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Applications of the degree theorem to absolute continuity on Wiener space

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Summary. Let (Ω, H, P) be an abstract Wiener space and define a shift on Ω by $T(\omega) = \omega + F(\omega)$ where F is an H-valued random variable. We study the absolute continuity of the measures $P \circ T^{-1}$ and $(\Lambda_F P) \circ T^{-1}$ with respect to P using the techniques of the degree theory of Wiener maps, where $\Lambda_F = \det_2(1 + \nabla F) \times \operatorname{Exp}\{-\delta F - \frac{1}{2}|F|^2\}$.

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1 Introduction

Consider the abstract Wiener space $\{\Omega, H, P_0\}$ where H is a separable Hilbert-space continuously and densely embedded in the Banach space Ω. Let $F(\omega)$ be a random variable taking values in H, set

$$T\omega = \omega + F(\omega)\,,$$

and let R denote the measure induced by T, i.e. $R(A) = P_0(T^{-1}A)$. The problem of the absolute continuity of $P_0 \circ T^{-1}$ with respect to P goes back to the early work of Cameron and Martin and was considered by many authors (cf. e.g. [1, 7, 10, 11, 13, 14] and the references therein, for the key results related to the Girsanov theorem in the adapted case, cf. e.g. [5, 6, 12]).

Consider now to the degree theorem let $y = \eta(x)$ be a smooth and proper map (i.e. the inverse image of compact sets is compact) from $\mathbb{R}^n$ to $\mathbb{R}^n$. The degree of this map at point y is defined to be the summation of sign $J_\eta(x)$ over all x such that $\eta(x) = y$; $J_\eta(x)$ denotes the determinant of the Jacobian matrix of the transformation η at x. The degree theorem in $\mathbb{R}^n$ (cf. e.g. p. 190 of [4]) states that for

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any smooth real valued function $u(x)$, $x \in \mathbb{R}^n$ with compact support:

$$\int_{\mathbb{R}^n} J_\eta(x) u(\eta(x))\, dx = \deg(\eta) \int_{\mathbb{R}^n} u(x)\, dx \tag{1.1}$$

where $\deg \eta$ is an integer, it is independent of u and for almost all y in $\mathbb{R}^n$:

$$\deg \eta = \deg \eta(y) = \sum_{x:\, \eta(x) = y} \operatorname{sign} J_\eta(x)\,. \tag{1.2}$$

This result was recently extended to the Wiener space by Getzler [3], (cf. also Kusuoka [8]). In this paper we apply the analysis of Getzler to the problem of absolute continuity associated with transformations on the Wiener space. The result of [3] is rederived here under somewhat weaker assumptions (except for Theorem 3.1, which is under the same assumptions and is rederived here in order to maintain the paper selfcontained). The results regarding the absolutely continuous transformation of measure include a rederivation of the Ramer result [13] under different conditions.

In the next section we derive some relations between the underlying measure P_0 and the two measures R and Q, where $R(A) = P_0(T^{-1}A)$ and a Girsanov measure Q possessing the property that the law of $T^{-1}A$ under Q is the same as the law of A under P i.e. $T\omega$ is a Wiener process under Q. Some technical results needed for the derivation of the degree theorem are also derived in the next section. Section 3 starts with the degree theorem of Getzler. It is then shown by an homotopy argument that under certain assumptions the degree is actually 1. These results are then applied to the derivation of new results on absolutely continuous transformations of measure.

2 Preliminaries

The following lemma is believed to be of independent interest (cf. also [10]).

Lemma 2.1 *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let T be a measurable transformation on Ω, set $R(A) = P(T^{-1}A)$, $A \in \mathscr{F}$. Suppose that there exists a (possibly signed) measure $Q(A)$, on $(\Omega, \mathscr{F})$ such that $Q(T^{-1}A) = P(A)$, then*

(a) *If the measures Q and P are mutually absolutely continuous, then Q, P, R are mutually absolutely continuous. If moreover, T possesses a left inverse, i.e. there exists a measurable transformation T_l^{-1} such that $T_l^{-1} T\omega = \omega$* a.s. *$P$, then*

$$\frac{dR}{dP}(T\omega) = \left(\frac{dQ}{dP}(\omega)\right)^{-1}. \tag{2.1}$$

(b) *Assume that $Q \ll P$ and $R \ll P$, further assume the existence of a left inverse T_l^{-1}, then the measures Q, P and R are equivalent.*
(c) *If R and P are mutually absolutely continuous and there exists a left inverse T_l^{-1}, then R, P and Q are equivalent.*
(d) *Assume that T is invertible ($T^{-1}T\omega = TT^{-1} = \omega$* a.s. *$P$), let R_T, Q_T ($R_{T^{-1}}$, $Q_{T^{-1}}$) denote the measures associated with T (T^{-1}) respectively. Assume that R_T, Q_T, P are mutually absolutely continuous, then $Q_{T^{-1}}$ exists and for every $A \in \mathscr{F}$*

$$R_{T^{-1}}(A) = Q_T(A) \tag{2.2}$$

$$Q_{T^{-1}}(A) = R_T(A)\,. \tag{2.3}$$

Remarks. (a) It was pointed out to us by E. Mayer-Wolf that the assumption on the existence of a left inverse is equivalent to the assumption that $\sigma\{T^{-1}A, A\in\mathscr{F}\}=\mathscr{F}$, where $\sigma\{T^{-1}A, A\in\mathscr{F}\}$ denotes the sigma field induced by $T^{-1}A$. (b) It was shown recently by D. Nualart that if P and R are mutually absolutely continuous, then Q exists.

Proof. (a) By the definition of the Q measure, $P(A)=0$ implies $Q(T^{-1}A)=0$ and since the measures are equivalent, it holds that $P(T^{-1}A)=0$, hence $R(A)=0$ and $R\ll P$. On the other hand, if $R(A)=0$, then by definition $P(T^{-1}A)=0$, hence by equivalence of measures $Q(T^{-1}A)=0$ and $P\ll R$. Turning to the proof of (2.1):

$$\begin{aligned}\int_{T^{-1}B}\frac{dR}{dP}(T\omega)\,dQ&=\int_B\frac{dR}{dP}(\omega)d(Q\circ T^{-1})\\&=R(B)=P(T^{-1}B)\\&=\int_{T^{-1}B}\frac{dP}{dQ}dQ\\&=\int_{T^{-1}B}\left(\frac{dP}{dQ}\right)^{-1}dQ\,. \end{aligned}\tag{2.4}$$

which completes the proof of (a). Turning to (b), set $T_l^*=(T_l^{-1})^{-1}$; by definition $Q(T^{-1}A)=P(A)$, hence $Q(A)=P(T_l^*A)$. Therefore $Q(A)=0$ implies $P(T_l^*A)=0$ and it follows by the absolute continuity assumption that $R(T_l^*A)=0$. Consequently $P(A)=0$ and $P\ll Q$. Therefore P and Q are equivalent and it follows from (a) that P and R are also equivalent. As for the existence of $Q_{T^{-1}}$ and the second equality — both follow directly from $R_T(TA)=PA$. To prove (c), note that by definition $P(A)=0$ implies $R(T_l^*)A=0$, hence by the equivalence of the measures $P(T_l^*A)=0$. Consequently, $Q(T^{-1}(T_l^*A))=Q(A)=0$ yielding $Q\ll P$ and $Q\sim P$ follows from part (b). Turning to (d), by the definitions $Q_T(T^{-1}A)=P(A)$ and $R_{T^{-1}}(A)=P(TA)$. Hence $Q_T(B)=R_{T^{-1}}(B)$. As for the existence of $Q_{T^{-1}}$ and the second equality – both follow directly from $R_T(TA)=P(A)$.

Next we summarize the notation of the Malliavin calculus and define the $\Lambda_F(\omega)$ functional (Eq. (2.8) below). For $h\in H^*=H$, $\langle h,\omega\rangle$ will denote the Wiener integral $W(h)$. Let X be a separable real Hilbert space; smooth X-valued functionals on (Ω, H, P_0) are functionals of the form

$$a(\omega)=\sum_1^N\eta_i(\langle h_1,\omega\rangle,\ldots,\langle h_m,\omega\rangle)x_i$$

with $x_i\in X$ and $\eta_i\in C_b^\infty(\mathbb{R}^m)$, $h_i\in\Omega^*\subset H^*$. For smooth X-valued functionals, define

$$\nabla a(\omega)=\sum_{i=1}^N\sum_{j=1}^m\partial_j\eta_i(\langle h_1,\omega\rangle,\ldots,\langle h_m,\omega\rangle)\cdot x_j\otimes h_i\,,$$

∇^k, $k=2,3,\ldots$ is defined recursively. The Sobolev spaces $\mathbb{D}_{p,k}(X)$ $p>1$, $k\in\mathbb{N}$ are the completion of X valued smooth functionals under the norm

$$\|a\|_{p,k}=\sum_{i=0}^k\|\nabla^i a\|_{L^p(P_0,X\otimes H^{\otimes i})}\,.\tag{2.5}$$

The derivative $\nabla\colon \mathbb{D}_{p,k}(X) \to \mathbb{D}_{p,k-1}(X \otimes H)$ denotes the closure of ∇ as defined above with respect to the norm given by (2.5); ∇a is considered as a mapping from H to X, $(\nabla a)^*$ will denote the mapping from X to H.
The adjoint of ∇ will be called the divergence and denoted by δ. Recall that if F is in $\mathbb{D}_{2,1}(H)$ then, for a.a. ω, ∇F is a Hilbert-Schmidt operator from H to H, and for any complete orthonormal basis of H, say $\{e_i, i = 1, 2, \ldots\}$, we have

$$\delta u = \sum_{i=1}^{\infty} (\langle F, e_i \rangle \cdot \langle e_i, \omega \rangle - \nabla_{e_i} \langle F, e_i \rangle)$$

we will denote the sum $\sum_{i=1}^{\infty} \langle F, e_i \rangle \langle e_i, \omega \rangle$ by $\delta \circ u$, hence

$$\delta \circ u =: \delta u + \operatorname{trace} \nabla F$$

whenever ∇F is of trace class. Note that this is always the case for smooth functionals. Let A denote a Hilbert-Schmidt operator from H to H and $\lambda_1, \lambda_2, \ldots$ its eigenvalues in decreasing magnitude and repeated according to their multiplicity, the Carleman-Fredholm determinant associated with A and denoted as $\det_2(I + A)$ is defined by

$$\det_2(I + A) = \prod_{1}^{\infty} (1 + \lambda_i) \operatorname{Exp} - \lambda_i\,,$$

(cf., e.g., Chap. XI.9 of [2] for a detailed treatment of this notion). The following inequalities will be particularly important later:

$$|\det_2(I + A)| \leqq \operatorname{Exp} \|A\|_2^2/2 \tag{2.6}$$

where $\|A\|_2$ denotes the Hilbert-Schmidt norm of A and

$$\|(\det_2(I + A)) \cdot (I + A)^{-1}\|_{\text{operator}} \leqq \operatorname{Exp} \tfrac{1}{2}(1 + \|A\|_2^2)\,. \tag{2.7}$$

For $F \in \mathbb{D}_{2,1}(H)$ set

$$\Lambda_F(\omega) = \det_2(1 + \nabla F) \operatorname{Exp}\{-\delta F - \tfrac{1}{2}|F|^2\}\,. \tag{2.8}$$

Note that $\Lambda_F(\omega)$ is defined without the "customary" absolute value on $\det_2$.

Lemma 2.2. *Let F and v denote smooth H-valued functionals, $T\omega = \omega + F(\omega)$. Then, with Λ_F as defined in* (2.8)

$$\delta\{\Lambda_F(\omega)(I + \nabla F)^{-1} v\} = \Lambda_F(\omega)(\delta \circ v + \langle v, F \rangle) - \operatorname{trace}\{\Lambda_F(\omega)(I + \nabla F)^{-1} \cdot \nabla v\} \tag{2.9}$$

a.s. P_0.

Remark 1 Note that $\det_2(I + \nabla F) \cdot (1 + \nabla F)^{-1}$ can be defined by continuity whether $(1 + \nabla F)$ is invertible or not [2, p. 1112] and consequently $\Lambda_F(1 + \nabla F)^{-1}$ is well-defined regardless of the invertibility of $(1 + \nabla F)$.

Remark 2 Lemma 2.2 plays a key role in what follows. In order to motivate it, let $T\omega = \omega + F(\omega)$ be a measurable transformation, such that the sigma fields induced by T^{-1} coincides with $\mathscr{F}$ i.e. $\sigma(T^{-1}A, A \in \mathscr{F}) = \mathscr{F}$. Further assume that Λ_F satisfies a Girsanov type identity $E(f \circ T \cdot \Lambda_F) = E[f]$ for any smooth $f(\omega)$. By the chain rule $(\nabla f) \circ T = ((I + \nabla F)^*)^{-1} \cdot \nabla (f \circ T)$. Consequently for any $v \in H$

$$
\begin{aligned}
E[f\delta v] = E[\langle v, \nabla f\rangle] &= E(\langle \nabla f \circ T, v\rangle \Lambda_F) \\
&= E[\langle (I + (\nabla F)^*)^{-1} \cdot \nabla (f \circ T), v\rangle \cdot \Lambda_F] \\
&= E[\langle \nabla (f \circ T), ((1 + \nabla F)^{-1} v)\rangle \Lambda_F] \\
&= E[(f \circ T)\delta\{\Lambda_F (I + \nabla F)^{-1} v\}] .
\end{aligned}
$$

On the other hand, $E(f\delta v) = E[f \circ T \cdot (\delta v) \circ T \cdot \Lambda_F]$. Comparing the two results for $E(f\delta v)$ yields

$$
\begin{aligned}
\delta\{\Lambda_F (I + \nabla F)^{-1} v\} &= \Lambda_F (\delta v) \circ T \\
&= \Lambda_F (\delta v + \langle v, F\rangle)
\end{aligned}
\tag{2.10}
$$

which is Eq. (2.9) for a non random v. This equation appears at the bottom of p. 400 of [3] however here and in several other formulas, the term $(I + \nabla F)^{-1}$ appears as $((I + \nabla F)^{-1})^*$ in [3], the difference between ∇a and $(\nabla a)^*$ was pointed out in Sect. 2. In the sequel we will show that (2.9) is "almost" sufficient for T to satisfy the Girsanov type identity $E[f \circ T \cdot \Lambda_F] = E[f]$.

Proof. Set

$$\alpha = \operatorname{Exp}(-\delta F - \tfrac{1}{2}|F|^2) .$$

Let $u(\omega)$ be a smooth H-valued functional, then

$$
\begin{aligned}
\delta\{\Lambda u\} &= \Lambda \delta u - \langle \nabla \Lambda, u\rangle \\
&= \Lambda \delta u - \alpha \langle \nabla \det{}_2 (1 + \nabla F), u\rangle - \Lambda \langle \nabla \log \alpha, u\rangle \\
&= \Lambda \delta u - \alpha \langle \nabla \det{}_2 (1 + \nabla F), u\rangle + \Lambda \langle \nabla_u F, F\rangle + \Lambda \langle \nabla \delta F, u\rangle .
\end{aligned}
$$

For $B \in \mathbb{D}_{2,1}(H)$ and $h \in H$

$$\nabla_h \delta B = \langle B, h\rangle + \delta \nabla_h B ,$$

and if, furthermore, ∇B is of trace class then

$$\nabla_h \delta B = \langle B, h\rangle + \delta \circ \nabla_h B - \nabla_h \operatorname{trace} \nabla B . \tag{2.11}$$

Note that (2.11) remains valid if h is replaced by smooth H-valued random functionals. Therefore

$$
\begin{aligned}
\delta(\Lambda_F u) = \Lambda \cdot \langle F, u + \nabla_u F\rangle &+ \Lambda \cdot \delta \circ [u + \nabla_u F] \\
&- \Lambda \operatorname{trace} \nabla u - \Lambda \nabla_u \operatorname{trace} \nabla F \\
&- \alpha \nabla_u \det{}_2 (1 + \nabla F) .
\end{aligned}
$$

By the lemma on p. 1110 of [2]

$$\nabla_u \det{}_2 (1 + \nabla F) = -\det{}_2 (1 + \nabla F) \cdot \operatorname{trace}[(1 + \nabla F)^{-1} \cdot \nabla F \cdot \nabla_u \nabla F] . \tag{2.12}$$

Set, now,

$$u = (1 + \nabla F)^{-1} v \tag{2.13}$$

then $\nabla (1 + \nabla F) u = \nabla v$, hence $\nabla (1 + \nabla F) \cdot u + (1 + \nabla F) \nabla u = \nabla v$.
Hence,

$$\nabla u = (1 + \nabla F)^{-1} [\nabla v - \nabla_u \nabla F] . \tag{2.14}$$

Substituting (2.12) and (2.13) into (2.14) yields

$$\delta(\Lambda(1+\nabla F)^{-1}v) = \Lambda\{\langle F, u + \nabla_u F\rangle + \delta\circ[u + \nabla_u F] - \mathrm{trace}(1+\nabla F)^{-1}\cdot\nabla v\}$$

where u and v are related by (2.13). Using $u + \nabla_u F = v$ to eliminate u from the last equation yields

$$\delta(\Lambda(1+\nabla F)^{-1}v) = \Lambda\{\langle F, v\rangle + \delta\circ v - \mathrm{trace}(I+\nabla F)^{-1}\nabla v\}$$

which is (2.9).

Proposition 2.1 *Assume that for some* $\gamma > 0$ *and* $r > (1+\gamma)/\gamma$, $F(\omega)\in\mathbb{D}_{r,2}(H)$ $\Lambda_F \in L^{1+\gamma}$, *and also* $\Lambda_F(I_H + \nabla F)^{-1}v \in L^{1+\gamma}(H)$ *with* $v\in H$ *non-random. Then*

$$\delta[\Lambda_F(I+\nabla F)^{-1}v] = \Lambda_F\langle T(\omega), v\rangle \tag{2.15}$$

a.s. P_0.

Remark. This result was proved in [3] under somewhat stronger assumptions, in particular we do not require F to be in $\bigcap_{p>1}\mathbb{D}_{p,2}(H)$.

Proof. Let $(h_n)\in\Omega$ be a C.O.N.S in H. Let $V_n = \sigma\{\delta h_1, \ldots, \delta h_n\}$, $n\in\mathbb{N}$ and let π_n denote the orthogonal projection on the span of $\{h_1, \ldots, h_n\}$. Consider

$$F_n(\omega) = E[\pi_n U_{1/n} F \mid V_n] = \Theta_n(\delta h_1, \ldots, \delta h_n)$$

where U_t denotes the Ornstein-Uhlenbeck semigroup on Ω. Since $F\in\mathbb{D}_{r,2}(H)$, $(r > (1+\gamma)/\gamma)$ it follows that $U_{1/n}F\in\mathbb{D}_{r,\infty}(H)$. Consequently, by the finite dimensional Sobolev embedding theorem, Θ_n can be selected to be a C^∞ function from $\mathbb{R}^n$ to H. Consequently, for all $v\in\Omega^*$

$$\Lambda_{F_n}(\omega)(I_H + \nabla F_n)^{-1}v \text{ and } \det{}_2(I_H + \nabla F_n)(1+\nabla F_n)^{-1}v$$

are Fréchet differentiable (as pointed out on p. 1112 of [2], this also holds by continuity on $\{\omega: \det_2(I_H + \nabla F_n) = 0\}$). Note that as $n\to\infty$ $F_n(\omega)\to F(\omega)$, $\nabla F_n(\omega \to \nabla F(\omega)$ and $\delta F_n \to \delta F$ a.s. and in $L^{(1+\gamma)/\gamma}(H)$, $L^{(1+\gamma)/\gamma}(H^{\otimes 2})$ and $L^{(1+\gamma)/\gamma}$ respectively, indeed, for $\rho > 1$

$$\begin{aligned} E|\delta F_n|^\rho &= E|E(\delta\pi_n U_{1/n}F\mid V_n)|^\rho \\ &= e^{\rho/n}\cdot E|E(U_{1/n}\delta\pi_n F\mid V_n)|^\rho \\ &\leqq e^{\rho/n} E U_{1/n} E(|\delta\pi_n F|^\rho \mid V_n) \\ &\leqq e^{\rho/n} E|\delta F|^\rho \end{aligned}$$

since $E[\delta\pi_n F\mid V_n] = E[\delta F\mid V_n]$. Similarly

$$E|\nabla F_n|^\rho \leqq e^{-\rho/n} E|\nabla F|^\rho .$$

Now by the capacity version of Egoroff's theorem [9], for every $\varepsilon > 0$ there exists a measurable set $A_\varepsilon \subset \Omega$ such that $\mathrm{Cap}_{(1+\gamma)/\gamma,1}(A_\varepsilon) < \varepsilon$ and outside A_ε, F_n, ∇F_n, δF_n converge uniformly as $n\to\infty$. Let $\phi_\varepsilon(\omega)$ be a smooth arbitrary function such that $\phi_\varepsilon(\omega) = 0$, for $\omega\in A_\varepsilon$. Then (denoting Λ_F, Λ_{F_n} by Λ and Λ_n respectively):

$$\phi_\varepsilon\delta[\Lambda_n(I+\nabla F_n)^{-1}v] = \delta[\phi_\varepsilon\Lambda_n(1+\nabla F_n)^{-1}v] - \langle\nabla\phi_\varepsilon, \nabla\Lambda_n(1+\nabla F_n)^{-1}\rangle .$$

By the uniform convergence of F_n, ∇F_n, δF_n on A_ε^c the right hand side of the last equation converges to $\delta[\phi_\varepsilon\Lambda(1+\nabla F)^{-1}v] - \langle\nabla\phi_\varepsilon, \nabla\Lambda(1+\nabla F)^{-1}v\rangle$ which equals

$$\phi_\varepsilon \delta[\Lambda(1 + \nabla F)^{-1} v]$$

since ϕ_ε is smooth and $\delta[\Lambda(1 + \nabla F)^{-1} v]$ is in $\mathbb{D}_{(1+\gamma), -1}$. Consequently, by (2.5) (with v non random) $\phi_\varepsilon \delta[\Lambda_n(1 + \nabla F_n)^{-1} v] = \phi_\varepsilon \cdot \Lambda_n \langle T_n \omega, v \rangle$, hence

$$\phi_\varepsilon \delta[\Lambda(1 + \nabla F)^{-1} v] = \phi_\varepsilon \Lambda \langle T(\omega), v \rangle . \tag{2.16}$$

The right hand side is the product of two random variables ϕ_ε and $\Lambda \langle T(\omega), v \rangle$. Had we known that $\delta(\Lambda(1 + \nabla F)^{-1} v)$ is also a random variable, then the result would have followed since $\varepsilon > 0$ and ϕ_ε are arbitrary. In order to overcome this somewhat delicate point, note that ϕ_ε can be chosen so that it converges to 1 in $\mathbb{D}_{(1+\gamma)/\gamma, 1}$ which is the dual space to $\mathbb{D}_{1+\gamma, -1}$ (this follows from a capacity argument, cf. Proposition 2.4 of [3]), consequently (2.16) holds with ϕ_ε replaced by any ϕ in $\mathbb{D}_{(1+\gamma)/\gamma, 1}$ and (2.13) follows.

3 Absolutely continuous transformations

Theorem 3.1 [3] *For all $f \in L^\infty(\Omega)$, under the assumptions of Proposition* 2.1.

$$E\{f(T\omega)\Lambda_F(\omega)\} = E\Lambda_F(\omega) Ef(\omega) . \tag{3.1}$$

Proof. Set $f_t(\omega) = \operatorname{Exp} it\langle v, \omega \rangle$ and $g_t(\omega) = f_t(T\omega) = \operatorname{Exp} it\langle v, \omega + F(\omega) \rangle$. Note that

$$\langle v, (I + \nabla F)^{-1} \cdot \nabla g_t(\omega) \rangle = it|v|^2 g_t(\omega) . \tag{3.2}$$

By Proposition 2.1

$$E[g_t(\omega) \cdot \delta\{\Lambda(\omega)(1 + \nabla F)^{-1*} v\}] = E\{\langle v, T\omega \rangle \Lambda(\omega) g_t(\omega)\} .$$

Integration by parts yields

$$E\{\langle v, (1 + \nabla F)^{-1*} \nabla g_t \rangle \Lambda(\omega)\} = E\{\langle v, T\omega \rangle \Lambda(\omega) g_t(\omega)\} .$$

Substituting from (3.2) yields

$$it|v|^2 E\Lambda(\omega) g_t(\omega) = \frac{1}{i} \frac{d}{dt} E\Lambda(\omega) g_t(\omega) .$$

Hence

$$\frac{d \log E\Lambda g_t(\omega)}{dt} = -t|v|^2$$

$$\log E\Lambda g_t(\omega) = \log E\Lambda g_0(\omega) - \frac{t^2|v|^2}{2}$$

consequently

$$E\Lambda(\omega) g_t(\omega) = E\Lambda(\omega) \cdot \operatorname{Exp} - \frac{t^2|v|^2}{2}$$

$$= E\Lambda \cdot E \operatorname{Exp} it\langle v, \omega \rangle ,$$

which proves (3.1) for $f(\omega) = \operatorname{Exp} i\langle v, \omega \rangle$. This result can now be extended to smooth cylinder functions by using standard Fourier transform arguments and then to all $f \in L^\infty(\Omega)$ by a dominated convergence argument.

From Theorem 3.1 we have the following Girsanov-type result.

Corollary 3.1 *Under the assumptions of Theorem* 3.1 *and further assuming* $E\Lambda_F \neq 0$, *set* $\lambda(\omega) = \Lambda_F(\omega)/E\Lambda_F(\omega)$ *and*

$$Q(A) = \int_A \lambda(\omega) P_0(d\omega), \quad A \in \mathscr{F} .$$

Then $T\omega$ *is Wiener under the measure* Q.

Proof. Set $f(\omega) = \mathbb{I}_A(\omega)$, substituting in (3.1) yields $Q(T^{-1}A) = P_0(A)$ i.e.

$$Q(\omega\colon T\omega \in A) = P_0(A) .$$

Remark. Note that this Girsanov-type result is obtained under relatively general assumptions, T does not have to be bijective and Q may turn out to be a signed measure and not necessarily a positive measure (however, the measure R induced by T will be positive). As a simple example for the case where Q is a signed measure consider the transformation

$$T\omega = \omega + f(\delta e) \cdot e$$

where $e \in H$, $|e| = 1$ and

$$f(x) = -2 \sin x , \quad |x| \leqq 2\pi$$

for $x > 2\pi$, $f(x)$ is assumed to decrease slowly to $-\frac{1}{2}$, similarly for $x < -2\pi$, $f(x)$ is assumed to decrease slowly from $\frac{1}{2}$. Then

$$\begin{aligned} \Lambda &= (1 + f'(\delta e)) \operatorname{Exp} - f'(\delta e) \operatorname{Exp} - (\delta[f(\delta e) e] - \tfrac{1}{2} f^2(\delta e)) \\ &= (1 + f'(\delta e)) \operatorname{Exp} - f(\delta e)\delta e - \tfrac{1}{2} f^2(\delta e) . \end{aligned}$$

Consider the measure p_0 induced on $\mathbb{R}_1$ by the projection $\langle \omega, e \rangle$ and by q the projection of $\langle T\omega, e \rangle$ then

$$q(dx) = (1 + f'(x)) \cdot \operatorname{Exp}[-x f(x) - \tfrac{1}{2} f^2(x)] \cdot p_0(dx)$$

where

$$p_0(dx) = \frac{1}{\sqrt{2\pi}} \operatorname{Exp} - \frac{x^2}{2} \cdot dx .$$

Since $(1 + f'(x))$ takes on negative (as well as positive) values, $q(dx)$ is not positive. However by Corollary 2.1 $q(\tau^{-1}A) = p_0(A)$ where $\tau x = x + f(x)$ and the Girsanov-type result holds.

As a second corollary to Theorem 2.1 we have the following weak form of Sard's lemma:

Corollary 3.2 *Let* $\sigma(T)$ *denote the* σ*-field induced by* $\{T^{-1}A, A \in \mathscr{F}\}$. *Let*

$$B = \{\omega\colon E\{\Lambda_F(\omega) | \sigma(T)\} = 0\} ,$$

then, under the assumptions of Theorem 2.1, *either*

(a) $E\Lambda_F(\omega) = 0$

or

(b) *There exists a set* $A \in \mathscr{F}$ *where* $\mathscr{F}$ *denotes the completed* σ*-field, such that* $B = T^{-1}A$ *and* $P_0(A) = 0$.

Proof. Since B is in $\sigma(T)$ it is in $\mathscr{F}$ and there exists an $A \in \mathscr{F}$ such that $B = T^{-1}A$. Therefore

$$\begin{aligned} 0 &= E\{\mathbb{I}_B(\omega) E(\Lambda(\omega) | \sigma(T))\} \\ &= E\{\mathbb{I}_B(\omega) \Lambda(\omega)\} \\ &= E\{\mathbb{I}_A(T\omega) \Lambda(\omega)\} . \end{aligned}$$

Therefore, by Theorem 3.1

$$0 = E\mathbb{I}_A(\omega) \cdot E\Lambda(\omega)$$

which proves that either $E\Lambda(\omega) = 0$ or $P_0(A) = 0$.

Theorem 3.2 *Assume that*

$$\operatorname{Exp}(-\delta F + \|\nabla F\|_2^2) \in L^{1+\gamma}(\Omega) \tag{3.3}$$

for some $\gamma > 0$ *and* $F \in \mathbb{D}_{2,r}$ *for some* $r > (\gamma + 2)/\gamma$. *Then* $E\Lambda_F = 1$.

Remarks. (a) By (2.7), condition (3.3) implies the conditions of Proposition 2.1. (b) Theorem 3.2 improves the results of Theorem 4.4 of [3], since in [3] it is also required that $F \in \bigcap_{r>1} \mathbb{D}_{r,1}$ and the conclusion being that $E\Lambda$ is an integer.

Proof. We show, first, that $E\Lambda_F$ is an integer. Let F_n denote an approximation to F introduced in the proof to Proposition 2.1, i.e.: $F_n = E(\pi_n U_{1/n} F | V_n)$. Set $\Lambda_{F_n} = \Lambda_n$. From the Sobolev injection theorem, F_n is of the form

$$F_n(\omega) = \sum f_i(\delta h_1, \ldots, \delta h_n) h_i$$

where f_i: $\mathbb{R}^n \to \mathbb{R}^n$ is C^∞. Denote by f the function $(f_1, \ldots, f_n)$. We have:

$$E[\Lambda_n] = \int_{\mathbb{R}^n} \det{}_2(I + Df(x)) \cdot \exp(-\delta f(x) - \tfrac{1}{2}|f(x)|_2^2) p(x)\, dx$$

where $p(x)$ is the standard Gaussian kernel on $\mathbb{R}^n$. First let us prove that $(\Lambda_n; n \in \mathbb{N})$ is bounded in $L^{1+\varepsilon}(P)$ for some $0 < \varepsilon < \gamma$. From the inequality (2.7), we have

$$\begin{aligned} E[\Lambda_n^{1+\varepsilon}] &\leq e^{\frac{1+\varepsilon}{2}} \cdot E[\exp(1+\varepsilon) \cdot (-\delta F_n + \tfrac{1}{2}\|\nabla F_n\|_2^2)] \qquad (3.4) \\ &= e^{\frac{1+\varepsilon}{2}} \cdot E[\exp(1+\varepsilon) \{\tfrac{1}{2}\|\nabla E[\pi_n U_{1/n} F | V_n]\|_2^2 - \delta E[\pi_n U_{1/n} F | V_n]\}] \\ &= e^{\frac{1+\varepsilon}{2}} \cdot E[\exp(1+\varepsilon) \{\tfrac{1}{2}\|e^{-1/n} E[\pi_n U_{1/n} \nabla F | V_n]\|_2^2 \\ &\quad - e^{1/n} E[U_{1/n} \delta F | V_n]\}] \\ &= e^{\frac{1+\varepsilon}{2}} \cdot E\left[\exp(1+\varepsilon) \left\{\frac{e^{-2/n}}{2} \|U_{1/n} E[\pi_n \cdot \nabla F | V_n] \cdot \|_2^2 \right.\right. \\ &\quad \left.\left. - e^{1/n} U_{1/n} E[\delta F | V_n]\right\}\right] . \end{aligned}$$

Replacing $e^{-2/n}$ by $e^{1/n}$ and applying Jensen's inequality:

$$\begin{aligned} E[\Lambda_n^{1+\varepsilon}] &\leqq e^{\frac{1+\varepsilon}{2}} E[\exp(1+\varepsilon)e^{1/n} U_{1/n} E[(\tfrac{1}{2}\|\pi_n \nabla F\|_2^2 - \delta F)| V_n]] \\ &\leqq e^{\frac{1+\varepsilon}{2}} E[U_{1/n} E[\exp\cdot(1+\varepsilon)e^{1/n}(\tfrac{1}{2}\|\nabla F\|_2^2 - \delta F)| V_n]] \\ &= e^{\frac{1+\varepsilon}{2}} E[\exp(1+\varepsilon)e^{1/n}(\tfrac{1}{2}\|\nabla F\|_2^2 - \delta F)] , \end{aligned}$$

it suffices to choose $(1+\varepsilon)e^{1/n} \leqq 1+\gamma$. From this majorization, we see that $E[\Lambda_n] \to E[\Lambda_F]$ as n tends to infinity, consequently, to show that $E[\Lambda_F]$ is an integer, it is sufficient to show that $E[\Lambda_n]$ is an integer for any $n \in \mathbb{N}$.

To show that $E[\Lambda_n]$ is an integer, let $B = \{x \in \mathbb{R}^n: \det_2(I + Df(x)) = 0\}$, from the implicit function theorem, there exists a sequence of compact sets $(K_n; n \in \mathbb{N})$ covering B^c, such that $T = \mathrm{Id}_{\mathbb{R}^n} + f$: $K_n \to T(K_n)$ is a diffeomorphism. Let $E_1 = K_1$, $E_{n+1} = K_{n+1} \backslash E_n$. For any $u \in C_b(\mathbb{R}^n)$, we have from Jacobi's theorem and Sard's lemma:

$$\begin{aligned} \int_{\mathbb{R}^n} u(Tx)|\tilde{\Lambda}_n(x)| p(x)\,dx &= \sum_n \int_{E_n} u(Tx)|\tilde{\Lambda}_n(x)| p(x)\,dx \\ &= \sum_n \int_{T(E_n)} u(x) p(x)\,dx \end{aligned}$$

where $\tilde{\Lambda}_n$ is defined by $\tilde{\Lambda}_n(\delta h_1, \ldots, \delta h_n) = \Lambda_n$. In particular, taking $u = 1$, we see that $\sum_n \int_{T(E_n)} p(x)\,dx$ is convergent. Similarly

$$\int_{\mathbb{R}^n} u(Tx)\tilde{\Lambda}_n(x) p(x)\,dx = \sum_n \operatorname{sign}(\tilde{\Lambda}_n; E_n) \int_{T(E_n)} u(x) p(x)\,dx ,$$

note that the sum at the right hand side is absolutely convergent. Also from Theorem 3.1, the above term is equal to $E[\Lambda_n] \cdot \int_{\mathbb{R}^n} u(x) p(x)\,dx$.

Comparing them, we conclude that

$$E[\Lambda_n] = \sum_{n=1}^{\infty} \mathbb{I}_{T(E_n)}(x) \cdot \operatorname{sign}(\tilde{\Lambda}_n; E_n) \quad \text{a.s.} \tag{3.5}$$

In order to show that $E\Lambda = 1$, note the following lemma, the proof of which is straightforward.

Lemma 3.1 *Let* $T_t = I + F_t$, *if* $E\Lambda_{F_t}$ *is continuous in* t *and an integer for every* $t \in [a, b]$, *then* $E\Lambda_{F_a} = E\Lambda_{F_b}$.

Therefore if $t \to F_t$ and $t \to \nabla F_t$ are continuous in probability on $[a, b]$ and

$$\sup_{a \leqq t \leqq b} E\,\mathrm{Exp}(1+\varepsilon)(\|\nabla F_t\|_2^2 - \delta F_t) < \infty \tag{3.6}$$

then $E\Lambda_{F_b} = E\Lambda_{F_a}$. Setting $F_t = t \cdot F$,

$$\begin{aligned} &\sup_{1 \leqq t \leqq 1} E\,\mathrm{Exp}(1+\varepsilon)(t^2 \|\nabla F\|_2^2 - t\delta F_t) \\ &\leqq \sup_{0 \leqq t \leqq 1} E\,\mathrm{Exp}(1+\varepsilon) \cdot t(\|\nabla F\|_2^2 - \delta F) \\ &\leqq \sup_{0 \leqq t \leqq 1} (E\,\mathrm{Exp}(1+\varepsilon)(\|\nabla F\|_2^2 - \delta F))^t < \infty \end{aligned} \tag{3.7}$$

and consequently $E\Lambda_F = 1$.

Theorem 3.3 *Under the assumptions of Theorem* 3.1 *and further assuming*: (i): $\det_2(1+\nabla F) \neq 0$ a.s. P_0 (*or, equivalently*, $(1+\nabla F)$ *is invertible* a.s. P_0) *and* (ii): *There exists a measurable transformation* T_l^{-1} *such that* $T_l^{-1}T_F\omega = \omega$ a.s. P_0 *we have*

(a) *Either* $\det_2(1+\nabla F) > 0$, P_0 a.s. *or also* $\det_2(1+\nabla F) < 0$, P_0 a.s. *and in any case* $E\Lambda_F \neq 0$, $(E\Lambda_F)^{-1}\Lambda_F > 0$ a.s. P_0.

(b) *The measures* Q, P_0 *and* R *are mutually absolutely continuous where* Q *is as defined in Corollary* 3.1. *Moreover*, Q *is a probability measure.*

(c) *If, moreover, the conditions of Theorem* 3.2 *are satisfied, then* $E\Lambda_F = 1$.

Proof. Let $A_1 = \{\omega: \Lambda > 0\}$, $A_2 = \{\omega: \Lambda < 0\}$ and let $f_i(\omega) = \mathrm{I\!I}_{A_i}(T_l^{-1}\omega)$ for $i = 1, 2$. Then by Theorem 4.1

$$E[\mathrm{I\!I}_{A_i}(\omega)\Lambda(\omega)] = E\Lambda \cdot E[f_i] \,. \tag{3.8}$$

Since $P_0(A_1) + P_0(A_2) = 1$, at least one of the two, $P(A_1)$, $P(A_2) \neq 0$ and consequently $E\Lambda \neq 0$. Assuming that $E\Lambda > 0$, Eq. (3.8) for $i = 2$ yields that the left hand side is non positive, while the right hand side is non negative, consequently $P(A_2) = 0$. Similarly assuming $E\Lambda < 0$ yields $P(A_1) = 0$, which proves part (a). Since $\det_2(1+\nabla F) \neq 0$ a.s. it follows that $P_0(A) > 0 \Rightarrow Q(A) > 0$ namely $Q(A) = 0 \Rightarrow P_0(A) = 0$. Consequently $P_0 \sim Q$ and $P_0 \sim R$ follows from (b) of Lemma 2.1. Part (c) follows directly from Theorem 3.2.

Theorem 3.4 *Under the assumptions of Theorem* 3.3 *and further assuming that there exists a measurable transformation* T^{-1} *such that* $T^{-1}T\omega = T \cdot T^{-1}\omega = \omega$ a.s. P_0

$$\frac{dR}{dP_0}(\omega) = C\det_2(I - \tilde{\nabla}(F \circ T^{-1})) \cdot \mathrm{Exp}\{\tilde{\delta}(F \circ T^{-1}) - \tfrac{1}{2}|F \circ T^{-1}|^2\} \tag{3.9}$$

where C *is the integration constant* $E\Lambda_F^{-1}$ (*and* $C = 1$ *under the assumptions of Theorem* 3.2), $\tilde{\delta}$ *is defined as follows: if* $F \circ T^{-1}$ *is in* $\mathbb{D}_{p,1}(H)$, *for some* $p > 1$, *then* $\tilde{\delta}(F \circ T^{-1}) = \delta(F \circ T^{-1})$, *otherwise fix a complete orthonormal base* $\{e_i, i = 1, 2, \ldots\}$ *in* H, *for* $\xi^N = \sum_1^N \langle \xi, e_i \rangle e_i$ *set*

$$\tilde{\delta}(\xi^N \circ T^{-1}) = \sum_{i=1}^{N} \{\langle \xi \circ T^{-1}, e_i \rangle \delta e_i - \tilde{\nabla}_{e_i}\langle \xi \circ T^{-1}, e_i \rangle\} \tag{3.10}$$

and define $\tilde{\delta}(f \circ T^{-1})$ *as the* P_0*-limit in the probability of* $\tilde{\delta}(\xi^N \circ T^{-1})$ *as* $N \to \infty$,. $\tilde{\nabla}(\xi \circ T)$ *is defined as the limit in probability of* $\nabla(\xi_n \circ T)$ *where* ξ_n *is a sequence of smooth* $\mathbb{D}_{p,1}$ *approximations to* ξ (cf. *Definition* 2.2 *of* [14]).

Proof. By Theorem 3.3 and Lemma 2.1

$$\frac{dR}{dP_0}(\omega) = \left(\frac{C}{\det_2(1+\nabla F)\mathrm{Exp}(-\delta F - \frac{1}{2}|F|^2)}\right) \circ T^{-1}\omega$$

(a different proof of this formula appears in Theorem 7.5 of [11]). Since $T\omega = \omega + F(\omega)$, $T^{-1}\omega = \omega - F(T^{-1}\omega)$. By Theorem 2.1 of [14] (cf. also [1])

$$(\delta F) \circ T^{-1}\omega = \tilde{\delta}(F(T^{-1}\omega)) - |F(T^{-1}\omega)|^2 - \mathrm{trace}(\nabla F \circ T^{-1}\omega \cdot \tilde{\nabla}(F(T^{-1}\omega))) \,.$$

Hence

$$\frac{dR}{dP_0}(\omega) = \frac{C}{(\det_2(1+\nabla F))\circ T^{-1}\omega} \cdot \tag{3.11}$$

$$\cdot\{\operatorname{Exp}[\tilde{\delta}(F(T^{-1}\omega)) - \tfrac{1}{2}|F(T^{-1}\omega)|^2 - \operatorname{trace}(\nabla F\circ T^{-1}\omega\cdot\tilde{\nabla}F(T^{-1}\omega))]\}\,.$$

Now $(1+A)^{-1} = I - (I+A)^{-1}A$, and since $\det_2(1+A)\det_2(1+B) = \det_2(1+A+B+AB)\cdot\operatorname{Exp}\operatorname{trace} AB$ it follows that

$$(\det_2(1+\nabla F))^{-1} = \det_2[(1+\nabla F)^{-1}]\cdot\operatorname{Exp} + \operatorname{trace}\{\nabla F(1+\nabla F)^{-1}\nabla F\}\,. \tag{3.12}$$

Also, note that $T^{-1}\omega = \omega - F\circ T^{-1}\omega$ therefore

$$\begin{aligned}(I - \nabla(F\circ T^{-1}))\circ T &= (I+\nabla F)^{-1}\\ &= I - (I+\nabla F)^{-1}\cdot\nabla F\,.\end{aligned}$$

Consequently

$$(\nabla(F\circ T^{-1}\omega))\circ T = (1+\nabla F)^{-1}\cdot\nabla F\,. \tag{3.13}$$

Substituting in (3.12) yields

$$\begin{aligned}&(\det_2(1+\nabla F)\circ T^{-1}\omega)^{-1}\\ &= \det_2(1+(\nabla F)\circ T^{-1}\omega)^{-1}\cdot\operatorname{Exp}\operatorname{trace}\{(\nabla F)\circ T^{-1}\cdot\nabla(F\circ T^{-1})\}\\ &= \det_2(1-\nabla(F\circ T^{-1}))\operatorname{Exp}\operatorname{trace}\{(\nabla F)\circ T^{-1}\cdot\nabla(F\circ T^{-1})\}\end{aligned} \tag{3.14}$$

and (3.9) follows from (3.11) and (3.14).

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