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Probability Theory and Related Fields

Random rotations of the Wiener path

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Summary. Let (W, H, μ) be an abstract Wiener space and let $R(w)$ be a strongly measurable random variable with values in the set of isometries on H. Suppose that ∇Rh is smooth in the Sobolev sense and that it is a quasi-nilpotent operator on H for every $h \in H$. It is shown that $\delta(R(w)h)$ is again a Gaussian $(0, |h|_H^2)$-random variable. Consequently, if $(e_i, i \in \mathbb{N}) \subset W^*$ is a complete, orthonormal basis of H, then $\tilde{w} = \sum_i (\delta R(w) e_i) e_i$ defines a measure preserving transformation, a "rotation", on W. It is also shown that if for some strongly measurable, operator valued (on H) random variable R, $\delta(R(w+k)h)$ is $(0, |h|_H^2)$-Gaussian for all $k, h \in H$, then R is an isometry and ∇Rh is quasi-nilpotent for all $h \in H$. The relation between the stochastic calculi for these Wiener paths w and $\tilde{w}$, as well as the conditions of the invertibility of the map $w \to \tilde{w}$ are discussed and the problem of the absolute continuity of the image of the Wiener measure μ under Euclidean motion on the Wiener space (i.e. $w \to \tilde{w}$ composed with a shift) is studied.

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1 Introduction

Let $\{W_t, t \in [0,1]\}$ be the standard Brownian motion on $[0,1]$. Consider the following random transformation on the Cameron–Martin space $R(w)$: $\int_0^\cdot h_s\, ds \to \int_0^\cdot h_s \operatorname{sign} W_s\, ds$, then $R(w)$ is a.s. an isometry, even a unitary transformation, on H. This transformation induces a transformation $w \to \tilde{w}$ of the Wiener space where $\tilde{w}$ is defined as

$$W(\tilde{w})_t = \int_0^t \operatorname{sign} W_s(w)\, dW_s(w)\,,$$

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where W_t is the coordinate or the evaluation map at $t \in [0,1]$. From the celebrated theorem of Paul Lévy, $W(\tilde{w})$ is again a standard Brownian motion and hence, $w \mapsto \tilde{w}$ preserves the Wiener measure μ. Consequently, $F(w) \mapsto F(\tilde{w})$ defines an isometry of $L^p(\mu)$ for any $p \geqq 1$. Note that $\sigma(\tilde{w})$ – the σ-field induced by $\tilde{w}$ on $C_0([0,1])$ is strictly smaller than the $\sigma(w)$ – the σ-field induced by w on $C_0([0,1])$, since $\tilde{w} = (-w)\tilde{}$. The purpose of this paper is to consider, in a general setup, isometries $R(w)$ on H and the induced transformations $w \rightarrow \tilde{w}$ and $F(w) \rightarrow F(\tilde{w})$. We will refer to this collection of topics as "random rotations". With the exception of the last section, the results that will be presented will, unfortunately, not include the particular case presented above where $R(w)$ is induced by sign W, since the analysis will be based on the Malliavin calculus and sign W does not possess the necessary smoothness needed for the analysis.

In the next section we consider the abstract Wiener space (W,H,μ) and show that if $R(w)$ is a random isometry on H, Rh is smooth in the Sobolev sense and ∇Rh is a quasi-nilpotent operator on H (a Hilbert–Schmidt operator A on H is quasi-nilpotent, if the only eigenvalues of A are zero, cf. e.g., [1,8] for characterizations of this notion), then the Skorohod integral δRh is a Gaussian $(0,|h|_H^2)$-random variable. Consequently, for $e_i \in W^*$, where $(e_i, i \in \mathbb{N})$ is a complete orthonormal basis of H (identifying the elements of W^* with their injection in $H^* = H$),

$$\tilde{w} = \sum_i (\delta R(w)e_i) \cdot e_i$$

defines a measure preserving transformation of W. Note that if $R(w)$ is an isometry and ∇Rh is quasi-nilpotent, then $R(w+k)$ has the same properties for any $k \in H$. In Sect. 3 it is shown that the assumptions that $R(w)$ is an isometry and ∇Rh is quasi-nilpotent, then they are "natural" in the sense that, under some smoothness conditions, if $\delta(R(w+k)h)$ is Gaussian $(0,|h|_H^2)$-distributed for all h and k in H, then R is an isometry and ∇Rh is quasi-nilpotent. The results of Sect. 2 yield two "coupled" abstract Wiener spaces (W,H,μ) and $(\tilde{W},H,\mu)$. Some relations between the stochastic calculi in W and $\tilde{W}$ are discussed in Sect. 4 as well as conditions for the invertibility of the map from W to $\tilde{W}$. Section 5 raises the following problem. Let $y = \tilde{w} + u(w)$ where u is an H-valued random variable and $\tilde{w}$ is the rotated path, consider the problem of the absolute continuity of the measure induced by y (on the underlying Banach space) with respect to the measure μ and the corresponding Radon–Nikodym derivative under the assumption that the σ-field induced by $\tilde{w}$ is strictly smaller than the one induced by w. In particular let

$$y_t = \int_0^t \operatorname{sign} W_s dW_s + \int_0^t a_s(w)\, ds$$

where a_s depends on W and not on $\tilde{W}$. And the problem is to find a condition for the absolute continuity of the measure induced by y on $C_0[0,1]$ with respect to the Wiener measure on the same space. This seems to be a delicate problem and a solution is presented, in a special case.

In the remaining part of this section we will summarize the notation of the stochastic calculus of variations and present some results for later reference.

Let (W,H,μ) be an abstract Wiener space, i.e., H is the Cameron–Martin space i.e., a separable Hilbert space which is densely and continuously injected into Banach space W. We assume that $H = H^*$ (its continuous dual), hence W^* is also densely and continuously injected into H. μ denotes the standard Gaussian measure on W.

A mapping φ from W into some separable Hilbert space $\mathscr{X}$ will be called a cylindrical function if it is of the form $\varphi(w) = f(\langle v_1, w\rangle, \dots, \langle v_n, w\rangle)$ where $f \in C_0^\infty(\mathbb{R}^n, \mathscr{X})$ $v_i \in W^*$ for $i = 1, \dots, n$.

For such a φ, we define $\nabla\varphi$ as

$$\nabla\varphi(w) = \sum_{i=1}^{n} \partial_i f(\langle v_1, w\rangle, \dots, \langle v_n, w\rangle)\tilde{v}_i\,,$$

where $\tilde{v}_i$ is the image of v_i under the injection $W^* \hookrightarrow H$.

It follows that ∇ is a closable operator on $L^p(\mu;\mathscr{X})$, $p \geqq 1$ and we will denote its closure with the same notation. The powers ∇^k of ∇ are defined by iteration. For $p > 1$, $k \geqq 1$, we denote by $D_{p,k}(\mathscr{X})$ the completion of $\mathscr{X}$-valued cylindrical functions with respect to the norm:

$$\|\varphi\|_{D_{p,k}(\mathscr{X})} \equiv \|\varphi\|_{p,k} = \sum_{i=0}^{k} \|\nabla^i\varphi\|_{L^p(\mu,\mathscr{X}\otimes H^{\otimes i})}\,.$$

Let us denote by δ the formal adjoint of ∇ with respect to the Wiener measure μ and define $\mathscr{L}$ as $\delta \cdot \nabla$. The well-known result of P. A. Meyer assures that the norm defined above is equivalent to

$$|||\varphi|||_{p,k} = \|(I + \mathscr{L})^{k/2}\varphi\|_{L^p(\mu,\mathscr{X})}\,,$$

where $\mathscr{L}$ is the Ornstein–Uhlenbeck operator or the number operator. Note that, due to its self adjointness, its non-integer powers are well-defined. Moreover we can also define $D_{p,k}(\mathscr{X})$ for negative k using the second norm and we denote by $\mathbb{D}(\mathscr{X}) = \bigcap_{p>1}\bigcap_{k\in N}\mathbb{D}_{p,k}(\mathscr{X})$ and, $\mathbb{D}'(\mathscr{X}) = \bigcup_{p>1}\bigcup_{k\in\mathbb{Z}}\mathbb{D}_{p,k}(\mathscr{X})$. In case $\mathscr{X} = \mathbb{R}$ we write simply $\mathbb{D}_{p,k}, \mathbb{D}, \mathbb{D}'$ instead of $\mathbb{D}_{p,k}(\mathbb{R}), \mathbb{D}(\mathbb{R}), \mathbb{D}'(\mathbb{R})$.

Let us recall that $\nabla : \mathbb{D}_{p,k}(\mathscr{X}) \to \mathbb{D}_{p,k-1}(\mathscr{X}\otimes H)$ and $\delta : \mathbb{D}_{p,k}(\mathscr{X}\otimes H) \to \mathbb{D}_{p,k-1}(\mathscr{X})$ are continuous linear operators for any $p > 1$, $k \in \mathbb{Z}$.

In the sequel we will derive H-valued random variables to obtain operator-valued random variables, and then apply then to the vectors of H. Since in general these operators will not be self-adjoint, we have to be careful about the order of these operations. Henceforth the following convention will be used: if ξ is an H-valued random variable (or Wiener functional) then $\nabla\xi$ will be a Hilbert-Schmidt operator and for h and k, we define

$$\begin{aligned}\frac{d}{dt}(\xi(w+th), k)|_{t=0} &= (\nabla\xi h, k)\\ &= (\nabla_h\xi, k)\\ &= (\nabla\xi, h\otimes k)_2\,,\end{aligned}$$

where $(\cdot , \cdot)_2$ denotes the inner product in the space of Hilbert–Schmidt operators. We conclude this section with the following lemma which will be needed in later sections.

Lemma 1.1 *Let $\xi \in \mathbb{D}_{p,2}(H)$, $\eta \in \mathbb{D}_{q,2}(H)$ with $p^{-1} + q^{-1} < 1$. Then we have*

$$\delta(\nabla_h \xi)|_{h=\eta} = \delta \nabla_\eta \xi + \text{trace}\, \nabla\xi \cdot \nabla\eta \tag{1.1}$$

and

$$\nabla_\eta \delta\xi = (\eta, \xi)_H + \delta\nabla_\eta \xi + \text{trace}\, (\nabla\xi \cdot \nabla\eta)\,. \tag{1.2}$$

Proof. For cylindrical ξ, η in $\mathbb{D}(H)$, we have

$$\delta(\nabla_h \xi)|_{h=\eta} = \sum_{i=1}^{\infty}((\nabla_\eta \xi, e_i)\delta e_i - \nabla_\eta \nabla_{e_i}(\xi, e_i))\,,$$

where $(e_i;\ i \in \mathbb{N})$ is a complete, orthonormal basis in H. Since

$$\sum_i \nabla_\eta \nabla_{e_i}(\xi, e_i) = \sum_i \nabla_{e_i}(\nabla_\eta \xi, e_i) - \text{trace}\, (\nabla\xi \cdot \nabla\eta)\,,$$

we have (1.1). The general case of (1.1) follows by a limiting procedure and (1.2) follows since

$$\nabla_\eta \delta\xi = \delta(\nabla_h \xi)_{h=\eta} + (\xi, \eta)_H\,. \qquad \square$$

2 Random rotations

Theorem 2.1 *Let $R(w)$ be a strongly measurable random variable on W with values in the space of bounded linear operators on H. Assume that R is a.s. an isometry on H ($|R(w)h|_H |h|_H$* a.s. *for all $h \in H$). Further assume that for some $p > 1$ and for all $h \in H$, $Rh \in \mathbb{D}_{p,2}(H)$ and $\nabla Rh \in \mathbb{D}_{p,1}(H \otimes H)$ is a quasi-nilpotent operator on H (i.e. $\lim_{n\to\infty} \|(\nabla Rh)^n\|_{L(H,H)}^{1/n} = 0$ a.s. or, equivalently,* trace *$(\nabla Rh)^n = 0$ a.s., for all $n \geqq 2$, cf.* [1,8]). *If, moreover, either*

(a) *$(I + i\nabla Rh)^{-1} \cdot Rh$ is in $L^q(\mu, H), q > 1$ for any $h \in H$ (here q may depend on $h \in H$) or,*

(b) *$Rh \in \mathbb{D}(H)$ for any $h \in H$.*

Then

$$E[\exp i\delta(Rh)] = \exp - \frac{1}{2}|h|_H^2\,. \tag{2.1}$$

A. Proof of Theorem 2.1 under assumption (a): For this purpose we state and prove the following three lemmas.

Lemma 2.1 *Let R be an isometry valued (on H) strongly measurable random variable on $(W, \mathscr{F}, \mu)$ where $\mathscr{F}$ is the Borel σ-field of W. Suppose that $Rh \in \mathbb{D}_{p,1}(H)$ and $(I + i\nabla Rh)$ is a.s. invertible for every $h \in H$. Then*

$$(Rh,\ (I + i\nabla Rh)^{-1}Rh)_H = |h|_H^2\,.$$

Proof of Lemma. Since $0 = \nabla_u(Rh, Rh)_H = 2(Rh, \nabla_u Rh)$ setting $u = (I + i\nabla Rh)^{-1}Rh$, we obtain (cf. the conventions preceding Lemma 1.1)

$$(Rh, (I + i\nabla Rh)^{-1}Rh)_H = (Rh, (I + i\nabla Rh)^{-1}Rh + i\nabla_{(I+i\nabla Rh)^{-1}Rh}Rh)_H$$

$$= (Rh, (I + i\nabla Rh)(I + i\nabla Rh)^{-1}Rh)_H = |h|_H^2 .$$

□

Lemma 2.2 *Let R be a strongly measurable isometry, $Rh \in \mathbb{D}_{p,2}(H)$. Further assume that for every $h \in H, \nabla Rh$ is a.s. quasi-nilpotent and $(I + i\nabla Rh)^{-1} Rh \in L^p(\mu, H)$ for some $p > 1$ which may depend on h. Then, a.s.*

$$\operatorname{trace}(\nabla Rh \cdot \nabla((I + i\nabla Rh)^{-1}Rh)) = 0 .$$

Proof of Lemma. The invertibility of $(I + i\nabla Rh)$ follows from the assumption that ∇Rh is quasi-nilpotent. Note that

$$\operatorname{trace}(\nabla Rh \cdot (I + i\nabla Rh)^{-1} \cdot \nabla Rh) = -i \operatorname{trace}(\nabla Rh \cdot (I - (I + i\nabla Rh)^{-1})) ,$$

where the last equality follows from the fact that $\lambda \mapsto \operatorname{trace}(\nabla Rh \cdot (I - (I + \lambda\nabla Rh)^{-1}))$ is analytic, hence we can calculate it with Taylor's series, whose coefficients are trace $(\nabla Rh \cdot (\nabla Rh)^k)$, $k \geqq 1$, and they are zero by the quasi-nilpotence hypothesis (cf. Theorem X1.6.25 of [1]). It remains then to prove that

$$\operatorname{trace}(\nabla Rh \cdot (I + i\nabla Rh)^{-1}\nabla^2 Rh(I + i\nabla Rh)^{-1}Rh) = 0 .$$

Since the ∇Rh is quasi-nilpotent, Carleman–Fredholm determinant $\det_2(I + \lambda\nabla Rh) = 1$ for any $\lambda \in \mathbb{C}$. By taking its Sobolev derivative in the direction of the vector field $u = (I + i\nabla Rh)^{-1}Rh$, we have

$$\operatorname{trace}[((I + \lambda\nabla Rh)^{-1} - I) \cdot \nabla^2 Rh(I + i\nabla Rh)^{-1}Rh] = 0 .$$

This completes the proof since

$$\lambda\nabla Rh \cdot (I + \lambda\nabla Rh)^{-1} = I - (I + \lambda\nabla Rh)^{-1} . \qquad \square$$

Lemma 3.3 *Let R satisfy the condition of Lemma 2.2, then*

$$\delta[e^{-i\delta Rh}(I + i\nabla Rh)^{-1} Rh] = e^{-i\delta Rh}(i|h|_H^2 + \delta Rh) . \tag{2.2}$$

Proof of Lemma. Applying Lemmas 2.1 and 2.2 yield,

$$\delta[e^{-i\delta Rh}(I + i\nabla Rh)^{-1}Rh]$$

$$= e^{-i\delta Rh}\{\delta[(I + i\nabla Rh)^{-1} Rh] + i(\nabla\delta Rh,\ (I + i\nabla Rh)^{-1} Rh)_H\}$$

$$= e^{-i\delta Rh}\{\delta[(I + i\nabla Rh)^{-1} Rh] + i(Rh,\ (I + i\nabla Rh)^{-1} Rh)_H$$

$$+ i\delta(\nabla_{(I+i\nabla Rh)^{-1}Rh}Rh)\} .$$

Hence, by Lemma 2.1,

$$\delta[e^{-i\delta Rh}(I + i\nabla Rh)^{-1}Rh] = e^{-i\delta Rh}\{i|h|_H^2 + \delta[(I + i\nabla Rh)(I + i\nabla Rh)^{-1} \cdot Rh]\},$$

which completes the proof of the lemma. □

Turning to the proof of Theorem 2.1 under (a):

Taking expectations on both sides of Eq. (2.2) yields

$$E[e^{-i\delta Rh}\delta Rh] = -i|h|_H^2 E[e^{-i\delta Rh}]$$

replacing h by th yields

$$\frac{d}{dt}E[e^{-it\delta Rh}] = -t|h|_H^2 E[e^{-it\delta Rh}],$$

and (2.1) follows. □

B. *Proof of Theorem 2.1 under assumption* (b): We prepare, first, the following two lemmas.

Lemma 2.4 *Assume that for all $h \in H$, $Rh \in \mathbb{D}_{p,1}(H)$ and ∇Rh is a.s. quasi-nilpotent. Then, for any n and $h_1,\dots,h_n \in H$,*

$$\operatorname{trace}(\nabla Rh_1 \cdot \nabla Rh_2 \cdots \nabla Rh_n) = 0 .$$

Proof of Lemma. For any $x_n \cdots x_n \in \mathbb{R}$,

$$\begin{aligned}\operatorname{trace}\{(\nabla R(x_1h_1 + \cdots + x_nh_n))^M\} &= \sum_{i_1+i_2+\cdots+i_n=M} \frac{M!}{i_1!\cdots i_n!}x_1^{i_1}\cdots x_n^{i_n} \\ &\quad \operatorname{trace}\{(\nabla Rh_1)^{i_1}\cdots(\nabla Rh_n)^{i_n}\} \\ &= 0 .\end{aligned}$$

Hence, the coefficients of this polynomial must be null and the result follows. □

Lemma 2.5 *Let $\xi \in D_{p,2}(H)$, $\eta \in D_{q,2}(H)$, $\alpha \in D_{r,2}(H)$ with $p^{-1} + q^{-1} + r^{-1} < 1$. Then we have*

$$\begin{aligned}\operatorname{trace}(\nabla\xi \cdot \nabla\nabla_\eta\alpha) &= \operatorname{trace}(\nabla\xi \cdot \nabla\eta \cdot \nabla\alpha) + \operatorname{trace}(\nabla\xi \cdot \nabla_\eta\nabla\alpha) \\ &= \operatorname{trace}(\nabla\xi \cdot \nabla\eta \cdot \nabla\alpha) + \nabla_\eta\operatorname{trace}(\nabla\xi \cdot \nabla\alpha) \\ &\quad - \operatorname{trace}(\nabla_\eta\nabla\xi \cdot \nabla\alpha) .\end{aligned}$$

Proof. It is sufficient to prove when ξ, η and α are in $D(H)$. Moreover, the second equality follows from the first one and from the fact that

$$\nabla_\eta \operatorname{trace}(\nabla\xi \cdot \nabla\alpha) = \operatorname{trace}(\nabla_\eta\nabla\xi \cdot \nabla\alpha) + \operatorname{trace}(\nabla\xi \cdot \nabla_\eta\nabla\alpha) .$$

For the first one, let (e_i) be a complete, orthonormal basis of H. Denote ∇_{e_i} by ∇_i and (ξ, e_j) by ξ_j.

Then

$$\begin{aligned}
\operatorname{trace}(\nabla\xi\cdot\nabla\nabla_\eta\alpha) &= \sum_{i,j=1}^{\infty}\nabla_i\xi_j\cdot\nabla_j(\nabla_\eta\alpha)_i \\
&= \sum_{i,j,k}\nabla_i\xi_j\cdot\nabla_j(\eta_k\nabla_k\alpha_i) \\
&= \sum_{i,j,k}\nabla_i\xi_j(\nabla_j\eta_k\nabla_k\alpha_i+\eta_k\nabla_k\nabla_j\alpha_i) \\
&= \operatorname{trace}(\nabla\xi\cdot\nabla\eta\cdot\nabla\alpha)+\sum_{i,j}\nabla_i\xi_j\cdot\nabla_\eta\nabla_j\alpha_i \\
&= \operatorname{trace}(\nabla\xi\cdot\nabla\eta\cdot\nabla\alpha)+\operatorname{trace}(\nabla\xi\cdot\nabla_\eta\nabla\alpha)\,. \qquad \square
\end{aligned}$$

To prove the theorem under the hypothesis that $Rh\in\mathbb{D}(H)$ and that ∇Rh is a.s. quasi-nilpotent, for any $h\in H$, we will show that $E[(\delta Rh)^n]=E[(\delta h)^n]$ for any $n\in\mathbb{N}$; we have, evidently $E[\delta Rh]=0$, moreover

$$\begin{aligned}
E[(\delta Rh)^2] &= |h|^2+E[\operatorname{trace}(\nabla Rh\cdot\nabla Rh)] \\
&= |h|^2\,,
\end{aligned}$$

since ∇Rh is quasi-nilpotent.

For $n=3$, we have

$$\begin{aligned}
E[(\delta Rh)^3] &= E[\delta Rh\cdot(\delta Rh)^2] \\
&= E[(Rh,\ \nabla(\delta Rh)^2)] \\
&= 2E[(Rh,\ \nabla\delta Rh)_H\delta Rh] \\
&= 2E[|h|^2\delta Rh+\delta\nabla_{Rh}Rh\cdot\delta Rh] \\
&= 2E[\delta\nabla_{Rh}Rh\cdot\delta Rh] \\
&= 2E[(\nabla_{Rh}Rh,Rh)]+2E[\operatorname{trace}(\nabla\nabla_{Rh}Rh\cdot\nabla Rh)] \\
&= 2E[\operatorname{trace}(\nabla\nabla_{Rh}Rh\cdot\nabla Rh)]\,, \qquad (2.3)
\end{aligned}$$

since $(\nabla_{Rh}Rh,Rh)=\frac{1}{2}\nabla_{Rh}(Rh,Rh)=\frac{1}{2}\nabla_{Rh}|h|^2=0$.

From Lemma 2.5, we have

$$\begin{aligned}
\operatorname{trace}(\nabla\nabla_{Rh}Rh\cdot\nabla Rh) &= \operatorname{trace}(\nabla Rh\cdot\nabla Rh\cdot\nabla Rh)+\operatorname{trace}(\nabla Rh\cdot\nabla_{Rh}\nabla Rh) \\
&= \operatorname{trace}(\nabla Rh\cdot\nabla_{Rh}\nabla Rh) \\
&= \frac{1}{2}\nabla_{Rh}\operatorname{trace}(\nabla Rh\cdot\nabla Rh) \\
&= 0
\end{aligned}$$

by the quasi-nilpotence of ∇Rh and by the fact that R preserves the norm in the Cameron–Martin space H. To complete the proof we will proceed by induction. Note first that

$$\begin{aligned}
E[(\delta Rh)^2] &= |h|^2\,, \qquad E[\delta\nabla_{Rh}Rh]=0\,, \\
E[(\delta Rh)^3] &= 0\,, \qquad E[\delta Rh\cdot\delta\nabla_{Rh}Rh]=0\,.
\end{aligned}$$

Suppose that $E[(\delta Rh)^n] = E[(\delta h)^n]$ and $E[(\delta Rh)^{n-2}\delta\nabla_{Rh} Rh] = 0$.

We will show that the same relations hold for $n+1$:

$$\begin{aligned} E[(\delta Rh)^{n+1}] &= E[\delta Rh(\delta Rh)^n] \\ &= nE[(Rh, \delta \otimes \nabla Rh + Rh)(\delta Rh)^{n-1}] \\ &= nE[|h|^2(\delta Rh)^{n-1}] + nE[\delta\nabla_{Rh} Rh(\delta Rh)^{n-1}] \\ &= n|h|^2 E[(\delta h)^{n-1}] + nE[\delta\nabla_{Rh} Rh \cdot (\delta Rh)^{n-1}] \\ &= E[(\delta h)^{n+1}] + nE[\delta\nabla_{Rh} Rh(\delta Rh)^{n-1}] . \end{aligned}$$

Let us define τ_n inductively as

$$\tau_1 = Rh , \quad \tau_{n+1} = \nabla_{\tau_n} Rh , \quad n \in \mathbb{N} .$$

We have

$$\begin{aligned} E[\delta\nabla_{Rh} Rh(\delta Rh)^{n-1}] &= E[\delta\nabla_{\tau_1} Rh \cdot (\delta Rh)^{n-1}] \\ &= (n-1)E[(\nabla_{\tau_1} Rh, Rh + \delta \otimes \nabla Rh)(\delta Rh)^{n-2}] \\ &= (n-1)E[(\nabla_{\tau_1} Rh, \delta \otimes \nabla Rh)(\delta Rh)^{n-2}] \\ &= (n-1)E[\delta\nabla_{\tau_2} Rh \cdot (\delta Rh)^{n-2}] . \end{aligned}$$

Since we have $\nabla_{\tau_1} Rh = \tau_2$, applying Lemma 1.1 yields.

$$(\tau_2, \delta \otimes \nabla Rh) = \delta\nabla_{\tau_2} Rh + \operatorname{trace}(\nabla Rh \cdot \nabla\tau_2)$$

and from Lemma 2.5,

$$\begin{aligned} \operatorname{trace}(\nabla Rh \cdot \nabla\tau_2) &= \operatorname{trace}(\nabla Rh \cdot \nabla\nabla_{\tau_1} Rh) \\ &= \operatorname{trace}(\nabla Rh \cdot \nabla\tau_1 \cdot Rh) + \operatorname{trace}(\nabla Rh \cdot \nabla_{\tau_1}\nabla Rh) \\ &= \operatorname{trace}(\nabla Rh \cdot \nabla Rh \cdot \nabla Rh) \\ &= 0 . \end{aligned}$$

This explains the last line of the above inequality.

Furthermore, continuing this way we also have

$$(\nabla_{\tau_n} Rh, \delta \otimes \nabla Rh) = \delta\nabla_{\tau_{n+1}} Rh$$

and

$$\begin{aligned} E[\delta\nabla_{Rh} Rh \cdot (\delta Rh)^{n-1}] &= (n-1)E[\delta\nabla_{\tau_2} Rh(\delta Rh)^{n-2}] \\ &= (n-1)(n-2)E[\delta\nabla_{\tau_3} Rh \cdot (\delta Rh)^{n-3}] \\ &= \vdots \\ &= (n-1)!E[\delta\nabla_{\tau_n} Rh] = 0 , \end{aligned}$$

hence, the induction hypothesis is satisfied. □

The same arguments as those used to prove part A of Theorem 2.1 also yield the following results which are of independent interest. The reader is referred to [5, 6] for a different approach.

Proposition 2.1 *Let* $\xi \in \mathbb{D}_{p,1}(H)$, $p > 1$ *and set*

$$\Lambda = \exp - \delta\xi - \frac{1}{2}|\xi|_H^2 .$$

Suppose that $\nabla\xi$ *is quasi-nilpotent and that*

- $E \exp \delta\xi < \infty$,
- $\Lambda \cdot (I + \nabla\xi)^{-1}\xi \in \mathbb{D}_{q,1}(H)$ *for some* $q > 1$,
- $(I + \nabla\xi)^{-1}\xi \in \mathbb{D}_{r,1}(H)$ *for some* $r > 1$.

Then $E\Lambda = 1$.

Proof. Using the same arguments as in the proof of part A of Theorem 2.1 yields

$$\delta[\Lambda(I + \nabla\xi)^{-1}\xi] = (\delta\xi + |\xi|^2)\Lambda .$$

Replace ξ by $t \cdot \xi$, $t \in \mathbb{R}$ and denote by Λ_t the corresponding exponential. Taking expectations yields

$$\frac{d}{dt}E\Lambda_t = E\{(-\delta\xi - t|\xi|_H^2)\Lambda_t\} ,$$

hence $dE\Lambda_t/dt = 0$ and $E\Lambda_t = E\Lambda_0 = 1$. □

Proposition 2.2 *Assume that for all* h: $Rh \in \mathbb{D}_{p,2}(H)$ *for some* $p > 1$, ∇Rh *is quasi-nilpotent and* $\nabla Rh \in L^\infty(\mu, H \otimes H)$, *where* $H \otimes H$ *denotes the space of Hilbert–Schmidt operators on H. Set*

$$\mathring{I}_p(R) = \{\xi \in \mathbb{D}_{p,1}(H) : \nabla\xi = \sum_1^n \alpha_i(w)\nabla Rh_i , \quad \alpha_i \in L^\infty(\mu) , \ h_i \in H , \ n \in \mathbb{N}\}$$

and by $I_p(R)$ *the closure of* $\mathring{I}_p(R)$ *in* $L^p(\mu, H)$. *Then, for any* $\eta \in I_p(R)$ *we have*

$$E[|\delta\eta|^p] \leqq c_p E[|\eta|^p] ,$$

where c_p *is independent of* η.

Proof. Let us note first that, by Lemma 2.4, if $\eta \in I_p(R)$ has a (Sobolev) derivative $\nabla\eta$ in some L^p-space, then the derivative is almost surely a quasi-nilpotent operator. By Theorem 3.1 of [6], if $\xi \in \mathring{I}_p(R)$ then

$$E\left[\exp t\delta\xi - \frac{t^2}{2}|\xi|_H^2\right] = 1$$

for small t. Differentiating p times with respect to t at $t = 0$, for p even, yields

$$E|\delta\xi|^p \leqq C_p \|\xi\|_{L^p(\mu, H)}^p .$$

Consequently, if $\eta \in I_p(R)$, let $\xi_n \in \mathring{I}_p(R)$ converge to η in $L^p(\mu, H)$, then $\delta\eta$ is in L^p, $\delta\xi_n \to \delta\eta$ in L^p. □

We conclude this section with an example of an isometry $R(w)$ satisfying the assumptions of Theorem 2.1.

Example. Let $(e_i) \in W^*$ be a complete orthonormal base for H. Set $R(w)$ as

$$R(w)e_1 = \cos\alpha(\delta e_3(w))e_1 - \sin\alpha(\delta e_3(w))e_2\,,$$
$$R(w)e_2 = \sin\alpha(\delta e_3(w))e_1 + \cos\alpha(\delta e_3(w))e_2\,,$$
$$R(w)e_k = e_k\,, \quad k \geqq 3\,,$$

where α is any nice function on $\mathbb{R}$. Then $R(w)$ satisfies the hypothesis of Theorem 2.1.

3 Quasi-invariance and a (partial) converse to Theorem 2.1

In general the shift $w \mapsto w + R(w)h,\ h \in H$, is not invertible, however we have a Girsanov-like result:

Proposition 3.1 *Under the assumptions of Theorem 2.1, for any $F \in C_b(W)$, $h \in H$, we have*

$$E\left[F(w + R(w)h)\exp - \delta Rh - \frac{1}{2}|h|^2\right] = E[F]\,.$$

Proof. From the density of the cylindrical functions in any $L^p(\mu)$ $(p \geqq 1)$ and Fourier transform, it is sufficient to prove the identity

$$E[e^{\delta k(w+Rh)}\mathscr{E}(-\delta Rh)] = e^{1/2|k|_H^2}$$

for any $k \in H$, where $\mathscr{E}(-\delta Rh)$ denotes $\exp(-\delta Rh - \frac{1}{2}|h|_H^2)$. Since

$$\delta k(w + Rh) = \delta k(w) + (Rh(w), k)\,,$$

this amounts to prove that $E[\exp\delta(Rh + k) - \frac{1}{2}|Rh + k|^2] = E[\mathscr{E}(\delta(Rh + k))] = 1$. We have, denoting by τ_k the map $\tau_k(w) = w + k$,

$$\begin{aligned} E[\mathscr{E}(\delta(Rh + k))] &= E[\mathscr{E}(\delta Rh)\circ\tau_k e^{-(Rh,k)\circ\tau_k}] \\ &= E\left[\exp\{\delta(Rh\circ\tau_k) + (Rh\circ\tau_k, k) - \frac{1}{2}|h|^2 - (Rh,k)\circ\tau_k\}\right] \\ &= E\left[\exp\delta(R\circ\tau_k h) - \frac{1}{2}|h|^2\right] = 1\,. \end{aligned}$$

where the last equality follows from Theorem 2.1, since $(R\circ\tau_k)h$ satisfies the same hypothesis of quasi-nilpotence as Rh. □

The following result is a partial converse to Theorem 2.1.

Proposition 3.2 *Suppose that $R : W \mapsto \mathscr{L}(H,H)$ be a strongly measurable random variable such that $Rh \in \mathbb{D}_{p,1}(H)$, for any $h \in H$ and for some $p > 1$. Assume moreover that $\nabla Rh \in L^\infty(\mu, H \otimes H)$ for any in $h \in H$. Then the following are equivalent*:

(i) *a.s. $R(w)$ is an isometry and ∇Rh is quasi-nilpotent for any $h \in H$.*

(ii) $E\exp\delta R(w + k)h = \exp\frac{1}{2}|h|^2$ *for all h, $k \in H$.*

Proof. (i) $\Rightarrow$ (ii) follows from the Theorem 2.1, since we have

$$\|(I+\nabla Rh)^{-1}\| \leqq e^{1+(1/2)\|\nabla Rh\|_2^2}$$

and $\nabla Rh \in L^\infty(\mu, H\otimes H)$. To show (ii) $\Rightarrow$ (i), note that the hypothesis of (ii) is equivalent to the Girsanov theorem. In fact, for k, $h \in H$,

$$\begin{aligned} E&\left[e^{(\delta k)(w+Rh)}\exp\left(-\delta Rh - \frac{1}{2}|h|^2\right)\right] \\ &= E\left[\exp \delta k + (k, Rh) - \delta Rh - \frac{1}{2}|h|^2\right] \\ &= E[e^{-\delta Rh(w+k)-\frac{1}{2}|h|^2+(k,\, R(w+k)h)}]e^{(1/2)|k|^2} \\ &= e^{(1/2)|k|^2}E[e^{-\delta R(w+k)h-\frac{1}{2}|h|^2}] \\ &= e^{(1/2)|k|^2}\,. \end{aligned}$$

On the other hand, for small $|h|$, we have (cf. [6]),

$$E\left[F(w+Rh)\cdot|\det_2(I+\nabla Rh)|\exp - \delta Rh - \frac{1}{2}|Rh|^2\right] = E[F]\,,$$

for any $F \in C_b(W)$, since the map $w \mapsto w + R(w)h$ is invertible when $\|\nabla Rh\| < 1$.
Comparing this expression with the one obtained above, i.e., with

$$E\left[F(w+Rh)\exp - \delta Rh - \frac{1}{2}|h|^2\right] = E[F]\,,$$

we find immediately that

$$|\det_2(I+\nabla Rh)|e^{-(1/2)|Rh|^2} = e^{-(1/2)|h|^2} \quad \text{a.s.} \tag{3.1}$$

For small $|h|$, $\det_2(I+\nabla Rh)$ has constant sign almost surely since $\|\nabla Rh\| \in L^\infty(\mu)$. Hence we can suppose that it is positive. Moreover, replacing h by th, $t \in [-\varepsilon, \varepsilon]$, we have

$$\det_2(I+t\nabla Rh)e^{-(t^2/2)|Rh|^2} = e^{-(t^2/2)|h|^2}\,.$$

For small t, we have (cf., [1])

$$\det_2(I+t\nabla Rh) = \exp\sum_{j=2}^{\infty}(-1)^{j+1}\frac{t^j}{j}\operatorname{trace}((\nabla Rh)^j)\,.$$

By comparison, we should have trace $((\nabla Rh)^j) = 0$ for any $j > 2$. Consequently, $(\nabla Rh)^2$ is a quasi-nilpotent operator. On the other hand, if λ is an eigenvalue of ∇Rh then λ^2 is an eigenvalue of $(\nabla Rh)^2$. Hence $\lambda = 0$, i.e. ∇Rh is also quasi-nilpotent. Consequently $\det_2(I+t\nabla Rh) = 1$ and hence $|Rh| = |h|$. □

4 Some stochastic calculus associated with rotations

Let $R(w)$ satisfy the assumptions of Theorem 2.1, and for reasons of simplicity we further assume that $Rh \in \mathbb{D}(H)$ for all $h \in H$. For any cylindrical functional F on W of the form

$$F(w) = f(\delta h_1, \dots, \delta h_n),$$

where $h_i \in H$, define

$$\hat{R}F = f(\delta Rh_1, \dots, \delta Rh_n).$$

Then $\hat{R}F$ extends to a *linear* isometry on $L^p(\mu)$ for any $p \geqq 1$.

Remark. Note that since $\hat{R}$ is an isometry, it always has a left inverse, in fact $\hat{R}^*\hat{R} = I$, where $\hat{R}^*$ is the formal adjoint of $\hat{R}$.

Let $(e_i) \subset W^*$ be a complete, orthonormal basis for H. Since, for $i \neq j$, δRe_i and δRe_j are independent, the series

$$\sum_{i=1}^{\infty} (\delta Re_i) \cdot e_i$$

converges almost surely in the strong topology of W (cf. [2]). We will denote the resulting path by $\tilde{R}w$, i.e.,

$$\tilde{R}w = \sum_{i=1}^{\infty} (\delta Re_i)(w) e_i .$$

If $v \in W^*$, then we have $\langle v, \tilde{R}w \rangle = \delta R(j(v))$, where $j(v)$ is the image of v in H under the injection $W^* \hookrightarrow H$. Consequently, $\tilde{R}w$, as defined above, is independent of the particular choice of the basis (e_i) and

$$\hat{R}F(w) = F(\tilde{R}w),$$

for $F \in L^1(\mu)$.

For F as above, we define the derivation operator D on the range of $\hat{R}$ by

$$\begin{aligned} D\hat{R}F &= \sum_{i=1}^{n} \partial_i f(\delta Rh_1, \dots, \delta Rh_n) h_i \\ &= \hat{R}\nabla F . \end{aligned}$$

Consequently D extends continuously from $\hat{R}\mathbb{D}_{p,k}$ to $\hat{R}\mathbb{D}_{p,k-1}(H)$, $p > 1$, $k \in \mathbb{N}$ and since $\hat{R}$ is injective $DF(\tilde{w}) = \hat{R}\nabla\hat{R}^* F(\tilde{w})$.

Let D^* denote the dual to D on the range of $\hat{R}$, then for $\varphi \in \mathbb{D}(H)$

$$\begin{aligned} E[\hat{R}F \cdot D^*\hat{R}\varphi] &= E[(D\hat{R}F, \hat{R}\varphi)] \\ &= E[(\hat{R}\nabla F, \hat{R}\varphi)] \\ &= E[(\nabla F, \varphi)] \\ &= E[F\delta\varphi] \\ &= E[\hat{R}F \ \hat{R}\delta\varphi] \end{aligned}$$

and consequently

$$D^*\hat{R}\varphi = \hat{R}\delta\varphi$$

and D^* extends on $\hat{R}\mathbb{D}_{p,k}(H)$ to $\hat{R}\mathbb{D}_{p,k-1}$, $p > 1$, $k \in \mathbb{N}$ and for any $\eta(\tilde{w}) \in \hat{R}\mathbb{D}_{p,k}(H)$

$$D^*\eta(\tilde{w}) = \hat{R}\delta\hat{R}^*\eta(\tilde{w}) .$$

In order to clarify the operations D and D^*, note that the map $\tilde{R} : W \to W$ constructed above preserves the Wiener measure μ. Let $\mathscr{F}$ denote the Borel σ-field on W and $\mathscr{B}$ and the subsigma field $\tilde{R}^{-1}(\mathscr{F})$. In general, $\mathscr{B}$ is strictly smaller than $\mathscr{F}$ and we have two different measure spaces $(W, \mathscr{F}, \mu)$ and $(W, \mathscr{B}, \mu)$ where $\mathscr{B} = \tilde{R}^{-1}(\mathscr{F})$. Denote by $\tilde{\nabla}$ the Sobolev derivative on $(W, \mathscr{B}, H, \mu)$ and setting $b = \tilde{R}w$:

$$\begin{aligned}
\tilde{\nabla}_h\varphi(b) &= \frac{d}{d\lambda}\varphi(b + \lambda h)\Big|_{\lambda=0} \\
&= \frac{d}{d\lambda}\varphi(\tilde{R}w + \lambda h)\Big|_{\lambda=0} \\
&= \frac{d}{d\lambda}\hat{R}\varphi(w + \lambda h)\Big|_{\lambda=0} \\
&= \hat{R}\nabla_h\varphi(w) \\
&= D_h\varphi(\tilde{R}w) .
\end{aligned}$$

Similarly we can verify that $\tilde{\delta} = (\tilde{\nabla})^* = D^*$. Consequently, as long as we restrict ourselves to $\mathscr{B}$-measurable random variables, the theory remains unchanged (just the notations are different).

As mentioned earlier, since $\hat{R}$ is an isometry, we have always $\hat{R}^*\hat{R} = Id$, however $\hat{R}\hat{R}^*$ is in general different from the identity map.

Proposition 4.1 *Let $\mathscr{B}$ be the σ-field $\tilde{R}^{-1}(\mathscr{F}) = \sigma\{\delta Rh;\ h \in H\}$. We have*

$$\hat{R}\hat{R}^*\varphi = E[\varphi|\mathscr{B}]$$

for any $\varphi \in L^1(\mu)$.

Proof. It is sufficient to prove the proposition for bounded φ.

Let $F(w) = f(\delta h_1, \ldots, \delta h_n)$, $f \in C_b^\infty(\mathbb{R}^n)$, $h_i \in H$.
We have

$$\begin{aligned}
E[\hat{R}\hat{R}^*\varphi \cdot \hat{R}F] &= E[\hat{R}^*\varphi \cdot F] \\
&= E[\varphi \cdot \hat{R}F] \\
&= E[E|\varphi|\mathscr{B}] \cdot \hat{R}F] .
\end{aligned}$$

Since, by definition $\hat{R}F$ is $\mathscr{B}$-measurable and the set of $\hat{R}F$, when F runs in the set of cylindrical random variables, is dense in $L^p(\mu, \mathscr{B})$, for any $p \geqq 1$, we have $E[\varphi|\mathscr{B}] = \hat{R}\hat{R}^*\varphi$. □

Corollary 4.1 *$\hat{R}$ is invertible on $L^p(\mu)$, $p \geqq 1$, i.e., there exists some $\hat{R}^{-1}$ such that $\hat{R}\hat{R}^{-1} = \hat{R}^{-1}\hat{R} = Id$, if and only if $\mathscr{B} = \mathscr{F}$ upto negligible sets. Moreover,*

$\tilde{R}$ is almost surely invertible on W if and only if, $\hat{R}$ is invertible and then $(\hat{R})^*F(w) = F(\tilde{R}^{-1}w) = \hat{R}^{-1}F(w)$ almost surely. Note, however, that in general even if the isometry R is unitary, i.e., on to H, this does not imply that $\mathscr{B} = \mathscr{F}$.

Proposition 4.2 *Suppose that R is also unitary, then $\mathscr{B} = \mathscr{F}$ (up to negligible sets) if and only if Rh and ∇Rh are $\mathscr{B}$-measurable for any $h \in H$.*

Proof. The necessity is evident. To prove the sufficiency, it is enough to show that, for any $h \in W^*$, δh is $\mathscr{B}$-measurable under the hypothesis that Rh and ∇Rh are $\mathscr{B}$-measurable.

We have

$$\begin{aligned} \delta h &= \delta RR^* h \\ &= \delta \left[R \sum_{i=1}^{\infty} (R^* h, e_i) e_i \right] , \end{aligned}$$

where $(e_i, i \in \mathbb{N}) \subset W^*$ is a complete orthonormal basis for H. We have then

$$\delta h = \sum_{i=1}^{\infty} [(h, Re_i)\delta Re_i - \nabla_{Re_i}(h, Re_i)] .$$

δRe_i is $\mathscr{B}$-measurable by definition, Re_i and $\nabla_{Re_i}(h, Re_i)$ are $\mathscr{B}$-measurable by hypothesis and the sum converges in probability. □

Let us denote by $A_{p,k}(\mathscr{X})$, $p > 1$, $k \in \mathbb{Z}$, the Sobolev spaces of $\mathscr{X}$-valued, $\mathscr{B}$-measurable random variables, where $\mathscr{X}$ is any separable Hilbert space. Let us recall that $A_{p,k}(\mathscr{X})$ is defined as the isometric image of $\mathbb{D}_{p,k}(\mathscr{X})$ under $\hat{R}$, or, which is equivalent, as the completion of $\{f(D^* h_1, \ldots, D^* h_n);\ h_i \in H,\ f \in C_0^\infty(\mathbb{R}^n, \mathscr{X}),\ n \in \mathbb{N}\}$ under the norm

$$\|\eta\|_{p,k} = \|(I + A)^{k/2}\eta\|_{L^p(\mu,\mathscr{X})} ,$$

where A is defined as $\hat{R} \circ \mathscr{L} = D^* \circ D \circ \hat{R} = A \circ \hat{R}$.

Lemma 4.1 *$E[\,\cdot\,|\mathscr{B}]$ is a bounded map from $\mathbb{D}_{p,k}$ into $A_{p,k}$, for any $p > 1$.*

Proof. Suppose that $\varphi \in \mathbb{D}_{p,k}$. We have $E[|\varphi|\mathscr{B}] = \hat{R}\hat{R}^* \varphi$. Hence it is sufficient to prove that $\hat{R}^* \varphi$ is in $\mathbb{D}_{p,k}$. Therefore it is sufficient to show that $(I + \mathscr{L})^{k/2} \circ \hat{R}^* \varphi$ is in $L^p(\mu)$. However, we have $(I + \mathscr{L})^{k/2} \circ \hat{R}^* = (\hat{R} \circ (I + \mathscr{L})^{k/2})^*$ and $\hat{R} \circ (I + \mathscr{L})^{k/2}$ is continuous from $\mathbb{D}_{r,l}$ into $A_{r,l-k}$ for any $r > 1$, $l \in \mathbb{Z}$, hence by duality, $(I + \mathscr{L})^{k/2} \circ \hat{R}^*$ is continuous from $A_{r,l}$ into $\mathbb{D}_{r,l-k}$. This shows, in particular, that $(I + \mathscr{L})^{k/2} \circ \hat{R}^* \varphi$ is $L^p(\mu)$. □

In order to be able to consider expressions of the form $\nabla \hat{R} F(w)$ and for other applications we have to consider an additional operation as follows.

Let R be a random rotation as in the preceding section. For the sake of simplicity, we shall suppose, unless the contrary is explicitly mentioned, that $Rh \in \mathbb{D}(H)$ for any $h \in H$. Since the Sobolev derivative ∇ commutes with the deterministic shifts, for any $k \in H$ and $t \in \mathbb{R}$, $w \mapsto R(w + tk)$ defines another rotation such that a.s., $\nabla R(w + tk)h$ is quasi-nilpotent for any $h \in H$. We denote by $\hat{R}_{tk}$ the isometry (on $L^p(\mu)$, $p \geqq 1$) corresponding to it.

If $F = f(\delta h_1, \dots, \delta h_n)$, $h_i \in H$ $f \in C_0^\infty(\mathbb{R})$, define

$$X_k F = \frac{d}{dt} \hat{R}_{tk} F \bigg|_{t=0} .$$

If G is another cylindrical function as F, from Theorem 2.1, we have

$$\begin{aligned} E[F \cdot G] &= E[\hat{R}_{tk}(F \cdot G)] \\ &= E[\hat{R}_{tk}(F) \cdot \hat{R}_{tk}(G)] , \end{aligned}$$

hence

$$E[X_k F \cdot \hat{R} G] + E[\hat{R} F \cdot X_k G] = 0 ,$$

and, in particular, setting $G = 1$ yields

$$E X_k F = 0 .$$

Returning to the definition of X_k, note that

$$\begin{aligned} X_k \delta h &= \delta \nabla_k R h \\ &= \nabla_k \delta R h - (Rh, k) \end{aligned}$$

and for $F = f(\delta h_1, \dots, \delta h_n)$ as above

$$\begin{aligned} X_k F &= \sum_{i=1}^{n} \partial_i f(\delta R h_1, \dots, \delta R h_n) \delta \nabla_k R h_i \\ &= \nabla_k \hat{R} F - \hat{R} \nabla_k F \\ &= \nabla_k \hat{R} F - D_{R^* k} \hat{R} F , \end{aligned}$$

consequently X_k is closable.

Proposition 4.3 *If $F \in \mathbb{D}$ and $\xi \in \mathbb{D}(H)$ are cylindrical Wiener functionals, then*

$$(XF, \xi) = trace((\delta \otimes \nabla R) \cdot (\hat{R} \nabla F \otimes \xi)) .$$

Proof. We have by definition,

$$(XF, \xi) = \sum_{j=1}^{\infty} X_{e_j} F(\xi, e_j) ,$$

where $(e_j,\ j \in \mathbb{N})$ is a complete, orthonormal basis of H. On the other hand, the following identity holds:

$$X_{e_j} = \sum_{i=1}^{\infty} \nabla_{e_i} F \circ \tilde{R} \delta(\nabla_{e_j} R e_i) .$$

Inserting the second one in the first gives the required result. □

Let X^* denote the formal adjoint to X defined for cylindrical functions. For cylindrical $F \in \mathbb{D}$ and $\eta \in \mathbb{D}(H)$, we have

$$\begin{aligned} E[F \cdot X^*\eta] &= E[(XF, \eta)] \\ &= E[(\nabla \hat{R}F - R\hat{R}\nabla F, \eta)] \\ &= E[F \cdot \hat{R}^* \delta\eta] - E[F \cdot \delta\hat{R}^* R^*\eta] , \end{aligned}$$

hence

$$X^*\eta = \hat{R}^* \delta\eta - \delta\hat{R}^* R^*\eta .$$

Proposition 4.4 *For any $p > 1$ and $k \in \mathbb{Z}$, X (respectively X^*) has a continuous extension from $\mathbb{D}_{p,k}$ into $\mathbb{D}_{p,k-1}(H)$ (respectively from $\mathbb{D}_{p,k}(H)$ to $\mathbb{D}_{p,k-1}$) and*

$$E[\delta\eta|\mathscr{B}] = \hat{R}X^*\eta + D^*E[R^*\eta|\mathscr{B}] ,$$

for any $\eta \in \mathbb{D}(H)$.

Proof. Suppose first that η is an H-valued, cylindrical random variable. If ψ is a real-valued cylindrical random variable, we have

$$\begin{aligned} \langle E[\delta\eta|\mathscr{B}], \hat{R}\psi\rangle &= \langle \delta\eta, \hat{R}\psi\rangle \\ &= \langle \eta, \nabla\hat{R}\psi\rangle \\ &= \langle \eta, X\psi + R\hat{R}\nabla\psi\rangle \\ &= \langle X^*\eta, \psi\rangle + \langle R^*\eta, \hat{R}\nabla\psi\rangle \\ &= \langle \hat{R}X^*\eta, \hat{R}\psi\rangle + \langle E[R^*\eta|\mathscr{B}], \hat{R}\nabla\psi\rangle \\ &= \langle \hat{R}X^*\eta, \hat{R}\psi\rangle + \langle E[R^*\eta|\mathscr{B}], D\hat{R}\psi\rangle \\ &= \langle \hat{R}X^*\eta, \hat{R}\psi\rangle + \langle D^*E[R^*\eta|\mathscr{B}], \hat{R}\psi\rangle . \end{aligned}$$

Therefore, for cylindrical η we have the identity

$$E[\delta\eta|\mathscr{B}] = \hat{R}X^*\eta + D^*E[R^*\eta|\mathscr{B}] .$$

Suppose now that (η_k) is a sequence of cylindrical, H-valued random variable, converging to η in $D_{p,k}(H)$, $p > 1$, $k \in \mathbb{Z}$. From Lemma 4.1, $E[\delta\eta_k|\mathscr{B}] \to E[\delta\eta|\mathscr{B}]$ and $D^*E[R^*\eta_k|\mathscr{B}] \to D^*E[R^*\eta|\mathscr{B}]$ in $A_{p,k-1}$. Consequently, $(\hat{R}X^*\eta_k; k \in \mathbb{N})$ is convergent in $A_{p,k-1}$. This implies that it is closed and $\hat{R} \circ X^*$ has a continuous extension from $\mathbb{D}_{p,k}(H)$ into $A_{p,k-1}$, for any $p > 1$, $k \in \mathbb{Z}$, hence X^* has a continuous extension from $\mathscr{D}_{p,k}(H)$ into $\mathbb{D}_{p,k-1}$ and, by duality X has a continuous extension from $\mathbb{D}_{p,k}$ into $\mathbb{D}_{p,k-1}(H)$. □

Corollary 4.2 *For any $h \in H$, we have*

$$E[\delta h|\mathscr{B}] = D^*E[R^*h|\mathscr{B}] .$$

in particular, if Rh is independent of $\mathscr{B}$, then

$$E[\delta h|\mathscr{B}] = D^*(E[R^*]h) .$$

In case $E[R] = 0$, the projection of the first chaos in $L^2(\mu, \mathscr{B})$ is zero.

Proof. We have $E[R^*] = E[R]^*$. □

5 Transformations of measure induced by Euclidean motions of the Wiener path

Let R be as in Theorem 2.1 and $\tilde{w}$ the rotated path. Consider the Euclidean motion (rotation plus shift):

$$v(w) = \tilde{w} + U(w)\,,$$

where U is an H-valued random variable. Let μ^v denote the measure induced by v on the Banach space W. The question arises regarding conditions for the absolute continuity of μ^v with respect to μ and the related Radon–Nikodym derivative. This problem for $\mathscr{B} \mp \mathscr{F}$, is not easy, we consider here the following case. Let $(W_t; t \in [0,1])$ be the standard d-dimensional Brownian motion and let $\mathscr{F}_t$ denote the σ-field generated by $\{W_\theta; \theta \in [0,t]\}$. Let $\{y_t; t \in [0,1]\}$ be a rotated and adapted d-dimensional Brownian motion i.e., we assume that $\{(y_t, \mathscr{F}_t);\ t \in [0,1]\}$ is a Wiener process (and therefore R plays no role in the assumptions). Set:

$$v_t = y_t + \int_0^t u_s(w)\,ds\,, \tag{5.1}$$

where the process $(u_s;\ s \in [0,1])$ takes values in $\mathbb{R}^d$, has measurable sample paths and is adapted to the filtration $(\mathscr{F}_t;\ t \in [0,1])$.

As the example $y_s = \int_0^s \operatorname{sign} w_\theta\ dw_\theta$ shows it is possible that the σ-field generated by $\{y_s,\ 0 \leqq s \leqq t\}$ is strictly smaller than the one generated by $\{W_s,\ 0 \leqq s \leqq t\}$ and then the problem of the absolute continuity of the measure on function space induced by v. With respect to the measure induced by w. (or y.) is not covered by previously known results.

Proposition 5.1 *Let W_t, y_t and u_t be as defined above. Assume that $E\int_0^1 |u_s|ds < \infty$. Let $\mathscr{F}_t^v$ denote the σ-field induced by $\{v_\theta,\ 0 \leqq \theta \leqq t\}$, set*

$$\hat{u}_s(v) = E\{u_s | \mathscr{F}_s^v\}\,.$$

Note that since $\hat{u}$ is a measurable function on $C_0([0,1])$ we can also consider $\hat{u}(w)$.

(a) *$\mu^v \ll \mu$ if and only if $\int_0^1 |\hat{u}_s(v)|^2 ds < \infty$ a.s.*

(b) *Assuming that $\hat{u}_s$ is ds-square integrable on the v-trajectories, i.e., $\int_0^1 |\hat{u}_s(v)|^2 ds < \infty$ a.s., set*

$$G = \{w : \int_0^1 |\hat{u}_s(w)|^2\ ds < \infty\}\,,$$

then

$$\frac{d\mu^v}{d\mu}(\eta) = \begin{cases} \exp \int_0^1 \hat{u}_s(\eta)\ d\eta_s - \frac{1}{2}\int_0^1 |\hat{u}_s(\eta)|^2\ ds\,, & \eta \in G\,, \\ 0, & \eta \in G^c \end{cases}$$

and $\mu^v \sim \mu$ if and only if $\mu(G) = 1$.

Proof. Rewrite (5.1) as

$$\begin{aligned} v_t &= y_t + \int_0^t (u_s - \hat{u}_s)\,ds + \int_0^t \hat{u}_s\,ds \\ &= \alpha_t + \int_0^t \hat{u}_s\,ds\,, \end{aligned}$$

where we have denoted $y_t + \int_0^t (u_s - \hat{u}_s)\,ds$ by α_t. Note that since $\hat{u}_t$ is $\mathscr{F}_t^v$ adapted, so is α_t. Next we are going to show that α_t is the innovation process for v_t i.e., α_t is Wiener on the $\mathscr{F}_t^v$ filtration (this argument goes back to Kailath, Frost, Shiryayev and Kallianpur, cf. [3] and the references therein). To show that α_t is Wiener on $\mathscr{F}_t^v$, note that

$$\begin{aligned} E(\alpha_{t+\tau} - \alpha_t | \mathscr{F}_t^v) &= E(y_{t+\tau} - y_t | \mathscr{F}_t^v) + \int_t^{t+\tau} E(u_s - \hat{u}_s | \mathscr{F}_t^v)\,ds \\ &= E\{E(y_{t+\tau} - y_t | \mathscr{F}_t) | \mathscr{F}_t^v\} + 0 \\ &= 0\,. \end{aligned}$$

Consequently, α_t is an $\mathscr{F}_t^v$ martingale whose increasing process is the same one as that of y, hence by Lévy's Theorem α is a Brownian motion. The problem reduces therefore to

$$v_t = \alpha_t + \int_0^t \hat{u}_s\,ds\,,$$

where both α_t and $\hat{u}_s$ are $\mathscr{F}_s^v$ adapted, and the results of the proposition follow now by standard arguments (cf. e.g., Theorem 2 of [4]). □

We shall now give a version of this result in the setting of abstract Wiener space without any requirement of non-anticipation and some consequences of it.

Lemma 5.1 *Let $u \in I_p(R)$ $(p > 1)$, where R is a random isometry of H satisfying one of the two hypothesis of the second section. Suppose moreover that $u \in \mathbb{D}_{p,1}(H)$, $w \mapsto \tau_u(w) = w + u(w)$ is almost surely invertible, with the inverse shift $\tau_v(w) = w + v(w)$ such that $v \in \mathbb{D}_{r,1}(H)$ for some $r > 1$. Assume that the shift τ_u satisfies the Girsanov identity*

$$E\left[F \circ \tau_u \cdot \exp - \delta u - \frac{1}{2}|u|^2\right] = E[F]\,,$$

for any (smooth) cylindrical function F on W. We have then

$$(\delta(R \circ \tau_v)h) \circ \tau_u = \delta Rh + (Rh, u)\,,$$

μ-almost surely for any $h \in H$.

Proof. We have (cf. [6])

$$[\delta(R \circ \tau_v)h] \circ \tau_u = \delta Rh + (Rh, u) + trace(\nabla u \cdot (\nabla(R \circ \tau_v)h) \circ \tau_u)\,.$$

Since $\nabla[(R\circ\tau_v)h] = (I+\nabla v)(\nabla R\circ\tau_v)h$, we have

$$\begin{aligned}\nabla[(R\circ\tau_v)h]\circ\tau_u &= (I+\nabla v)\nabla Rh\\ &= (I+\nabla u)^{-1}\nabla Rh\,.\end{aligned}$$

Since $u\in I_p(R)$, $\mathrm{trace}(\nabla u\cdot(I+\nabla u)^{-1}\nabla Rh)=0$, hence the lemma follows. □

Lemma 5.2 *Suppose that the hypothesis of Lemma 5.1 hold and assume that ∇v has an essentially bounded Hilbert–Schmidt norm. We then have*

$$E\left[F(\tilde{R}w+R^*u(w))\exp-\delta u-\frac{1}{2}|u|^2\right]=E[F]\,,$$

for any cylindrical function F on W.

Proof. Let $(e_i)\subset W^*$ be a complete, orthonormal basis of H. We have

$$\tilde{R}w=\sum_{i=1}^{\infty}(\delta Re_i)(w)e_i$$

for almost all $w\in W$. If $F=f(\delta e_1,\dots,\delta e_n)$, then, from Lemma 5.1, it follows that

$$\begin{aligned}F(\tilde{R}w+R^*u(w)) &= f(\delta e_1(\tilde{R}w+R^*u(w)),\dots,\delta e_n(\tilde{R}w+R^*u(w)))\\ &= f(\delta Re_1+(Re_1,u)+\dots,\delta Re_n+(Re_n,u))\\ &= f(\delta((R\circ\tau_v)e_1),\dots,\delta((R\circ\tau_v)e_n)))\circ\tau_n\,.\end{aligned}$$

From the Girsanov identity, we obtain

$$\begin{aligned}&E\left[F(\tilde{R}w+R^*u(w))\exp-\delta u-\frac{1}{2}|u|^2\right]\\ &\quad=E[f(\delta((R\circ\tau_v)e_1),\dots,\delta((R\circ\tau_v)e_n))]\,.\end{aligned}$$

Now remark that

$$\nabla(R\circ\tau_v)h=(I+\nabla v)\nabla Rh\circ\tau_v\,,$$

hence

$$\begin{aligned}(\nabla(R\circ\tau_v)h)\circ\tau_u &= (I+\nabla v)\circ\tau_u\nabla Rh\\ &= (I+\nabla v)^{-1}\nabla Rh\,.\end{aligned}$$

Consequently, $(\nabla(R\circ\tau_v)h)\circ\tau_u$ is a quasi-nilpotent operator, hence so is $\nabla(R\circ\tau_v)h$. Therefore we can apply Theorem 2.1 to obtain

$$\begin{aligned}E[f(\delta((R\circ\tau_v)e_1),\dots,\delta((R\circ\tau_v)e_n))] &= E[f(\delta e_1,\dots,\delta e_n)]\\ &= E[F]\,.\qquad\square\end{aligned}$$

Theorem 5.1 *Suppose that R is a rotation as before and moreover that ∇Rh has an essentially bounded Hilbert–Schmidt norm, for any $h \in H$. Then we have*

$$E[(\nabla F(\tilde{R}w), R^* u)] = E[\delta u \cdot F(\tilde{R}w)] ,$$

for any smooth cylindrical function F and $u \in I_p(R)$ $(p > 1)$.

Proof. Let (u_n) be a sequence from $\mathring{I}_p(R)$ such that $u_n \mapsto u$ in $L^p(\mu, H)$. By definition,

$$\nabla u_n = \sum_{i=1}^{k} \alpha_i \nabla R h_i ,$$

with $h_i \in H$ and $\alpha_i \in L^\infty(\mu)$, consequently the map $\tau_{\lambda u_n}(w) = w + \lambda u_n(w)$ satisfies all the hypothesis of this section for small $\lambda > 0$ and from Lemma 5.2, we obtain

$$E\left[F(\tilde{R}w + \lambda R^* u_n(w)) \exp -\lambda \delta u_n - \frac{\lambda^2}{2}|u_n|^2\right] = E[F] .$$

Differentiating both sides, we obtain the claimed identity for u_n, then we can pass to the limit. □

Corollary. *Under the hypothesis of the theorem, we have*

$$X^* u = 0 .$$

for any $u \in I_p(R)$, $p > 1$.

Proof. From the theorem, we have

$$E[(\hat{R}\nabla F, R^* u)] = E[\hat{R}F \cdot \delta u] .$$

Moreover

$$\begin{aligned} E[\hat{R}F \cdot \delta u] &= E[(\nabla \hat{R}F, u)] \\ &= E[(u, XF + R\hat{R}\nabla F)] \\ &= E[X^* u \cdot F] + E[(\hat{R}\nabla F, R^* u)] . \end{aligned}$$

Hence $E[X^* u \cdot F] = 0$ for any cylindrical function F. □

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