N} \tag{6.26}$$

such that for all $r\in\mathbb{N}$

$$P\{\sup_{t\geqq n_r}(t\,\mathrm{L}_2\,t)^{-1/2}\|\sum_{k\leqq t}\xi_k - W_r(t)\|\geqq 2r^{-2}\}\leqq 2r^{-2}, \tag{6.27}$$

where $n_r=\sum_{j=1}^{r} m_j\in M(r)$. We define inductively

$$W(t):=\begin{cases} W_1(t) & \text{if } 0\leqq t\leqq n_1,\\ W(n_r)+W_r(t)-W_r(n_r) & \text{if } n_r<t\leqq n_{r+1}\end{cases} \tag{6.28}$$

($r\in\mathbb{N}$). In view of (6.24), it is clear that W is a Brownian motion with $W(0)=0$ and covariance function C of $W(1)$. It remains to prove (6.25). Using (6.26) and (6.27), we see that on a set of probability $\geqq 1-2r^{-2}$

$$\left\|\sum_{k=n_r+1}^{n_{r+1}}\xi_k-(W_r(n_{r+1})-W_r(n_r))\right\|\leqq\varepsilon_r(m_{r+1}\,\mathrm{L}_2\,m_{r+1})^{1/2},$$

where $\varepsilon_r=o(1)$ (as $r\to\infty$). Hence

$$\sum_{r=1}^{\infty}P\left\{\left\|\sum_{k=n_r+1}^{n_{r+1}}\xi_k-(W_r(n_{r+1})-W_r(n_r))\right\|>\varepsilon_r(m_{r+1}\,\mathrm{L}_2\,m_{r+1})^{1/2}\right\}<\infty,$$

which, together with the Borel-Cantelli lemma and (6.26), leads to

$$\left\|\sum_{k=1}^{n_r}\xi_k - W(n_r)\right\| = o((n_r\,\mathrm{L}_2\,n_r)^{1/2}) \quad \text{a.s.} \tag{6.29}$$

In conjunction with the Borel-Cantelli lemma, the estimate

$$P\left\{\sup_{n_r\leqq t\leqq n_{r+1}}(t\,\mathrm{L}_2\,t)^{-1/2}\left\|\sum_{k=n_r+1}^{[t]}\xi_k-(W(t)-W(n_r))\right\|\geqq 4r^{-2}\right\}\leqq 2r^{-2} \tag{6.30}$$

($r\in\mathbb{N}$), implied by (6.27), completes the proof of the lemma.

7. An Approximation for R.V.'s Taking Values in a Sequence Space

In this section we shall derive an approximation theorem for partial sums of stationary ergodic sequences taking values in $\mathbb{R}^{\mathbb{N}}$ (equipped with the usual product topology and the corresponding Borel σ-field). This result is of preparatory nature; it will, however, constitute the core of the proofs of Theorems 3.1 and 3.2.

For $\iota\in\mathbb{N}$, we use p_ι (A_ι, resp.) to denote the projection $\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ ($\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^\iota$, resp.) of $\mathbb{R}^{\mathbb{N}}$ onto the ι-th component (the first ι components, respectively).

Now let $\mathfrak{G}=((\Omega,\mathscr{F},P),\tau,\mathscr{F}_0,Y_0)$ be the germ of an $\mathbb{R}^{\mathbb{N}}$-valued stationary ergodic F-sequence, and let $(Y_k,\mathscr{F}_k)_{k\in\mathbb{Z}}$ be the stationary ergodic F-sequence induced by $\mathfrak{G}$. We write

$$T_0:=0\,(\in\mathbb{R}^{\mathbb{N}})\quad\text{and, for } n\in\mathbb{N},\ T_n:=\sum_{j=1}^{n} Y_j.$$

Definition 7.1. Suppose

$$E|p_\iota(Y_0)|<\infty\qquad\text{for all } \iota\in\mathbb{N}.$$

Then $(Y_k,\mathscr{F}_k)_{k\in\mathbb{Z}}$ is said to be a *stationary ergodic martingale difference sequence* if and only if, for all $\iota\in\mathbb{N}$, the sequences $(p_\iota(Y_k),\mathscr{F}_k)_{k\in\mathbb{Z}}$ are stationary ergodic martingale difference sequences.

Definition 7.2. Suppose

$$E|p_\iota(Y_0)|^2<\infty\qquad\text{for all } \iota\in\mathbb{N}.$$

Then $(Y_k,\mathscr{F}_k)_{k\in\mathbb{Z}}$ is said to have the *weak M_2-property* if and only if, for all $\iota\in\mathbb{N}$, the sequences $(p_\iota(Y_k),\mathscr{F}_k)_{k\in\mathbb{Z}}$ have the M_2-property.

Proposition 7.1. *Notation is as above. Moreover, let $\delta\in\mathbb{R}^+$, and let $(\kappa(r))_{r\in\mathbb{N}}$ be a nondecreasing sequence in $\mathbb{N}$. Suppose:*

$$E|p_\iota(Y_0)|^2<\infty\qquad\textit{for all } \iota\in\mathbb{N};\tag{7.1}$$

$$(Y_k,\mathscr{F}_k)_{k\in\mathbb{Z}}\qquad\textit{has the weak } M_2\textit{-property.}\tag{7.2}$$

Then the limits

$$C_r:=\lim_{n\to\infty}\frac{1}{n}\operatorname{Cov}(A_{\kappa(r)}T_n)\tag{7.3}$$

exist for all $r\in\mathbb{N}$, and, without changing its distribution, one can redefine the sequence $(Y_k)_{k\in\mathbb{N}}$ on a new probability space $(\bar\Omega,\bar{\mathscr{F}},\bar P)$ on which there exists a sequence $(\tilde W_r)_{r\in\mathbb{N}}$ of Brownian motions $\tilde W_r=\{\tilde W_r(t);\,t\in[0,\infty)\}$ with values in $\mathbb{R}^{\kappa(r)}$ such that

$$\tilde W_r(0)=0\quad\textit{and}\quad\operatorname{Cov}(\tilde W_r(1))=C_r\tag{7.4}$$

having the following properties:

(i) *there is a sequence $(l(r))_{r\in\mathbb{N}}$ in $\mathbb{N}$ with*

$$l(r+1)/l(r)\in\mathbb{N}\tag{7.5}$$

such that, for each $(m, r) \in \mathbb{N}^2$, *the* σ*-fields*

$$\mathscr{A}(\tilde{W}_s(t);\ 1 \leqq s \leqq r,\ t \leqq m\,l(r))$$

and

$$\mathscr{A}(\tilde{W}_r(t) - \tilde{W}_r(m\,l(r));\ t \geqq m\,l(r))$$

are independent;

(ii) $\bar{P}\{\limsup_{t\to\infty} (t\,\mathrm{L}_2\,t)^{-1/2}\, \|\tilde{W}_r(t) - A_{\kappa(r)}\, T_{[t]}\|$

$$\geqq 1/(2\delta r^2)\} \leqq r^{-2} \qquad \text{(for each } r \in \mathbb{N}\text{)}. \tag{7.6}$$

Proof. The result is established in six steps S.1–S.6.

S.1 (reduction to the case of a stationary ergodic martingale difference sequence). According to Proposition 2.1, the r.v.'s Y_k $(k \in \mathbb{Z})$ can be written in the form

$$Y_k = \tilde{Y}_k + R_{k-1} - R_k \quad \text{a.s.,} \tag{7.7}$$

where

$$(\tilde{Y}_j, \mathscr{F}_j)_{j\in\mathbb{Z}} \text{ is a stationary ergodic martingale difference sequence such that } E|p_\iota(\tilde{Y}_0)|^2 < \infty \quad \text{for all } \iota \in \mathbb{N} \tag{7.8}$$

and

$$(R_j)_{j\in\mathbb{Z}} \text{ is a stationary sequence of r.v.'s such that } E|p_\iota(R_0)|^2 < \infty \quad \text{for all } \iota \in \mathbb{N}. \tag{7.9}$$

For $n \in \mathbb{N}$, we write $\tilde{T}_n := \sum_{j=1}^{n} \tilde{Y}_j$. Then

$$T_n = \tilde{T}_n + R_0 - R_n \quad \text{a.s.} \qquad \text{(for all } n \in \mathbb{N}\text{)}. \tag{7.10}$$

In view of (7.8) and (7.9), this implies that the limits in (7.3) exist and that, for each $r \in \mathbb{N}$ and all $n \in \mathbb{N}$,

$$C_r = \operatorname{Cov}(A_{\kappa(r)}\, \tilde{Y}_0) = \frac{1}{n} \operatorname{Cov}(A_{\kappa(r)}\, \tilde{T}_n). \tag{7.11}$$

Furthermore, by (7.9) and (7.10),

$$\lim_{n\to\infty} n^{-1/2}\, \|A_{\kappa(r)}(T_n - \tilde{T}_n)\| = 0 \quad \text{a.s.} \tag{7.12}$$

(for each $r \in \mathbb{N}$). Consequently, it is enough to establish the assertion of Proposition 7.1 with $T_{[t]}$ in (7.6) replaced by $\tilde{T}_{[t]}$ (where $\tilde{T}_0 := 0 \in \mathbb{R}^{\mathbb{N}}$).

S.2 (an application of the conditional central limit theorem). Let $r \in \mathbb{N}$ be arbitrarily fixed. (7.8) entails that, for each $k \in \mathbb{Z}$,

$$E(A_{\kappa(r)}\, \tilde{Y}_k \,|\, \mathscr{G}_{k-1}^{(r)}) = 0 \quad \text{a.s.,} \tag{7.13}$$

where $\mathscr{G}_{k-1}^{(r)} := \mathscr{A}(A_{\kappa(r)} \tilde{Y}_j;\ -\infty < j \leqq k-1)$. In order to apply the conditional central limit theorem stated in Proposition 6.2, we consider the coordinate representation process corresponding to the sequence $(A_{\kappa(r)} \tilde{Y}_j)_{j\in\mathbb{Z}}$, i.e., the process $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}, (\hat{\xi}_j)_{j\in\mathbb{Z}})$ defined by $\hat{\Omega} := (\mathbb{R}^{\kappa(r)})^{\mathbb{Z}}$, $\hat{\mathscr{F}} :=$ Borel σ-field on $\hat{\Omega}$ induced by the product topology, $\hat{P} := P \circ ((A_{\kappa(r)} \tilde{Y}_j)_{j\in\mathbb{Z}})^{-1}$, $\hat{\xi}_j := j$-th projection $\hat{\Omega} \to \mathbb{R}^{\kappa(r)}$. By (7.8) and (7.13), the process $(\hat{\xi}_j)_{j\in\mathbb{Z}}$ satisfies the hypotheses of Proposition 6.2. Now let $\hat{\mu} := \hat{P} \circ ((\hat{\xi}_j)_{j\leqq 0})^{-1}$, and let $\hat{Q}(\hat{x}, \cdot)$ be a regular conditional probability on $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P})$ given $(\hat{\xi}_j)_{j\leqq 0} = \hat{x}$. For $\hat{x} \in \prod_{j\leqq 0} \mathbb{R}^{\kappa(r)}$ and $m \in \mathbb{N}$, put

$$Q_m(\hat{x}, A) := \hat{Q}\left(\hat{x}, \left\{m^{-1/2} \sum_{j=1}^{m} \hat{\xi}_j \in A\right\}\right) \quad (\text{for } A \in \mathscr{B}(\mathbb{R}^{\kappa(r)})).$$

Then

$$\begin{aligned} &Q_m(\hat{x}, \cdot) \text{ is a regular conditional distribution for} \\ &m^{-1/2} A_{\kappa(r)} \tilde{T}_m \quad \text{given} \quad (A_{\kappa(r)} \tilde{Y}_j)_{j\leqq 0} = \hat{x} \\ &(\hat{x} \in \prod_{j\leqq 0} \mathbb{R}^{\kappa(r)}). \end{aligned} \tag{7.14}$$

Moreover, according to Proposition 6.2, there is a $\hat{\mu}$-null set $N \in \mathscr{B}(\prod_{j\leqq 0} \mathbb{R}^{\kappa(r)})$ such that for each $\hat{x} \in (\prod_{j\leqq 0} \mathbb{R}^{\kappa(r)}) \setminus N$

$$Q_m(\hat{x}, \cdot) \to N(0, C_r) \quad \text{weakly as } m \to \infty. \tag{7.15}$$

(Note that, by (7.11), $C_r = \operatorname{Cov}(A_{\kappa(r)} \tilde{Y}_0)$.)

S.3 (enlargement of the underlying probability space). Let $(\Omega_1, \mathscr{F}_1, P_1)$ be a probability space on which there is defined a family $\{U_{\alpha,k}^{(r)};\ k, r \in \mathbb{N},\ \alpha \in \{1, 2\}\}$ of i.i.d. r.v.'s $U_{\alpha,k}^{(r)}: \Omega_1 \to (0, 1)$ with uniform distribution on $(0, 1)$, and let $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ be the product space $(\Omega \times \Omega_1, \mathscr{F} \otimes \mathscr{F}_1, P \otimes P_1)$. In the sequel, all r.v.'s originally defined on Ω (Ω_1, resp.) will tacitly be regarded as r.v.'s on $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ by identifying a r.v. X (Y, resp.) defined on Ω (Ω_1, resp.) with the r.v. $\bar{X}$ ($\bar{Y}$, resp.) defined by $\bar{X}(\omega, \omega_1) := X(\omega)$ ($\bar{Y}(\omega, \omega_1) := Y(\omega_1)$, resp.); note that $\bar{P} \circ \bar{X}^{-1} = P \circ X^{-1}$ and $\bar{P} \circ \bar{Y}^{-1} = P_1 \circ Y^{-1}$. We shall also make repeated use of the fact that, under the above hypotheses,

$$\bar{P} \circ (\bar{X}, \bar{Y})^{-1} = (P \circ X^{-1}) \otimes (P_1 \circ Y^{-1}). \tag{7.16}$$

S.4 (approximation of a single partial sum). By (7.11), we have, for each $r \in \mathbb{N}$ and all $m \in \mathbb{N}$,

$$m^{-1} \int \|A_{\kappa(r)} \tilde{T}_m\|^2 \, d\bar{P} = \operatorname{trace}(C_r). \tag{7.17}$$

Combining (7.14), (7.15), (7.17), and Proposition 6.1, we can find a sequence $(l(r))_{r\in\mathbb{N}}$ in $\mathbb{N}$ with

$$l(r+1)/l(r) \in \mathbb{N} \tag{7.18}$$

and, for each $r \in \mathbb{N}$, a Borel measurable mapping

$$A_r: \mathbb{R}^{\kappa(r)} \times (\prod_{j \leqq 0} \mathbb{R}^{\kappa(r)}) \times (0, 1) \to \mathbb{R}^{\kappa(r)} \tag{7.19}$$

such that the r.v.

$$Z_1^{(r)} := A_r(A_{\kappa(r)} \tilde{T}_{l(r)}, (A_{\kappa(r)} \tilde{Y}_j)_{j \leqq 0}, U_{1,1}^{(r)}) \tag{7.20}$$

has the following properties:

$$P \circ (Z_1^{(r)})^{-1} = N(0, l(r) C_r), \tag{7.21}$$

$$Z_1^{(r)} \text{ is independent of } (A_{\kappa(r)} \tilde{Y}_j)_{j \leqq 0}, \tag{7.22}$$

$$\int \| A_{\kappa(r)} \tilde{T}_{l(r)} - Z_1^{(r)} \|^2 \, d\bar{P} \leqq l(r)/(8 \delta^2 r^6). \tag{7.23}$$

S.5 (approximation of a single block). Let $r \in \mathbb{N}$ be arbitrarily fixed, and let $l(r)$ and $Z_1^{(r)}$ be as in S.4. For $a \in \mathbb{R}^+$, let $C_{\mathbb{R}^{\kappa(r)}}[0, a]$ denote the separable Banach space of all continuous functions f: $[0, a] \to \mathbb{R}^{\kappa(r)}$, equipped with the norm $\| \cdot \|_{(a)}$ defined by $\| f \|_{(a)} := \sup \{ \| f(x) \| : x \in [0, a] \}$. According to Lemma 5.1, there is a Borel measurable mapping

$$\Psi_r: \mathbb{R}^{\kappa(r)} \times (0, 1) \to C_{\mathbb{R}^{\kappa(r)}}[0, l(r)]$$

such that

$$V_1^{(r)} := \Psi_r(Z_1^{(r)}, U_{2,1}^{(r)})$$

is a Brownian motion with

$$V_1^{(r)}(0) = 0 \quad \text{and} \quad \operatorname{Cov}(V_1^{(r)}(1)) = C_r$$

satisfying

$$V_1^{(r)}(l(r)) = Z_1^{(r)} \quad \bar{P}\text{-a.s.}$$

S.6 (conclusion of the proof). Let $(l(r))_{r \in \mathbb{N}}$, $(A_r)_{r \in \mathbb{N}}$, and $(\Psi_r)_{r \in \mathbb{N}}$ be as in S.4 and S.5. Our next aim is to construct a sequence $(\tilde{W}_r)_{r \in \mathbb{N}}$ of Brownian motions $\tilde{W}_r = \{ \tilde{W}_r(t); t \in [0, \infty) \}$ with values in $\mathbb{R}^{\kappa(r)}$ satisfying (7.4) and the following four conditions:

(a) for all $k \in \mathbb{N}$,

$$\begin{aligned} &\mathscr{A}(\tilde{W}_r((k-1)\, l(r) + t) - \tilde{W}_r((k-1)\, l(r));\, t \in [0, l(r)]) \\ &\quad \subset \mathscr{A}(A_{\kappa(r)}(\tilde{T}_{k l(r)} - \tilde{T}_{(k-1) l(r)}), (A_{\kappa(r)} \tilde{Y}_j)_{j \leqq (k-1) l(r)}, U_{1,k}^{(r)}, U_{2,k}^{(r)}); \end{aligned}$$

(b) for all $k \in \mathbb{N}$, the σ-fields

$$\mathscr{A}(\tilde{W}_r((k-1)\, l(r) + t) - \tilde{W}_r((k-1)\, l(r));\, t \in [0, l(r)])$$

and

$$\mathscr{A}(A_{\kappa(r)} \tilde{Y}_j, U_{1,m}^{(r)}, U_{2,m}^{(r)};\, -\infty < j \leqq (k-1)\, l(r),\, 1 \leqq m \leqq k-1)$$

are independent;

(c) for all $k\in\mathbb{N}$,

$$\int \|(\tilde{W}_r(kl(r))-\tilde{W}_r((k-1)\,l(r)))-A_{\kappa(r)}(\tilde{T}_{kl(r)}-\tilde{T}_{(k-1)l(r)})\|^2\,d\bar{P} \leqq l(r)/(8\delta^2 r^6);$$

(d) $((\tilde{W}_r(kl(r))-\tilde{W}_r((k-1)\,l(r)))-A_{\kappa(r)}(\tilde{T}_{kl(r)}-\tilde{T}_{(k-1)l(r)}))_{k\in\mathbb{N}}$ is a stationary sequence of r.v.'s.

(Here and below, $\tilde{T}_0:=0\in\mathbb{R}^{\mathbb{N}}$.) To this end, we put

$$Z_k^{(r)}:=\Lambda_r(A_{\kappa(r)}(\tilde{T}_{kl(r)}-\tilde{T}_{(k-1)l(r)}),\,(A_{\kappa(r)}\tilde{Y}_j)_{j\leqq(k-1)l(r)},\,U_{1,k}^{(r)})$$

and

$$V_k^{(r)}:=\Psi_r(Z_k^{(r)},\,U_{2,k}^{(r)})$$

for $k, r\in\mathbb{N}$. The Brownian motions $\tilde{W}_r$ ($r\in\mathbb{N}$) are then defined inductively by setting $\tilde{W}_r(0):=0$ and

$$\tilde{W}_r(t):=\tilde{W}_r((k-1)\,l(r))+V_k^{(r)}(t-(k-1)\,l(r))$$

for $t\in((k-1)\,l(r),\,kl(r)]$ and $k\in\mathbb{N}$. It is obvious from S.4, S.5, and the stationarity of the sequence $(\tilde{Y}_j)_{j\in\mathbb{Z}}$ that the thus obtained sequence $(\tilde{W}_r)_{r\in\mathbb{N}}$ satisfies (7.4) and (a)–(d). Moreover,

(e) $\bar{P}\{\limsup_{t\to\infty}(t\,\mathrm{L}_2\,t)^{-1/2}\,\|\tilde{W}_r(t)-A_{\kappa(r)}\,\tilde{T}_{[t]}\|\geqq 1/(2\delta r^2)\}\leqq r^{-2}$

for each $r\in\mathbb{N}$.

To prove (e), let $r\in\mathbb{N}$ be arbitrarily fixed, and let the sequences $(M_{\alpha,k}^{(r)})_{k\in\mathbb{N}}$ ($\alpha\in\{1,2\}$) be defined by

$$M_{1,k}^{(r)}:=\max_{(k-1)\,l(r)<m\leqq kl(r)}\|A_{\kappa(r)}(\tilde{T}_m-\tilde{T}_{(k-1)l(r)})\|$$

and

$$M_{2,k}^{(r)}:=\max_{0\leqq t\leqq 1}\|\tilde{W}_r((k-1+t)\,l(r))-\tilde{W}_r((k-1)\,l(r))\|.$$

It suffices to show that

$$\bar{P}\{\limsup_{m\to\infty}(ml(r)\,\mathrm{L}_2(ml(r)))^{-1/2}\,\|A_{\kappa(r)}\,\tilde{T}_{ml(r)} - \tilde{W}_r(ml(r))\|\geqq 1/(2\delta r^2)\}\leqq r^{-2} \tag{7.24}$$

and that

$$\lim_{k\to\infty} k^{-1/2}\,M_{\alpha,k}^{(r)}=0\quad \bar{P}\text{-a.s.}\qquad (\alpha\in\{1,2\}). \tag{7.25}$$

Taking (c) and (d) into account, it follows from Proposition 6.3 that

$$\limsup_{m\to\infty}(ml(r)\,\mathrm{L}_2(ml(r)))^{-1/2}\,\|A_{\kappa(r)}\,\tilde{T}_{ml(r)} - \tilde{W}_r(ml(r))\|\leqq 2^{1/2}\,\eta_r\quad \bar{P}\text{-a.s.},$$

where η_r is a real-valued r.v. with $\eta_r \geqq 0$ and $E\eta_r^2 \leqq 1/(8\delta^2 r^6)$. Using Chebyshev's inequality, this leads to (7.24). (7.25) is an immediate consequence of the fact that the sequences $(M_{\alpha,k}^{(r)})_{k\in\mathbb{N}}$ $(\alpha\in\{1,2\})$ are stationary and satisfy $\int \|M_{\alpha,1}^{(r)}\|^2 d\bar{P} < \infty$.

To complete the proof, it remains only to demonstrate that the above defined sequence $(\tilde{W}_r)_{r\in\mathbb{N}}$ has the properties (i) and (ii) stated in Proposition 7.1. But (i) can easily be deduced from (a), (b), (7.18), and the fact that the sequence $(\kappa(r))_{r\in\mathbb{N}}$ is nondecreasing, while (ii) is implied by (7.12) and (e).

8. Proof of Theorem 3.2

8.1. Preliminaries. As already mentioned in Sect. 3, Theorem 3.1 is an immediate consequence of Proposition 7.1 and Lemma 6.4. The last mentioned two results have already been established, and so the proof of Theorem 3.1 too is complete.

The existence of the limits in (3.7) follows directly from relation (3.3) in Theorem 3.1 (case $d\in\{1,2\}$). We also mention that the case $d=1$ of Theorem 3.1 together with the law of the iterated logarithm for a real-valued Brownian motion entails that

$$\limsup_{n\to\infty} (2n\mathrm{L}_2 n)^{-1/2} f(S_n) = C(f,f)^{1/2} \quad \text{a.s.}$$
$$\text{for all } f\in\mathbb{B}^*. \tag{8.1}$$

Next we need to introduce some facts and notations concerning Gaussian measures in Banach spaces (for further details, see Kuelbs [25], Lemma 2.1). Let $C\colon \mathbb{B}^*\times\mathbb{B}^*\to\mathbb{R}$ be as in Theorem 3.2. C is a nonnegative definite, symmetric, bilinear functional. Suppose there exists a mean zero Gaussian measure ν (on $\mathscr{B}(\mathbb{B})$) with covariance function C. To avoid trivial complications, assume that $C\neq 0$. Since ν is a mean zero Gaussian measure, we have $\int x\,\nu(dx)=0$ and $\int \|x\|_{\mathbb{B}}^2\,\nu(dx)<\infty$ (see [25], Lemma 2.1 (vi)). Thus ν satisfies the hypotheses of Lemma 2.1 in [25]. Let the linear operator $S\colon \mathbb{B}^*\to\mathbb{B}$ be defined by the Bochner integral $Sf:=\int x f(x)\,\nu(dx)$, and let $\mathbb{H}_\nu$ denote the completion of the range of S with respect to the norm $\|\cdot\|_\nu$ induced by the inner product $\langle Sf, Sg\rangle_\nu := C(f,g)$ $(f,g\in\mathscr{B}^*)$. The continuous extensions of $\langle\cdot,\cdot\rangle_\nu$ and $\|\cdot\|_\nu$ from the range of S to all of $\mathbb{H}_\nu$ will be denoted by the same symbols again. $\mathbb{H}_\nu$ can be realized as a subset of $\mathbb{B}$, and, writing $c_\nu := (\int \|y\|_{\mathbb{B}}^2\,\nu(dy))^{1/2}$, one has

$$\|x\|_{\mathbb{B}} \leqq c_\nu \|x\|_\nu \qquad \text{for all } x\in\mathbb{H}_\nu. \tag{8.2}$$

$(\mathbb{H}_\nu, \langle\cdot,\cdot\rangle_\nu)$ is a separable Hilbert space. Proceeding from a weak-star dense subset $F=\{f_k\colon k\in\mathbb{N}\}$ of the unit ball of $\mathbb{B}^*$ (note that $\mathbb{B}$ is separable), a complete orthonormal system of $\mathbb{H}_\nu$ can be constructed as follows:

Put $F_0 := \{f_k\in F\colon C(f_k,f_k)\neq 0$ and $f_k\notin \operatorname{span}\{f_j\colon j<k\}\}$, $D_\nu := \{l\in\mathbb{N}\colon 1\leqq l \leqq \dim\mathbb{H}_\nu\}$, and let $\{\alpha_k\colon k\in D_\nu\}$ be an orthonormal system obtained from F_0 by applying the Gram-Schmidt orthonormalization procedure with respect to

the inner product $C(\cdot,\cdot)$. Then $\{S\alpha_k : k \in D_v\}$ is a complete orthonormal system of $\mathbb{H}_v$. Moreover, the linear operators $\Pi_N : \mathbb{B} \to \mathbb{B}$ $(N \in D_v)$ defined by

$$\Pi_N x := \sum_{k=1}^{N} \alpha_k(x) S \alpha_k \tag{8.3}$$

are continuous. When restricted to $\mathbb{H}_v$, the Π_N's are orthogonal projections onto their ranges.

8.2. An intermediate result. In a sense, the following proposition is obtained by translating Proposition 7.1 into a Banach space result.

Proposition 8.1. *Under the basic hypotheses of Theorem* 3.2, *suppose there exists a mean zero Gaussian measure* v *with covariance function* C. *Also assume that* $C \neq 0$, *and that for each* $r \in \mathbb{N}$ *there is an* $N(r) \in D_v$ *such that for all* $N \in D_v$ *with* $N \geqq N(r)$

$$\limsup_{n\to\infty} (n \mathrm{L}_2 n)^{-1/2} \left\| \sum_{k=1}^{n} (X_k - \Pi_N X_k) \right\|_{\mathbb{B}} \leqq \frac{1}{4} r^{-2} \quad \text{a.s.} \tag{8.4}$$

Then, without changing its distribution, one can redefine the sequence $(X_k)_{k\in\mathbb{N}}$ *on a new probability space* $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ *on which there exists a sequence* $(W_r)_{r\in\mathbb{N}}$ *of* $\mathbb{B}$*-valued Brownian motions with* $W_r(0)=0$ *and covariance function* C *of* $W_r(1)$ *such that the assumptions of Lemma* 6.4, *with* $(\xi_k)_{k\in\mathbb{N}}$ *replaced by* $(X_k)_{k\in\mathbb{N}}$, *are satisfied for an appropriate system* $\mathscr{M}$ *of infinite subsets of* $\mathbb{N}$.

Proof. For $\iota \in D_v$, let α_ι be as above, and let the operators $\tilde{\Pi}_\iota : \mathbb{B} \to \mathbb{R}^\iota$ be defined by

$$\tilde{\Pi}_\iota x := (\alpha_1(x), \ldots, \alpha_\iota(x)).$$

Moreover, let the projections $p_\iota : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ and $A_\iota : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^\iota$ $(\iota \in \mathbb{N})$ be defined as in Sect. 7. We associate with $(X_k)_{k\in\mathbb{Z}}$ the sequence $(Y_k)_{k\in\mathbb{Z}}$ of r.v.'s with values in $\mathbb{R}^{\mathbb{N}}$ determined by

$$p_\iota(Y_k) = \begin{cases} \alpha_\iota(X_k) & \text{if } \iota \in D_v, \\ 0 & \text{if } \iota \in \mathbb{N} \setminus D_v. \end{cases} \tag{8.5}$$

It is obvious that the thus obtained sequence $(Y_k, \mathscr{F}_k)_{k\in\mathbb{Z}}$ satisfies the hypotheses of Proposition 7.1. As in Sect. 7, we write

$$T_0 := 0 \, (\in \mathbb{R}^{\mathbb{N}}) \quad \text{and} \quad T_n := \sum_{j=1}^{n} Y_j \quad \text{for } n \in \mathbb{N}.$$

Next, invoking Lemma 4.2 of Kuelbs and Philipp [27], we choose a nondecreasing sequence $(\kappa(r))_{r\in\mathbb{N}}$ in $\mathbb{N}$ such that:

(a) $N(r) \leqq \kappa(r) \leqq \dim \mathbb{H}_v$ for all $r \in \mathbb{N}$;

(b) if $W^* = \{W^*(t); t \in [0, \infty)\}$ is a $\mathbb{B}$-valued Brownian motion with $W^*(0)=0$ and covariance function C of $W^*(1)$, then

$$\limsup_{t\to\infty} (t \mathrm{L}_2 t)^{-1/2} \| W^*(t) - \Pi_{\kappa(r)} W^*(t) \|_{\mathbb{B}} \leqq \tfrac{1}{4} r^{-2} \quad \text{a.s.} \tag{8.6}$$

for each $r \in \mathbb{N}$.

(The formulation of Lemma 4.2 in [27] is somewhat misleading; one should replace "there is an N such that" by "there is an $N(\eta) \in \mathbb{N}$ with $N(\eta) \leqq \dim H_\mu$ such that for all $N \in \mathbb{N}$ with $N(\eta) \leqq N \leqq \dim H_\mu$".)

By virtue of (8.5) and the definition of the linear functionals α_ι, it follows from Proposition 7.1 that

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{Cov}(A_{\kappa(r)} T_n) = I_{\kappa(r)} \qquad \text{for each } r \in \mathbb{N}.$$

Furthermore, using Proposition 7.1 and (e.g.) Lemma 5.1, we can, without changing its distribution, redefine the sequence $(X_k)_{k\in\mathbb{N}}$ on a new probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ on which there exists a sequence $(\tilde{W}_r)_{r\in\mathbb{N}}$ of Brownian motions $\tilde{W}_r = \{\tilde{W}_r(t);\ t \in [0, \infty)\}$ with values in $\mathbb{R}^{\kappa(r)}$ such that

$$\tilde{W}_r(0) = 0 \quad \text{and} \quad \operatorname{Cov}(\tilde{W}_r(1)) = I_{\kappa(r)}$$

having the following properties:

(c) there is a sequence $(l(r))_{r\in\mathbb{N}}$ in $\mathbb{N}$ with

$$l(r+1)/l(r) \in \mathbb{N} \tag{8.7}$$

such that, for each $(m, r) \in \mathbb{N}^2$, the σ-fields

$$\mathscr{A}(\tilde{W}_s(t);\ 1 \leqq s \leqq r,\ t \leqq m\, l(r))$$

and

$$\mathscr{A}(\tilde{W}_r(t) - \tilde{W}_r(m\, l(r));\ t \geqq m\, l(r))$$

are independent;

(d) for each $r \in \mathbb{N}$,

$$\bar{P}\{\limsup_{t\to\infty} (t \mathrm{L}_2 t)^{-1/2} \| \tilde{W}_r(t) - A_{\kappa(r)} T_{[t]} \| \geqq 1/(2 c_\nu r^2)\} \leqq r^{-2}. \tag{8.8}$$

(Here, c_ν is as in (8.2).)

We now proceed to construct the Brownian motions W_r ($r \in \mathbb{N}$) occurring in Proposition 8.1. To this end, we may assume that the probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{P})$ is so rich that there also exists a family $\{U_k^{(r)};\ (k, r) \in \mathbb{N}^2\}$ of i.i.d. r.v.'s $U_k^{(r)}: \bar{\Omega} \to (0, 1)$ with uniform distribution on $(0, 1)$ which is independent of $((X_k)_{k\in\mathbb{N}}, (\tilde{W}_r)_{r\in\mathbb{N}})$. For $k, r \in \mathbb{N}$, and $t \in [0, l(r)]$, let

$$\tilde{V}_k^{(r)}(t) := \tilde{W}_r((k-1)\, l(r) + t) - \tilde{W}_r((k-1)\, l(r)). \tag{8.9}$$

According to Lemma 5.1, there is a sequence $(H_r)_{r\in\mathbb{N}}$ of Borel measurable mappings

$$H_r\colon C_{\mathbb{R}^{\kappa(r)}}[0, l(r)] \times (0, 1) \to C_{\mathbb{B}}[0, l(r)]$$

such that

$$V_k^{(r)} := H_r(\tilde{V}_k^{(r)}, U_k^{(r)}) \tag{8.10}$$

is a Brownian motion with $V_k^{(r)}(0)=0$ and covariance function C of $V_k^{(r)}(1)$ satisfying

$$\tilde{\Pi}_{\kappa(r)} V_k^{(r)}(t) = \tilde{V}_k^{(r)}(t) \qquad \text{for } t\in[0, l(r)]. \tag{8.11}$$

The Brownian motions W_r ($r\in\mathbb{N}$) are then defined inductively by setting $W_r(0):=0$ and

$$W_r(t) := W_r((k-1)\,l(r)) + V_k^{(r)}(t-(k-1)\,l(r)) \tag{8.12}$$

for $t\in((k-1)\,l(r), k\,l(r)]$ and $k\in\mathbb{N}$.

It remains to show that the thus obtained Brownian motions have the desired properties. We first notice that the mapping $\tilde{\Pi}_{\kappa(r)} \mid \Pi_{\kappa(r)}\mathbb{B}\colon \Pi_{\kappa(r)}\mathbb{B} \to \mathbb{R}^{\kappa(r)}$ is an isometry from the first space, regarded as a subspace of the Hilbert space $(\mathbb{H}_\nu, \langle\cdot,\cdot\rangle_\nu)$, onto the second space (equipped with the Euclidean norm $\|\cdot\|$). Hence and from (8.2), (8.5), (8.8), (8.11), and (8.12), it follows that, for each $r\in\mathbb{N}$,

$$\bar{P}\{\limsup_{t\to\infty} (t\,\mathrm{L}_2\, t)^{-1/2}\, \|\Pi_{\kappa(r)}(W_r(t)-S_{[t]})\|_{\mathbb{B}} \geqq \tfrac{1}{2} r^{-2}\} \leqq r^{-2}, \tag{8.13}$$

so that

$$\bar{P}\{\limsup_{t\to\infty} (t\,\mathrm{L}_2\, t)^{-1/2}\, \|W_r(t)-S_{[t]}\|_{\mathbb{B}} \geqq r^{-2}\} \leqq r^{-2} \tag{8.14}$$

(by (8.4) and (8.6)). Moreover, writing

$$\mathscr{A}^*_{m,r} := \mathscr{A}(W_s(t);\ 1\leqq s\leqq r,\ t\leqq m\,l(r))$$

and

$$\mathscr{A}^{**}_{m,r} := \mathscr{A}(W_r(t)-W_r(m\,l(r));\ t\geqq m\,l(r))$$

for $(m, r)\in\mathbb{N}^2$, we have

$$\mathscr{A}^*_{m,r} \subset \mathscr{A}(\tilde{W}_s(t), U_k^{(s)};\ 1\leqq s\leqq r,\ k\leqq m\,l(r)/l(s),\ t\leqq m\,l(r))$$

and

$$\mathscr{A}^{**}_{m,r} \subset \mathscr{A}(\tilde{W}_r(t)-\tilde{W}_r(m\,l(r)), U_k^{(r)};\ k>m,\ t\geqq m\,l(r)),$$

which together with (c) and the choice of the family $\{U_k^{(r)};\ (k, r)\in\mathbb{N}^2\}$ implies that

$$\mathscr{A}^*_{m,r} \quad \text{and} \quad \mathscr{A}^{**}_{m,r} \text{ are independent.} \tag{8.15}$$

Combining (8.14) and (8.15) (and taking (8.7) into account), we see that the proof of the proposition is complete.

8.3. Conclusion of the proof of Theorem 3.2. In case $C=0$, our task reduces to proving the equivalence of the following two statements:

(a′) The sequence $((n\mathrm{L}_2 n)^{-1/2} S_n)_{n\in\mathbb{N}}$ is with probability one conditionally $\|\cdot\|_{\mathbb{B}}$-compact;

(b′) $\|S_n\|_{\mathbb{B}}=o((n\mathrm{L}_2 n)^{1/2})$ a.s. (as $n\to\infty$).

The implication (b′)⇒(a′) is trivial. To prove the converse implication, it is enough to observe that, by (8.1), $f(S_n)=o((n\mathrm{L}_2 n)^{1/2})$ a.s. (as $n\to\infty$) for all $f\in\mathbb{B}^*$. (Note that the weak-star topology of $\mathbb{B}^*$ is separable, and that a conditionally $\|\cdot\|_{\mathbb{B}}$-compact sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$ satisfying

$g(x_n)\to 0$ (as $n\to\infty$) for all g in a weak-star dense subset of $\mathbb{B}^*$

is strongly convergent to 0.)

We now proceed to the case $C\neq 0$. First we consider the implication (a)⇒(b) of Theorem 3.2. In view of Lemma 6.4 and Proposition 8.1, we need only verify (8.4). Combining (a), Lemma 2.1 in [25], and (8.1), we see that the conditions of the first part of Theorem 3.1 in [25] are satisfied with

$$Y_n := n^{-1/2} S_n \quad \text{and} \quad \phi_n := (2\mathrm{L}_2 n)^{1/2} \qquad (n\in\mathbb{N}).$$

Hence

$$\lim_{n\to\infty} d((2n\mathrm{L}_2 n)^{-1/2} S_n, K)=0 \quad \text{a.s.},$$

where, for $x\in\mathbb{B}$, $d(x,K):=\inf\{\|x-y\|_{\mathbb{B}}: y\in K\}$ and K is the unit ball of the Hilbert space $\mathbb{H}_\nu$ introduced in Subsection 8.1. But this means that the proof of Lemma 4.1 in Kuelbs and Philipp [27] remains applicable to establish (8.4).

Let us turn to the converse implication (b)⇒(a). From Theorem 4.1 in [26], applied to the sequence $(W(n)-W(n-1))_{n\in\mathbb{N}}$, and (3.8) we conclude that the sequence $((n\mathrm{L}_2 n)^{-1/2} S_n)_{n\in\mathbb{N}}$ is with probability one conditionally $\|\cdot\|_{\mathbb{B}}$-compact. The proof that (b) entails the existence of a mean zero Gaussian measure ν with covariance function C, and that the covariance function of $W(1)$ is necessarily equal to C, can be adapted from Sect. 5 in [27], the only difference being that one has to employ relation (8.1) instead of the Hartman-Wintner law of the iterated logarithm.

9. Proof of Proposition 2.2

We shall only prove that (i)⇔(ii). (The proof that (iii)⇔(iv) proceeds analogously.) Let

$$Q := \sum_{m=1}^{\infty} \limsup_{n\to\infty} E\left(\left(\sum_{l=m}^{n} x_{-l}\right)^2\right).$$

We begin by demonstrating that the following three facts (F1)–(F3) hold true under (i) as well as (ii):

(F1) For all $m, n \in \mathbb{N}$ with $m \leqq n$, one has

$$E\left(E\left(\sum_{k=m}^{n} \xi_k \mid \mathscr{F}_0\right)^2\right) = \sum_{l=m}^{\infty} E\left(\sum_{k=l}^{n-m+l} x_{-k}\right)^2.$$

(F2) The sequence $\left(\sum_{l=1}^{n} x_{-l}\right)_{n \in \mathbb{N}}$ is Cauchy in $\mathscr{L}_2$.

(F3) $Q = \sum_{m=1}^{\infty} E\left(\left(\sum_{l=m}^{\infty} x_{-l}\right)^2\right)$ (irrespective of whether Q is finite or infinite).

To verify (F1), we first recall that

$$E(\xi_{k+l} \mid \mathscr{F}_l) = E(\xi_k \mid \mathscr{F}_0) \circ \tau^l \quad \text{a.s.} \qquad \text{for all } k, l \in \mathbb{Z}. \tag{9.1}$$

Next we observe that each of the assumptions (i) and (ii) entails that

$$E(\xi_0 \mid \mathscr{F}_{-k}) \xrightarrow[\mathscr{L}_2]{} 0 \quad \text{as} \quad k \to \infty. \tag{9.2}$$

In the case (i) this is evident; in the case (ii) it follows from the martingale convergence theorem (note that $E(\xi_0 \mid \mathscr{F}_{-\infty}) = 0$ a.s.). From (9.1) and (9.2) we infer that, again in both cases (i) and (ii),

$$E\left(\sum_{k=m}^{n} \xi_k \mid \mathscr{F}_0\right) = \sum_{l=-\infty}^{0} \sum_{k=m}^{n} (E(\xi_k \mid \mathscr{F}_l) - E(\xi_k \mid \mathscr{F}_{l-1})) \quad \text{a.s.}$$

(for all $m, n \in \mathbb{N}$ with $m \leqq n$), where the infinite series converges in $\mathscr{L}_2$. Hence

$$E\left(E\left(\sum_{k=m}^{n} \xi_k \mid \mathscr{F}_0\right)^2\right) = \sum_{l=-\infty}^{0} E\left[\sum_{k=m}^{n} (E(\xi_k \mid \mathscr{F}_l) - E(\xi_k \mid \mathscr{F}_{l-1}))\right]^2$$

$$= \sum_{l=m}^{\infty} E\left(\sum_{k=l}^{n-m+l} x_{-k}\right)^2 \qquad \text{(using (9.1))},$$

which gives (F1). (F2) is an immediate consequence of (i) and (F1); on the other hand it can also be obtained from the relation

$$\lim_{m \to \infty} \limsup_{n \to \infty} E\left(\sum_{k=m}^{n} x_{-k}\right)^2 = 0,$$

implied by (ii). Finally, it is clear that (F2)$\Rightarrow$(F3).

We now consider the implication (i)$\Rightarrow$(ii). Suppose that (i) is fulfilled. From the martingale convergence theorem and (9.2), we conclude that $E(\xi_0 \mid \mathscr{F}_{-\infty}) = 0$ a.s. Using (i) and (F1), we find that for each $\varepsilon \in \mathbb{R}^+$ there is an $N_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{l=m}^{\infty} E\left(\sum_{k=l}^{n-m+l} x_{-k}\right)^2 \leqq \varepsilon \qquad \text{for all } n, m \geqq N_0(\varepsilon) \text{ with } n \geqq m.$$

Because of (F2) and (F3), this is only possible if $Q<\infty$.

The converse implication (ii)⇒(i) follows by combining (F1), the inequality

$$\sum_{l=m}^{\infty} E\left(\sum_{k=l}^{n-m+l} x_{-k}\right)^2 \leqq 2\sum_{l=m}^{\infty} E\left(\sum_{k=l}^{\infty} x_{-k}\right)^2 + 2\sum_{l=n+1}^{\infty} E\left(\sum_{k=l}^{\infty} x_{-k}\right)^2 \tag{9.3}$$

(valid for all $m, n\in\mathbb{N}$ with $m\leqq n$) and the fact that, on account of (F3) and since $Q<\infty$, the right-hand side of (9.3) becomes arbitrarily small if m and n are chosen sufficiently large.

10. Proof of Corollary 4.1 and Remark 4.1

10.1. Proof of corollary 4.1. In the following we shall use the symbol $\|Y\|_p$ to denote the p-norm ($p\in[1,\infty)$) of a r.v. $Y:\Omega\to\mathbb{R}$ with $E|Y|^p<\infty$, i.e., $\|Y\|_p := (E|Y|^p)^{1/p}$. For $m, n\in\mathbb{N}$ with $m\leqq n$, we write $T_{m,n} := \sum_{k=m}^{n} X_k$.

We first consider the cases A.1–A.4. To this end, let $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}, (\hat{X}_k)_{k\in\mathbb{Z}})$ be the coordinate representation process corresponding to the sequence $(X_k)_{k\in\mathbb{Z}}$, i.e., $\hat{\Omega} := \mathbb{B}^{\mathbb{Z}}$, $\hat{\mathscr{F}} :=$ Borel σ-field induced by the product topology on $\mathbb{B}^{\mathbb{Z}}$, $\hat{P} := P\circ((X_k)_{k\in\mathbb{Z}})^{-1}$, $\hat{X}_k := k$-th projection $\hat{\Omega}\to\mathbb{B}$. Without loss of generality we may suppose that $(\Omega, \mathscr{F}, P, (X_k)_{k\in\mathbb{Z}}) = (\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}, (\hat{X}_k)_{k\in\mathbb{Z}})$. Let $\mathscr{F}_0 := \mathscr{A}(X_{-j}; j\in\mathbb{N}_0)$, and let $\tau:\Omega\to\Omega$ be the shift transformation (i.e., $\tau((x_j)_{j\in\mathbb{Z}}) := (x_{j+1})_{j\in\mathbb{Z}}$ for $(x_j)_{j\in\mathbb{Z}}\in\Omega$). Then $X_k = X_0\circ\tau^k$ and $\mathscr{F}_k := \tau^{-k}\mathscr{F}_0 = \mathscr{A}(X_j; j\leqq k)$ for all $k\in\mathbb{Z}$. Each of the assumptions A.1–A.4 entails that the sequence $(X_k)_{k\in\mathbb{Z}}$ is ergodic (see, e.g., Ibragimov and Linnik [20]). Hence τ is also ergodic. Summarizing, it follows that, with the above notations and conventions, $(X_k, \mathscr{F}_k)_{k\in\mathbb{Z}}$ is the stationary ergodic F-sequence induced by $((\Omega, \mathscr{F}, P), \tau, \mathscr{F}_0, X_0)$. Therefore, it suffices to show that

$$(X_k, \mathscr{F}_k)_{k\in\mathbb{Z}} \quad \text{has the weak } M_2\text{-property}. \tag{10.1}$$

It is obvious that (10.1) holds under A.1. So it remains only to investigate the cases A.2–A.4. For $k\in\mathbb{N}_0$, X_{-k} is $\mathscr{F}_0$-measurable. Hence, and in view of Minkowski's inequality, it is enough to demonstrate that, for each $f\in\mathbb{B}^*$,

$$\sum_{k=1}^{\infty} \max_{1\leqq l\leqq 2^k} \|E(f(T_{2^k+1,2^k+l})\,|\,\mathscr{F}_0)\|_2 < \infty. \tag{10.2}$$

But this is immediate from the following estimates (valid for all $m, n\in\mathbb{N}$ with $m\leqq n$).

Under A.2: $\|E(f(T_{m,n})\,|\,\mathscr{F}_0)\|_2 \leqq 5\alpha(m)^{\delta/(4+2\delta)} \|f(T_{m,n})\|_{2+\delta}$
(cf. McLeish [30], Lemma 3.5),

$\|f(T_{m,n})\|_{2+\delta} \leqq (n-m+1)\,\|f(X_1)\|_{2+\delta}$
(by Minkowski's inequality).

Under A.3: $\|E(f(T_{m,n})\,|\,\mathscr{F}_0)\|_2 \leqq \rho(m)\,\|f(T_{m,n})\|_2$
(note that $\|E(f(T_{m,n})\,|\,\mathscr{F}_0)\|_2^2$
$= E(f(T_{m,n})\,E(f(T_{m,n})\,|\,\mathscr{F}_0))$

$$\leqq \rho(m) \|E(f(T_{m,n}) | \mathscr{F}_0)\|_2 \|f(T_{m,n})\|_2),$$

$$\|f(T_{m,n})\|_2 \leqq c_1(f)(n-m+1)^{1/2}$$

(cf. Ibragimov [21] and Bradley [3]).

Under A.4: $\|E(f(T_{m,n}) | \mathscr{F}_0)\|_2 \leqq 2\phi(m)^{(1+\delta)/(2+\delta)} \|f(T_{m,n})\|_{2+\delta}$

(cf. [30], Lemma 3.5),

$$\|f(T_{m,n})\|_{2+\delta} \leqq c_2(f)(n-m+1)^{1/2}$$

if $\limsup_{l \to \infty} E(f(T_{1,l})^2) = \infty$ and $\delta \in [0, 1]$

(cf. [21] and [3]),

$$\|E(f(T_{m,n}) | \mathscr{F}_0)\|_2 \leqq 2\phi(m)^{1/2} c_3(f)$$

if $\limsup_{l \to \infty} E(f(T_{1,l})^2) < \infty$

(cf. [30], Lemma 3.5).

(Here $c_1(f)$, $c_2(f)$, and $c_3(f)$ are positive constants.)

Let us now turn to the cases A.5–A.7. In these cases we may, without loss of generality, assume that $(\Omega, \mathscr{F}, P, (Y_k)_{k \in \mathbb{Z}})$ is equal to the coordinate representation process pertaining to the sequence $(Y_k)_{k \in \mathbb{Z}}$. We choose τ to be the shift transformation on Ω, and $\mathscr{F}_k := \mathscr{A}(Y_j; j \leqq k)$ (for $k \in \mathbb{Z}$). Following the lines of the above proof in the cases A.1–A.4 and using some arguments from McLeish [30], p. 177, the details for proving that the assertion of the corollary holds under A.5 and A.6 are straightforward and hence omitted. Finally, the proof that the conclusion of the corollary also obtains under A.7 is immediate from Theorems 3.1 and 3.2 in conjunction with Proposition 2.2.

10.2. Proof of Remark 4.1. For $n \in \mathbb{N}$, let $T_{1,n}$ be defined as above. Again, we begin by considering the cases A.1–A.4. Let $\mathscr{F}_0 := \mathscr{A}(X_j; j \leqq 0)$. Since X_0 is $\mathscr{F}_0$-measurable, we have

$$U_n(f, g) = E(f(X_0) g(X_0)) + E(f(X_0) E(g(T_{1,n}) | \mathscr{F}_0)) + E(g(X_0) E(f(T_{1,n}) | \mathscr{F}_0)),$$

so that the convergence of the sequence $(U_n(f, g))_{n \in \mathbb{N}}$ follows from the fact that, by (10.2), the sequences $(E(h(T_{1,n}) | \mathscr{F}_0))_{n \in \mathbb{N}}$ $(h \in \mathbb{B}^*)$ are Cauchy in $\mathscr{L}_2$. It remains to demonstrate that

$$U(f, g) := \lim_{n \to \infty} U_n(f, g) = C(f, g) \qquad \text{for all } f, g \in \mathbb{B}^*.$$

As the mappings $(f, g) \mapsto U(f, g)$ and $(f, g) \mapsto C(f, g)$ are symmetric bilinear functionals, the polarization formula tells us that we need only show that $U(f, f) = C(f, f)$ for all $f \in \mathbb{B}^*$. Now

$$E(f(T_{1,n})^2) = E(f(X_0)^2) + \sum_{k=1}^{n-1} U_k(f, f) \qquad \text{for all } n \in \mathbb{N}, f \in \mathbb{B}^*.$$

Since $\lim_{m \to \infty} U_m(f, f) = U(f, f)$, this entails that

$$C(f, f) = \lim_{n \to \infty} \frac{1}{n} E(f(T_{1,n})^2) = U(f, f),$$

as desired.

If one of the assumptions A.5 and A.6 holds, the convergence of the sequences $(U_n(f, g))_{n \in \mathbb{N}}$ can easily be obtained by adapting the reasoning in Ibragimov and Linnik [20], p. 352ff. The rest of the proof is as above. Finally, if A.7 is satisfied, the assertion concerning the convergence of $(V_n(f, g))_{n \in \mathbb{N}}$ to $C(f, g)$ follows directly from Proposition 2.2, Remark 3.5, and the proof of Proposition 2.1.

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