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Addendum to "A general minimax theorem based on connectedness"

By

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The present note wants to answer the main question which had been left open in the paper of the title [9] and, as far as we know, in the subsequent literature. It is whether the typical dichotomy "finite intersections are connected in one variable, all intersections are connected in the other variable" is in the nature of the matter or not. The answer will be that the dichotomy cannot be avoided; the surprise is that there exists a simple example on classical soil. This underlines the definitive character of our work [9].

1. The situation. Minimax theorems consider functions $f: X \times Y \to \overline{\mathbb{R}}$ on the product of nonvoid sets X and Y, and the formations

$$f_* := \operatorname*{Sup}_{x \in X} \operatorname*{Inf}_{y \in Y} f(x, y) \leqq \operatorname*{Inf}_{y \in Y} \operatorname*{Sup}_{x \in X} f(x, y) =: f^*.$$

The assertions are that the minimax relation $f_* = f^*$ holds true under the respective assumptions. The present context arose from the classical

Theorem of Sion ([13]). *Let X and Y be compact convex subsets of topological vector spaces. Assume that $f: X \times Y \to \overline{\mathbb{R}}$ fulfills*

$$\forall y \in Y: f(\cdot, y) \in \operatorname{USC}(X) \quad (:= \textit{upper semicontinuous}),$$

$$\forall x \in X: f(x, \cdot) \in \operatorname{LSC}(Y) \quad (:= \textit{lower semicontinuous});$$

$$\forall y \in Y: f(\cdot, y) \textit{ quasiconcave, that is } [f(\cdot, y) \geqq \lambda] \subset X \textit{ is convex } \forall \lambda \in \mathbb{R},$$

$$\forall x \in X: f(x, \cdot) \textit{ quasiconvex, that is } [f(x, \cdot) \leqq \lambda] \subset Y \textit{ is convex } \forall \lambda \in \mathbb{R}.$$

Then $f_* = f^*$.

The question arose how to abandon the vector space structure, as in the classical quantitative minimax theorem of Ky Fan and its successors. The basic idea was due to Wu [15]: For a set system $\mathfrak{S}$ in a topological vector space one has the implications

$$\text{all } S \in \mathfrak{S} \text{ are convex} \Rightarrow \bigcap_{S \in \mathfrak{T}} S \text{ is connected for all nonvoid } \mathfrak{T} \subset \mathfrak{S}$$

$$\Rightarrow \bigcap_{S \in \mathfrak{T}} S \text{ is connected for all nonvoid finite } \mathfrak{T} \subset \mathfrak{S}.$$

Thus we are led to consider functions $f: X \times Y \to \overline{\mathbb{R}}$ on the product of topological spaces X and Y, and to define f to fulfill the [finite] condition $X(\geqq)$ iff

$$X(H, \lambda, \geqq) := \bigcap_{y \in H} [f(\cdot, y) \geqq \lambda] := \{x \in X : f(x, y) \geqq \lambda \, \forall \, y \in H\} \subset X$$

is connected for all nonvoid [finite] subsets $H \subset Y$ and (justified by success) for all real $\lambda > f_*$, and the obvious variant $X(>)$. Likewise we define f to fulfill the [finite] condition $Y(\leqq)$ iff

$$Y(H, \lambda, \leqq) := \bigcap_{x \in H} [f(x, \cdot) \leqq \lambda] := \{y \in Y : f(x, y) \leqq \lambda \, \forall \, x \in H\} \subset Y$$

is connected for all nonvoid [finite] subsets $H \subset X$ and all real $\lambda > f_*$, and the obvious variant $Y(<)$. Our main result in [9] then reads as follows.

Theorem. *Let X and Y be topological spaces with Y compact and X connected, and let $f: X \times Y \to \overline{\mathbb{R}}$ fulfill the continuity condition*

(C) $f(\cdot, y) \in \mathrm{USC}(X) \, \forall \, y \in Y$ and $f(x, \cdot) \in \mathrm{LSC}(Y) \, \forall \, x \in X$, or the variant (C′) $f \in \mathrm{LSC}(X \times Y)$.

Then $f_* = f^*$ whenever f fulfills some combination of one of the conditions $X(\geqq)$ and *$X(>)$ with one of the finite conditions $Y(\leqq)$ and $Y(<)$, provided that for the combination of $X(>)$ with $Y(<)$ the space Y is Hausdorff.*

The assumption that X be connected (which takes care of the empty intersection) cannot be dropped, as shown by the trivial example

$$X = Y = \{0, 1\} \quad \text{and} \quad f(x, y) = \begin{Bmatrix} 0 & \text{if} & x = y \\ \infty & \text{if} & x \neq y \end{Bmatrix}.$$

However, the connectedness of both X and Y is implied if one assumes $f: X \times Y \to \mathbb{R}$ and formulates the connectedness conditions with $\lambda \in \mathbb{R}$ instead of $\lambda > f_*$.

The main steps on the road from Sion [13] to [9] were Tuy [14], Geraghty-Lin [2], Kindler-Trost [6], Komiya [7], Simons [11], Horvath [3]. A posteriori it turned out that the above theorem could have been deduced from the results of [11] and [3], and Jürgen Kindler pointed out to the author that the same holds true for [6]. However, as noted in [9], the possible existence of results like the present one remained unnoticed in all these previous papers. For the historical context, also for the other types of minimax theorems, we refer to the survey article of Simons [12]. We now have to add Komiya [8], simultaneous with [9], which presents an almost equal but weaker result: when one strips off the robe of convexities one finds the connectedness conditions as above, but with variable $\lambda \in \mathbb{R}$ instead of fixed $\lambda > f_*$.

In conclusion we want to remark that instead of the (partial) Hausdorff condition in the above theorem a much weaker separation axiom called wHausdorff is sufficient. It is one of the numerous unconventional separation axioms considered in topology; see for example Dunham [1]. A topological space is called wHausdorff iff each pair of points which can be distinguished by an open set can be separated by a disjoint pair of open sets. An example of a non-wHausdorff topology is the so-called cofinite topology, the collection of all subsets with finite complement plus $\varnothing$, on an infinite set. We list a few simple but notable properties.

1) Each Hausdorff space and each regular space is wHausdorff. 2) A normal space need not be wHausdorff, but a normal wHausdorff space is regular. 3) A compact wHausdorff space is normal and hence regular.

Remark. Let $\mathfrak{S}$ be a collection of closed compact subsets of a wHausdorff space. If $\bigcap_{S\in\mathfrak{T}} S$ is connected for all nonvoid finite $\mathfrak{T}\subset\mathfrak{S}$ then for all nonvoid $\mathfrak{T}\subset\mathfrak{S}$ too.
This is what in fact had been proved in [9] Remark 2.3. It follows that its main results remain true with wHausdorff instead of Hausdorff.

2. The example for unsymmetry. The above theorem raises the question whether it suffices to assume the finite conditions $X(\geqq)$ and $X(>)$ instead of the full conditions, as on the other side, or whether this is true at least for some variants or under additional assumptions like Hausdorff or wHausdorff. This is the main problem which had been left open in [9], and it remained open in the subsequent papers Kindler [4], [5] and Ricceri [10] and in Komiya [8]. We shall present a negative example below; it is much simpler than suspected.

Example. Let E be an infinite-dimensional real Hilbert space. We consider

$$X := \{x\in E: \|x\| \geqq 1\}, \text{ equipped with the norm topology,}$$
$$Y := \{y\in E: \|y\| \leqq 1\}, \text{ equipped with the weak topology,}$$

so that Y is compact and convex. For $f: X\times Y\to\mathbb{R}$ we take the simplest possible choice $f(x,y)=\langle x,y\rangle$. Then we have the properties which follow.

1) f is continuous in the product topology of $X\times Y$. In fact, for $(a,b),(x,y)\in X\times Y$ we have

$$f(x,y)-f(a,b)=\langle x-a,y\rangle+\langle a,y-b\rangle$$

and hence

$$|f(x,y)-f(a,b)|\leqq\|x-a\|+|\langle a,y-b\rangle|.$$

2) The minimax relation $f_*=f^*$ is violated, because

$$\operatorname*{Inf}_{y\in Y} f(x,y) = -\|x\| \quad \forall x\in X, \qquad \text{and hence } f_*=-1,$$
$$\operatorname*{Sup}_{x\in X} f(x,y)=\begin{Bmatrix}\infty & \text{if} & y\neq 0\\ 0 & \text{if} & y=0\end{Bmatrix}\forall y\in Y, \quad \text{and hence } f^*=0.$$

3) For $\lambda\in\mathbb{R}$ and $x\in X$ the sets $[f(x,\cdot)\leqq\lambda]$ and $[f(x,\cdot)<\lambda]\subset Y$ are convex. Therefore for $\lambda\in\mathbb{R}$ and nonvoid $H\subset X$ the sets $Y(H,\lambda,\leqq)$ and $Y(H,\lambda,<)\subset Y$ are convex and hence connected. Thus even the full conditions $Y(\leqq)$ and $Y(<)$ are fulfilled.

4) The finite conditions $X(\geqq)$ and $X(>)$ are fulfilled. We present the proof for $X(\geqq)$, for $X(>)$ it is identical. Let

$$D := X(H,\lambda,\geqq)=\bigcap_{l=1}^{r}[f(\cdot,y_l)\geqq\lambda] = \{x\in E: \|x\|\geqq 1 \text{ and } \langle x,y_l\rangle\geqq\lambda\ (l=1,\dots,r)\}$$

with

$$H=\{y_1,\dots,y_r\}\subset Y \quad\text{and}\quad \lambda\in\mathbb{R},$$

and fix $u, v \in D$. We choose $a \in E$ with $\| a \| = 1$ orthogonal to $u, v, y_1, \ldots, y_r$. Then on the one hand

$$\text{for } t \in \mathbb{R}: \quad \| u + t a \| \geqq \| u \| \geqq 1, \quad \text{and hence } u + t a \in D,$$
$$\| v + t a \| \geqq \| v \| \geqq 1, \quad \text{and hence } v + t a \in D,$$

and on the other hand

$$\text{for } 0 \leqq t \leqq 1: \quad \|(1-t)(u+a) + t(v+a)\| = \|((1-t)u + tv) + a\| \geqq \| a \| = 1,$$

and hence $(1-t)(u+a) + t(v+a) \in D$. Therefore the path which consists of the line segments from u to $u + a$, from $u + a$ to $v + a$, and from $v + a$ to v, connects u and v in D, so that D is even arcwise connected. The same with $H = \varnothing$ shows that X itself is arcwise connected.

5) It is now clear from our theorem that the full conditions $X(\geqq)$ and $X(>)$ cannot be fulfilled. But for the sake of illustration we present an explicit example. Fix $a \in E$ with $\| a \| = 1$ and form $H := \{y \in Y : \langle y, a \rangle = 0\} \subset Y$. For $\lambda \in \mathbb{R}$ one computes

$$X(H, \lambda, \geqq) = \{x \in X : \langle x, y \rangle \geqq \lambda \, \forall y \in H\} = \{x \in X : \|x - \langle x, a \rangle a\| \leqq -\lambda\},$$

which for $-1 < \lambda \leqq 0$ is not connected since it decomposes into its subsets $\langle x, a \rangle > 0$ and < 0. Likewise one verifies that $X(H, \lambda, >)$ is not connected for $-1 \leqq \lambda < 0$.

Thus we see that in our theorem the finite conditions $X(\geqq)$ and $X(>)$ cannot replace the respective full conditions, even under the smoothest circumstances.

3. A symmetrical special case. One could be suspicious that the unsymmetry obtained in the last section comes from the unsymmetry in the initial assumption, where we imposed compactness on Y but not on X. This turns out to be true. In the present section we shall derive from [9] a symmetric minimax theorem under compactness assumptions on both X and Y. A similar result is in Komiya [8], but with the unusually strong continuity assumption $f \in C(X \times Y)$.

Lemma. *Assume that X is compact wHausdorff and $f: X \times Y \to \overline{\mathbb{R}}$ satisfies $f(\cdot, y) \in \mathrm{USC}(X) \, \forall y \in Y$. Then the finite condition $X(>)$ implies the finite condition $X(\geqq)$.*

Proof after [9] Consequence 2.4: Consider the set

$$X(H, \lambda, \geqq) = \bigcap_{l=1}^{r} [f(\cdot, y_l) \geqq \lambda] \subset X \quad \text{with } H = \{y_1, \ldots, y_r\} \quad \text{and} \quad \lambda > f_*.$$

Fix a sequence of numbers $\lambda > \lambda_n > f_*$ with $\lambda_n \uparrow \lambda$. Then

$$X(H, \lambda, \geqq) \subset D_n := \bigcap_{l=1}^{r} [f(\cdot, y_l) > \lambda_n] \subset \bar{D}_n \subset \bigcap_{l=1}^{r} [f(\cdot, y_l) \geqq \lambda_n] = X(H, \lambda_n, \geqq),$$

the last inclusion because $X(H, \lambda_n, \geqq)$ is closed. For $n \to \infty$ we have $X(H, \lambda_n \geqq) \downarrow X(H, \lambda, \geqq)$ and hence $\bar{D}_n \downarrow X(H, \lambda, \geqq)$. Now by assumption the D_n are connected, hence the $\bar{D}_n$ are connected. The assertion follows from the remark in the first section.

Theorem. *Let X and Y be compact topological spaces with X wHausdorff and connected, and let $f: X \times Y \to \overline{\mathbb{R}}$ fulfill condition* (C). *Then $f_* = f^*$ whenever f satisfies some combination of one of the finite conditions $X(\geqq)$ and $X(>)$ with one of the finite conditions $Y(\leqq)$ and $Y(<)$.*

Proof. We note two facts. i) In view of the remark in the first section and of $f(\cdot, y) \in \mathrm{USC}(X)\ \forall y \in Y$ the finite conditions $X(\geqq)$ implies the full condition $X(\geqq)$. ii) By the above lemma the finite condition $X(>)$ implies the finite condition $X(\geqq)$. Therefore each of the two finite conditions $X(\geqq)$ and $X(>)$ implies the full condition $X(\geqq)$. The assertion then follows from our main theorem.

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